

GRÖBNER-SHIRSHOV BASES FOR TEMPERLEY-LIEB ALGEBRAS OF THE COMPLEX REFLECTION GROUP OF TYPE $G(d, 1, n)$

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ABSTRACT. We construct a Gröbner-Shirshov basis of the Temperley-Lieb algebra $\mathcal{T}(d, n)$ of the complex reflection group $G(d, 1, n)$, inducing the standard monomials expressed by the generators $\{E_i\}$ of $\mathcal{T}(d, n)$. This result generalizes the one for the Coxeter group of type B_n in [13]. We also give a combinatorial interpretation of the standard monomials of $\mathcal{T}(d, n)$, relating to the fully commutative elements of the complex reflection group $G(d, 1, n)$. In this way, we obtain the dimension formula of $\mathcal{T}(d, n)$.

1. INTRODUCTION

The Temperley-Lieb algebra appears originally in the context of statistical mechanics [26], and later its structure has been studied in connection with knot theory, where it is known to be a quotient of the Hecke algebra of type A in [10].

Our approach to understanding the structure of Temperley-Lieb algebras is from the noncommutative Gröbner basis theory, or the *Gröbner-Shirshov basis theory* more precisely, which provides a powerful tool for understanding the structure of (non-)associative algebras and their representations, especially in computational aspects. With the ever-growing power of computers, it is now viewed as a universal engine behind algebraic or symbolic computation.

The main interest of the notion of Gröbner-Shirshov bases stems from Shirshov's Composition Lemma and his algorithm [22] for Lie algebras and independently from Buchberger's algorithm [4] of computing Gröbner bases for commutative algebras. In [2], Bokut applied Shirshov's method to associative algebras, and Bergman mentioned the diamond lemma for ring theory [1]. The main idea of the Composition-Diamond lemma is to establish an algorithm for constructing standard monomials of a quotient algebra by a two-sided ideal generated by a set of relations called Gröbner-Shirshov basis. Our set of standard monomials in this algorithm is a minimal set of monomials which are indivisible

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by any leading monomial of the polynomials in the Gröbner-Shirshov basis. The details on the Gröbner-Shirshov basis theory are given in Section 2.

The Gröbner-Shirshov bases for Coxeter groups of classical and exceptional types were completely determined by Bokut, Lee *et al.* in [3, 16, 18, 19, 25]. The cases for Hecke algebras and Temperley-Lieb algebras of type A as well as for Ariki-Koike algebras were calculated by Lee *et al.* in [11, 12, 17].

This paper consists of two principal parts as follows :

1) In the first part of this paper, extending the result for type B_n in [13], we construct a Gröbner-Shirshov basis for the Temperley-Lieb algebra $\mathcal{T}(d, n)$ of the complex reflection group of type $G(d, 1, n)$ and compute the dimension of $\mathcal{T}(d, n)$, by enumerating the standard monomials which are in bijection with the fully commutative elements.

The main Theorem goes as follows :

Theorem 1.1 (Main theorem 4.3). *The algebra $\mathcal{T}(d, n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(d, n)}$ with respect to our monomial order $<$ (i.e. degree-lexicographic order with $E_0 < E_1 < \cdots < E_{n-1}$) :*

$$\begin{aligned} & E_0^d - (d-1)\delta E_0, \\ & E_i E_0 - E_0 E_i && \text{for } 1 < i \leq n-1, \\ & E_0 E_1 E_0^k E_1 - (k+1)E_0 E_1 && \text{for } 1 \leq k < d, \\ & E_1 E_0^k E_1 E_0 - (k+1)E_1 E_0 && \text{for } 1 \leq k < d, \\ \widehat{R}_{\mathcal{T}(d, n)} : & E_{i,1} E_0^k E^{1,j} E_i - E_{i-2,1} E_0^k E^{1,j} E_i && \text{for } i > j+1 \geq 1, \\ & E_i^2 - \delta E_i && \text{for } 1 \leq i \leq n-1, \\ & E_i E_j - E_j E_i && \text{for } i > j+1 > 1, \\ & E_{i,j} E_i - E_{i-2,j} E_i && \text{for } i > j > 0, \\ & E_j E_{i,j} - E_j E_{i,j+2} && \text{for } i > j > 0. \end{aligned}$$

The cardinality of the set of $\widehat{R}_{\mathcal{T}(d, n)}$ -standard monomials is

$$\dim \mathcal{T}(d, n) = (d-1)(\mathfrak{F}_{n, n-1}(d) - 1) + dC_n$$

where $\mathfrak{F}_{n, k}(x) = \sum_{s=0}^k C(n, s)x^{k-s}$ is the (n, k) th Catalan triangle polynomial, introduced in [20, §2.3].

We remark here that by specializing $d = 2$, we recover the formula for the Temperley-Lieb algebra $\mathcal{T}(B_n)$ of the B_n .

2) In the second part of this paper, we try to understand some combinatorial aspects on the dimension of the Temperley-Lieb algebra $\mathcal{T}(d, n)$.

In [14, 15], Kleshchev and Ram constructed a class of representations called homogeneous representations of the KLR algebra (or a quiver Hecke algebra) and showed that the homogeneous representations can be parametrized by the set of *fully commutative*

elements of the corresponding Coxeter group. These elements were studied by Fan [5] and Stembridge [23] for Coxeter groups and parametrize the bases of the corresponding Temperley-Lieb algebras.

Motivated by the bijective correspondance of the homogeneous representations of KLR algebras (or a quiver Hecke algebras) with fully commutative elements, Feinberg and Lee studied the fully commutative elements of the Coxeter groups of types *A* and *D* in their papers [7] and [8] and obtained a dimension formula of the homogeneous representations by using Dyck paths.

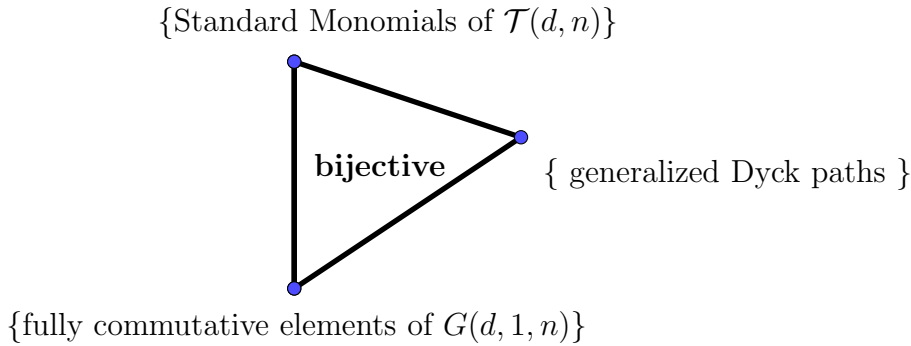


FIGURE 1. These three sets are bijective for $G(d, 1, n)$

Their combinatorial strategy is as follows :

For type *A*, we decompose the fully commutative elements into natural subsets according to the lengths of fully commutative elements and show that the fully commutative elements of a given length k can be parametrized by the Dyck paths of semi-length n with the property that (the sum of peak heights)–(the number of peaks)= k using the canonical form of reduced words for fully commutative elements. For type *D*, from the set of fully commutative element written in canonical form, we decompose the fully commutative elements according to the same type of prefixes and call the set with exactly the same prefix a *collection*. Then they prove that some collections have the same number of elements by showing that those collections contain the same set of prefixes. We then group those collections together and call the group a *Packet*. This decomposition process is called the *packet decomposition*.

In the article [6], we generalize the above strategy to the complex reflection group of type $G(d, 1, n)$ for the enumeration of the fully commutative elements of $G(d, 1, n)$. The main result on the combinatorial aspects in Section 5 is that there is a bijection between the standard monomials of $\mathcal{T}(d, n)$ in the first part of this paper and the fully commutative elements of $G(d, 1, n)$ in the second part of this paper. In this way, we realize an explicit computation of the dimension of $\mathcal{T}(d, n)$.

Our canonical forms for the reduced elements will be slightly in different form from the ones in [6] and so is the packet decomposition process. Though, the fully commutative

elements in this paper are better adapted for the standard monomials induced from a Gröbner-Shirshov basis.

2. PRELIMINARIES

2.1. Gröbner-Shirshov basis. We recall a basic theory of *Gröbner-Shirshov bases* for associative algebras so as to make the paper self-contained. Some properties listed below without proofs are well-known and the readers are invited to see the references cited next to each claim for further detailed explanations.

Let X be a set and let $\langle X \rangle$ be the free monoid of associative words on X . We denote the empty word by 1 and the *length* (or *degree*) of a word u by $l(u)$. We define a total-order $<$ on $\langle X \rangle$, called a *monomial order* as follows ;

$$\text{if } x < y \text{ implies } axb < ayb \text{ for all } a, b \in \langle X \rangle.$$

Fix a monomial order $<$ on $\langle X \rangle$ and let $\mathbb{F}\langle X \rangle$ be the free associative algebra generated by X over a field \mathbb{F} . Given a nonzero element $p \in \mathbb{F}\langle X \rangle$, we denote by \bar{p} the monomial (called the *leading monomial*) appearing in p , which is maximal under the ordering $<$. Thus $p = \alpha\bar{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in \langle X \rangle$, $\alpha \neq 0$ and $w_i < \bar{p}$ for all i . If $\alpha = 1$, p is said to be *monic*.

Let S be a subset of monic elements in $\mathbb{F}\langle X \rangle$, and let I be the two-sided ideal of $\mathbb{F}\langle X \rangle$ generated by S . Then we say that the algebra $A = \mathbb{F}\langle X \rangle/I$ is *defined by* S .

Definition 2.1. Given a subset S of monic elements in $\mathbb{F}\langle X \rangle$, a monomial $u \in \langle X \rangle$ is said to be *S -standard* (or *S -reduced*) if u cannot be expressed as $a\bar{s}b$, that is $u \neq a\bar{s}b$, for any $s \in S$ and $a, b \in \langle X \rangle$. Otherwise, the monomial u is said to be *S -reducible*.

Lemma 2.2 ([1, 2]). *Every $p \in \mathbb{F}\langle X \rangle$ can be expressed as*

$$(2.1) \quad p = \sum \alpha_i a_i s_i b_i + \sum \beta_j u_j,$$

where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, u_j \in \langle X \rangle$, $s_i \in S$, $a_i \bar{s}_i b_i \leq \bar{p}$, $u_j \leq \bar{p}$ and u_j are *S -standard*.

Remark. The term $\sum \beta_j u_j$ in the expression (2.1) is called a *normal form* (or a *remainder*) of p with respect to the subset S (and with respect to the monomial order $<$). In general, a normal form is not unique.

As an immediate corollary of Lemma 2.2, we obtain:

Proposition 2.3. *The set of S -standard monomials spans the algebra $A = \mathbb{F}\langle X \rangle/I$ defined by the subset S , as a vector space over \mathbb{F} .*

Let p and q be monic elements in $\mathbb{F}\langle X \rangle$ with leading monomials \bar{p} and \bar{q} . We define the *composition* of p and q as follows.

Definition 2.4. (a) If there exist a and b in $\langle X \rangle$ such that $\bar{p}a = b\bar{q} = w$ with $l(\bar{p}) > l(b)$, then we define $(p, q)_w := pa - bq$, called the *composition of intersection*.

(b) If there exist a and b in $\langle X \rangle$ such that $a \neq 1$, $a\bar{p}b = \bar{q} = w$, then we define $(p, q)_{a,b} := apb - q$, called the *composition of inclusion*.

Let $p, q \in \mathbb{F}\langle X \rangle$ and $w \in \langle X \rangle$. We define the *congruence relation* on $\mathbb{F}\langle X \rangle$ as follows: $p \equiv q \pmod{(S; w)}$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{F}$, $a_i, b_i \in \langle X \rangle$, $s_i \in S$, and $a_i \bar{s}_i b_i < w$.

Definition 2.5. A subset S of monic elements in $\mathbb{F}\langle X \rangle$ is said to be *closed under composition* if

$$(p, q)_w \equiv 0 \pmod{(S; w)} \text{ and } (p, q)_{a,b} \equiv 0 \pmod{(S; w)} \text{ for all } p, q \in S, a, b \in \langle X \rangle \\ \text{whenever the compositions } (p, q)_w \text{ and } (p, q)_{a,b} \text{ are defined.}$$

The following theorem is a main tool for our results in the subsequent sections.

Theorem 2.6 (Composition Lemma [1, 2]). *Let S be a subset of monic elements in $\mathbb{F}\langle X \rangle$. Then the following conditions are equivalent:*

- (a) S is closed under composition.
- (b) For each $p \in \mathbb{F}\langle X \rangle$, a normal form of p with respect to S is unique.
- (c) The set of S -standard monomials forms a linear basis of the algebra $A = \mathbb{F}\langle X \rangle / I$ defined by S .

Definition 2.7. A subset S of monic elements in $\mathbb{F}\langle X \rangle$ satisfying one of the equivalent conditions in Theorem 2.6 is called a *Gröbner-Shirshov basis* for the algebra A defined by S .

3. TEMPERLEY-LIEB ALGEBRAS OF TYPES A_{n-1} AND B_n

3.1. Temperley-Lieb algebra of type A_{n-1} . First, we review the results on Temperley-Lieb algebras $\mathcal{T}(A_{n-1})$ ($n \geq 2$). Define $\mathcal{T}(A_{n-1})$ to be the associative algebra over the complex field \mathbb{C} , generated by $X = \{E_1, E_2, \dots, E_{n-1}\}$ with defining relations:

$$\begin{aligned} E_i^2 &= \delta E_i & \text{for } 1 \leq i \leq n-1, & \text{ (idempotent relations)} \\ R_{\mathcal{T}(A_{n-1})} : E_i E_j &= E_j E_i & \text{for } i > j+1, & \text{ (commutative relations)} \\ E_i E_j E_i &= E_i & \text{for } j = i \pm 1, & \text{ (untwisting relations)} \end{aligned}$$

where $\delta \in \mathbb{C}$ is a parameter. We call the first and second relations to be the quadratic and commutative relations, respectively. Our monomial order $<$ is taken to be the degree-lexicographic order with

$$E_1 < E_2 < \dots < E_{n-1}.$$

We write $E_{i,j} = E_i E_{i-1} \dots E_j$ for $i \geq j$ (hence $E_{i,i} = E_i$). By convention $E_{i,i+1} = 1$ for $i \geq 1$.

$$(3.4) \quad \begin{array}{l} \text{- quadratic relations : } s_i^2 = 1 \text{ for } 0 \leq i \leq n-1 \\ \text{- braid relations : } \begin{cases} s_i s_j = s_j s_i & \text{for } |i-j| > 1 \text{ } (i, j = 1, \dots, n), \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } 1 \leq i < n-1, \\ s_1 s_0 s_1 s_0 = s_0 s_1 s_0 s_1. \end{cases} \end{array}$$

Then $W(B_n)$ is the Weyl group of type B_n , which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$.

Let $\mathcal{H} := \mathcal{H}(B_n)$ be the Iwahori-Hecke algebra of type B_n , which is the associative algebra over $A := \mathbb{Z}[q, q^{-1}]$, generated by $\{T_i\}_{0 \leq i < n}$ with defining relations :

$$\begin{array}{l} \text{- quadratic relations : } (T_i - q)(T_i + q^{-1}) = 0 \text{ for } 0 \leq i \leq n-1, \\ \text{- braid relations : } \begin{cases} T_i T_j = T_j T_i & \text{for } |i-j| > 1 \text{ } (i, j = 0, \dots, n-1), \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \leq i < n-1, \\ T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1 \end{cases} \end{array}$$

where q is a parameter and $T_i := T_{s_i}$, the generator corresponding to the reflection s_i .

Let $\mathcal{T}(B_n)$ ($n \geq 2$) be the Temperley-Lieb algebra of type B_n , which is a quotient of the Hecke algebra $\mathcal{H}(B_n)$. $\mathcal{T}(B_n)$ is the associative algebra over the complex field \mathbb{C} , generated by $X = \{E_0, E_1, \dots, E_{n-1}\}$ with defining relations :

$$(3.5) \quad R_{\mathcal{T}(B_n)} : \begin{array}{l} E_i^2 = \delta E_i \quad \text{for } 0 \leq i \leq n-1, \\ E_i E_j = E_j E_i \quad \text{for } i > j+1, \\ E_i E_j E_i = E_i \quad \text{for } j = i \pm 1, \ i, j > 0, \\ E_i E_j E_i E_j = 2E_i E_j \quad \text{for } \{i, j\} = \{0, 1\}, \end{array}$$

where $\delta \in \mathbb{C}$ is a parameter. Fix our monomial order $<$ to be the degree-lexicographic order with

$$E_0 < E_1 < \dots < E_{n-1}.$$

We write $E_{i,j} = E_i E_{i-1} \dots E_j$ for $i \geq j \geq 0$, and $E^{i,j} = E_i E_{i+1} \dots E_j$ for $i \leq j$. By convention, $E_{i,i+1} = 1$ and $E^{i+1,i} = 1$ for $i \geq 0$. We can easily prove the following lemma.

Lemma 3.3. ([13, §4]) *The following relations hold in $\mathcal{T}(B_n)$:*

$$E_{i,0} E^{1,j} E_i = E_{i-2,0} E^{1,j} E_i$$

for $i > j+1 \geq 1$.

Let $\widehat{R}_{\mathcal{T}(B_n)}$ be the collection of defining relations (3.5) combined with (3.1) and the relations in Lemma 3.3. From this, we denote by $M_{\mathcal{T}(B_n)}$ the set of $\widehat{R}_{\mathcal{T}(B_n)}$ -standard monomials. Among the monomials in $M_{\mathcal{T}(B_n)}$, we consider the monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$ -standard. That is, we take only $\widehat{R}_{\mathcal{T}(B_n)}$ -standard monomials which are not of the

form (3.2). This set is denoted by $M_{\mathcal{T}(B_n)}^0$. Note that each monomial in $M_{\mathcal{T}(B_n)}^0$ contains E_0 . We decompose the set $M_{\mathcal{T}(B_n)}^0$ into two parts as follows :

$$M_{\mathcal{T}(B_n)}^0 = M_{\mathcal{T}(B_n)}^{0+} \amalg M_{\mathcal{T}(B_n)}^{0-}$$

where the monomials in $M_{\mathcal{T}(B_n)}^{0+}$ are of the form

$$E_0 E_{i_1, j_1} E_{i_2, j_2} \cdots E_{i_p, j_p} \quad (0 \leq p \leq n-1)$$

with

$$\begin{aligned} 1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n-1, \\ i_1 \geq j_1, \quad i_2 \geq j_2, \quad \dots, \quad i_p \geq j_p, \quad \text{and} \\ j_k > 0 \quad (1 \leq k < p) \quad \text{implies} \quad j_k < j_{k+1} \end{aligned}$$

(the case of $p = 0$ is the monomial E_0), and the monomials in $M_{\mathcal{T}(B_n)}^{0-}$ are of the form

$$E'_{i_1, j_1} E_{i_2, j_2} \cdots E_{i_p, j_p} \quad (1 \leq p \leq n-1)$$

with

$$E'_{i, j} = E_{i, 0} E^{1, j}$$

and the same restriction on i 's and j 's as above. It can be easily checked that $M_{\mathcal{T}(B_n)}^0$ is the set of $\widehat{R}_{\mathcal{T}(B_n)}$ -standard monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$ -standard.

Counting the number of elements in $M_{\mathcal{T}(B_n)}^0$, we obtain the following theorem.

Theorem 3.4. ([13, §4]) *The algebra $\mathcal{T}(B_n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(B_n)}$ with respect to our monomial order $<$:*

$$\begin{aligned} \widehat{R}_{\mathcal{T}(B_n)} : \quad & E_i^2 - \delta E_i && \text{for } 0 \leq i \leq n-1, \\ & E_i E_j - E_j E_i && \text{for } i > j + 1, \\ & E_{i, j} E_i - E_{i-2, j} E_i && \text{for } i > j > 0, \\ & E_j E_{i, j} - E_j E_{i, j+2} && \text{for } i > j > 0. \\ & E_i E_j E_i E_j - 2 E_i E_j && \text{for } \{i, j\} = \{0, 1\}, \\ & E_{i, 0} E^{1, j} E_i - E_{i-2, 0} E^{1, j} E_i && \text{for } i > j + 1 \geq 1. \end{aligned}$$

The cardinality of the set $M_{\mathcal{T}(B_n)}$, i.e. the set of $\widehat{R}_{\mathcal{T}(B_n)}$ -standard monomials, is

$$\dim \mathcal{T}(B_n) = (n+2)C_n - 1.$$

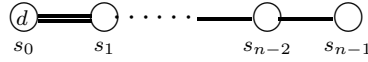
Example 3.5. For $n = 3$, we have $|M_{\mathcal{T}(B_3)}| = (3+2)C_3 - 1 = 24$.

We enumerate the standard monomials in $M_{\mathcal{T}(B_3)}^0 = M_{\mathcal{T}(B_3)} \setminus M_{\mathcal{T}(A_3)}$, that is, the standard monomials containing E_0 , of cardinality $24 - 5 = 19$ as follows (using the notation $E'_{i, j} := E_{i, 0} E^{1, j}$) :

$$\begin{aligned}
 & E_0, E_0E_{1,0}, E_{1,0}, E_0E_1, E'_1 = E_1E_0E_1, E_0E_{2,0}, E_{2,0}, E_0E_{2,1}, \\
 & E'_{2,1} = E_2E_1E_0E_1, E_0E_2, E'_2 = E_2E_1E_0E_1E_2, E_0E_{1,0}E_{2,0}, E_{1,0}E_{2,0}, \\
 & E_0E_{1,0}E_{2,1}, E_{1,0}E_{2,1}, E_0E_{1,0}E_2, E_{1,0}E_2, E_0E_1E_2, E'_1E_2 = E_1E_0E_1E_2.
 \end{aligned}$$

4. GRÖBNER-SHIRSHOV BASES FOR TEMPERLEY-LIEB ALGEBRAS OF THE COMPLEX REFLECTION GROUP OF TYPE $G(d, 1, n)$

In this section, we try to generalize the result of the previous section to the complex reflection group of type $G(d, 1, n)$, where $d \geq 2$ and $n \geq 2$. Let $W_n := G(d, 1, n)$ be the wreath product $(\mathbb{Z}/d\mathbb{Z})^n \rtimes \mathfrak{S}_n$ of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ and the symmetric group \mathfrak{S}_n , with the following diagram :



In other words, W_n is the group with generators $\{s_i\}_{0 \leq i < n}$ and the following defining relations :

$$\begin{aligned}
 & s_i^2 = 1 \text{ for } 1 \leq i \leq n-1 \quad \text{and } s_0^d = 1, \\
 \text{braid relations : } & \begin{cases} s_i s_j = s_j s_i & \text{if } |i-j| > 1 \quad (i, j = 0, \dots, n-1), \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{if } 1 \leq i < n-1, \\ s_1 s_0 s_1 s_0 = s_0 s_1 s_0 s_1 \end{cases}
 \end{aligned}$$

Let $\mathcal{H}_n := \mathcal{H}(W_n)$ be the Ariki-Koike algebra of type $G(d, 1, n)$, cyclotomic Hecke algebra which is the associative algebra over $A := \mathbb{Z}[\zeta, q, q^{-1}]$, generated by $\{T_i\}_{0 \leq i < n}$ with defining relations :

$$\begin{aligned}
 & (T_i - q)(T_i + q^{-1}) = 0 \text{ for } 0 \leq i \leq n-1, \\
 & (T_0 - q^{m_0})(T_0 - q^{m_1}\zeta)(T_0 - q^{m_2}\zeta^2) \cdots (T_0 - q^{m_{d-1}}\zeta^{d-1}) = 0, \\
 \text{braid relations : } & \begin{cases} T_i T_j = T_j T_i & \text{for } |i-j| > 1 \quad (i, j = 0, \dots, n-1), \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \leq i < n-1, \\ T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1 \end{cases}
 \end{aligned}$$

where ζ is a primitive d^{th} root of unity and $T_i := T_{s_i}$, the generator corresponding to the reflection s_i , and $m_i \in \mathbb{N}$.

Similarly, we define the Temperley-Lieb algebra of $G(d, 1, n)$ ($d, n \geq 2$), denoted by $\mathcal{T}(d, n)$, as a quotient of the Ariki-Koike algebra of $G(d, 1, n)$ ($d, n \geq 2$).

Definition 4.1. The Temperley-Lieb algebra $\mathcal{T}(d, n)$ is the associative algebra over the complex field \mathbb{C} , generated by $X = \{E_0, E_1, \dots, E_{n-1}\}$ with defining relations:

$$(4.1) \quad R_{\mathcal{T}(d,n)} : \begin{aligned} E_i^2 &= \delta E_i && \text{for } 1 \leq i \leq n-1, \\ E_0^d &= (d-1)\delta E_0, \\ E_i E_j &= E_j E_i && \text{for } i > j+1, \\ E_i E_j E_i &= E_i && \text{for } j = i \pm 1, \ i, j > 0, \\ E_0 E_1 E_0^k E_1 &= (k+1)E_0 E_1 && \text{for } 1 \leq k < d, \\ E_1 E_0^k E_1 E_0 &= (k+1)E_1 E_0 && \text{for } 1 \leq k < d, \end{aligned}$$

where $\delta \in \mathbb{C}$ is a parameter.

Fix the same monomial order $<$ as in the previous section, i.e. the degree-lexicographic order with $E_0 < E_1 < \dots < E_{n-1}$.

Lemma 4.2. *The following relations hold in $\mathcal{T}(d, n)$;*

$$E_{i,1} E_0^k E^{1,j} E_i = E_{i-2,1} E_0^k E^{1,j} E_i$$

for $i > j+1 \geq 1$ and $1 \leq k < d$.

Proof. Since $2 \leq i \leq n-1$ and $0 \leq j \leq i-2$, we calculate that

$$E_{i,1} E_0^k E^{1,j} E_i = (E_i E_{i-1} E_i) E_{i-2,1} E_0^k E^{1,j} = E_i E_{i-2,1} E_0^k E^{1,j} = E_{i-2,1} E_0^k E^{1,j} E_i$$

by the commutative relations and $E_i E_{i-1} E_i = E_i$. \square

The set of defining relations (4.1) combined with (3.1) and the relations in Lemma 4.2 is denoted by $\widehat{R}_{\mathcal{T}(d,n)}$. We enumerate $\widehat{R}_{\mathcal{T}(d,n)}$ -standard monomials containing E_0 considering the following two types of standard monomials.

The first type of standard monomials is :

$$(4.2) \quad (E_{i_1,1} E_0^{k_1}) (E_{i_2,1} E_0^{k_2}) \cdots (E_{i_q,1} E_0^{k_q}) E_{i_{q+1}, j_{q+1}} \cdots E_{i_p, j_p} \quad (1 \leq p \leq n-1, 1 \leq q \leq p)$$

with

$$\begin{aligned} 1 \leq k_1, k_2, \dots, k_q < d, \quad 0 \leq i_1 < i_2 < \cdots < i_q < i_{q+1} < \cdots < i_p \leq n-1, \\ 1 \leq j_{q+1} < j_{q+2} < \cdots < j_p \leq n-1, \quad i_{q+1} \geq j_{q+1}, \ i_{q+2} \geq j_{q+2}, \ \dots, \ i_p \geq j_p. \end{aligned}$$

In a similar way as we did in the proof of [13, Theorem 4.2], to count the number of standard monomials of the above form (4.2), we associate those standard monomials bijectively to the paths defined below.

To each monomial

$$(E_{i_1,1} E_0^{k_1}) \cdots (E_{i_q,1} E_0^{k_q}) E_{i_{q+1}, j_{q+1}} \cdots E_{i_p, j_p},$$

we associate a unique path, which we call a $G(d, 1, n)$ -Dyck path,

$$(i_1, 0)^{k_1} \rightarrow \cdots \rightarrow (i_q, 0)^{k_q} \rightarrow (i_{q+1}, j_{q+1}) \rightarrow \cdots \rightarrow (i_p, j_p) \rightarrow (n, n).$$

Here, a path consists of moves to the east or to the north, *not above the diagonal* in the lattice plane. The move from (i, j) to (i', j') ($i < i'$ and $j < j'$) is a concatenation of eastern moves followed by northern moves. As an example, the monomial $E_0^2 E_{1,0} E_{2,1}$ in $\mathcal{T}(3, 3)$ corresponds to the figure below

$$\mathbf{A}(0, 0)^2 \rightarrow \mathbf{B}(1, 0) \rightarrow \mathbf{C}(2, 1) \rightarrow (3, 3).$$

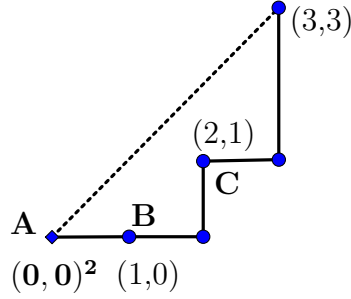


TABLE 1. The path corresponding to $E_0^2 E_{1,0} E_{2,1}$ in $\mathcal{T}(3, 3)$

We note that in the move from $B(1, 0)$ to $C(2, 1)$, there is a concatenation of an eastern move followed by a northern move as well as in the move from $C(2, 1)$ to the plot $(3, 3)$ as shown in Table 1.

The second type of standard monomials is:

$$(4.3) \quad (E_{i_1,1} E_0^k E^{1,j_1}) E_{i_2,j_2} E_{i_3,j_3} \cdots E_{i_p,j_p} \quad (1 \leq p \leq n-1)$$

with

$$\begin{aligned} 1 \leq k < d, \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \\ 1 \leq j_1 < j_2 < \cdots < j_p \leq n-1, \quad i_1 \geq j_1, \quad i_2 \geq j_2, \quad \dots, \quad i_p \geq j_p. \end{aligned}$$

Now we let $M_{\mathcal{T}(d,n)}$ be the set of standard monomials of the forms (4.2) and (4.3) combined with (3.2).

The following is our main theorem.

Theorem 4.3. *The algebra $\mathcal{T}(d, n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(d, n)}$ with respect to our monomial order $<$ (i.e. degree-lexicographic order with $E_0 < E_1 < \cdots < E_{n-1}$) :*

$$\begin{aligned} & E_0^d - (d-1)\delta E_0, \\ & E_i E_0 - E_0 E_i && \text{for } 1 < i \leq n-1, \\ & E_0 E_1 E_0^k E_1 - (k+1)E_0 E_1 && \text{for } 1 \leq k < d, \\ & E_1 E_0^k E_1 E_0 - (k+1)E_1 E_0 && \text{for } 1 \leq k < d, \\ \widehat{R}_{\mathcal{T}(d, n)} : & E_{i,1} E_0^k E^{1,j} E_i - E_{i-2,1} E_0^k E^{1,j} E_i && \text{for } i > j+1 \geq 1, \\ & E_i^2 - \delta E_i && \text{for } 1 \leq i \leq n-1, \\ & E_i E_j - E_j E_i && \text{for } i > j+1 > 1, \\ & E_{i,j} E_i - E_{i-2,j} E_i && \text{for } i > j > 0, \\ & E_j E_{i,j} - E_j E_{i,j+2} && \text{for } i > j > 0. \end{aligned}$$

The cardinality of the set $M_{\mathcal{T}(d, n)}$, i.e. the set of $\widehat{R}_{\mathcal{T}(d, n)}$ -standard monomials, is

$$\dim \mathcal{T}(d, n) = (d-1)(\mathfrak{F}_{n, n-1}(d) - 1) + dC_n$$

where $\mathfrak{F}_{n, k}(x) = \sum_{s=0}^k C(n, s)x^{k-s}$ is the (n, k) th Catalan triangle polynomial, introduced in [20, §2.3].

Proof. Since the set of relations (3.1) which do not contain E_0 is already closed under composition, we have only to check for the relations containing E_0 , to show that $\widehat{R}_{\mathcal{T}(d, n)}$ is closed under composition.

(I) First, consider the compositions between $E_0^d - (d-1)\delta E_0$ and another relation.

- 1) $(E_i E_0 - E_0 E_i) E_0^{d-1} - E_i (E_0^d - (d-1)\delta E_0)$
 $= -E_0 E_i E_0^{d-1} + (d-1)\delta E_i E_0 = -E_i (E_0^d - (d-1)\delta E_0) = 0.$
- 2) $(E_0^d - (d-1)\delta E_0) E_1 E_0^k E_1 - E_0^{d-1} (E_0 E_1 E_0^k E_1 - (k+1)E_0 E_1)$
 $= -(d-1)\delta E_0 E_1 E_0^k E_1 + (k+1)E_0^d E_1$
 $= -(d-1)\delta(k+1)E_0 E_1 + (k+1)(d-1)\delta E_0 E_1 = 0.$
- 3) $(E_1 E_0^k E_1 E_0 - (k+1)E_1 E_0) E_0^{d-1} - E_1 E_0^k E_1 (E_0^d - (d-1)\delta E_0)$
 $= -(k+1)E_1 E_0^d + (d-1)\delta E_1 E_0^k E_1 E_0$
 $= -(k+1)(d-1)\delta E_1 E_0 + (d-1)\delta(k+1)E_1 E_0 = 0.$

(II) Next, we take the relation $E_i E_0 - E_0 E_i$ and another relation.

- 1) $(E_i^2 - \delta E_i) E_0 - E_i (E_i E_0 - E_0 E_i) = -\delta E_i E_0 + E_i E_0 E_i = E_0(-\delta E_0 + E_i^2) = 0.$
- 2) $(E_i E_0 - E_0 E_i) E_1 E_0^k E_1 - E_i (E_0 E_1 E_0^k E_1 - (k+1)E_0 E_1)$
 $= -E_0 E_i E_1 E_0^k E_1 + (k+1)E_i E_0 E_1 = -E_i (E_0 E_1 E_0^k E_1 - (k+1)E_0 E_1) = 0.$
- 3) $(E_{i,1} E_0^k E^{1,j} E_i - E_{i-2,1} E_0^k E^{1,j} E_i) E_0 - E_{i,1} E_0^k E^{1,j} (E_i E_0 - E_0 E_i)$
 $= -E_{i-2,1} E_0^k E^{1,j} E_i E_0 + E_{i,1} E_0^k E^{1,j} E_0 E_i$
 $= (-E_{i-2,1} E_0^k E^{1,j} E_i + E_{i,1} E_0^k E^{1,j} E_i) E_0 = 0.$
- 4) $(E_i E_j - E_j E_i) E_0 - E_i (E_j E_0 - E_0 E_j) = -E_j E_i E_0 + E_i E_0 E_j = 0.$

- 5) $(E_{i,j}E_i - E_{i-2,j}E_i)E_0 - E_{i,j}(E_iE_0 - E_0E_i)$
 $= -E_{i-2}E_iE_0 + E_{i,j}E_0E_i = (-E_{i-2}E_i + E_{i,j}E_i)E_0 = 0.$
- 6) $(E_jE_{i,j} - E_jE_{i,j+2})E_0 - E_jE_{i,j+1}(E_jE_0 - E_0E_j)$
 $= -E_jE_{i,j+2}E_0 + E_jE_{i,j+1}E_0E_j = (-E_jE_{i,j+2} + E_jE_{i,j})E_0 = 0.$

(III) Calculate for the relation $E_0E_1E_0^kE_1 - (k+1)E_0E_1$ and another relation.

- 1) $(E_0E_1E_0^kE_1 - (k+1)E_0E_1)E_0^kE_1 - E_0E_1E_0^{k-1}(E_0E_1E_0^kE_1 - (k+1)E_0E_1)$
 $= -(k+1)E_0E_1E_0^kE_1 + (k+1)E_0E_1E_0^kE_1 = 0.$
- 2) $(E_0E_1E_0^kE_1 - (k+1)E_0E_1)E_0 - E_0(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -(k+1)E_0E_1E_0 + (k+1)E_0E_1E_0 = 0.$
- 3) $(E_0E_1E_0^kE_1 - (k+1)E_0E_1)E_0^kE_1E_0 - E_0E_1E_0^k(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -(k+1)E_0E_1E_0^kE_1E_0 + (k+1)E_0E_1E_0^kE_1E_0 = 0.$
- 4) $(E_1E_0^kE_1E_0 - (k+1)E_1E_0)E_0^{k-1}E_1 - E_1E_0^{k-1}(E_0E_1E_0^kE_1 - (k+1)E_0E_1)$
 $= -(k+1)E_1E_0^kE_1 + (k+1)E_1E_0^kE_1 = 0.$
- 5) $(E_1E_0^kE_1E_0 - (k+1)E_1E_0)E_1E_0^kE_1 - E_1E_0^kE_1(E_0E_1E_0^kE_1 - (k+1)E_0E_1)$
 $= -(k+1)E_1E_0E_1E_0^kE_1 + (k+1)E_1E_0^kE_1E_0E_1$
 $= -(k+1)E_1(k+1)E_0E_1 + (k+1)^2E_1E_0E_1 = 0.$
- 6) $(E_0E_1E_0^kE_1 - (k+1)E_0E_1)E_1 - E_0E_1E_0^k(E_1^2 - \delta E_1)$
 $= -(k+1)E_0E_1^2 + \delta E_0E_1E_0^kE_1 = -(k+1)\delta E_0E_1 + \delta(k+1)E_0E_1 = 0.$
- 7) $(E_0E_1E_0^kE_1 - (k+1)E_0E_1)E_{i,1} - E_0E_1E_0^k(E_1E_{i,1} - E_1E_{i,3})$
 $= -(k+1)E_0E_1E_{i,1} + E_0E_1E_0^kE_1E_{i,3}$
 $= -(k+1)E_0E_1E_{i,1} + (k+1)E_0E_1E_{i,3} = -(k+1)E_0(E_1E_{i,1} - E_1E_{i,3}) = 0.$

(IV) Check for the relation $E_1E_0^kE_1E_0 - (k+1)E_1E_0$ and another relation.

- 1) $(E_1E_0^kE_1E_0 - (k+1)E_1E_0)E_0^{k-1}E_1E_0 - E_1E_0^k(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -(k+1)E_1E_0^kE_1E_0 + (k+1)E_1E_0^kE_1E_0 = 0.$
- 2) $(E_1^2 - \delta E_1)E_0^kE_1E_0 - E_1(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -\delta E_1E_0^kE_1E_0 + (k+1)E_1^2E_0 = -\delta(k+1)E_1E_0 + (k+1)\delta E_1E_0 = 0.$
- 3) $(E_iE_1 - E_1E_i)E_0^kE_1E_0 - E_i(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -E_1E_iE_0^kE_1E_0 + (k+1)E_iE_1E_0 = -E_i(E_1E_0^kE_1E_0 - (k+1)E_1E_0) = 0.$
- 4) $(E_1E_{i,1} - E_1E_{i,3})E_0^kE_1E_0 - E_1E_{i,2}(E_1E_0^kE_1E_0 - (k+1)E_1E_0)$
 $= -E_1E_{i,3}E_0^kE_1E_0 + (k+1)E_1E_{i,1}E_0 = -E_{i,3}(E_1E_0^kE_1E_0 - (k+1)E_1E_0) = 0.$

(V) Finally, consider $E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i$ and another relation.

- 1) $(E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i)E_i - E_{i,1}E_0^kE^{1,j}(E_i^2 - \delta E_i)$
 $= -E_{i-2,1}E_0^kE^{1,j}E_i^2 + \delta E_{i,1}E_0^kE^{1,j}E_i = \delta(-E_{i-2,1}E_0^kE^{1,j}E_i + E_{i,1}E_0^kE^{1,j}E_i) = 0.$
- 2) $(E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i)E_j - E_{i,1}E_0^kE^{1,j}(E_iE_j - E_jE_i)$
 $= -E_{i-2,1}E_0^kE^{1,j}E_iE_j + E_{i,1}E_0^kE^{1,j}E_jE_i$
 $= (-E_{i-2,1}E_0^kE^{1,j}E_i + E_{i,1}E_0^kE^{1,j}E_i)E_j = 0.$
- 3) $(E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i)E_{i-1,j}E_i - E_{i,1}E_0^kE^{1,j}(E_{i,j}E_i - E_{i-2,j}E_i)$
 $= -E_{i-2,1}E_0^kE^{1,j}E_{i,j}E_i + E_{i,1}E_0^kE^{1,j}E_{i-2,j}E_i$
 $= (-E_{i-2,1}E_0^kE^{1,j}E_i + E_{i,1}E_0^kE^{1,j}E_i)E_{i-2,j} = 0.$

$$\begin{aligned}
4) & (E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i)E_{\ell,i} - E_{i,1}E_0^kE^{1,j}(E_iE_{\ell,i} - E_iE_{\ell,i+2}) \\
& = -E_{i-2,1}E_0^kE^{1,j}E_iE_{\ell,i} + E_{i,1}E_0^kE^{1,j}E_iE_{\ell,i+2} \\
& = (-E_{i-2,1}E_0^kE^{1,j}E_i + E_{i,1}E_0^kE^{1,j}E_i)E_{\ell,i+2} = 0. \\
5) & (E_jE_{i,j} - E_jE_{i,j+2})E_{j+1,1}E_0^kE^{1,j}E_i - E_j(E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i) \\
& = -E_jE_{i,j+2}E_{j+1,1}E_0^kE^{1,j}E_i + E_jE_{i-2,1}E_0^kE^{1,j}E_i \\
& = -E_j(E_{i,1}E_0^kE^{1,j}E_i - E_{i-2,1}E_0^kE^{1,j}E_i) = 0.
\end{aligned}$$

Hence Theorem 2.6 yields that $\widehat{R}_{\mathcal{T}(d,n)}$ is a Gröbner-Shirshov basis for $\mathcal{T}(d,n)$. \square

Remark. We can easily check that all $\widehat{R}_{\mathcal{T}(d,n)}$ -standard monomials are the ones in $M_{\mathcal{T}(d,n)}$. Following the same procedure as in the proof of [13, Theorem 4.2], the number of monomials of the form (4.2) is

$$(d-1) \sum_{s=0}^{n-1} C(n,s)d^{m-1-s} = (d-1)\mathfrak{F}_{n,n-1}(d)$$

by counting the monomials in (4.2) according to $i_q (= n-1-s) = 0, 1, 2, \dots, n-1$ via $G(d, 1, n)$ -Dyck paths.

More precisely, if $i_q = 0$ then the monomials are of the form $E_0^k E_{i_2, j_2} \cdots E_{i_p, j_p}$, so the number is $(d-1)C_n = (d-1)C(n, n-1)$. If $i_q = 1$ then the number of the monomials is $d(d-1)C(n, n-2)$. If $i_q = 2$ then the number is $d^2(d-1)C(n, n-3)$. In general, if $i_q = \ell \leq n-1$ then the number is $d^\ell(d-1)C(n, n-1-\ell)$.

On the other hand, the number of monomials of the form (4.3) is

$$(d-1)(C_n - 1).$$

Thus the cardinality of the set $M_{\mathcal{T}(d,n)}$ is

$$\begin{aligned}
\dim \mathcal{T}(d,n) & = |M_{\mathcal{T}(d,n)}| = C_n + (d-1)\mathfrak{F}_{n,n-1}(d) + (d-1)(C_n - 1) \\
& = (d-1)(\mathfrak{F}_{n,n-1}(d) - 1) + dC_n.
\end{aligned}$$

In particular, with specialisation $d = 2$, we recover the result for type B_n :

$$\dim \mathcal{T}(B_n) = (n+2)C_n - 1.$$

Example 4.4. Consider the cases of $n = 3$. The the dimension of $\mathcal{T}(d, 3)$ is

$$(d-1)(\mathfrak{F}_{3,2}(d) - 1) + dC_n = (d-1)(d^2 + 3d + 4) + 5d = d^3 + 2d^2 + 6d - 4.$$

Notice that, for $n = 3$ and $d = 3$, $|M_{\mathcal{T}(3,3)} \setminus M_{\mathcal{T}(B_3)}| = 59 - 24 = 35$. The explicit and complete list of the standard monomials in $M_{\mathcal{T}(3,3)} \setminus M_{\mathcal{T}(B_3)}$, that is, the standard

monomials containing E_0^2 , is as follows:

$$\begin{aligned}
 & E_0^2, E_0^2 E_1, E_0^2 E_1 E_2, E_0^2 E_{2,1}, E_0^2 E_2, \\
 & E_0^2 E_{1,0}, E_0^2 E_{1,0} E_{2,1}, E_0^2 E_{1,0} E_2, E_1 E_0^2, (E_1 E_0^2) E_{2,1}, (E_1 E_0^2) E_2, \\
 & E_0(E_1 E_0^2), E_0(E_1 E_0^2) E_{2,1}, E_0(E_1 E_0^2) E_2, E_0^2(E_1 E_0^2), E_0^2(E_1 E_0^2) E_{2,1}, E_0^2(E_1 E_0^2) E_2, \\
 & E_0^2 E_{2,0}, E_0^2 E_{1,0} E_{2,0}, (E_1 E_0^2) E_{2,0}, E_0(E_1 E_0^2) E_{2,0}, E_0^2(E_1 E_0^2) E_{2,0}, \\
 & E_{2,1} E_0^2, E_0(E_{2,1} E_0^2), E_0^2(E_{2,1} E_0^2), E_{1,0}(E_{2,1} E_0^2), (E_1 E_0^2)(E_{2,1} E_0^2), E_0 E_{1,0}(E_{2,1} E_0^2), \\
 & E_0^2 E_{1,0}(E_{2,1} E_0^2), E_0(E_1 E_0^2)(E_{2,1} E_0^2), E_0^2(E_1 E_0^2)(E_{2,1} E_0^2), \\
 & E_1 E_0^2 E_1, (E_1 E_0^2 E_1) E_2, E_{2,1} E_0^2 E_1, E_{2,1} E_0^2 E_1^{1,2}.
 \end{aligned}$$

5. COMBINATORIAL ASPECTS - CONNECTIONS TO FULLY COMMUTATIVE ELEMENTS AND DYCK PATHS

In this section, we introduce some combinatorial tools which are useful for the enumeration of standard monomials induced from a *Gröbner-Shirshov basis* for a Temperley-Lieb algebra of Coxeter groups. We later try to set up a bijective correspondence between the standard monomials and the fully commutative elements.

Let W be a Coxeter group. An element $w \in W$ is said to be *fully commutative* if any reduced word for w can be obtained from any other by interchanges of adjacent commuting generators. The fully commutative elements play an important role particularly for computing the dimension of the Temperley-Lieb algebra of the Coxeter groups.

Stembridge [23] classified all of the Coxeter groups which have finitely many fully commutative elements. His results completed the work of Fan [5], which was done only for the simply-laced types. In the same paper [5], Fan showed that the fully commutative elements parameterize natural bases corresponding to certain quotients of Hecke algebras. In type A_n , these give rise to the Temperley-Lieb algebras (see [10]). Fan and Stembridge also enumerated the set of fully commutative elements. In particular, they showed the following property.

Proposition 5.1 ([5, 24]). *Let C_n be the n^{th} Catalan number, i.e. $C_n = \frac{1}{n+1} \binom{2n}{n}$. Then the numbers of fully commutative elements in the Coxeter group of types A_n , D_n and B_n respectively are given as follows:*

$$\begin{cases} C_{n+1} & \text{if the type is } A_n, \\ \frac{n+3}{2} \times C_n - 1 & \text{if the type is } D_n, \\ (n+2) \times C_n - 1 & \text{if the type is } B_n. \end{cases}$$

Remark. There are several combinatorial ways to realize the Catalan number C_n . Among those, we consider the Dyck paths in $n \times n$ -lattice plane starting from the point at $(0, 0)$ and ending at the point at (n, n) using only with northern and eastern directions at each step

and never passing above the diagonal line. Seeing that the dimension of the Temperley-Lieb algebra $\mathcal{T}(A_{n-1})$ of the Coxeter group of type A_{n-1} is the Catalan number C_n which is the number of fully commutative elements, we can realize a bijective correspondence between the Dyck paths and the fully commutative elements for $\mathcal{T}(A_{n-1})$ as we can see in the article [8].

The number of fully commutative elements in Coxeter groups of type B and D respectively in Proposition 5.1 above can also be understood in pure combinatorial way by using the notions of Catalan’s triangle and Packets as mentioned in the articles [7] and [6]. In this section, we recall the definitions and some combinatorial properties of Catalan numbers and Catalan triangles as well as the main results in [7] and [6].

The Catalan numbers can be defined in a recursive way as follows : we set the first entry $C(0, 0) = 1$. For $n \geq 0$ and $0 \leq k \leq n$, we denote $C(n, k)$ ($0 \leq k \leq n$) the entry in the n^{th} row and k^{th} column of Table 2 below, called the *Catalan’s Triangle*, verifying the following recursive formula.

$$C(n, k) = \begin{cases} 1 & \text{if } n = 0; \\ C(n, k - 1) + C(n - 1, k) & \text{if } 0 < k < n; \\ C(n - 1, 0) & \text{if } k = 0; \\ C(n, n - 1) = C(n, n) = C_n & \text{if } k = n. \\ 0 & \text{if } k > n. \end{cases}$$

1							
1	1(= C_1)						
1	2	2(= C_2)					
1	3	5	5(= C_3)			$C(n, k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!}$	
1	4	9	14	14(= C_4)			
1	5	14	28	42	42(= C_5)		
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots		\ddots

TABLE 2. Catalan’s Triangle

Remark. We consider the Dyck paths in $n \times n$ -lattice plane starting from the point at $(0, 0)$ and ending at the point at (n, n) using only with northern and eastern directions at each step and never passing above the diagonal line. Then the coefficient $C(n, k)$ in the Catalan’s triangle can be understood as the number of these Dyck paths passing through exactly the point (n, k) ($k \leq n$).

In [7], Feinberg and Lee showed that the set of fully commutative elements can be decomposed into subsets called *collections* depending on the shape of the suffix of each reduced element written in canonical form. Then they proved that some collections have

the same set of prefixes, i.e. the same cardinality. Now, we group those collections together and call the group a *packet* and we call this process the *packet decomposition*.

Following the notations in (3.3) and (3.4), for $0 \leq i < n$, we define the words s_{ij} and $s^{j,i}$ by:

$$s_{ij} = \begin{cases} [i, i-1, \dots, j] := s_i \cdots s_j & \text{if } i > j, \\ [i] := s_i & \text{if } i = j, \\ [] := 1 & \text{if } i < j. \end{cases} \quad s^{j,i} = \begin{cases} s_j s_{j+1} \cdots s_i & \text{if } i > j, \\ [i] := s_i & \text{if } i = j, \\ [] := 1 & \text{if } i < j. \end{cases}$$

Lemma 5.2. ([12, Proposition 2.3], cf. [3, Lemma 4.2]) *Any element of the Coxeter group of type B_n can be uniquely written in the reduced form*

$$s_{0,a_0}^{(k_0)} s_{1,a_1}^{(k_1)} s_{2,a_2}^{(k_2)} \cdots s_{n-1,a_{n-1}}^{(k_{n-1})}$$

where

$$1 \leq a_i \leq i+1, \quad k_i = 0 \text{ or } 1, \quad \text{and } s_{ij}^k = \begin{cases} s_{ij} & \text{if } k = 0, \\ s_{i,0} s^{1,j-1} & \text{if } k = 1. \end{cases}$$

Remark. Notice that there are $i+1$ choices for each a_i since $1 \leq a_i \leq i+1$ while $i = 0, 1, \dots, n-1$, thus $n!$ choices for the values of a_i 's altogether. There are 2 ways for each exponent k_i of a_i , and thus there are 2^n choices for the exponents. Thus we have $n! \cdot 2^n$ elements in the canonical reduced form. We recall that there are the same number of elements in the type B_n Coxeter group.

Among the above canonical elements, since, for $1 \leq i_1 < i_2$ and $j_1 > 1$, any element

$$s_{i_1, j_1}^{(1)} s_{i_2, j_2}^{(1)} = s_{i_1, 0} s^{1, j_1-1} s_{i_2, 0} s^{1, j_2-1} = s_{i_1, 0} s^{1, j_1-2} s_{i_2, j_1+1} (s_{j_1-1} s_{j_1} s_{j_1-1}) s_{j_1-2, 0} s^{1, j_2-1}$$

contains a braid relation $s_{j_1-1} s_{j_1} s_{j_1-1}$, thus any element containing this form is not fully commutative. Also, for $j > 1$, any element containing the form

$$s_0 s_{i,j}^{(1)} = s_0 s_{i,0} s^{1,j-1} = s_{i,2} (s_0 s_1 s_0 s_1) s^{2,j-1}$$

is non-fully-commutative element.

Now we consider only the elements of the form

$$(5.1) \quad s_{i_1, 0} s_{i_2, 0} \cdots s_{i_\ell, 0} s_{i_\ell+1, a_{i_\ell+1}} \cdots s_{n-1, a_{n-1}}$$

where $0 \leq i_1 < i_2 < \cdots < i_\ell < n$ for $\ell \geq 0$, or

$$(5.2) \quad (s_{i,0} s^{1,a_i}) s_{i+1, a_{i+1}} \cdots s_{n-1, a_{n-1}}$$

with $i \geq 1$.

The left factors $s_{i_1, 0} s_{i_2, 0} \cdots s_{i_\ell, 0}$ in (5.1), and $s_{i,0}$ in (5.2) will be called the *prefix*. Similarly the right factors $s_{i_\ell+1, a_{i_\ell+1}} \cdots s_{n-1, a_{n-1}}$ in (5.1), and $s^{1,a_i} s_{i+1, a_{i+1}} \cdots s_{n-1, a_{n-1}}$ in (5.2) will be called the *suffix* of the reduced word. Given a reduced word \mathbf{w} in the extracted canonical form, we will denote by \mathbf{w}_p the prefix and by \mathbf{w}_s the suffix of \mathbf{w} and we can write, in a unique way, $\mathbf{w} = \mathbf{w}_p \mathbf{w}_s$.

We notice that every prefix is a fully commutative element. Some of the collections in the set of *fully commutative* elements of $W(B_n)$ have the same number of elements and thus we group them together into a set called a (n, k) -*packet* depending on the form of the prefixes as follows :

Definition 5.3. For $0 \leq k \leq n$, we define the (n, k) -*packet* of collections:

- The $(n, 0)$ -packet is the set of collections labeled by prefixes of the form

$$s_{i_1,0}s_{i_2,0} \cdots s_{i_\ell,0}s_{n-1,0} \quad (\ell \geq 1).$$

- The (n, k) -packet, $1 \leq k \leq n - 2$, is the set of collections labeled by $s_{n-k,0}$ or prefixes of the form

$$s_{i_1,0}s_{i_2,0} \cdots s_{i_\ell,0}s_{n-k-1,0} \quad (\ell \geq 1).$$

- The $(n, n-1)$ -packet contains only the collection labeled by $s_0 = [0]$ or $s_{1,0} = [1, 0]$.
- The (n, n) -packet contains only the collection labeled by the empty prefix $[\]$.

We will denote the (n, k) -packet by $\mathcal{P}_B(n, k)$. As an example, Table 3 shows all of collections of the packets for type B_3 .

Proposition 5.4 ([6]). *The cardinality of the packet $\mathcal{P}_B(n, k)$ for type B_n is*

$$|\mathcal{P}_B(n, k)| = \begin{cases} 2^{n-1} - 1 & \text{if } k = 0, \\ 2^{n-k-1} & \text{if } 1 \leq k \leq n - 2, \\ 2 & \text{if } k = n - 1, \\ 1 & \text{if } k = n. \end{cases}$$

Therefore $\sum_{k=0}^n |\mathcal{P}_B(n, k)| = 2^n$. We remark that for a fixed k with $0 \leq k \leq n$ ($n \geq 3$), each collection with same prefix in the packet $\mathcal{P}_B(n, k)$ has the same cardinality which is $C(n, k)$.

Corollary 5.5. *For $n \geq 3$, we obtain the identity:*

$$(5.3) \quad \sum_{k=0}^n C(n, k) |\mathcal{P}_B(n, k)| = (n + 2) \times C_n - 1.$$

We generalize the above method for the complex reflection group of type $G(d, 1, n)$.

Lemma 5.6. (Canonical Form for $G(d, 1, n)$ [12, Proposition 2.3]) *Any element of the Coxeter group of type $G(d, 1, n)$ can be uniquely written in the reduced form*

$$s_{0,a_0}^{(k_0)} s_{1,a_1}^{(k_1)} s_{2,a_2}^{(k_2)} \cdots s_{n-1,a_{n-1}}^{(k_{n-1})}$$

where

$$1 \leq a_i \leq i + 1, \quad 0 \leq k_i \leq d - 1, \quad \text{and } s_{ij}^k = \begin{cases} s_{ij} & \text{if } k = 0, \\ s_{i,1} s_0^k s^{1,j-1} & \text{if } k = 1, 2, \dots, d - 1. \end{cases}$$

Remark. Notice that there are $n!$ choices of a_i 's ($i = 0, 1, \dots, n-1$), and there are d^n ways to choose k_i 's. Therefore we have $n! \cdot d^n$ elements in the canonical reduced form. We recall that there are the same number of elements in the complex reflection group of type $G(d, 1, n)$.

The fully commutative elements of the complex reflection group of type $G(d, 1, n)$ are of the form

$$(5.4) \quad s_{i_1,1} s_0^{k_1} s_{i_2,1} s_0^{k_2} \cdots s_{i_q,1} s_0^{k_q} s_{i_{q+1},j_{q+1}} \cdots s_{i_p,j_p}$$

with

$$\begin{aligned} 0 \leq p \leq n, \quad 0 \leq q \leq p, \quad 1 \leq k_1, k_2, \dots, k_q < d, \\ 0 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 0 < j_{q+1} < \cdots \leq j_p \leq n-1, \quad \text{and} \\ i_{q+1} \geq j_{q+1}, \quad \dots, \quad i_p \geq j_p, \end{aligned}$$

or

$$(5.5) \quad (s_{i_1,1} s_0^k s^{1,j_1}) s_{i_2,j_2} s_{i_3,j_3} \cdots s_{i_p,j_p}$$

with

$$\begin{aligned} 1 \leq p \leq n-1, \quad 1 \leq k < d, \\ 1 \leq i_1 < i_2 < i_3 < \cdots < i_p \leq n-1, \quad 1 \leq j_1 < j_2 < j_3 < \cdots < j_p \leq n-1, \quad \text{and} \\ i_1 \geq j_1, \quad i_2 \geq j_2, \quad \dots, \quad i_p \geq j_p. \end{aligned}$$

The left factor $s_{i_1,1} s_0^{k_1} s_{i_2,1} s_0^{k_2} \cdots s_{i_q,1} s_0^{k_q}$ with $q \geq 0$ in (5.4) and (5.5) will be called the *prefix* of the fully commutative element of type $G(d, 1, n)$.

Definition 5.7. For $0 \leq s \leq n$, we define the (n, s) -*packet* of collections:

- The $(n, 0)$ -packet is the set of collections labeled by prefixes of the form

$$s_{i_1,1} s_0^{k_1} s_{i_2,1} s_0^{k_2} \cdots s_{i_{q-1},1} s_0^{k_{q-1}} s_{n-1,1} s_0^{k_q} \quad (q \geq 2).$$

- The (n, s) -packet, $1 \leq s \leq n-2$, is the set of collections labeled by $s_{n-s,1} s_0^k$ or prefixes of the form

$$s_{i_1,1} s_0^{k_1} s_{i_2,1} s_0^{k_2} \cdots s_{i_{q-1},1} s_{n-s-1,1} s_0^{k_q} \quad (q \geq 2).$$

- The $(n, n-1)$ -packet contains only the collections labeled by $s_0^k = [0^k]$ or $s_1 s_0^k = [1, 0^k]$.
- The (n, n) -packet contains only the collection labeled by the empty prefix $[\]$.
These elements cover the elements of $\mathfrak{S}_n \subset G(d, 1, n)$.

We will denote the (n, s) -packet by $\mathcal{P}(n, s)$. For clarifying the packet decomposition, we denote (n, s) -packet of type $G(d, 1, n)$ which is not contained in the (n, s) -packet of type B_n by

$$\mathcal{P}(n, s)' = \mathcal{P}(n, s) \setminus \mathcal{P}_B(n, s).$$

Proposition 5.8. (cf. [6, §4])

- (a) *Every collection in the packet $\mathcal{P}(n, s)$ has $C(n, s)$ elements.*

(b) The size of the packet $\mathcal{P}(n, s)$ is

$$|\mathcal{P}(n, s)| = \begin{cases} (d-1)(d^{n-1}-1) & \text{if } s = 0, \\ (d-1)d^{n-1-s} & \text{if } 1 \leq s \leq n-2, \\ 2(d-1) & \text{if } s = n-1, \\ 1 & \text{if } s = n. \end{cases}$$

Proof. (a) The collections in $\mathcal{P}(n, 0)$ have no other element than the prefix. It's also clear that the collections in $\mathcal{P}(n, n-1)$ have $C_n = C(n, n-1)$ elements and the collection in $\mathcal{P}(n, n)$ has $C_n = C(n, n)$ elements.

For the collections in $\mathcal{P}(n, s)$ with $1 \leq s \leq n-2$, the number of suffixes attached to the given prefix is exactly $C(n, s)$ by counting the Dyck paths from $(n-s, 0)$ to (n, n) via the Dyck paths, reflected relative to $y = n-x$ in the xy -plane, from $(0, 0)$ to (n, s) .

(b) Counting the number of prefixes in each packet, we get its packet size. \square

In the same way as the formula (5.3), we obtain the identity for the complex reflection group of type $G(d, 1, n)$:

$$\begin{aligned} \sum_{s=0}^n C(n, s) |\mathcal{P}(n, s)| &= (d-1)(d^{n-1}-1) + \sum_{s=1}^{n-2} C(n, s)(d-1)d^{n-1-s} + 2(d-1)C_n + C_n \\ &= (d-1)(\mathfrak{F}_{n, n-1}(d) - 1) + dC_n, \end{aligned}$$

which is equal to $\dim \mathcal{T}(d, n)$.

As an example, Table 3 shows all of collections of the packets for type $G(3, 1, 3)$.

6. TEMPERLEY-LIEB ALGEBRAS OF TYPES $G(d, d, n)$ AND $G(d, r, n)$

Since we obtain a \mathbb{C} -basis of $\mathcal{T}(d, n)$ and its multiplication structure, we define the subalgebras $\mathcal{T}(d, d, n)$ and $\mathcal{T}(d, r, n)$ in general as follows.

Definition 6.1.

(a) The subalgebra $\mathcal{T}(d, d, n)$ is a subalgebra of $\mathcal{T}(d, n)$, whose \mathbb{C} -basis consists of

$$\begin{cases} \text{the elements in (3.2),} \\ \text{the elements in (4.2) satisfying } k_1 + \cdots + k_q \equiv 0 \pmod{d}, \\ \text{the elements in (4.3) with } k = 1. \end{cases}$$

We remark that

$$\begin{aligned} \dim \mathcal{T}(d, d, n) &= C_n + (d-1)\mathfrak{F}_{n, n-2}(d) + C_n - 1 \\ &= (d-1)\mathfrak{F}_{n, n-2}(d) + 2C_n - 1. \end{aligned}$$

In particular, for $d = 2$, we recover the same formula as we had in [13] :

$$\dim \mathcal{T}(2, 2, n) = \dim \mathcal{T}(D_n) = \frac{n+3}{2}C_n - 1.$$

	Standard Monomials	Fully Commutative Elements
$M_{\mathcal{T}(A_2)}$	$1, E_1, E_2, E_{2,1} = E_2 E_1, E_1 E_2$	$\mathcal{P}(3, 3) = \mathcal{C}_{[\]} = \{id, s_1, s_2, s_2 s_1, s_1 s_2\}$
$M_{\mathcal{T}(B_3)}^0$	$\begin{cases} E_0, E_0 E_2, E_0 E_1 E_2, E_0 E_1, E_0 E_2 E_1, \\ E_{1,0}, E_{1,0} E_2, E_{1,0} E_1 E_2, E_{1,0} E_1, E_{1,0} E_{2,1} \\ \\ E_{2,0}, E_{2,0} E_1, E_{2,0} E_1^{1,2}, \\ E_0 E_{1,0}, E_0 E_{1,0} E_2, E_0 E_{1,0} E_{2,1} \\ \\ E_0 E_{2,0}, \\ E_{1,0} E_{2,0}, \\ E_0 E_{1,0} E_{2,0} \quad \text{Monomials containing } E_0 \end{cases}$	$\begin{aligned} \mathcal{P}_B(3, 2) &= \begin{cases} \mathcal{C}_{[0]} = \{\mathbf{s}_0, \mathbf{s}_0 s_2, \mathbf{s}_0 s_1 s_2, \mathbf{s}_0 s_1, \mathbf{s}_0 s_2 s_1\} \\ \mathcal{C}_{[1,0]} = \{\mathbf{s}_1 \mathbf{s}_0, \mathbf{s}_1 \mathbf{s}_0 s_2, \mathbf{s}_1 \mathbf{s}_0 s_1 s_2, \mathbf{s}_1 \mathbf{s}_0 s_1, \mathbf{s}_1 \mathbf{s}_0 s_2 s_1\} \end{cases} \\ \mathcal{P}_B(3, 1) &= \begin{cases} \mathcal{C}_{[2,1,0]} = \{\mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0, \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0 s_1, \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0 s_1 s_2\} \\ \mathcal{C}_{[0,1,0]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0, \mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0 s_2, \mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0 s_2 s_1\} \end{cases} \\ \mathcal{P}_B(3, 0) &= \begin{cases} \mathcal{C}_{[0,2,1,0]} = \{\mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[1,0,2,1,0]} = \{\mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[0,1,0,2,1,0]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \end{cases} \end{aligned}$
$M_{\mathcal{T}(3,3)}^{0^2}$	$\begin{cases} E_0^2, E_0^2 E_2, E_0^2 E_1^{1,2}, E_0^2 E_1, E_0^2 E_{2,1}, \\ E_1 E_0^2, E_1 E_0^2 E_2, E_1 E_0^2 E_1^{1,2}, E_1 E_0^2 E_1, E_1 E_0^2 E_{2,1} \\ \\ E_{2,1} E_0^2, E_{2,1} E_0^2 E_1, E_{2,1} E_0^2 E_1^{1,2}, \\ E_0^2 E_{1,0}, E_0^2 E_{1,0} E_2, E_0^2 E_{1,0} E_{2,1}, \\ E_0 E_1 E_0^2, E_0 E_1 E_0^2 E_2, E_0 E_1 E_0^2 E_{2,1}, \\ E_0^2 E_1 E_0^2, E_0^2 E_1 E_0^2 E_2, E_0^2 E_1 E_0^2 E_{2,1} \\ \\ E_0^2 E_{2,0}, \\ E_1 E_0^2 E_{2,0}, \\ E_0^2 E_{1,0} E_{2,0}, \\ E_0 (E_1 E_0^2) E_{2,0}, \\ E_0^2 (E_1 E_0^2) E_{2,0}, \\ E_0 (E_{2,1} E_0^2), \\ E_0^2 (E_{2,1} E_0^2), \\ E_{1,0} (E_{2,1} E_0^2), \\ (E_1 E_0^2) (E_{2,1} E_0^2), \\ E_0 E_{1,0} (E_{2,1} E_0^2), \\ E_0^2 E_{1,0} (E_{2,1} E_0^2), \\ E_0 (E_1 E_0^2) (E_{2,1} E_0^2), \\ E_0^2 (E_1 E_0^2) (E_{2,1} E_0^2) \quad \text{Monomials containing } E_0^2 \end{cases}$	$\begin{aligned} \mathcal{P}(3, 2)' &= \begin{cases} \mathcal{C}_{[0^2]} = \{\mathbf{s}_0^2, \mathbf{s}_0^2 s_2, \mathbf{s}_0^2 s_1 s_2, \mathbf{s}_0^2 s_1, \mathbf{s}_0^2 s_2 s_1\} \\ \mathcal{C}_{[1,0^2]} = \{\mathbf{s}_1 \mathbf{s}_0^2, \mathbf{s}_1 \mathbf{s}_0^2 s_2, \mathbf{s}_1 \mathbf{s}_0^2 s_1 s_2, \mathbf{s}_1 \mathbf{s}_0^2 s_1, \mathbf{s}_1 \mathbf{s}_0^2 s_2 s_1\} \end{cases} \\ \mathcal{P}(3, 1)' &= \begin{cases} \mathcal{C}_{[2,1,0^2]} = \{\mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2, \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2 s_1, \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2 s_1 s_2\} \\ \mathcal{C}_{[0^2,1,0]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0, \mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0 s_2, \mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0 s_2 s_1\} \\ \mathcal{C}_{[0,1,0^2]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0^2, \mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0^2 s_2, \mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0^2 s_2 s_1\} \\ \mathcal{C}_{[0^2,1,0^2]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0^2, \mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0^2 s_2, \mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0^2 s_2 s_1\} \end{cases} \\ \mathcal{P}(3, 0)' &= \begin{cases} \mathcal{C}_{[0^2,2,1,0]} = \{\mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[1,0^2,2,1,0]} = \{\mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[0^2,1,0,2,1,0]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[0,1,0^2,2,1,0]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[0^2,1,0^2,2,1,0]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0\} \\ \mathcal{C}_{[0,2,1,0^2]} = \{\mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[0^2,2,1,0^2]} = \{\mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[1,0,2,1,0^2]} = \{\mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[1,0^2,2,1,0^2]} = \{\mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[0,1,0,2,1,0^2]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[0^2,1,0,2,1,0^2]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[0,1,0^2,2,1,0^2]} = \{\mathbf{s}_0 \mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \\ \mathcal{C}_{[0^2,1,0^2,2,1,0^2]} = \{\mathbf{s}_0^2 \mathbf{s}_1 \mathbf{s}_0^2 \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_0^2\} \end{cases} \end{aligned}$

 TABLE 3. Bijection between Standard Monomials and Fully Commutative Elements of $G(3, 1, 3)$

- (b) More generally, the subalgebra $\mathcal{T}(d, r, n)$ ($r|d$ and $r \geq 2$) is a subalgebra of $\mathcal{T}(d, n)$, whose \mathbb{C} -basis consists of

$$\left\{ \begin{array}{l} \text{the elements in (3.2),} \\ \text{the elements in (4.2) satisfying } k_1 + \cdots + k_q \equiv 0 \pmod{r}, \\ \text{the elements in (4.3) with } k \in \{1, r, 2r, \dots, d - 2r, d - r\}. \end{array} \right.$$

We remark that

$$\begin{aligned} \dim \mathcal{T}(d, r, n) &= C_n + \frac{d}{r}(d-1)\mathfrak{F}_{n, n-2}(d) + \frac{d}{r}(C_n - 1) \\ &= \frac{d}{r}(d-1)\mathfrak{F}_{n, n-2}(d) + \left(1 + \frac{d}{r}\right)C_n - \frac{d}{r}. \end{aligned}$$

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