Growing Graceful Trees

Edinah K. Gnang [*], Isaac Wass [†]

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Abstract

We describe constructions based on Gaussian elimination for listing and enumerating special induced edge label sequences of graphs. Our enumeration construction settles in the affirmative a conjecture raised by Whitty in [W08] on the existence of matrix constructions whose determinant enumerate gracefully labeled trees. We also describe and algorithm for obtaining all graceful labelings of a given graphs. We conclude the paper with a conjugation algorithm which determines the set of graphs on n vertices having no isolated vertices which admit no graceful labeling.

1 Introduction

The Kotzig-Ringel conjecture, also known as the *Graceful Labeling Conjecture* (or GLC for short) [AlS03, Gal05] asserts that every tree admits a graceful labeling. A labeling is graceful if it injectively assigns to vertices of a graph an equal number of distinct consecutive integers so as to also induce an injective map from edges to integers. In the induced edge labeling, each edge is mapped to the absolute difference of its spanning vertex labels. For convenience we reformulate the GLC in terms of functions in $\{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ which have a fixed point which is attractive over the domain. We compactly express the attractive fixed point condition for a given $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$, via the size of the image by $f^{(n-1)}$ of $\{0, \cdots, n-1\}$ as follows

$$\left| f^{(n-1)} \left(\{0, \cdots, n-1\} \right) \right| = 1, \tag{1}$$

where

$$\forall\, i \in \{0, \cdots, n-1\}\,,\ f^{(0)}\left(i\right) := i,\ \text{and}\ \forall\, k \geq 0,\ f^{(k+1)}\left(i\right) = f^{(k)}\left(f\left(i\right)\right) = f\left(f^{(k)}\left(i\right)\right).$$

To an arbitrary $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ we associate a *functional directed graph*

$$G_f := \left(V = \{0, \cdots, n-1\}\,,\ E = \{(i,\ f\left(i\right))\}_{0 \leq i < n}\right).$$

The GLC is equivalent to the assertion that for any function f subject to the attractive fixed point condition (1), there exist at a least one choice of fixed permutations σ and γ of integers in the domain of f such that

$$f\left(i\right) \in \sigma^{-1}\left(\sigma\left(i\right) \pm \gamma\left(\sigma\left(i\right)\right)\right), \quad \forall\ 0 \leq i < n. \tag{2}$$

More generally, the functional directed graph G_g associated with $g \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ is graceful if there exist fixed permutations σ and γ of integers in the domain of g such that

$$g\left(i\right) \in \sigma^{-1}\left(\sigma\left(i\right) \pm \gamma\left(\sigma\left(i\right)\right)\right), \quad \forall\ 0 \leq i < n.$$

[*]Department of Applied Mathematics and Statistics, Johns Hopkins, egnang1@jhu.edu
[†]Department of Mathematics, Iowa State University icwass@iastate.edu

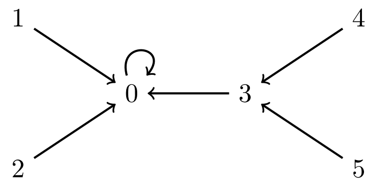

Figure 1: A functional directed graph on 6 vertices.

We denote by $\mathrm{GrL}(G_g)$ the set of distinct graceful labelings of the graph G_g. Throughout this discussion, the induced edge label sequence of a graph refers to the non-decreasing sequence of edge labels obtained by taking absolute differences of vertex labels spanned by each edge. For instance the function in Figure 1

$$f\,:\,\{0,1,2,3,4,5\}\rightarrow\{0,1,2,3,4,5\}$$

defined by

$$f\,(0)=0,\;f\,(1)=0,\;f\,(2)=0,\;f\,(3)=0,\;f\,(4)=3,\;f\,(5)=3,$$

is a functional spanning subtree of the complete graph on 6 vertices (or functional tree for short). The attractive fixed point condition (1) is met since $f^{(5)}\left(\{0,1,2,3,4,5\}\right)=\{0\}$. The edge set of G_f is $\{(0,0)\,,\,(1,0)\,,\,(2,0)\,,\,(3,0)\,,\,(4,3)\,,\,(5,3)\}$ and the corresponding induced edge label sequence is $(0,1,1,2,2,3)$.

The GLC is easily verified for the families of star and path functional trees respectively illustrated by

$$f,\,g\,:\,\{0,\cdots,n-1\}\rightarrow\{0,\cdots,n-1\}\,,$$

$$\forall\,0\le i<n,\quad f\,(i)=0\;\text{ and }\;g\,(i)=\begin{cases}0 & \text{if } i=0\\ i-1 & \text{otherwise}\end{cases}.$$

In particular $\left|\mathrm{GrL}\left(G_f\right)\right|=\left\lfloor\frac{n}{2}\right\rfloor+\left(n-2\left\lfloor\frac{n}{2}\right\rfloor\right)$. Our main results are constructions based on Gaussian elimination for listing and enumerating induced edge label sequences of graphs. Our enumeration construction settles in the affirmative a conjecture raised by Whitty in [W08] on the existence of matrix constructions whose determinant enumerate gracefully labeled trees. We also describe an algorithm for obtaining all graceful labelings of a given graph. We conclude the paper with a conjugation algorithm which determines the set of graphs on n vertices having no isolated vertices which admit no graceful labeling.

This article is accompanied by an extensive SageMath[S18] graceful graph package which implements the symbolic constructions described here. The package is made available at the link:
`https://github.com/gnang/Graceful-Graphs-Package`

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2 Counting gracefully labeled functional digraphs.

There are $n!$ gracefully labeled undirected graphs on n vertices with n edges. The probability that an undirected graph on n vertices with n edges chosen uniformly at random, is gracefully labeled is

$$\frac{n!}{n\binom{\binom{n}{2}}{n-1}}=\frac{(n-1)!}{\binom{\binom{n}{2}}{n-1}}.$$

The enumeration of gracefully labeled functional directed graph is more difficult. The trivial upper bound of $n!2^{n-1}$ follows from the count of undirected gracefully labeled graphs. The permutation characterization in (2) permits an improvement to this upper bound while simultaneously providing a lower bound. Recall that a functional directed graph G_f associated with $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$, is graceful if there exists fixed permutations $\sigma \in {}^{\mathrm{S}_n}/_{\mathrm{Aut}G_f}$ and γ such that

$$f(i) \in \sigma^{-1}(\sigma(i) \pm \gamma(\sigma(i))), \quad \forall\, 0 \le i < n,$$

$$\implies \sigma f \sigma^{-1}(i) \in i \pm \gamma(i), \quad \forall\, 0 \le i < n.$$

Consequently, G_f is gracefully labeled if σ is the identity permutation noted $\sigma \in \mathrm{Aut}G_f$. Assume that $f(0) = 0$, to ensure that the gracefully labeled graph G_f has no isolated vertices. The permutation γ must be chosen such that

$$\forall\, 0 < i < n, \quad |\{i \pm \gamma(i)\} \cap \{0, \cdots, n-1\}| > 0 \Rightarrow \begin{cases} \gamma(i) \le i \\ \text{or} \\ \gamma(i) < n - i \end{cases} \forall\, 0 < i < n\,. \tag{3}$$

Alternatively, the function f associated with a functional directed graph G_f is gracefully labeled iff

$$f(i) = i + \mathfrak{s}(i)\,\gamma(i), \quad \forall\, 0 < i < n,$$

where $\mathfrak{s} \in \{-1, 1\}^{\{0,\cdots,n-1\}}$. The following proposition follows from the permutation criterion (3).

Proposition 1 : The number of permutations of labels $\{0, \cdots, n-1\}$ subject to the restriction (3) for $n > 2$ is given by

$$|\{\gamma \in \mathrm{S}_n \text{ such that } \gamma(0) = 0 \text{ and } \forall\, 0 < i < n, \text{ either } \gamma(i) < n - i \text{ or } \gamma(i) \le i\}| = \left(\left\lfloor \frac{n-1}{2} \right\rfloor !\right) \cdot \left(\left\lceil \frac{n-1}{2} \right\rceil !\right).$$

The corresponding sequence appears in the OEIS database as [A010551].

Proof : The proof follows by observing that for such a permutation γ there is a single choice for the pre-image of $(n-1)$ and this choice is determined by

$$\gamma(n-1) = n - 1 \Rightarrow \mathfrak{s}(n-1) = -1.$$

Following this choice there are two possible choices for the pre-image of $(n-2)$ as determined by

$$(\gamma(1) = n - 2 \;\Rightarrow \mathfrak{s}(1) = 1) \;\text{ or }\; (\gamma(n-2) = n - 2 \;\Rightarrow \mathfrak{s}(n-2) = -1)\,. \tag{4}$$

Following the first two choices, there will be three remaining choices for the pre-image of $(n-3)$. The possible choices for the pre-image of $(n-3)$ (not accounting for the pre-image choices already made for $(n-1)$ and $(n-2)$) are determined by

$$(\gamma(1) = n - 3 \;\Rightarrow \mathfrak{s}(1) = 1)$$

or

$$(\gamma(2) = n - 3 \;\Rightarrow \mathfrak{s}(2) = 1)$$

or

$$(\gamma(n-3) = n - 3 \;\Rightarrow \mathfrak{s}(n-3) = -1)$$

or

$$(\gamma(n-2) = (n-3) \Rightarrow \mathfrak{s}(n-2) = -1).$$

Similarly following these three choices there will be four remaining choices for the pre-image of $(n-4)$. All the possible choices (not accounting for the pre-images choices already made for $(n-1)$, $(n-2)$ and $(n-3)$) are determined by

$$(\gamma(1) = n-4 \Rightarrow \mathfrak{s}(1) = 1)$$

or

$$(\gamma(2) = n-4 \Rightarrow \mathfrak{s}(2) = 1)$$

or

$$(\gamma(3) = n-4 \Rightarrow \mathfrak{s}(3) = 1)$$

or

$$(\gamma(n-4) = (n-4) \Rightarrow \mathfrak{s}(n-4) = -1)$$

or

$$(\gamma(n-3) = (n-4) \Rightarrow \mathfrak{s}(n-3) = -1)$$

or

$$(\gamma(n-2) = (n-4) \Rightarrow \mathfrak{s}(n-2) = -1)$$

The argument proceeds similarly all the way up to the choices for the pre-images of $\left\lceil \frac{n-1}{2} \right\rceil$. These possible pre-image assignments account for the factorial factor $\left\lfloor \frac{n-1}{2} \right\rfloor!$. Note that for each one of these pre-image choices, the output of the sign function $\mathfrak{s}$ is uniquely determined. Finally, the remaining factorial factor arises from taking all possible permutations of the remaining integers thus completing the proof.$\square$

Proposition 1 yields the following upper and lower bounds for the number of gracefully labeled directed functional graphs on n vertices having no isolated vertices.

$$\left(\left\lfloor \frac{n-1}{2} \right\rfloor!\right)\left(\left\lceil \frac{n-1}{2} \right\rceil!\right) 2 \le \left|\left\{ f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \text{ s.t. } \begin{array}{c} \{0,\cdots,n-1\} = \{|f(i)-i|\}_{0\le i<n} \\ 1 = \left|\{f(i)=i\}_{0\le i<n}\right| \end{array} \right\}\right|$$

and

$$\left|\left\{ f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \text{ s.t. } \begin{array}{c} \{0,\cdots,n-1\} = \{|f(i)-i|\}_{0\le i<n} \\ 1 = \left|\{f(i)=i\}_{0\le i<n}\right| \end{array} \right\}\right| \le \left(\left\lfloor \frac{n-1}{2} \right\rfloor!\right)\left(\left\lceil \frac{n-1}{2} \right\rceil!\right) n\, 2^{\left\lceil \frac{n-1}{2} \right\rceil}.$$

The extra factor of n in the upper bound accounts for alternative placements of the loop edge. The proof argument of Proposition 1 describes an optimal algorithm for constructing the set of signed permutations noted SP_n (used to construct gracefully labeled functional directed graphs having no isolated vertices) defined by

$$\mathrm{SP}_n :=$$

$$\left\{ g \text{ s.t. } \forall\, i \in \{0,\cdots,n-1\},\ 0 \le (i+g(i)) = (i+\mathfrak{s}(i)\,\gamma(i)) < n, \text{ for some } \mathfrak{s} \in \{-1,1\}^{\{0,\cdots,n-1\}} \text{ and } \gamma \in \mathrm{S}_n \begin{array}{c} \gamma \in \mathrm{S}_n \\ \gamma(0) = 0 \end{array} \right\}. \tag{5}$$

A determinantal sieve construction follows from the directed Matrix Tree Theorem [Z85]. The edge set for distinct gracefully labeled functional trees rooted at 0 make up distinct monomial terms of the polynomial identity below

$$\left(\sum_{\substack{f^{(n-1)}\left(\{0,\cdots,n-1\}\right)=\{0\}\\ \left\{|f(i)-i|\right\}_{0\le i<n}=\{0,\cdots,n-1\}}}\prod_{0\le i<n}\mathbf{A}\left[i,f\left(i\right)\right]\right)=$$

$$\sum_{g\in\mathrm{SP}_n}\mathbf{A}\left[0,0\right]\det\left\{\left(\mathrm{diag}\left(\sum_{0\le i<n}\mathbf{A}\left[i,i+g\left(i\right)\right]\mathbf{I}\left[:,i\right]\mathbf{I}\left[i+g\left(i\right),:\right]\mathbf{1}_{n\times 1}\right)-\sum_{0\le i<n}\mathbf{A}\left[i,i+g\left(i\right)\right]\mathbf{I}\left[:,i\right]\mathbf{I}\left[i+g\left(i\right),:\right]\right)\left[1:,1:\right]\right\}.$$

We are using in the expression above the colon notation. Recall that in the colon notation $\mathbf{A}\left[1:,1:\right]$ refers to the $(n-1)\times(n-1)$ matrix obtained by deleting row 0 and column 0 of the matrix. Similarly $\mathbf{I}\left[:,i\right]$ denotes the i-th column of the identity matrix and $\mathbf{A}$ denotes a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{A}\left[i,j\right]=a_{ij},\quad \forall\ 0\le i,j<n.$$

3 Generatingfunctionology of induced edge labelings

We derive here various generating functions whose coefficient enumerate special functional directed graphs which have a given induced edge label sequence. Our first construction is obtained by computing a permanent as showcased by the following proposition

Proposition 2 : Let $\mathbf{A}$ denote a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges. Let $\mathcal{B}\left(\mathbf{A}\right)$ denotes a symbolic $n^2\times n^2$ matrix derived from $\mathbf{A}$ with entries given by

$$\mathcal{B}\left(\mathbf{A}\right)\left[n\cdot u+v,\,n\cdot i+j\right]=\begin{cases}\mathbf{A}\left[u,v\right] & \text{if}\quad u=n\cdot i+j\\ 0 & \text{if}\quad n\cdot i+j\ne u\text{ and }n\cdot i+j<n,\\ 1 & \text{otherwise}\end{cases}\quad\forall\ \begin{array}{c}0\le u,v<n\\ 0\le i,j<n\end{array},$$

then

$$\frac{\mathrm{Per}\left\{\mathcal{B}\left(\mathbf{A}\right)\right\}}{\left(n^2-n\right)!}=\prod_{0\le i<n}\left(\sum_{0\le j<n}\mathbf{A}\left[i,j\right]\right)=\left(\sum_{f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}}\prod_{0\le i<n}\mathbf{A}\left[i,f\left(i\right)\right]\right).\tag{6}$$

Proof : Recall that

$$\mathrm{Per}\left\{\mathcal{B}\left(\mathbf{A}\right)\right\}=\sum_{\sigma\in\mathrm{S}_{n^2}}\prod_{0\le i,j<n}\mathcal{B}\left(\mathbf{A}\right)\left[n\cdot i+j,\,\sigma\left(n\cdot i+j\right)\right].$$

The only terms which contribute to the sum are terms where exactly one entry of each row $\mathbf{A}$ is selected. This ensures that the edges in the subgraph which make up each monomial term are such that each vertex has out degree one. The $\left(n^2-n\right)!$ factor follows from the $n^2\times\left(n^2-n\right)$ matrix block whose entries are equal to 1, thus completing the proof.$\square$

For example, in the case $n=3$ we have

$$\mathbf{A}=\left(\begin{array}{ccc} a_{00} & a_{01} & a_{02}\\ a_{10} & a_{11} & a_{12}\\ a_{20} & a_{21} & a_{22} \end{array}\right),\quad \mathcal{B}\left(\mathbf{A}\right)=\left(\begin{array}{ccccccccc} a_{00} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ a_{01} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ a_{02} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & a_{10} & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & a_{11} & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & a_{12} & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & a_{20} & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & a_{21} & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & a_{22} & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right),$$

and

$$\frac{\text{Per}\left\{\mathcal{B}\left(\mathbf{A}\right)\right\}}{(n^2-n)!}=\left(a_{00}+a_{01}+a_{02}\right)\left(a_{10}+a_{11}+a_{12}\right)\left(a_{20}+a_{21}+a_{22}\right).$$

We devise from the factored form in (6) an efficient algorithm for expressing a generating function whose coefficients enumerate the number of distinct functional directed graphs which have the same given induced edge label sequence.

Corollary 3 : Let $\mathbf{X}$ denote a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{X}\left[i,j\right]=x^{\left((n+1)^{|j-i|}\right)},\quad \forall\, 0\le i,j<n$$

then the polynomial $\frac{\text{Per}\{\mathcal{B}(\mathbf{X})\}}{(n^2-n)!}$ in the variable x corresponds to the generating function whose coefficients enumerate the number of distinct functional directed graphs on n vertices which have the same given induced edge label sequence.

Proof : It follows from Proposition 2 that

$$\frac{\text{Per}\left\{\mathcal{B}\left(\mathbf{X}\right)\right\}}{(n^2-n)!}=\prod_{0\le i<n}\left(\sum_{0\le j<n} x^{(n+1)^{|j-i|}}\right)=\left(\sum_{f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}}\prod_{0\le i<n} x^{(n+1)^{|f(i)-i|}}\right).$$

Consequently, the coefficient of the monomial $\prod_{0<j\le n} x^{(n+1)^{n-j}b_{n-j}}$ enumerates the number of distinct functional directed graphs whose induced edge label sequence is such that b_{n-j} of its n edges are labeled $(n-j)$.□

Note that for the monomial $\prod_{0<j\le n} x^{(n+1)^{n-j}b_{n-j}}$ to have non-vanishing coefficient in the generating function $\frac{\text{Per}\{\mathcal{B}(\mathbf{X})\}}{(n^2-n)!}$ it is necessary that

$$n=\left(\sum_{0<j\le n} b_{n-j}\right)\ \text{ and }\ 0\le b_{n-j}\le\min\left\{2j,\,n\right\},\quad \forall\, 0<j\le n.$$

Therefore, to upper bound the number of terms with non-vanishing coefficient in the generating function it suffices to count the number of non negative integer solutions to the constraints

$$n=\sum_{0<j\le n} b_{n-j}\ \text{ subject to }\ \left\{\begin{array}{cc} 0\le b_{n-j}\le 2j & \text{if } 0<j<\left\lceil\frac{n}{2}\right\rceil\\ b_{n-j}\ge 0 & \text{otherwise}\end{array}\right.,$$

Recall that there are $\binom{2n-1}{n}$ non negative integer solutions to the constraints

$$n=\left(\sum_{0<j\le n} b_{n-j}\right),\text{ such that }\; b_{n-j}\ge 0\;\forall\, 0<j\le n.$$

We can somewhat account for the condition $0 \le b_{n-j} \le 2j$ for all $0 < j < \left\lceil \frac{n}{2} \right\rceil$ by subtracting from the previous count the number of non negative integer solutions to the constraints

$$n=\sum_{0<j\le n} b_{n-j},\;\text{ subject to }\;\begin{cases} b_{n-j}>2j & \text{if } 0<j<\left\lceil\frac{n}{2}\right\rceil \\ b_{n-j}\ge 0 & \text{otherwise}\end{cases}.$$

Recall that the number of non negative integer solutions to the latter set of constraints is $\binom{n+\lfloor\frac{n}{2}\rfloor}{n}$. It therefore follows that the number of non vanishing terms in the generating function is upper bounded by $\binom{2n-1}{n}-\binom{n+\lfloor\frac{n}{2}\rfloor}{n}$. This yields the asymptotic upper bound of $O\left(\frac{4^n}{\sqrt{n}}\right)$ for the number of non vanishing terms. The generating function is thus efficiently determined via Lagrange interpolation using up to $O\left(\frac{4^n}{\sqrt{n}}\right)$ evaluations of the variable x.

The term of lowest degree in the generating function is associated with the identity function specified by

$$f(i)=i\;\forall\, 0\le i<n,$$

and the corresponding term is $x^{(n+1)^0\, n}$. On the other hand, the terms of largest degree is

$$\prod_{\lfloor\frac{n-1}{2}\rfloor\le i<n} x^{2(n+1)^i}\text{ if }\;(n-1)\equiv 1 \mod 2$$

or

$$x^{(n+1)^{\frac{n-1}{2}}}\prod_{\lfloor\frac{n-1}{2}\rfloor\le i<n} x^{2(n+1)^i}\text{ if } n-1\equiv 0 \mod 2$$

When n is even the corresponding functional directed graph is associated with the function

$$f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}\;\text{ such that }\; f(i)=\begin{cases} n-1 & \text{if } 0\le i\le\left\lfloor\frac{n-1}{2}\right\rfloor \\ 0 & \text{if } \left\lceil\frac{n-1}{2}\right\rceil\le i<n\end{cases}.$$

When n is odd there are two such functional directed graphs respectively associated with the functions by

$$f,\,g\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$$

such that

$$f(i)=\begin{cases} n-1 & \text{if } 0\le i<\frac{n-1}{2} \\ 0 & \text{if } \frac{n-1}{2}\le i<n\end{cases}\quad\text{and } g(i)=\begin{cases} n-1 & \text{if } 0\le i\le\frac{n-1}{2} \\ 0 & \text{if } \frac{n-1}{2}<i<n\end{cases}.$$

We summarize these observations by stating that for the monomial $\prod_{0<j\le n} x^{(n+1)^{n-j}b_{n-j}}$ to have non-vanishing coefficient in the generating function it is necessary that

$$n=\sum_{0<j\le n} b_{n-j},$$

$$0 \leq b_{n-j} \leq \min\left\{2j,\, n\right\}, \quad \forall\, 0 < j \leq n,$$

and

$$n \leq \sum_{0<j\leq n} (n+1)^{n-j}\, b_{n-j} \leq \begin{cases} 2\left(\sum_{\lfloor \frac{n-1}{2} \rfloor \leq i < n} (n+1)^{i}\right) & \text{if } (n-1) \equiv 1 \mod 2 \\ (n+1)^{\frac{n-1}{2}} + 2\left(\sum_{\lfloor \frac{n-1}{2} \rfloor \leq i < n} (n+1)^{i}\right) & \text{if } n-1 \equiv 0 \mod 2 \end{cases}$$

The next proposition refines the construction to obtain instead the generating function whose coefficients enumerate the number of distinct functional trees which have the same given induced edge label sequence. We will use here a variant of Gantmacher's notation where the matrix

$$\mathbf{M}\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]$$

denotes the sub-matrix obtained by deleting the i-th row and i-th column of the $n \times n$ matrix $\mathbf{M}$

Proposition 4 : Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{X}\left[i, j\right] = x^{\left(n^{|j-i|}\right)}, \quad \forall\, 0 \leq i, j < n,$$

then the polynomial

$$\sum_{0\leq i<n} \mathbf{X}\left[i,i\right] \det\left\{\left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n\times 1}\right) - \mathbf{X}\right)\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]\right\},$$

in the variable x corresponds to the generating function whose coefficients enumerate the number of distinct functional trees on n vertices having the same given induced edge label sequence.

Proof : The proof easily follows from Tutte's directed Matrix Tree Theorem [Z85] which asserts that

$$\sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right|=1} \prod_{0\leq i<n} \mathbf{A}\left[i, f\left(i\right)\right] = \sum_{0\leq i<n} \mathbf{A}\left[i,i\right] \det\left\{\left(\operatorname{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right)\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]\right\},$$

where the matrix

$$\left(\operatorname{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right)\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]$$

denotes the i-th principal sub-matrix (the matrix resulting from the removal of the i-th row and i-th column) of the directed Laplacian matrix $\operatorname{diag}\left(\mathbf{A}\cdot\mathbf{1}_{n\times 1}\right) - \mathbf{A}$. The desired result follows by substituting the symbolic matrix $\mathbf{A}$ by the symbolic matrix $\mathbf{X}$. Hence

$$\sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right|=1} \prod_{0\leq i<n} x^{n^{|f(i)-i|}} = x \sum_{0\leq i<n} \det\left\{\left(\operatorname{diag}\left(\mathbf{X}\cdot\mathbf{1}_{n\times 1}\right) - \mathbf{X}\right)\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]\right\}.$$

thus completing the proof□.

The lowest degree terms in the new generating function is associated with paths where every non loop edge of the functional tree is labeled with integer 1. The corresponding monomial is $x^{1\,n^{0}+(n-1)\,n^{1}}$. On the other hand, terms of largest degree are associated with double star functional trees such as the functions illustrated below

When n is even an example of functional double star functional tree is associated with

$$f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}\;\text{ such that }\; f(i)=\begin{cases} n-1 & \text{if } 0\le i\le\left\lfloor\frac{n-1}{2}\right\rfloor \;\text{ or }\; i=n-1\\ 0 & \text{if } \left\lceil\frac{n-1}{2}\right\rceil\le i<n-1\end{cases}.$$

When n is odd an example of functional double star tree is associated with

$$f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$$

$$\text{such that}$$

$$f(i)=\begin{cases} n-1 & \text{if } 0\le i<\frac{n-1}{2} \;\text{ or }\; i=n-1\\ 0 & \text{if } \frac{n-1}{2}\le i<n-1\end{cases}$$

The corresponding terms are respectively

$$x^{n^0}x^{n^{n-1}}\prod_{\left\lfloor\frac{n-1}{2}\right\rfloor\le i<n-1}x^{2\,n^i}\;\text{if }\;(n-1)\equiv 1\mod 2$$

$$\text{or}$$

$$x^{n^0}x^{n^{n-1}}x^{n^{\frac{n-1}{2}}}\prod_{\frac{n-1}{2}\le i<n-1}x^{2\,n^i}\;\text{if }\; n-1\equiv 0\mod 2$$

In summary, for the monomial $\prod_{0<j\le n}x^{b_{n-j}n^{n-j}}$ to have non-vanishing coefficient in the new generating function it is necessary that

$$n=\sum_{0<j\le n}b_{n-j},$$

$$0\le b_{n-j}\le\min\{2j,\,n\},\quad\forall\,0<j\le n$$

$$\text{and}$$

$$1+(n-1)\,n\le\sum_{0<j\le n}b_{n-j}n^{n-j}\le\begin{cases} n^0+n^{(n-1)}+2\left(\sum_{\left\lfloor\frac{n-1}{2}\right\rfloor\le i<n-1}n^i\right) & \text{if }\;(n-1)\equiv 1\mod 2\\ n^0+n^{\left(\frac{n-1}{2}\right)}+n^{(n-1)}+2\left(\sum_{\frac{n-1}{2}\le i<n}n^i\right) & \text{if }\; n-1\equiv 0\mod 2\end{cases}.$$

The new generating function is also efficiently determined via Lagrange interpolation using at most $O\left(\frac{4^n}{\sqrt{n}}\right)$ evaluations of x. By subtracting the polynomial resulting from the second generating function construction from the polynomial associated with the first generating function construction we obtain the generating function

$$\sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right|>1}\prod_{0\le i<n}x^{(n+1)^{|f(i)-i|}}=\frac{\text{Per}\{\mathcal{B}(\mathbf{X})\}}{(n^2-n)!}-\sum_{0\le i<n}x\det\left\{(\text{diag}(\mathbf{X}\cdot\mathbf{1}_{n\times1})-\mathbf{X})\begin{bmatrix}0,\cdots,i-1,i+1,\cdots,n-1\\0,\cdots,i-1,i+1,\cdots,n-1\end{bmatrix}\right\}.$$

whose coefficients enumerate the number of functional directed graphs which admit at least one cycle of length >1, where

$$\mathbf{X}[i,j]=x^{(n+1)^{|j-i|}},\forall\,0\le i,j<n.$$

In particular, the generating function whose coefficients enumerated the number of distinct functional directed graphs which correspond to a spanning union of cycles is

$$\mathrm{Per}\left(\mathbf{X}\right) = \sum_{\sigma \in \mathrm{S}_n} \prod_{0 \le i < n} x^{(n+1)^{|i-\sigma(i)|}}$$

The number of gracefully labeled paths is thus obtained by evaluating at $x = 1$ the Hadamard product

$$\mathrm{Per}\left(\mathbf{X}\right) \circledast \left(x^{\frac{(n+1)^n - (n+1)}{(n+1)-1}} \sum_{0 < i < n-1} x^{(n+1)^i} \right), \tag{7}$$

where the operation $\circledast$ denotes the Hadamard product of polynomials. Fortunately the coefficients of the polynomial described in (7) can be also be found using determinants instead of a permanents.

Theorem 5 : Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\mathbf{X}\left[i, j\right] = x^{\left((n+1)^{|j-i|}\right)}, \quad \forall\, 0 \le i, j < n$$

then the number of gracefully labeled paths on n vertices is the sum of the absolute values of the coefficients of the Hadamard product

$$\det\left(\mathbf{X}\right) \circledast \left(x^{\frac{(n+1)^n - (n+1)}{(n+1)-1}} \sum_{0 < i < n-1} x^{(n+1)^i} \right) \text{ if } n \text{ is odd,}$$

and

$$\det\left\{ \mathbf{X} \circ \left(\sum_{i<j} \mathbf{I}_n\left[:, i\right] \mathbf{I}_n\left[j, :\right] - \mathbf{I}_n\left[:, j\right] \mathbf{I}_n\left[i, :\right] \right) \right\} \circledast \left(x^{\frac{(n+1)^n - (n+1)}{(n+1)-1}} \sum_{0 < i < n-1} x^{(n+1)^i} \right) \text{ if } n \text{ is even,}$$

where $\circ$ denotes the Hadamard matrix product.

Proof : Let us start with the case where n is odd, then

$$\det\left(\mathbf{X}\right) = \sum_{\sigma \in \mathrm{S}_n} \mathrm{sgn}\left(\sigma\right) \prod_{0 \le i < n} x^{(n+1)^{|i-\sigma(i)|}},$$

Note that the terms with non vanishing coefficient in the Hadamard product

$$\det\left(\mathbf{X}\right) \circledast \left(x^{\frac{(n+1)^n - (n+1)}{(n+1)-1}} \sum_{0 < i < n-1} x^{(n+1)^i} \right)$$

corresponds to terms associated with the closing up of gracefully labeled paths on n vertices to form a spanning directed cycle. Such an odd cycles yields an even permutation. Consequently, the corresponding permutation and its inverse are encoded by the same term. On the other hand, if n is even the sign of the permutation will be opposite to the sign of its inverse and therefore the coefficient of the corresponding term vanishes. To address this, we consider instead of $\mathbf{X}$ a variant of the Tutte's skew symmetric matrix construction given by

$$\mathbf{X} \circ \left(\sum_{i<j} \mathbf{I}_n\left[:, i\right] \mathbf{I}_n\left[j, :\right] - \mathbf{I}_n\left[:, j\right] \mathbf{I}_n\left[i, :\right] \right).$$

In this setting the sign change is compensated by the edge sign change due to the skew symmetry. As a result, terms associated with closings of gracefully labeled paths no longer cancel out, thereby completing our proof.□

Let $\mathbf{X}$ denote a symbolic $n \times n$ matrix whose entries are given by

$$\mathbf{X}[i,j] = x^{(n+1)^{|j-i|}} \quad \forall\, 0 \le i,j < n,$$

as a corollary, the generating function whose coefficients enumerate the number of distinct functional directed graphs which are neither functional trees nor a spanning union of cycles having the same given induced edge label sequence is given by

$$\left(\sum_{\substack{f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \backslash \mathrm{S}_n \\ \left|f^{(n-1)}\left(\{0,\cdots,n-1\}\right)\right| > 1}} \prod_{0 \le i < n} x^{(n+1)^{|i-f(i)|}} \right) =$$

$$\frac{\operatorname{Per}\{\mathcal{B}(\mathbf{X})\}}{(n^2-n)!} - \operatorname{Per}(\mathbf{X}) - x \sum_{0 \le i < n} \det\left\{ \left(\operatorname{diag}\left(\mathbf{X} \cdot \mathbf{1}_{n \times 1}\right) - \mathbf{X}\right) \left[\begin{array}{c} 0,\cdots,i-1,i+1,\cdots,n-1 \\ 0,\cdots,i-1,i+1,\cdots,n-1 \end{array} \right] \right\}.$$

Theorem 6 : Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\forall\, 0 \le i,j < n, \quad \mathbf{X}[i,j] = y_i \, x^{\left(n^{|j-i|}\right)} \, y_j$$

then the generating function whose coefficient enumerate the number of distinct k-arry functional trees (for $k > 1$) having the same given induced edge label sequence is

$$\left(x \sum_{0 \le i < n} \det\left\{ \left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n \times 1}\right) - \mathbf{X}\right) \left[\begin{array}{c} 0,\cdots,i-1,i+1,\cdots,n-1 \\ 0,\cdots,i-1,i+1,\cdots,n-1 \end{array} \right] \right\} \bmod \left\{ y_j^{k+1} \right\}_{0 \le j < n} \right) \bmod \left\{ y_j - 1 \right\}_{0 \le j < n},$$

Proof : Reducing the polynomial

$$x \sum_{0 \le i < n} \det\left\{ \left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n \times 1}\right) - \mathbf{X}\right) \left[\begin{array}{c} 0,\cdots,i-1,i+1,\cdots,n-1 \\ 0,\cdots,i-1,i+1,\cdots,n-1 \end{array} \right] \right\}$$

modulo the algebraic relations $\left\{ y_j^{k+1} \equiv 0 \right\}_{0 \le j < n}$ ensures that only trees whose vertex degree is less or equal to k are retained in the expansion. The variable y_i keeps track of the degree of vertex labeled i in the tree. Note that the loop edge does not contribute to the degree. Finally reducing modulo the relations $\{y_j - 1\}_{0 \le j < n}$ is equivalent to assigning the value 1 to each variable y_j in the resulting expression, thus completes the proof.□

The generating function is efficiently determined by interpolating over $O\left(\frac{4^n}{\sqrt{n}}\right)$ evaluations of x. An illustration for a functional path is given by the functional tree associated with

$$f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \text{ such that } \forall\, 0 \le i < n, \quad f(i) = \left\{ \begin{array}{cc} 0 & \text{if } i = 0 \\ i-1 & \text{otherwise} \end{array} \right. .$$

Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\forall\, 0 \le i, j < n, \quad \mathbf{X}[i,j] = y_i\, x^{\left(n^{|j-i|}\right)}\, y_j.$$

As a corollary of Theorem 6 it follows that for

$$F_2(x) = \left(x \det\left\{\left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n\times 1}\right) - \mathbf{X}\right)[1:,1:]\right\} \bmod \left\{x^{1+\frac{n^n-1}{n-1}}\right\} \cup \left\{y_j^3\right\}_{0\le j<n}\right) \bmod \left\{y_j - 1\right\}_{0\le j<n}.$$

and from the fact that $F_2(1) = \frac{n!}{2}$, the following exact formula for the number of gracefully labeled paths is given by

$$\lim_{x\to\infty} \left(\frac{F_2(x)}{x^{\frac{n^n-1}{n-1}}}\right) = \left\lfloor \frac{F_2\left(\frac{n!}{2}\right)}{\left(\frac{n!}{2}\right)^{\frac{n^n-1}{n-1}}} \right\rfloor.$$

Note that M. Adamaszek proved an exponential lower bound for for the number of gracefully labeled paths in [Ad06]. Similar expressions enumerate gracefully labeled k-arry functional trees as follows

$$F_k(x) = \left(x \det\left\{\left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n\times 1}\right) - \mathbf{X}\right)[1:,1:]\right\} \bmod \left\{x^{1+\frac{n^n-1}{n-1}}\right\} \cup \left\{y_j^{k+1}\right\}_{0\le j<n}\right) \bmod \left\{y_j - 1\right\}_{0\le j<n}.$$

The number of gracefully labeled k-arry functional trees is given by

$$\lim_{x\to\infty} \left(\frac{F_k(x)}{x^{\frac{n^n-1}{n-1}}}\right) = \left\lfloor \frac{F_k\left(\binom{n-2}{k-1,\, k-1,\, \cdots,\, k-1,\, k-1}\right)}{\binom{n-2}{k-1,\, k-1,\, \cdots,\, k-1,\, k-1}^{\frac{n^n-1}{n-1}}} \right\rfloor,$$

where the multinomial factor $\binom{n-2}{k-1,\, k-1,\, \cdots,\, k-1,\, k-1}$ enumerates the number of rooted trees k-arry tree. The construction can be further modified to list k-arry trees. Using Gaussian elimination it is possible to construct a polynomial whose distinct terms describe edges of gracefully labeled functional k-arry trees. For this purpose, Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\forall\, 0 \le i, j < n, \quad \mathbf{X}[i,j] = y_i\, a_{ij} x_{|i-j|}\, y_j$$

The listing of gracefully labeled k-arry trees is therefore given by

$$G_k\left(a_{00}, \cdots, a_{n-1,n-1}, x_0, \cdots, x_{n-1}, y_0, \cdots, y_{n-1}\right) =$$

$$x_0 \sum_{0\le i<n} \det\left\{\left(\operatorname{diag}\left(\mathbf{X}\mathbf{1}_{n\times 1}\right) - \mathbf{X}\right)\left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array}\right]\right\} \bmod \left(\left\{x_j^2,\, y_j^{k+1}\right\}_{0\le j<n}\right)$$

Note that the corresponding polynomial may be efficiently obtained via Gaussian elimination provided that throughout the steps of the Gaussian elimination procedure we mod out any term where any of the variable x_i appears with a degree > 2 as well as any term where any variable y_i appears with a degree $> k$. This variant of the Gaussian elimination procedure is a special case of a more general method discussed in the next section.

4 Some combinatorial variants of Immanants via Gaussian elimination.

The generating function constructions described in the previous section motivate the following variants of the permanent, determinant and conjecturally of immanant polynomials associated with a $n \times n$ matrix $\mathbf{A}$ and a function $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$

$$\mathrm{mPer}_f(\mathbf{A}) = \sum_{\sigma \in \mathrm{S}_n/\mathrm{Aut}G_f} \prod_{0\le i<n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right],$$

$$\mathrm{mDet}_f(\mathbf{A}) = \sum_{\sigma \in \mathrm{S}_n/\mathrm{Aut}G_f} \mathrm{sgn}\left(\sigma^{-1} f \sigma\right) \prod_{0 \le i < n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right],$$

where $\mathrm{sgn}\left(\sigma^{-1} f \sigma\right)$ denotes the number of inversion incurred by the function $\sigma^{-1} f \sigma$. Conjecturally a modified immanant should be of the form

$$\mathrm{mImm}_{\lambda, f}(\mathbf{A}) = \sum_{\sigma \in \mathrm{S}_n/\mathrm{Aut}G_f} \chi_\lambda\left(\sigma^{-1} f \sigma(i)\right) \prod_{0 \le i < n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right],$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ denotes a partition of n and let χ_λ relates to the corresponding irreducible representation-theoretic character. Let $\mathcal{C}_n$ denote an arbitrary choice of one of the largest subset of S_n whose elements are pairwise non-isomorphic as functional directed graphs. The relationship between the modified permanent/determinant and their classical counterparts is given by

$$\mathrm{Per}(\mathbf{A}) = \sum_{f \in \mathcal{C}_n} \mathrm{mPer}_f(\mathbf{A}) \quad \text{and} \quad \det(\mathbf{A}) = \sum_{f \in \mathcal{C}_n} \mathrm{mDet}_f(\mathbf{A}). \tag{8}$$

For $n = 3$ a representative choice for $\mathcal{C}_3$ is

$$\mathcal{C}_3 = \left\{\left\{(0,0), (1,1), (2,2)\right\}, \left\{(0,1), (1,0), (2,2)\right\}, \left\{(0,1), (1,2), (2,0)\right\}\right\}.$$

Similarly, let $\mathcal{T}_n$ denote an arbitrary choice of one of the largest subset of functional trees whose elements are pairwise non-isomorphic as functional directed graphs then we have

$$\sum_{f \in \mathcal{T}_n} \mathrm{mPer}_f(\mathbf{A}) = \sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right| = 1} \prod_{0 \le i < n} \mathbf{A}[i, f(i)].$$

$$\sum_{f \in \mathcal{T}_n} \mathrm{mDet}_f(\mathbf{A}) = \sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right| = 1} \mathrm{sgn}(f) \prod_{0 \le i < n} \mathbf{A}[i, f(i)].$$

For $n = 3$ a representative choice for $\mathcal{T}_3$ is given by

$$\mathcal{T}_3 = \left\{\left\{(0,0), (1,2), (2,0)\right\}, \left\{(0,0), (1,0), (2,0)\right\}\right\}.$$

Furthermore, Tutte's directed Matrix Tree Theorem relates the modified permanent to modified determinant as follows

$$\sum_{f \in \mathcal{T}_n} \mathrm{mPer}_f(\mathbf{A}) = \sum_{0 \le i < n} \mathbf{A}[i,i] \sum_{f \in \mathcal{C}_n} \mathrm{mDet}_f \left\{ \left(\mathrm{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right) \left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array} \right] \right\}. \tag{9}$$

Let $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ subject to $\left|f^{(n-1)}(\{0, \cdots, n-1\})\right| = 1$ and let $\mathbf{A}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\forall\, 0 \le i, j < n, \quad \mathbf{A}[i,j] = a_{ij}.$$

It follows from the identity

$$\mathbf{A}[i,i] \det \left\{ \left(\mathrm{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right) \left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array} \right] \right\} = \sum_{f^{(n-1)}(\{0,\cdots,n-1\}) = \{i\}} \prod_{0 \le j < n} \mathbf{A}[j, f(j)],$$

that the polynomial $\mathrm{mPer}_f(\mathbf{A})$ may be alternatively derived by a variant of Gaussian elimination procedure. The summands of $\mathrm{mPer}_f(\mathbf{A})$ are obtained via row operations. Each row linear combination step is followed by setting to zero terms which appear in the numerators of intermediary rational functions. More precisely we set to zero any term in the numerator which is made up of edges of a subgraph of the complete graph which is not sub-isomorphic G_f. Note that by symmetry it follows that

$$\sum_{\left|f^{(n-1)}(\{0,\cdots,n-1\})\right| = 1} \frac{\left|\mathrm{Aut}G_f\right| \mathrm{mPer}_f(\mathbf{A})}{n!} = \sum_{0 \le i < n} \mathbf{A}[i,i] \det \left\{ \left(\mathrm{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right) \left[\begin{array}{c} 0, \cdots, i-1, i+1, \cdots, n-1 \\ 0, \cdots, i-1, i+1, \cdots, n-1 \end{array} \right] \right\} \tag{10}$$

5 Node splitting and edge contraction labelings

In the previous section we described how a variant of the Gaussian elimination procedure can be used to obtain all the graceful labelings of a given functional tree. In this section we describe a different algorithm for obtaining all graceful labelings of an arbitrary input functional directed graph. The node splitting procedure devises from an input gracefully labeled undirected graph G new gracefully labeled undirected graphs on $n+1$ vertices and n non-loop edges as follows :

Step 1 : Delete an arbitrary but fixed edge subset of G.

Step 2 : Split the vertex labeled 0 into the new edge $(n,\, 0)$.

Step 3 : Output all gracefully labeled graphs obtained by placing new edges spanning appropriate vertex pairs.

On the other hand the edge contraction procedure takes as input a gracefully labeled graph on n vertices and devises gracefully labeled graphs on $n-1$ vertices as follows :

Step 1 : Delete a fixed edge subset of G which must include all the edges adjacent to the vertex labeled $(n-1)$.

Step 2 : Discard the vertex labeled $(n-1)$.

Step 3 : Output all gracefully labeled graphs obtained by placing new edges spanning appropriate vertex pairs.

Each graph obtained by the edge contraction procedure is a direct parent of the input graph G. More generally, a graph H is a parent (not necessarily a direct parent) of G if G can be obtained from H by a sequence of node splitting procedure. By the same token H is a descendent of G. For example all gracefully labeled graphs on three vertices or more are descendent of the gracefully labeled graph on two vertices.

5.1 The contraction/splitting labeling procedure

We co-opt the edge contraction and node splitting constructions to devise a graceful relabelings procedure. It is easy to see that an input graph G_f associated with $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$ admits a graceful labeling iff there exist a solution to the system of algebraic equations in the variables $\{x_0, \cdots, x_{n-1}\}$ given by

$$\forall\, z \in \mathbb{C}, \quad z^{2n-1} - 1 = \prod_{0\le f(i)=i<n} \left(z - \frac{x_i}{x_{f(i)}}\right)^{|\{0\le f(i)=i<n\}|^{-1}} \prod_{0\le f(i)\ne i<n} \left(z - \frac{x_i}{x_{f(i)}}\right)\left(z - \frac{x_{f(i)}}{x_i}\right).$$

Which is equivalent to the assertion that the system

$$\left\{1 + \sum_{0\le i\ne f(i)<n} \left(\frac{x_{f(i)}}{x_i}\right)^{-j} + \left(\frac{x_{f(i)}}{x_i}\right)^{j} = 0\right\}_{0<j\le 2n-1}. \tag{11}$$

admits a solution. Note that each constraint in (11) is invariant to arbitrary edge orientation changes. Consequently, the system 11 is just as well be associated with the undirected graph $\widetilde{G}_f$ devised from the functional directed graph G_f. Let

$$\mathrm{Aut}\widetilde{G}_f := \left\{\sigma \in \mathrm{S}_n \;\text{ s.t. } \sum_{0\le i\ne f(i)<n} \left(\frac{x_{f(i)}}{x_i}\right)^{-1} + \left(\frac{x_{f(i)}}{x_i}\right) = \sum_{0\le j\ne f(j)<n} \left(\frac{x_{\sigma^{-1}f\sigma(j)}}{x_j}\right)^{-1} + \frac{x_{\sigma^{-1}f\sigma(j)}}{x_j}\right\}.$$

The undirected edge $\{(i, f(i)), (f(i), i)\}$ and $\{(j, f(j)), (f(j), j)\}$ for $0 < i < j < n$ both lie in a common edge orbit in $\widetilde{G}_f$ induced by the action of S_n on the vertex set if

$$\exists\, \sigma \in \mathrm{S}_n, \text{ s.t. } \left(\left(\frac{x_{\sigma^{-1} f \sigma(i)}}{x_i} \right)^{-1} + \frac{x_{\sigma^{-1} f \sigma(i)}}{x_i} \right) = \left(\left(\frac{x_{f(j)}}{x_j} \right)^{-1} + \frac{x_{f(j)}}{x_j} \right).$$

Similarly, two vertices x_i and x_j for $0 \le i < j < n$ lie in a common vertex orbit induced by the action of S_n if

$$\exists\, \sigma \in \mathrm{Aut}\widetilde{G}_f, \text{ s.t. } x_{\sigma(i)} = x_j.$$

Having specified the notation, we now describe the edge contraction/splitting labeling procedure. The edge contraction labeling procedure finds all graceful relabeling of $\widetilde{G}_f$. As a result the procedure can be used to certify that an input graph admits no graceful labeling. The contraction/splitting labeling algorithm proceeds via two complementary subroutines. The first subroutine is an edge contraction routine. It determines the set of possible parent graphs which result from sequences of edge contractions. The edge contraction subroutine terminates once it reaches a star tree. A single iteration of the edge contraction subroutine an input graph $\widetilde{G}_f$ on n vertices proceeds as follows :

Step 1 : If none of the edges of $\widetilde{G}_f$ spans specially marked blue and red vertices, then select a non-isolated edge representative of an edge orbit of $\widetilde{G}_f$. The procedure separately selects one non-isolated edge per edge orbit in $\widetilde{G}_f$. On the other hand if $\widetilde{G}_f$ already has a special selected edge which spans specially marked blue and red vertices then the procedure skips Step 2 and moves onto Step 3.

Step 2 : Arbitrarily mark blue and red the vertices spanned by the special selected edge in Step 1.

Step 3 : Remove from $\widetilde{G}_f$ all unselected edges incident to the red vertex.

Step 4 : Contract the special selected edge into the specially marked blue vertex. If the specially marked red vertex was a leaf node, the iteration outputs a set of candidate parent graphs resulting from all the possible ways of selecting a new specially marked red vertex among the vertices adjacent to the specially marked blue vertex. For efficiency, only one red vertex is chosen per orbit of vertices adjacent to the blue vertex. Otherwise, if the selected edge is not a leaf edge then the iteration proceeds to Step 5.

Step 5 : Output a set of candidate parent graphs resulting from all the ways of replacing edges removed in Step 3 with new edges incident to the specially marked blue vertex. If n is odd, then at most one of the new edges can span the specially marked blue vertex and a vertex previously adjacent to the contracted red vertex. In which case this particular vertex is to be assigned the label $\left(\frac{n-1}{2}\right)$. Otherwise, if n is even then none of the new edges can be incident to a vertex which was previously adjacent to the contracted red vertex.

Repeatedly applying the edge contraction routine eventually leads to a terminating star tree. The second subroutine is a constrained node splitting routine. The constrained node splitting routine recovers possible vertex relabelings via sequences of node splittings. As input, the second subroutine takes a gracefully labeled star tree. The input star tree corresponds to one of the terminating star trees obtained by the edge contraction subroutine. The constrained node splitting routine prunes candidate parent graphs in order to attempt to reverse the steps of the edge contractions subroutine, thereby uncovering graceful labelings when they exist.

It follows that a given input graph admits no graceful labeling if every sequence of node splittings avoids the input graph. The contraction/splitting labeling procedure can thus be viewed as a special purpose elimination procedure. The simplest family of functional directed graph having no isolated vertices which admit no graceful labeling belongs to the family of graphs defined for $n \geq 5$ by

$$f : \{0, \cdots, n-1\} \to \{0, \cdots, n-1\}$$

such that

$$f(i) = \begin{cases} i+1 \mod 3 & \text{if } i \in \{0,1,2\} \\ 3 & \text{otherwise} \end{cases}.$$

The case $n = 5$ yields the smallest $f \in \{0,1,2,3,4\}^{\{0,1,2,3,4\}} \backslash \mathrm{S}_5$ which admits no graceful labeling. Note that the presence of a cycle in a functional directed graph which has no isolated vertex does not necessarily prevent the existence of a graceful labeling as illustrated by the functional directed graph prescribed by

$$f : \{0, \cdots, 6\} \to \{0, \cdots, 6\}$$

such that

$$f(i) = \begin{cases} i+1 \mod 3 & \text{if } i \in \{0,1,2\} \\ 3 & \text{if } i = 3 \\ i-1 & \text{otherwise} \end{cases},$$

admits a graceful labeling.

5.2 Proof of correctness of the contraction/splitting labeling procedure

We now discuss the proof of correctness of the edge contraction labeling algorithm. The fact that the algorithm terminates for any input functional graph follows from the fact that the number of vertices decrease by one at each iteration. Consequently, the edge contraction subroutine requires for an input functional graph on n vertices at most $n-3$ iterations to reach a terminating star tree. Similarly, at most $n-3$ iterations of the node splitting subroutine determine the graceful relabeling of the input graph. On the other hand the fact that procedure indeed identifies all graceful labeling follows from the following property of the edge contractions subroutine. For an undirected functional graph $\widetilde{G}_f$ associated with $f \in \{0, \cdots, n-1\}^{\{0,\cdots,n-1\}}$, the existence of solutions to the system of $2(n-1)$ equations in n variables

$$\left\{ \sum_{0<f(i)\neq i<n} \left(\frac{x_{f(i)}}{x_i}\right)^{-j} + \left(\frac{x_{f(i)}}{x_i}\right)^{j} = -1 \right\}_{1\leq j<2n-1}, \tag{12}$$

for odd n, reduces to the assertion that there exist an index $0 \leq k < n$ and a subset S (assume without loss of generality that $f^2(k) = f(k)$) prescribed by

$$S \subset \left\{\frac{n-1}{2}\right\} \cup \left(\{i \text{ s.t. } 0 \leq i < n, \text{ and } f(i) \neq i\} \backslash \left(\{k\} \cup f^{-1}(k) \cup f^{-1}(f(k))\right)\right), \quad |S| = \left|f^{-1}(k)\right| \geq 1,$$

such that the smaller system of $2n-3$ equations in $n-1$ variables admits a solution

$$\left\{ \sum_{i\in S} \left(\frac{x_{f(k)}}{x_i}\right)^{-j} + \left(\frac{x_{f(k)}}{x_i}\right)^{j} + \sum_{i\in\{0,\cdots,k-1,k+1,\cdots,n-1\}\backslash(f^{-1}(k)\cup S)} \left(\frac{x_{f(i)}}{x_i}\right)^{-j} + \left(\frac{x_{f(i)}}{x_i}\right)^{j} = -1 \right\}_{0<j<2(n-1)-1}.$$

In this setting the vertex associated with the variable x_k corresponds to the specially marked blue vertex and $x_{f(k)}$ corresponds to the variable with the specially marked red vertex. On the other hand if n is even, the assertion that 12 reduces to the assertion that there exist an index $0 \leq k < n$ and a subset S prescribed by

$$S \subset \{i \text{ s.t. } 0 \leq i < n, \text{ and } f(i) \neq i\} \backslash \left(\{k\} \cup f^{-1}(k) \cup f^{-1}(f(k))\right), \quad |S| = \left|f^{-1}(k)\right| \geq 1,$$

such that the smaller system of $2n-3$ equations in $n-1$ variables admits a solution

$$\left\{\sum_{i\in S}\left(\frac{x_{f(k)}}{x_i}\right)^{-j}+\left(\frac{x_{f(k)}}{x_i}\right)^{j}+\sum_{i\in\{0,\cdots,k-1,k+1,\cdots,n-1\}\backslash(f^{-1}(k)\cup S)}\left(\frac{x_{f(i)}}{x_i}\right)^{-j}+\left(\frac{x_{f(i)}}{x_i}\right)^{j}=-1\right\}_{0<j<2(n-1)-1}.$$

This follows from the fact that any subset of missing edge weights can always be recovered by selecting some subset of vertices to be made adjacent to the vertex labeled 0. In the edge contraction procedure the specially marked blue vertex stands for the vertex to be labeled 0. Similarly the specially marked red vertex is associated with the vertex to be assign the largest integer label among the remaining vertices. The contraction subroutine crucially exploit the fact that in a graceful labeling, the vertex with the largest label must necessarily be adjacent to the vertex with the smallest label. Since the edge contraction always terminates at star tree (whether or not the input graph is graceful) it is the node spitting procedure initiated on a gracefully labeled star tree which determine whether the input functional graph can be recovered by attempting to reverse the sequence of the edge contraction iteration.

6 Taking the GLC to the limit.

The GLC is equivalent to the assertion that given

$$f:\left\{\frac{0}{n},\cdots,\frac{n-1}{n}\right\}\rightarrow\left\{\frac{0}{n},\cdots,\frac{n-1}{n}\right\}$$

subject to

$$\left|f^{(n-1)}\left(\left\{\frac{0}{n},\cdots,\frac{n-1}{n-1}\right\}\right)\right|=1,$$

there exist fixed permutations σ and γ of $\left\{\frac{0}{n},\cdots,\frac{n-1}{n}\right\}$ such that

$$f\left(\frac{i}{n}\right)=\sigma^{-1}\left(\sigma\left(\frac{i}{n}\right)+\sqrt{\gamma\left(\sigma\left(\frac{i}{n}\right)\right)^2}\right).$$

We therefore take the limit of functional trees to be functions $f\in[0,1)^{[0,1)}$ for which there exist $\rho\in[0,1)$ such that $|f(x)-f(y)|\leq\rho\,|x-y|$ for all x,$y\in[0,1)$. The simplest illustrations of such functions includes the family of constant functions of the form

$$f(x)=c,\quad\forall\,0\leq c<1,$$

family of monomial functions of the form

$$f(x)=x^m,\quad\forall\,m\in\mathbb{N}\ \text{ and }\ m>1,$$

as well as family of exponential functions

$$f(x)=a^x,\quad\forall\,0\leq a<1.$$

In the limit, the GLC problem asks to determine limiting functional trees $f\in[0,1)^{[0,1)}$ which admits an expansion of the form

$$f(x)=g^{-1}\left(g(x)+\sqrt{h(g(x))^2}\right),\tag{13}$$

for some invertible functions g, h. In which case we say that the limiting functional tree f is graceful. Equivalently, the limiting functional tree $f \in [0,1)^{[0,1)}$is graceful if there exist there exist invertible functions g, h such that

$$(h(x))^2 = (g(f(x)) - g(x))^2 .$$

Proposition 7 : Let h be an arbitrary invertible function then the function

$$f(x) = x\, e^{\sqrt{h(\ln x)^2}}$$

is graceful.

Proof : The proof immediately follows from the observation that

$$f(x) = \exp\left\{\ln x + \sqrt{h(\ln x)^2}\right\},$$

$$\Longrightarrow (\ln(f(x)) - \ln x)^2 = (h(x))^2$$

consequently here $g(x) = \ln x$, thus completing the proof$\square$.

As a corollary of Proposition 7, the family of monomial functions

$$f(x) = x^m, \quad \forall\, m \in \mathbb{N} \text{ and } m > 1,$$

are graceful. In particular we will have

$$f(x) = \exp\left\{\ln(x) + \sqrt{(m-1)^2 \ln\left(x^{\ln(x)}\right)}\right\}$$

$$\Longrightarrow g(x) = \ln(x) \text{ and } h(x) = (m-1)\ln x.$$

It is also clear that the family of constant functions (limiting functions for the star trees) of the form

$$f(x) = c, \quad \forall\, 0 \le c < 1,$$

are graceful, for it suffices to choose invertible functions g, h such that $h(x)^2 = (g(c) - g(x))^2$. Similarly, the family of exponential functions

$$f(x) = a^x, \quad \forall\, 0 \le a < 1.$$

are graceful. This follows from the identity

$$f(x) = \ln\left\{e^x + \sqrt{\left(e^{(a^x)} - e^x\right)^2}\right\}$$

hence $g = e^x$ and $h(x) = e^{(a^x)} - e^x$. There are other more difficult example of graceful functions such as

$$f(x) = \sqrt{2x\sqrt{1-x^2}}.$$

for which $g(x) = g^{-1}(x) = \sqrt{1-x^2}$ and $h(x) = -x$ as seen by the following derivation

$$f(x) = \sqrt{1 - \left(\sqrt{1-x^2} + \sqrt{(-x)^2}\right)^2},$$

$$\implies f(x)=\sqrt{1-\left(\sqrt{1-x^2}-x\right)^2},$$

$$\implies f(x)=\sqrt{1-\left(\left(\sqrt{1-x^2}\right)^2-2x+x^2\right)},$$

$$\implies f(x)=\sqrt{1-\left(1-x^2-2x\sqrt{1-x^2}+x^2\right)},$$

$$\implies f(x)=\sqrt{1-\left(1-x^2-2x\sqrt{1-x^2}+x^2\right)},$$

$$\implies f(x)=\sqrt{2x\sqrt{1-x^2}}.$$

7 The method of creative stabilizing.

In this section we investigate the extent to which symmetries reveal properties of induced edge labelings. Intricate relations between symmetries and induced edge labelings of a functional directed graph G_f associated with $f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ are illustrated by the following bounds

$$0 \le |\mathrm{GrL}\,(G_f)| \le |^{(\mathrm{S}_n/\mathcal{I}_n)}/_{\mathrm{Aut}G_f}|\,,$$

where ${}^{\mathrm{S}_n}/\mathcal{I}_n$ denotes the set of representative of the equivalence class defined in terms of any element $\sigma \in \mathrm{S}_n$ associated with the pairing

$$\{(\sigma(0),\cdots,\sigma(i),\cdots,\sigma(n-1))\,,\,(n-1-\sigma(0),\cdots,n-1-\sigma(i),\cdots,n-1-\sigma(n-1))\}\,.$$

These sharp bounds suggests a heuristic relationship between $|\mathrm{GrL}\,(G_f)|$ and $|^{(\mathrm{S}_n/\mathcal{I}_n)}/_{\mathrm{Aut}G_f}|$ for functional graceful graphs. In this regard we may consider two extreme examples at opposing ends of one another. Theses examples are associated with functional star and path trees respectively illustrated by

$$f,\,g\,:\,\{0,\cdots,n-1\}\rightarrow\{0,\cdots,n-1\}\,,$$

$$\forall\, 0<i<n,\quad f(i)=0 \text{ and } g(i)=\left\{\begin{array}{cc} 0 & \text{if } i=0 \\ i-1 & \text{otherwise}\end{array}\right. .$$

For $n>3$, we formally define the set of functional stars to be the set of functional directed graphs whose edge set make up distinct terms of the multivariate polynomial

$$\sum_{0\le i<n}\mathbf{A}[i,i]\det\left\{(\mathrm{diag}\,(\mathbf{A}\mathbf{1}_{n\times 1})-\mathbf{A})\left[\begin{array}{c}0,\cdots,i-1,i+1,\cdots,n-1\\0,\cdots,i-1,i+1,\cdots,n-1\end{array}\right]\right\}$$

$$\mathrm{mod}\left\{\begin{array}{c}\mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(3)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(3)}\right]\\ \mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(2)}\right]\end{array}\right\}_{\substack{|\{i_0,i_1,i_2\}|=3\\ \varrho\in\mathrm{S}_3}}.$$

where $\mathbf{A}$ denotes a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{A}[i,j]=a_{ij},\quad \forall\, 0\le i,j<n.$$

In the case $n=4$ the multivariate polynomial whose terms describe list all functional stars is given by

$$\sum_{0\le i<4}\mathbf{A}[i,i]\det\left\{(\mathrm{diag}\,(\mathbf{A}\mathbf{1}_{4\times 1})-\mathbf{A})\left[\begin{array}{c}0,\cdots,i-1,i+1,\cdots,3\\0,\cdots,i-1,i+1,\cdots,3\end{array}\right]\right\}$$

$$\text{mod}\left\{\begin{array}{c}\mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(3)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(3)}\right]\\ \\ \mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(2)}\right]\end{array}\right\}_{\substack{|\{i_0,i_1,i_2\}|=3\\ \varrho\in\mathrm{S}_3}}$$

$$=$$

$$a_{00}a_{10}a_{20}a_{30}+a_{01}a_{11}a_{20}a_{30}+a_{02}a_{10}a_{22}a_{30}+a_{00}a_{13}a_{23}a_{30}+a_{00}a_{10}a_{21}a_{31}+a_{01}a_{11}a_{21}a_{31}+a_{01}a_{12}a_{22}a_{31}+a_{03}a_{11}a_{23}a_{31}+$$

$$a_{00}a_{12}a_{20}a_{32}+a_{02}a_{11}a_{21}a_{32}+a_{02}a_{12}a_{22}a_{32}+a_{03}a_{13}a_{22}a_{32}+a_{03}a_{10}a_{20}a_{33}+a_{01}a_{13}a_{21}a_{33}+a_{02}a_{12}a_{23}a_{33}+a_{03}a_{13}a_{23}a_{33}.$$

Similarly, for $n>3$ we formally define functional paths to be the set of all functional directed graphs whose edge set make up distinct terms of the multivariate polynomial

$$\sum_{0\le i<n}\left(\mathbf{A}\left[i,i\right]\det\left\{\left(\text{diag}\left(\mathbf{A}\mathbf{1}_{n\times1}\right)-\mathbf{A}\right)\left[\begin{array}{c}0,\cdots,i-1,i+1,\cdots,n-1\\0,\cdots,i-1,i+1,\cdots,n-1\end{array}\right]\right\}\text{mod}\left\{y_j^3\right\}_{0\le j<n}\right)\text{mod}\left\{y_k-1\right\}_{0\le k<n},$$

where $\mathbf{A}$ denote a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{A}\left[i,j\right]=\begin{cases}y_i\,a_{ij}\,y_j & \text{if } i\neq j\\ a_{ii} & \text{if } i=j\end{cases}\quad\forall\,0\le i,j<n.$$

In the case $n=4$ the multivariate polynomial whose terms describe the set of all functional path is

$$\sum_{0\le i<4}\left(\mathbf{A}\left[i,i\right]\det\left\{\left(\text{diag}\left(\mathbf{A}\mathbf{1}_{4\times1}\right)-\mathbf{A}\right)\left[\begin{array}{c}0,\cdots,i-1,i+1,\cdots,3\\0,\cdots,i-1,i+1,\cdots,3\end{array}\right]\right\}\text{mod}\left\{y_j^3\right\}_{0\le j<4}\right)\text{mod}\left\{y_k-1\right\}_{0\le k<4}=$$

$$a_{00}a_{12}a_{20}a_{30}+a_{00}a_{13}a_{20}a_{30}+a_{00}a_{10}a_{21}a_{30}+a_{01}a_{11}a_{21}a_{30}+a_{02}a_{11}a_{21}a_{30}+a_{00}a_{13}a_{21}a_{30}+a_{01}a_{12}a_{22}a_{30}+a_{02}a_{12}a_{22}a_{30}+$$

$$a_{02}a_{13}a_{22}a_{30}+a_{00}a_{10}a_{23}a_{30}+a_{01}a_{11}a_{23}a_{30}+a_{00}a_{12}a_{23}a_{30}+a_{00}a_{10}a_{20}a_{31}+a_{01}a_{11}a_{20}a_{31}+a_{03}a_{11}a_{20}a_{31}+a_{00}a_{12}a_{20}a_{31}+$$

$$a_{02}a_{11}a_{21}a_{31}+a_{03}a_{11}a_{21}a_{31}+a_{02}a_{10}a_{22}a_{31}+a_{02}a_{12}a_{22}a_{31}+a_{03}a_{12}a_{22}a_{31}+a_{00}a_{10}a_{23}a_{31}+a_{01}a_{11}a_{23}a_{31}+a_{02}a_{11}a_{23}a_{31}+$$

$$a_{00}a_{10}a_{20}a_{32}+a_{01}a_{11}a_{20}a_{32}+a_{00}a_{13}a_{20}a_{32}+a_{00}a_{10}a_{21}a_{32}+a_{01}a_{11}a_{21}a_{32}+a_{03}a_{11}a_{21}a_{32}+a_{02}a_{10}a_{22}a_{32}+a_{03}a_{10}a_{22}a_{32}+$$

$$a_{01}a_{12}a_{22}a_{32}+a_{03}a_{12}a_{22}a_{32}+a_{01}a_{13}a_{22}a_{32}+a_{02}a_{13}a_{22}a_{32}+a_{03}a_{12}a_{20}a_{33}+a_{01}a_{13}a_{20}a_{33}+a_{03}a_{13}a_{20}a_{33}+a_{03}a_{10}a_{21}a_{33}+$$

$$a_{02}a_{13}a_{21}a_{33}+a_{03}a_{13}a_{21}a_{33}+a_{02}a_{10}a_{23}a_{33}+a_{03}a_{10}a_{23}a_{33}+a_{01}a_{12}a_{23}a_{33}+a_{03}a_{12}a_{23}a_{33}+a_{01}a_{13}a_{23}a_{33}+a_{02}a_{13}a_{23}a_{33}.$$

Note that if G_f is a functional star then

$$\left|\text{Aut}G_f\right|\in\left\{(n-1)!,\,(n-2)!\right\}.$$

On the other hand if G_g is a functional path then

$$\left|\text{Aut}G_g\right|\in\left\{1,\,2\right\}.$$

It is easy to see that the set of induced edge label sequences is the same across all functional paths and similarly the set of induced edge label sequence is also the same across all functional stars. Furthermore for every $n>3$ functional stars and functional paths respectively minimize and maximize the cardinality of the set of induced edge label sequences. In particular for

$$f,\,g\,\in\,\left\{0,\cdots,n-1\right\}^{\left\{0,\cdots,n-1\right\}},$$

$$\forall\,0<i<n,\quad f\left(i\right)=0\ \text{ and }\ g\left(i\right)=\begin{cases}0 & \text{if } i=0\\ i-1 & \text{otherwise}\end{cases},$$

$$\left|\text{Aut}G_f\right|=(n-1)!\ \text{ and } 1=\left|\text{Aut}G_g\right|.$$

The following proposition determines the set of induced edge label sequences of a functional stars.

Proposition 8 : Let $\mathbf{X}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries given by

$$\mathbf{X}[i,j] = x^{\left(n^{|j-i|}\right)}, \quad \forall\, 0 \le i,j < n,$$

then the coefficients of the polynomial

$$\text{mPer}_f(\mathbf{X}) = \begin{cases} \left(\prod_{0\le i<n} x^{n^{\left|\frac{n-1}{2}-i\right|}}\right) + 2 \sum_{0\le t<\left\lfloor \frac{n}{2}\right\rfloor} \prod_{0\le i<n} x^{n^{|t-i|}} & \text{if } n \equiv 1 \mod 2 \\ 2 \sum_{0\le t<\left\lceil \frac{n}{2}\right\rceil} \prod_{0\le i<n} x^{n^{|t-i|}} & \text{if } n \equiv 0 \mod 2 \end{cases}$$

enumerate the number of distinct vertex relabelings of a given functional star which have the same induced edge label sequence.

Proof : Since all functional trees have the same set of induced edge label sequences, without loss of generality it suffices to consider the n constant functions

$$f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \text{ subject to } \forall\, 0 < i < n, \quad f(i) = k.$$

We observe that an induced edge label sequence of G_f is completely determined by the label of the vertex of highest degree. The factor 2 follows from the invariance of the induced edge labeling label sequence to replacing each vertex label i with the new label $(n-1)-i$ for all $0 \le i < n$. In particular, if the entries of the adjacency matrix are given by

$$\mathbf{A}[i,j] = a_{i,j}\, x^{\left(n^{|j-i|}\right)}, \quad \forall\, 0 \le i,j < n,$$

then for any constant function

$$f \in \{0,\cdots,n-1\}^{\{0,\cdots,n-1\}} \text{ subject to } \forall\, 0 < i < n, \quad f(i) = k.$$

we have

$$\text{mPer}_f(\mathbf{A}) = \begin{cases} \left(\prod_{0\le i<n} a_{\frac{n-1}{2},i} x^{n^{\left|\frac{n-1}{2}-i\right|}}\right) + \sum_{0\le t<\left\lfloor \frac{n}{2}\right\rfloor} \left(\prod_{0\le i<n} a_{t,i} + \prod_{0\le i<n} a_{n-1-t,n-1-i}\right) \prod_{0\le i<n} x^{n^{|t-i|}} & \text{if } n \equiv 1 \mod 2 \\ \sum_{0\le t<\left\lceil \frac{n}{2}\right\rceil} \left(\prod_{0\le i<n} a_{t,i} + \prod_{0\le i<n} a_{n-1-t,n-1-i}\right) \prod_{0\le i<n} x^{n^{|t-i|}} & \text{if } n \equiv 0 \mod 2 \end{cases}.$$

Consequently, assigning 1 to each variable in the set $\{a_{ij}\}_{0\le i,j<n}$, yields the desired result thus completing the proof. □

We now describe the *conjugation algorithm* which enables us to generate the whole orbit of functional graphs which admit a graceful labeling. The conjugation algorithm resembles the Buchberger's algorithm by its emphasis on orderings, monomial ideals as well as the fact the algorithm consists in reducing multivariate polynomials modulo an increasing number of algebraic relations. For any multivariate polynomial Q, TermSet(Q) denote the set

$$\{t : \text{ is a nonzero term in the canonical expansion of } Q\}.$$

For example, for

$$Q(x,y,z) = (x+y+z)^2 + (2x-y-2z)\,x \implies \text{TermSet}(Q) = \left\{3\,x^2, 2\,yz, xy, z^2, y^2\right\}.$$

Let $\mathbf{A}$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges with entries given by

$$\mathbf{A}[i,j] = a_{ij}\, x_{|j-i|}, \quad \forall\, 0 \le i,j < n.$$

Let $\mathcal{B}(\mathbf{A})$ denote a symbolic $n^2 \times n^2$ matrix derived from $\mathbf{A}$ with entries given by

$$\mathcal{B}(\mathbf{A})[n \cdot u + v,\, n \cdot i + j] = \begin{cases} \mathbf{A}[u,v] & \text{if} \quad u = n \cdot i + j \\ 0 & \text{if} \quad n \cdot i + j \ne u \text{ and } n \cdot i + j < n, \\ 1 & \text{otherwise} \end{cases} \quad \forall \begin{array}{c} 0 \le u,v < n \\ 0 \le i,j < n \end{array},$$

Assuming the elements of S_n to be lexicographically ordered, the conjugation algorithm is prescribed by a set recurrence whose initial conditions are prescribed by

$$\mathrm{Lm}_0 = \mathrm{Lm}_{\mathrm{lex(id)}} = \mathrm{TermSet}\left(\frac{\mathrm{Per}\left\{\mathcal{B}(\mathbf{A})\right\}}{(n^2-n)!} \mathrm{mod} \left\{x_j^2\right\}_{0 \le j < n}\right),$$

and the recurrence is prescribed for any $\sigma \in \mathrm{S}_n \backslash \{\mathrm{id}\}$ by

$$\mathrm{Lm}_{\mathrm{lex}(\sigma)} = \mathrm{Lm}_{\mathrm{lex}(\sigma)-1} \cup \mathrm{TermSet}\left(\sum_{\prod_{0 \le i < n} \mathbf{A}[i,f(i)] \in \mathrm{Lm}_0} \prod_{0 \le i < n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right]\right).$$

Consequently, the set

$$\mathrm{Lm}_{n!-1} = \mathrm{Lm}_{\mathrm{lex}(n-1,n-2,\cdots,1,0)} = \bigcup_{\sigma \in \mathrm{S}_n} \mathrm{TermSet}\left(\sum_{\prod_{0 \le i < n} \mathbf{A}[i,f(i)] \in \mathrm{Lm}_0} \prod_{0 \le i < n+1} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right]\right).$$

lists all functional directed graphs which admit a graceful labeling. Note that the lexicographic ordering of the permutation is used here for notational convenience any ordering in which the identity permutation is the minimum element would do just as well. A similar monomial set recurrence list graceful trees. The initial monomial set for the recurrence which list graceful trees is

$$\mathrm{Lm}_0 = \mathrm{Lm}_{\mathrm{lex(id)}} = \mathrm{TermSet}\left(\sum_{0 \le i < n} \mathbf{A}[i,i] \det\left\{\left(\mathrm{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right)\left[\begin{array}{c} 0,\cdots,i-1,i+1,\cdots,n-1 \\ 0,\cdots,i-1,i+1,\cdots,n-1 \end{array}\right]\right\} \mathrm{mod} \left\{x_j^2\right\}_{0 \le j < n}\right).$$

and the recurrence prescribed for any $\sigma \in \mathrm{S}_n \backslash \{\mathrm{id}\}$ by

$$\mathrm{Lm}_{\mathrm{lex}(\sigma)} = \mathrm{Lm}_{\mathrm{lex}(\sigma)-1} \cup \mathrm{TermSet}\left(\sum_{\prod_{0 \le i < n} \mathbf{A}[i,f(i)] \in \mathrm{Lm}_0} \prod_{0 \le i < n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right]\right).$$

Consequently the set

$$\mathrm{Lm}_{n!-1} = \mathrm{Lm}_{\mathrm{lex}(n-1,n-2,\cdots,1,0)} = \bigcup_{\sigma \in \mathrm{S}_n} \mathrm{TermSet}\left(\sum_{\prod_{0 \le i < n} \mathbf{A}[i,f(i)] \in \mathrm{Lm}_0} \prod_{0 \le i < n} \mathbf{A}\left[i, \sigma^{-1} f \sigma(i)\right]\right).$$

list all graceful trees. The GLC is thus equivalent to the assertion that

$$\mathrm{Lm}_{n!-1} = \mathrm{TermSet}\left(\sum_{0\le i<n} \mathbf{A}\left[i,i\right] \det\left\{\left(\mathrm{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right)-\mathbf{A}\right)\left[\begin{array}{c} 0,\cdots,i-1,i+1,\cdots,n-1 \\ 0,\cdots,i-1,i+1,\cdots,n-1 \end{array}\right]\right\}\right).$$

The conjugation algorithm also enables us to list undirected graphs on n vertices having $n-1$ edges and no isolated vertices which admit no graceful labeling. Let $\mathbf{A}$ denote a symbolically weighted adjacency matrix for an undirected graph on $2n+1$ vertices allowing for loop edges, with entries specified by

$$\mathbf{A}\left[i,j\right] = \left\{\begin{array}{cc} a_{\min(i,j)\max(i,j)} & \text{if } 0\le i,j<n \\ 0 & \text{otherwise} \end{array}\right. , \ \forall\, 0\le i,j<2n+1.$$

We now describe a polynomial construction for listing edges set of distinct graceful sub-graphs of the complete undirected graph on n vertices having no isolated vertices. The following polynomial construction determines undirected graceful subgraphs

$$\sum_{\prod_{0\le t<n}\mathbf{A}[i_t,j_t]\in\bigcup_{\sigma\in\mathrm{S}_n}\mathrm{TermSet}\left(\prod_{0\le i<n}\sum_{0\le j\le i}\mathbf{A}[\sigma(n-1-i+j),\sigma(j)]\right)} \prod_{0\le t<n}\mathbf{A}\left[i_t,j_t\right].$$

To simplify the notation we define the following injective map from S_n to S_{2n-1} prescribed as follows

$$\forall\,\sigma\in\mathrm{S}_n,\ \text{we associate } \widetilde{\sigma}\in\mathrm{S}_{2n-1}\ \text{ such that }\ \widetilde{\sigma}\left(i\right) = \left\{\begin{array}{cc} \sigma\left(i\right) & \text{if } 0\le i<n \\ i & \text{if } n\le i<2n+1 \end{array}\right. .$$

The polynomial construction is thus equivalently expressed as

$$\sum_{\prod_{0\le t<n}\mathbf{A}[i_t,j_t]\in\bigcup_{\sigma\in\mathrm{S}_n}\mathrm{TermSet}\left(\prod_{0\le i<n}\sum_{0\le j<n}\mathbf{A}[\widetilde{\sigma}(n-1-i+\sigma^{-1}(j)),j]\right)} \prod_{0\le t<n}\mathbf{A}\left[i_t,j_t\right].$$

Consequently graceful graphs on n vertices having $(n-1)$ non loop edges and no isolated vertices are given by

$$L = \bigcup_{\substack{\sigma\in\mathrm{S}_n \\ f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}}} \left\{\prod_{0\le i<n}\mathbf{A}\left[\widetilde{\sigma}\left(n-1-i+\sigma^{-1}f\left(i\right)\right),f\left(i\right)\right]\right\}.$$

On the other hand, the set of graphs on n vertices having $(n-1)$ non loop edges and no isolated vertices which admit no graceful labeling are described by the terms in the set

$$\mathrm{TermSet}\left(\left(\sum_{0\le k<n}\mathbf{A}\left[k,k\right]\right)\sum_{\substack{0<ni_1+j_1<\cdots<ni_{n-1}+j_{n-1}<n^2-1 \\ 0\le i_t<j_t<n \\ 0<t<n}}\prod_{0<t<n}\mathbf{A}\left[i_t,j_t\right]\right)\backslash L \tag{14}$$

which in the case $n=5$ yields the terms in the polynomial

$$a_{00}a_{04}a_{12}a_{13}a_{23}+a_{04}a_{11}a_{12}a_{13}a_{23}+a_{00}a_{02}a_{03}a_{14}a_{23}+a_{00}a_{01}a_{04}a_{14}a_{23}+a_{02}a_{03}a_{11}a_{14}a_{23}+$$
$$a_{01}a_{04}a_{11}a_{14}a_{23}+a_{04}a_{12}a_{13}a_{22}a_{23}+a_{02}a_{03}a_{14}a_{22}a_{23}+a_{01}a_{04}a_{14}a_{22}a_{23}+a_{00}a_{01}a_{03}a_{13}a_{24}+$$
$$a_{00}a_{02}a_{04}a_{13}a_{24}+a_{01}a_{03}a_{11}a_{13}a_{24}+a_{02}a_{04}a_{11}a_{13}a_{24}+a_{00}a_{03}a_{12}a_{14}a_{24}+a_{03}a_{11}a_{12}a_{14}a_{24}+$$
$$a_{01}a_{03}a_{13}a_{22}a_{24}+a_{02}a_{04}a_{13}a_{22}a_{24}+a_{03}a_{12}a_{14}a_{22}a_{24}+a_{04}a_{12}a_{13}a_{23}a_{33}+a_{02}a_{03}a_{14}a_{23}a_{33}+$$
$$a_{01}a_{04}a_{14}a_{23}a_{33}+a_{01}a_{03}a_{13}a_{24}a_{33}+a_{02}a_{04}a_{13}a_{24}a_{33}+a_{03}a_{12}a_{14}a_{24}a_{33}+a_{00}a_{01}a_{02}a_{12}a_{34}+$$
$$a_{00}a_{03}a_{04}a_{12}a_{34}+a_{01}a_{02}a_{11}a_{12}a_{34}+a_{03}a_{04}a_{11}a_{12}a_{34}+a_{00}a_{02}a_{13}a_{14}a_{34}+a_{02}a_{11}a_{13}a_{14}a_{34}+$$
$$a_{01}a_{02}a_{12}a_{22}a_{34}+a_{03}a_{04}a_{12}a_{22}a_{34}+a_{02}a_{13}a_{14}a_{22}a_{34}+a_{00}a_{01}a_{23}a_{24}a_{34}+a_{01}a_{11}a_{23}a_{24}a_{34}+$$
$$a_{01}a_{22}a_{23}a_{24}a_{34}+a_{01}a_{02}a_{12}a_{33}a_{34}+a_{03}a_{04}a_{12}a_{33}a_{34}+a_{02}a_{13}a_{14}a_{33}a_{34}+a_{01}a_{23}a_{24}a_{33}a_{34}+$$
$$a_{04}a_{12}a_{13}a_{23}a_{44}+a_{02}a_{03}a_{14}a_{23}a_{44}+a_{01}a_{04}a_{14}a_{23}a_{44}+a_{01}a_{03}a_{13}a_{24}a_{44}+a_{02}a_{04}a_{13}a_{24}a_{44}+$$
$$a_{03}a_{12}a_{14}a_{24}a_{44}+a_{01}a_{02}a_{12}a_{34}a_{44}+a_{03}a_{04}a_{12}a_{34}a_{44}+a_{02}a_{13}a_{14}a_{34}a_{44}+a_{01}a_{23}a_{24}a_{34}a_{44}.$$

We now describe a construction which expresses the orbit of terms which correspond to graceful directed subgraphs of the complete graph which have no isolated vertices (not necessarily functional). Let $\mathbf{A}$ denote a symbolically weighted adjacency matrix for a directed graph on $2n+1$ vertices allowing for loop edges, with entries specified by

$$\mathbf{A}\left[i,j\right]=\begin{cases} a_{ij} & \text{if } 0\le i,j<n \\ 0 & \text{otherwise} \end{cases},\ \forall\, 0\le i,j<2n+1.$$

The edges set of distinct graceful sub-graphs of the complete directed subgraph on n vertices which have no isolated vertices are listed by the following polynomial construction

$$\sum_{\prod_{0\le t<n}\mathbf{A}[i_t,j_t]\in\bigcup_{\sigma\in\mathrm{S}_n}\mathrm{TermSet}\left(\prod_{0\le i<n}\sum_{0\le j\le i}\left(\mathbf{A}[\sigma(n-1-i+j),\sigma(j)]+\mathbf{A}[\sigma(j),\sigma(n-1-i+j)]\right)\right)}\prod_{0\le t<n}\mathbf{A}\left[i_t,j_t\right].$$

Similarly to the undirected setting, we use the injective map from S_n to S_{2n-1} prescribed as follows

$$\forall\,\sigma\in\mathrm{S}_n,\ \text{we associate }\ \widetilde{\sigma}\in\mathrm{S}_{2n-1}\ \text{ such that }\ \widetilde{\sigma}\left(i\right)=\begin{cases}\sigma\left(i\right) & \text{if } 0\le i<n\\ i & \text{if } n\le i<2n+1\end{cases}.$$

Consequently the desired construction is given by

$$L=\bigcup_{\substack{\sigma\in\mathrm{S}_n\\ f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}\\ 0\le j_0,\cdots,j_{n-1}<2}}\left(2^{-1}\prod_{0\le i<n}\mathbf{A}\left[\widetilde{\sigma}\left(n-1-i+f\left(i\right)\right),\sigma f\left(i\right)\right]^{j_i}\mathbf{A}\left[\sigma f\left(i\right),\widetilde{\sigma}\left(n-1-i+f\left(i\right)\right)\right]^{1-j_i}\right)$$

Consequently, directed subgraphs of the complete graph which admit no graceful labeling are described by terms in the set

$$\mathrm{TermSet}\left(\left(\sum_{0\le k<n}\mathbf{A}\left[k,k\right]\right)\sum_{0<n\cdot i_1+j_1<\cdots<n\cdot i_{n-1}+j_{n-1}<n^2-1}\prod_{0<t<n}\mathbf{A}\left[i_t,j_t\right]\right)\backslash L. \tag{15}$$

Signed permutations introduced in (5), enables us to enumerate labeled trees which admit a graceful labeling. Let $\mathbf{X}$ and $\mathbf{B}$ denote a symbolically weighted adjacency matrix for the complete graph on n vertices allowing for loop edges with entries given by

$$\mathbf{X}[i,j]=\begin{cases} 0 & \text{if } i=j \text{ and } i>0 \\ x^{n^{|j-i|}} & \text{otherwise} \end{cases},\ \forall\, 0\le i,j<n.$$

Consequently, the count for the number of labeled trees which admit a graceful relabeling is given by

$$n!\,x^{\frac{n^n-1}{n-1}}\sum_{g\in\mathrm{SP}_n}\frac{x\det\left\{\left(\mathrm{diag}\left(\sum_{0\le i<n}\mathbf{I}[:,i]\,\mathbf{I}[i+g(i),:]\,\mathbf{1}_{n\times1}\right)-\sum_{0\le i<n}\mathbf{I}[:,i]\,\mathbf{I}[i+g(i),:]\right)[1:,1:]\right\}}{|\mathrm{Aut}G_{\mathrm{id}+g}|\left((-1)\ \mathrm{mPer}_{\mathrm{id}+g}(\mathbf{X})\bmod\left(x^{\frac{n^n-1}{n-1}}\right)+\mathrm{mPer}_{\mathrm{id}+g}(\mathbf{X})\right)}. \tag{16}$$

Let $\mathbf{A}$ denote a symbolically weighted adjacency matrix for the complete graph on n vertices allowing for loop edges with entries given by

$$\mathbf{A}[i,j]=a_{ij},\ \forall\, 0\le i,j<n,$$

then the listing of graceful spanning subgraphs of the complete graph having no isolated vertices is listed by the terms in the sum

$$\sum_{g\in\mathrm{SP}_n}\ \sum_{f\in\left\{\sigma^{-1}(\mathrm{id}+g)\sigma\right\}_{\sigma\in\mathrm{S}_n/\mathrm{Aut}G_{\mathrm{id}+g}}\setminus\mathrm{SP}_n}\ \prod_{0\le i<n}\mathbf{A}[i,f(i)]. \tag{17}$$

We conclude with the following conjecture generalizing the GLC

Conjecture 9 : Let $\mathbf{A}$ denote a symbolically weighted $n\times n$ adjacency matrix for the directed complete graph on n vertices which includes loop edges, with entries respectively given by

$$\mathbf{A}[i,j]=a_{ij},\quad \forall\, 0\le i,j<n,$$

then $f\in\{0,\cdots,n-1\}^{\{0,\cdots,n-1\}}$ is a functional star i.e. subject to

$$\prod_{0\le i<n}\mathbf{A}[i,f(i)]\in\mathrm{TermSet}\left\{\sum_{0\le i<n}\mathbf{A}[i,i]\det\left((\mathrm{diag}(\mathbf{A}\mathbf{1}_{n\times1})-\mathbf{A})\begin{bmatrix}0,\cdots,i-1,i+1,\cdots,n-1\\0,\cdots,i-1,i+1,\cdots,n-1\end{bmatrix}\right)\right.$$

$$\left.\bmod\left\{\begin{array}{c}\mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(3)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(3)}\right]\\ \mathbf{A}\left[i_{\varrho(0)},i_{\varrho(1)}\right]\mathbf{A}\left[i_{\varrho(1)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(2)},i_{\varrho(2)}\right]\mathbf{A}\left[i_{\varrho(3)},i_{\varrho(2)}\right]\end{array}\right\}_{\substack{|\{i_0,i_1,i_2\}|=3\\ \varrho\in\mathrm{S}_3}}\right\}$$

$$\Longleftrightarrow$$

$$0=\sum_{\sigma\in(\mathrm{S}_n/\mathcal{I}_n)/\mathrm{Aut}G_f}\ \sum_{g\in\mathcal{T}_n}\ \prod_{\gamma\in(\mathrm{S}_n/\mathcal{I}_n)/\mathrm{Aut}G_g}\left(\prod_{0\le i<n}2^{n^{\left|\sigma^{-1}f\sigma(i)-i\right|}}-\prod_{0\le j<n}2^{n^{\left|\gamma^{-1}g\gamma(j)-j\right|}}\right)$$

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