SET-PARTITION TABLEAUX AND REPRESENTATIONS OF DIAGRAM ALGEBRAS

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Abstract. The partition algebra is an associative algebra with a basis of set-partition diagrams and multiplication given by diagram concatenation. It contains as subalgebras a large class of diagram algebras including the Brauer, planar partition, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, planar rook monoid, and symmetric group algebras. We give a construction of the irreducible modules of these algebras in two isomorphic ways: first, as the span of symmetric diagrams on which the algebra acts by conjugation twisted with an irreducible symmetric group representation and, second, on a basis indexed by set-partition tableaux such that diagrams in the algebra act combinatorially on tableaux. The first representation is analogous to the Gelfand model and the second is a generalization of Young's natural representation of the symmetric group on standard tableaux. The methods of this paper work uniformly for the partition algebra and its diagram subalgebras. As an application, we express the characters of each of these algebras as nonnegative integer combinations of symmetric group characters whose coefficients count fixed points under conjugation.

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1. Introduction

The partition algebra $\mathbf{P}_k(n)$ for $k \in \mathbb{Z}_{\geq 0}$ is a unital, associative algebra over $\mathbb{C}$ (or any field of characteristic 0) and is semisimple for all $n \in \mathbb{C} \setminus \{0, 1, \ldots, 2k-2\}$. It has a basis of set-partition diagrams and multiplication given by diagram concatenation. This algebra arose in the work of P.P. Martin [Mar1, Mar3] and V. Jones [Jon] in the study of the Potts model, a k-site, n-state lattice model in statistical mechanics. For $k, n \in \mathbb{Z}_{\geq 1}$ the partition algebra $\mathbf{P}_k(n)$ and the symmetric group $\mathbf{S}_n$ are in Schur-Weyl duality on the k-fold tensor product $\mathsf{V}_n^{\otimes k}$ of the n-dimensional permutation module V_n of the symmetric group $\mathbf{S}_n$, and when $n \geq 2k$, $\mathbf{P}_k(n)$ is isomorphic to the centralizer algebra of $\mathbf{S}_n$ on $\mathsf{V}_n^{\otimes k}$. This allows information to flow back and forth between $\mathbf{P}_k(n)$ and $\mathbf{S}_n$.

The partition algebra $\mathbf{P}_k(n)$ contains as subalgebras a large class of diagram algebras including the Brauer, planar partition, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, planar rook monoid, and symmetric group algebras. Each of these subalgebras arises as the span of restricted types of set-partition diagrams (see Section 2.3). If $\mathbf{A}_k$ is the partition algebra or one of its diagram subalgebras, then the irreducible $\mathbf{A}_k$-modules can be indexed by a subset $\Lambda_n^{\mathbf{A}_k} \subseteq \{\lambda \vdash n\}$ of the integer partitions of n. In this paper we give two explicit combinatorial constructions of the irreducible modules $\mathbf{A}_k^\lambda$ for $\lambda \in \Lambda_n^{\mathbf{A}_k}$. The first construction is by conjugation on a basis of symmetric m-diagrams (Definition 3.1) that is twisted by a symmetric group representation. This method is analogous to the Gelfand models for diagram algebras found in [HR3] and [KM]. A nice feature of the construction here is that we isolate each irreducible module, rather than constructing a (multiplicity-free) sum of irreducible modules.

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The second method of constructing $\mathbf{A}_k^\lambda$ is on a basis of set-partition tableaux. In [BH2, BH3] and [OZ], it is shown that the dimension of the irreducible partition algebra module $\mathbf{P}_k^\lambda$ equals the number of standard set-partition tableaux of shape λ. We give a combinatorial action of the diagrams in $\mathbf{P}_k(n)$ on these tableaux, which naturally index a basis for $\mathbf{P}_k^\lambda$. This representation is a generalization of Young's natural representation of the symmetric group on a basis of standard Young tableaux. In fact, if λ has k boxes below the first row, when restricted to the symmetric group algebra $\mathbb{C}\mathbf{S}_k \subseteq \mathbf{P}_k(n)$ we exactly recover Young's representation.

A surprising feature of the methods in this paper is that, by restriction, they work uniformly for the partition algebra and all of the diagram subalgebras listed above. For each of the subalgebras we construct $\mathbf{A}_k^\lambda$ as the span of restricted types of set-partition tableaux, obtaining a complete set of analogs of Young's natural representation for the partition, Brauer, planar partition, rook monoid, rook-Brauer, Temperley-Lieb, Motzkin, and planar rook monoid algebras. In the case of the non-planar algebras – partition, Brauer, rook monoid, rook-Brauer – we obtain new constructions of the irreducible modules on symmetric diagrams and on set-partition tableaux. In the case of the planar algebras – planar partition, Temperley-Lieb, Motzkin, and planar rook monoid – our methods specialize to known constructions.

In Section 5, we use our explicit construction of the irreducible modules on symmetric diagrams to write the irreducible characters of each $\mathbf{A}_k$ into a nonnegative integer sum of characters of the symmetric groups $\mathbf{S}_m$, for $0 \leq m \leq k$. We prove that if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell] \vdash n$ with $\lambda^* = [\lambda_2, \lambda_3, \cdots, \lambda_\ell] \vdash m$, then the value of the irreducible $\mathbf{A}_k$ character on a diagram γ_κ of cycle type $\kappa \vdash k$ (see (5.2)) is given by

$$\chi^\lambda_{\mathbf{A}_k}(\gamma_\kappa) = \sum_{\mu \vdash m} \mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k}\, \chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu), \tag{1.1}$$

where $\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k} \in \mathbb{Z}_{\geq 0}$ and $\chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu)$ is the symmetric group character indexed by λ^* on the conjugacy class of cycle type $\mu \vdash m$. By counting fixed points under conjugation, we obtain a closed formula for the coefficients $\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k}$. For example, we prove in Proposition 5.19 that for the partition algebra $\mathbf{P}_k(n)$,

$$\mathsf{F}^{\mu,\kappa}_{\mathbf{P}_k(n)} = \sum_{\nu | \kappa} \prod_i \sum_t \left\{ \begin{matrix} \mathsf{m}_i(\nu) \\ t \end{matrix} \right\} \binom{t}{\mathsf{m}_i(\mu)} i^{\mathsf{m}_i(\nu) - t}, \tag{1.2}$$

where $\nu|\kappa$ means that ν is a divisor of κ (see Definition 5.16) and $\mathsf{m}_i(\nu)$ denotes the number of parts of ν equal to i. In this formula $\left\{ \begin{smallmatrix} a \\ b \end{smallmatrix} \right\}$ is the Stirling number of the second kind and $\binom{a}{b}$ is the binomial coefficient. The coefficient in (1.2) specializes to the diagram subalgebras giving new character formulas for the partition, Brauer, and rook-Brauer algebras and known formulas for the rook monoid, Temperley-Lieb, Motzkin, and planar rook monoid algebras.

For further background on partition algebras see [Jon], [Mar3, Mar4, Mar2], [ME], [MS1, MS2], [DW], [HR2], [BH3]. Representing the irreducible modules on a basis of set-partition tableaux is new for all of these algebras. The construction on symmetric diagrams for the partition algebra is closely related to the work in [MS1] and [DW] and for the Brauer algebra to the work in [HW]. The construction of the irreducible modules of the planar algebras on symmetric diagrams is identical to the construction in the Gelfand models of [HR3] and [KM] and is isomorphic to known representations of the Temperley-Lieb [Wes], Motzkin [BH], and planar partition [FHH] algebras. The representations constructed in this paper are different from the seminormal representations constructed for the partition [Eny], Brauer [Naz], rook-Brauer [Hd], rook monoid [Hal2], and Temperley-Lieb [HMR] algebras.

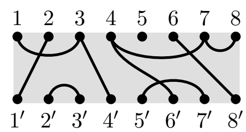
2. Partition Algebras

2.1. **Set-partition diagrams.** We let Π_{2k} denote the set of set partitions of $\{1,\dots,k,1',\dots,k'\}$ and refer to the subsets of a set partition as *blocks*. For example,

$$\{1',2 \mid 2',3' \mid 4',1,3 \mid 5',7' \mid 6',4,7,8 \mid 8',6 \mid 5\} \tag{2.1}$$

is a set partition in Π_{16} with 7 blocks. The number of set partitions in Π_{2k} with t blocks is given by the Stirling number of the second kind $\left\{{2k \atop t}\right\}$, and thus Π_{2k} has order equal to the Bell number $\mathsf{B}(2k) = \sum_t \left\{{2k \atop t}\right\}$.

A diagram d of a set partition $\pi \in \Pi_{2k}$ consists of two rows of k vertices labeled $1',\dots,k'$ on the bottom row and $1,\dots,k$ on the top row. Edges are drawn such that the connected components of d equal π. For example, the set partition in (2.1) is represented by



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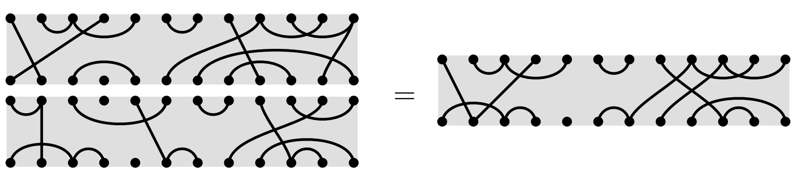
The way the edges are drawn is immaterial; what matters is that the connected components of the diagram correspond to the blocks of the set partition. Thus, d represents the equivalence class of all diagrams with connected components equal to the blocks of π. We define

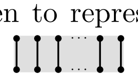
$$\mathcal{P}_k = \{d \mid d \text{ is the diagram of a set partition in } \Pi_{2k}\}. \tag{2.2}$$

Concatenation $d_1 \circ d_2$ of two diagrams d_1, d_2 is accomplished by placing d_1 above d_2, identifying the vertices in the bottom row of d_1 with those in the top row of d_2, concatenating the edges, and deleting all connected components that lie entirely in the middle row of the joined diagrams. For example,

(2.3) $d_1 =$, $d_2 =$



$= d_1 \circ d_2.$

It is easy to confirm that concatenation depends only on the underlying set partitions and is independent of the diagrams chosen to represent them. Concatenation makes $\mathcal{P}_k$ an associative monoid with identity element $\mathbf{1}_k =$  corresponding to the set partition $\{1,1' \mid \cdots \mid k,k'\}$.

Let $\mathbf{P}_0(n) = \mathbb{C}$. For $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{C}$, the *partition algebra* $\mathbf{P}_k(n)$ is the associative algebra over $\mathbb{C}$ with basis $\mathcal{P}_k$,

$$\mathbf{P}_k(n) := \mathbb{C}\mathcal{P}_k = \mathbb{C}\text{-span}\{d \mid d \in \mathcal{P}_k\}, \tag{2.4}$$

such that multiplication in $\mathbf{P}_k(n)$ is defined on basis diagrams $d_1, d_2 \in \mathcal{P}_k$ as

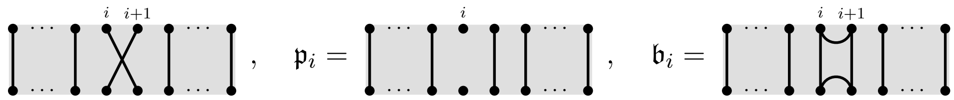
$$d_1 d_2 = n^{\ell(d_1,d_2)}\, d_1 \circ d_2, \tag{2.5}$$

where $\ell(d_1,d_2)$ is the number of connected components that were deleted from the middle row in the concatenation $d_1 \circ d_2$. For example, the product of the two diagrams in (2.3) is $d_1 d_2 = n^2 d_1 \circ d_2$. Since the basis of $\mathbf{P}_k(n)$ corresponds to set partitions in Π_{2k} we have $\dim \mathbf{P}_k(n) = |\mathcal{P}_k| = \mathsf{B}(2k)$.

The partition algebra is semisimple for all $n \in \mathbb{C}$ such that $n \notin \{0,1,\dots,2k-2\}$ (see [MS2], [HR2, Thm. 3.27]), and the partition algebras $\mathbf{P}_k(n)$ are isomorphic to one another for all choices of the parameter n such that $\mathbf{P}_k(n)$ is semisimple. For this reason, we will assume that $n \in \mathbb{Z}$ such that $n \geq 2k$ so that we can take advantage of the Schur-Weyl duality between $\mathbf{P}_k(n)$ and $\mathbf{S}_n$ (see Section 2.5).

2.2. **Generators and relations.** For $k \in \mathbb{Z}_{\geq 1}$, the partition algebra $\mathbf{P}_k(n)$ has a presentation by the generators

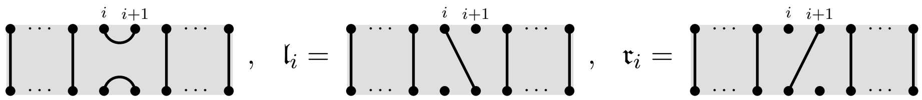
(2.6) $\mathfrak{s}_i =$, $\mathfrak{p}_i =$, $\mathfrak{b}_i =$



$1 \leq i \leq k-1$ $\quad 1 \leq i \leq k$ $\quad 1 \leq i \leq k-1$

and the relations found in [HR2, Thm. 1.11]. It is useful in generating diagram subalgebras to define the elements $\mathfrak{e}_i = \mathfrak{b}_i\mathfrak{p}_i\mathfrak{p}_{i+1}\mathfrak{b}_i$, $\mathfrak{l}_i = \mathfrak{s}_i\mathfrak{p}_i$, and $\mathfrak{r}_i = \mathfrak{p}_i\mathfrak{s}_i$, so that

(2.7) $\mathfrak{e}_i =$, $\mathfrak{l}_i =$, $\mathfrak{r}_i =$.



$1 \leq i \leq k-1$ $\quad 1 \leq i \leq k-1$ $\quad 1 \leq i \leq k-1$

2.3. **Subalgebras.** For $k, n \in \mathbb{Z}_{\geq 1}$ with $n \geq 2k$ the following are semisimple subalgebras of the partition algebra $\mathbf{P}_k(n)$:

$$\begin{aligned}
\mathbb{C}\mathbf{S}_k &= \mathbb{C}\text{-span}\left\{ d \in \mathcal{P}_k \;\middle|\; \begin{array}{l} \text{all blocks of } d \text{ have exactly one vertex in } \{1, \ldots k\} \\ \text{and exactly one vertex in } \{1', \ldots k'\} \end{array} \right\}, \\
\mathbf{R}_k &= \mathbb{C}\text{-span}\left\{ d \in \mathcal{P}_k \;\middle|\; \begin{array}{l} \text{all blocks of } d \text{ have at most one vertex in } \{1, \ldots k\} \\ \text{and at most one vertex in } \{1', \ldots k'\} \end{array} \right\}, \\
\mathbf{B}_k(n) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size } 2\}, \\
\mathbf{RB}_k(n) &= \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid \text{all blocks of } d \text{ have size 1 or } 2\}.
\end{aligned}$$

Here, $\mathbb{C}\mathbf{S}_k$ is the group algebra of the symmetric group, $\mathbf{B}_k(n)$ is the Brauer algebra [Bra], $\mathbf{R}_k$ is the rook monoid algebra [Sol], and $\mathbf{RB}_k(n)$ is the rook-Brauer algebra [Hd], [MM].

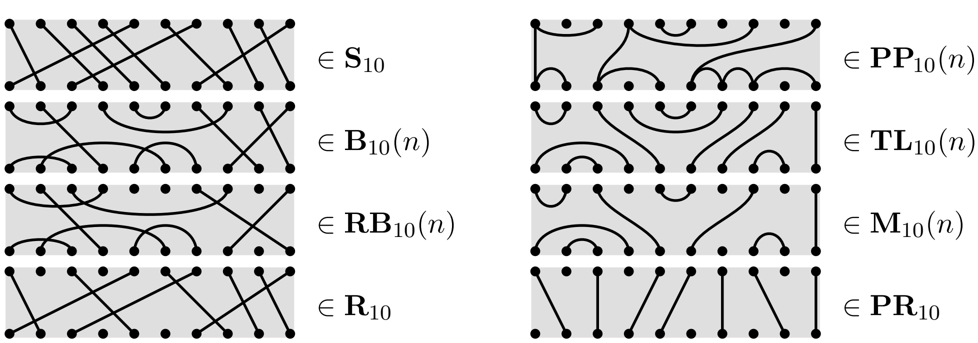
A set partition is *planar* if it can be represented as a diagram without edge crossings inside of the rectangle formed by its vertices. The planar partition algebra [Jon] is defined as

$$\mathbf{PP}_k(n) = \mathbb{C}\text{-span}\{\, d \in \mathcal{P}_k \mid d \text{ is planar }\},$$

and following are the planar subalgebras of $\mathbf{P}_k(n)$, which are also semisimple:

$$\begin{aligned}
\mathbb{C}\mathbf{1}_k &= \mathbb{C}\mathbf{S}_k \cap \mathbf{PP}_k(n), & \mathbf{TL}_k(n) &= \mathbf{B}_k(n) \cap \mathbf{PP}_k(n), \\
\mathbf{PR}_k &= \mathbf{R}_k \cap \mathbf{PP}_k(n), & \mathbf{M}_k(n) &= \mathbf{RB}_k(n) \cap \mathbf{PP}_k(n).
\end{aligned}$$

Here, $\mathbf{TL}_k(n)$ is the Temperley-Lieb algebra [TL], $\mathbf{PR}_k$ is the planar rook monoid algebra [FHH], and $\mathbf{M}_k(n)$ is the Motzkin algebra [BH]. There is an algebra isomorphism $\mathbf{PP}_k(n) \cong \mathbf{TL}_{2k}(n)$ (see [Jon] or [HR2]) and we forgo discussion of the planar partition algebra in favor of the Temperley-Lieb algebra. The parameter n does not arise when multiplying symmetric group diagrams (as there are never middle blocks to be removed). The following displays examples from each of these subalgebras:



Each diagram algebra $\mathbf{A}_k$ is generated as a unital subalgebra $\mathbf{A}_k \subseteq \mathbf{P}_k(n)$ of the partition algebra using a subset of the generators $\mathfrak{s}_i, \mathfrak{b}_i, \mathfrak{e}_i, \mathfrak{l}_i, \mathfrak{r}_i$ for $1 \leq i \leq k-1$ and $\mathfrak{p}_i$ for $1 \leq i \leq k$ as shown in the following table.

Algebra	Generators	Algebra	Generators	Algebra	Generators
$\mathbf{P}_k(n)$	$\mathfrak{s}_i, \mathfrak{b}_i, \mathfrak{p}_i$	$\mathbf{B}_k(n)$	$\mathfrak{s}_i, \mathfrak{e}_i$	$\mathbf{TL}_k(n)$	$\mathfrak{e}_i$
$\mathbb{C}\mathbf{S}_k$	$\mathfrak{s}_i$	$\mathbf{RB}_k(n)$	$\mathfrak{s}_i, \mathfrak{e}_i, \mathfrak{p}_i$	$\mathbf{M}_k(n)$	$\mathfrak{e}_i, \mathfrak{l}_i, \mathfrak{r}_i$
$\mathbf{R}_k$	$\mathfrak{s}_i, \mathfrak{p}_i$	$\mathbf{PP}_k(n)$	$\mathfrak{p}_i, \mathfrak{b}_i$	$\mathbf{PR}_k$	$\mathfrak{l}_i, \mathfrak{r}_i$

Typically the rook monoid and planar rook monoid algebras do not have the parameter n [Sol],[Hal2], and are recovered by replacing the generator $\mathfrak{p}_i$ with $\frac{1}{n}\mathfrak{p}_i$.

2.4. **Basic construction.** Let $\mathbf{A}_k \subseteq \mathbf{P}_k(n)$ be the partition algebra or one of the subalgebras described in Section 2.3 and let $\mathcal{A}_k \subseteq \mathcal{P}_k$ be its diagram basis. We have a natural embedding of $\mathbf{A}_{r-1}$ as a subalgebra of $\mathbf{A}_r$ by placing an identity edge to the right of any diagram in $\mathbf{A}_{r-1}$ thus forming a tower of algebras: $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \cdots \subseteq \mathbf{A}_{k-1} \subseteq \mathbf{A}_k$.

A block in a diagram $d \in \mathcal{A}_k$ is a *propagating block* if it contains vertices from both the top and bottom row, and the *rank* (also called the propagating number) of d, denoted $\mathsf{rank}(d)$, is the number of propagating blocks of d. For $d_1, d_2 \in \mathcal{A}_k$ we have $\mathsf{rank}(d_1 \circ d_2) \leq \min(\mathsf{rank}(d_1), \mathsf{rank}(d_2))$, so that the multiplication of diagrams can never increase the rank. It follows that

$$\mathbf{J}_m := \mathbb{C}\text{-span}\{d \in \mathcal{A}_k \mid \mathsf{rank}(d) \leq k\}, \qquad 0 \leq m \leq k, \tag{2.8}$$

is a two-sided ideal in $\mathbf{A}_k$ and we have the filtration

$$\mathbf{J}_0 \subseteq \mathbf{J}_1 \subseteq \mathbf{J}_2 \subseteq \cdots \subseteq \mathbf{J}_{k-1} \subseteq \mathbf{J}_k = \mathbf{P}_k(n). \tag{2.9}$$

In the case of the Brauer algebra $\mathbf{B}_k(n)$ and the Temperley-Lieb algebra $\mathbf{TL}_k(n)$ we have $\mathbf{J}_{k-1} = \mathbf{J}_k$, $\mathbf{J}_{k-3} = \mathbf{J}_{k-2}$, and so on, since the rank of diagrams in these algebras have the same parity as k.

For each $0 \leq m \leq k$ we have

$$\mathbf{A}_m \cong \mathbf{J}_{m-1} \oplus \mathbf{C}_m, \tag{2.10}$$

where $\mathbf{C}_m$ is the span of the diagrams of rank exactly equal to k. The isomorphism in (2.10) is the Jones basic construction for $\mathbf{A}_k$. In our examples,

$$\begin{array}{ll} \mathbf{C}_m \cong \mathbb{C}\mathbf{S}_m & \text{when } \mathbf{A}_k \text{ is one of the non-planar algebras } \mathbf{P}_k(n), \mathbf{B}_k(n), \mathbf{RB}_k(n), \text{ or } \mathbf{R}_k, \\ \mathbf{C}_m \cong \mathbb{C}\mathbf{1}_m & \text{when } \mathbf{A}_k \text{ is one of the planar algebras } \mathbf{TL}_k(n), \mathbf{M}_k(n), \text{ or } \mathbf{PR}_k. \end{array} \tag{2.11}$$

We let $\Gamma_{\mathbf{A}_k}$ denote the set of possible diagram ranks in $\mathbf{A}_k$, so that

$$\Gamma_{\mathbf{A}_k} = \begin{cases} \{m \mid 0 \leq m \leq k\}, & \text{if } \mathbf{A}_k \text{ equals } \mathbf{P}_k(n), \mathbf{RB}_k(n), \mathbf{R}_k, \mathbf{M}_k(n), \text{ or } \mathbf{PR}_k, \\ \{k - 2\ell \mid 0 \leq \ell \leq \lfloor k/2 \rfloor\}, & \text{if } \mathbf{A}_k \text{ equals } \mathbf{B}_k(n) \text{ or } \mathbf{TL}_k(n). \end{cases} \tag{2.12}$$

If $\Lambda^{\mathbf{A}_k}$ indexes the irreducible modules for $\mathbf{A}_k$, then it follows from the basic construction (see [HR3, Sec. 4.2]) that

$$\Lambda^{\mathbf{A}_k} = \bigsqcup_{m \in \Gamma_{\mathbf{A}_k}} \Lambda^{\mathbf{C}_m} = \begin{cases} \displaystyle\bigsqcup_{m \in \Gamma_{\mathbf{A}_k}} \{\mu \vdash m\}, & \text{if } \mathbf{A}_k \text{ is non-planar}, \\ \Gamma_{\mathbf{A}_k}, & \text{if } \mathbf{A}_k \text{ is planar}, \end{cases} \tag{2.13}$$

where the second equality comes from (2.11) and the fact that the irreducible modules for the group algebra $\mathbb{C}\mathbf{S}_m$ of the symmetric group are indexed by the set $\{\mu \vdash m\}$ of integer partitions of m.

2.5. **Schur-Weyl duality.** For $k, n \in \mathbb{Z}_{\geq 1}$ the partition algebra $\mathbf{P}_k(n)$ and the symmetric group $\mathbf{S}_n$ are in Schur-Weyl duality on the k-fold tensor product $\mathsf{V}_n^{\otimes k}$ of the n-dimensional permutation module V_n of the symmetric group $\mathbf{S}_n$ (see [Jon] or [HR2]). In particular, there is a surjective algebra homomorphism $\mathbf{P}_k(n) \to \mathsf{End}(\mathsf{V}_n^{\otimes k})$ such that the actions of $\mathbf{P}_k(n)$ and $\mathbf{S}_n$ on $\mathsf{V}^{\otimes k}$ commute. When $n \geq 2k$ the representation of $\mathbf{P}_k(n)$ on $\mathsf{V}_n^{\otimes k}$ is faithful and $\mathbf{P}_k(n) \cong \mathsf{End}_{\mathbf{S}_n}(\mathsf{V}_n^{\otimes k})$, the centralizer algebra of $\mathbf{S}_n$ on $\mathsf{V}_n^{\otimes k}$.

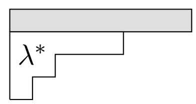
For $n \geq 2k$, the decomposition of $\mathsf{V}_n^{\otimes k}$ as a bimodule for $(\mathbf{P}_k(n), \mathbb{C}\mathbf{S}_n)$ is given by

$$\mathsf{V}_n^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_{k,n}} \mathbf{P}_k^\lambda \otimes \mathbf{S}_n^\lambda, \tag{2.14}$$

where $\Lambda_{k,n}$ indexes the irreducible $\mathbf{S}_n$ modules that appear as constituents of $\mathsf{V}_n^{\otimes k}$. Since irreducible $\mathbf{S}_n$ modules are indexed by partitions of n we have $\Lambda_{k,n} \subseteq \{\lambda \vdash n\}$, and it is easy to show by induction on k (see, for example [HR2, BH3]), that

$$\Lambda_{k,n} = \{\lambda \vdash n \mid 0 \leq |\lambda^*| \leq k\}, \tag{2.15}$$

where if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ is an integer partition of n then $\lambda^* = [\lambda_2, \ldots, \lambda_\ell]$ is the partition λ with its first part removed as illustrated here

(2.16) $\lambda =$  .

We now have two ways to index the irreducible $\mathbf{P}_k(n)$-modules: from the basic construction $\Lambda^{\mathbf{P}_k(n)} = \{\mu \vdash m \mid 0 \leq |\mu| \leq k\}$ and from Schur-Weyl duality $\Lambda_{k,n} = \{\lambda \vdash n \mid 0 \leq |\lambda^*| \leq k\}$. When $n \geq 2k$, they are in bijection by identifying $\lambda \in \Lambda_{k,n}$ with $\lambda^* \in \Lambda^{\mathbf{P}_k(n)}$. The set-partition tableaux that we use in Section 4 require partitions of n, so we use $\Lambda_{k,n}$ for the remainder of this paper. To this end, for each $\mathbf{A}_k$ we add a first row of size $n-m$ to the partitions in $\Lambda^{\mathbf{A}_k}$ to get the partitions in $\Lambda_n^{\mathbf{A}_k}$ so that

$$\Lambda_n^{\mathbf{A}_k} = \left\{\lambda \vdash n \mid \lambda^* \in \Lambda^{\mathbf{A}_k}\right\}. \tag{2.17}$$

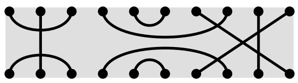
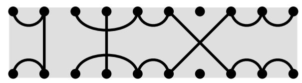
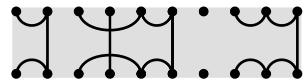
These sets are given below for each of the diagram algebras. To unify our notation we view $\mathbb{C}\mathbf{1}_m$ as the trivial subalgebra of $\mathbb{C}\mathbf{S}_m$ and label its irreducible representation with the partition $[m]$, the index of the trivial module $\mathbf{S}_m^{[m]}$.

$\mathbf{A}_k$	$\Lambda^{\mathbf{A}_k}$	$\Lambda_n^{\mathbf{A}_k}$
$\mathbf{P}_k(n), \mathbf{RB}_k(n), \mathbf{R}_k$	$\{\mu \vdash m \mid 0 \leq m \leq k\}$	$\{\lambda \vdash n \mid \lvert\lambda^*\rvert = m, 0 \leq m \leq k\}$
$\mathbf{B}_k(n)$	$\{\mu \vdash k - 2\ell \mid 0 \leq \ell \leq \lfloor k/2 \rfloor\}$	$\{\lambda \vdash n \mid \lvert\lambda^*\rvert = k - 2\ell, 0 \leq \ell \leq \lfloor k/2 \rfloor\}$
$\mathbf{M}_k(n), \mathbf{PR}_k$	$\{m \mid 0 \leq m \leq k\}$	$\{[n-m, m] \mid 0 \leq m \leq k\}$
$\mathbf{TL}_k(n)$	$\{k - 2\ell \mid 0 \leq \ell \leq \lfloor k/2 \rfloor\}$	$\{[n-m, m] \mid m = k - 2\ell, 0 \leq \ell \leq \lfloor k/2 \rfloor\}$

3. Irreducible Modules

In this section, for each $\lambda \in \Lambda_n^{\mathbf{A}_k}$ with $|\lambda^*| = m$, we construct the irreducible module $\mathbf{A}_k^\lambda = \mathbf{W}_{\mathcal{A}_k}^m \otimes \mathbf{S}_m^{\lambda^*}$ where $\mathbf{W}_{\mathcal{A}_k}^m$ is the span of symmetric m-diagrams in $\mathcal{A}_k$ that $\mathbf{A}_k$ acts on by conjugation and $\mathbf{S}_m^{\lambda^*}$ is an irreducible symmetric group module. When a diagram $d \in \mathcal{A}_k$ conjugates a symmetric m-diagram w it permutes the m fixed points of of w by a permutation $\sigma_{d,w} \in \mathbf{S}_m$ which in turn acts on $\mathbf{S}_m^{\lambda^*}$. We view this as conjugation that is "twisted" by the module $\mathbf{S}_m^{\lambda^*}$. This construction is similar to the Gelfand model for diagram algebras in [HR3] and [KM]. In those papers, a larger class of symmetric diagrams is used and it is always twisted by the sign representation of the symmetric group, and the result is a (multiplicity-free) direct sum of each irreducible $\mathbf{A}_k$-module.

3.1. **Symmetric diagrams.** For $d \in \mathcal{A}_k$, let $d^T \in \mathcal{A}_k$ be the diagram obtained by reflecting d over the horizontal axis. We say that a diagram is *symmetric* if $d = d^T$. For example, the following are symmetric diagrams in $\mathcal{P}_{10}$,

$d_1 =$  , $d_2 =$  , $d_3 =$  .

For a symmetric diagram $d = d^T$, let $\pi(d)$ and $\pi'(d)$ denote the propagating blocks in the top and bottom rows of d, respectively. In the examples above, $\pi(d_1) = \{2 \mid 7 \mid 9 \mid 10\}$ and $\pi(d_2) = \pi(d_3) = \{1, 2 \mid 4 \mid 3, 5, 6 \mid 8, 9, 10\}$, and observe that in a symmetric diagram $\pi'(d)$ is always equal to $\pi(d)$ with the vertices primed.

Definition 3.1. A diagram $d \in \mathcal{A}_k$ is a *symmetric m-diagram* if (1) d is symmetric; (2) $\mathsf{rank}(d) = m$; and (3) each of the m propagating blocks in $\pi(d)$ is connected to its mirror image in $\pi'(d)$.

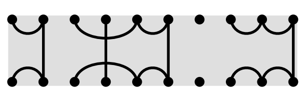
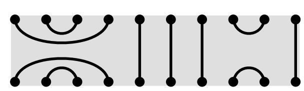
In the examples above, d_3 is a symmetric 4-diagram, but d_1 and d_2 are not since they each have a propagating block not connected to its mirror image. For any of the diagram algebras $\mathbf{A}_k$, let

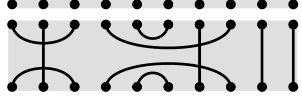
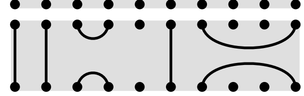
$$\mathcal{W}^m_{\mathcal{A}_k} = \{d \in \mathcal{A}_k \mid d \text{ is a symmetric } m\text{-diagram}\}. \tag{3.2}$$

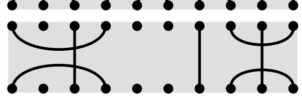
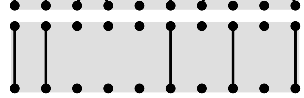
A simple counting argument can be used to determine the number of symmetric m-diagrams $|\mathcal{W}^m_{\mathcal{A}_k}|$ for each diagram algebra $\mathbf{A}_k$:

$\mathbf{A}_k$	$\lvert\mathcal{W}^m_{\mathcal{A}_k}\rvert$	$\mathbf{A}_k$	$\lvert\mathcal{W}^m_{\mathcal{A}_k}\rvert$
$\mathbf{P}_k(n)$	$\sum_t \left\{{k \atop t}\right\}\binom{t}{m}$	$\mathbf{TL}_k(n)$	$\binom{k}{\frac{k-m}{2}} - \binom{k}{\frac{k-m}{2}-1}$
$\mathbf{B}_k(n)$	$\binom{k}{m}(k-m-1)!!$	$\mathbf{M}_k(n)$	$\sum_t \binom{k}{m+2t}\left(\binom{m+2t}{t} - \binom{m+2t}{t-1}\right)$
$\mathbf{RB}_k(n)$	$\sum_t \binom{k}{m}\binom{k-m}{2t}(2t-1)!!$	$\mathbf{R}_k, \mathbf{PR}_k$	$\binom{k}{m}$

The corresponding integer triangles can be found in [OEIS] A049020, A008313, A096713, A064189, A111062, and A007318, respectively. In the case of the planar algebras, the symmetric m-diagrams are exactly equal to the rank-m symmetric diagrams used in the Gelfand models in [HR3] and [KM], and in the case of the non-planar algebras, the symmetric m-diagrams are a subset of the rank-m symmetric diagrams. Below are examples from these algebras.

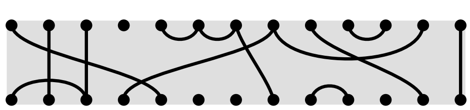
 $\in \mathcal{W}^4_{\mathcal{P}_{10}}$  $\in \mathcal{W}^4_{\mathcal{TL}_{10}}$

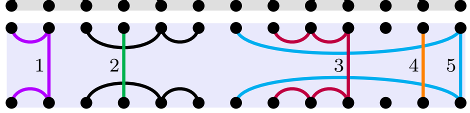
 $\in \mathcal{W}^4_{\mathcal{B}_{10}}$  $\in \mathcal{W}^3_{\mathcal{M}_{10}}$

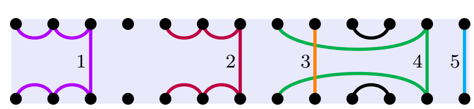
 $\in \mathcal{W}^3_{\mathcal{RB}_{10}}$  $\in \mathcal{W}^5_{\mathcal{R}_{10}} = \mathcal{W}^5_{\mathcal{PR}_{10}}$

For $d, w \in \mathcal{A}_k$, we say that $d \circ w \circ d^T$ is the *conjugate* of w by d. For example, below is the conjugation $d \circ w \circ d^T$ of diagrams $d \in \mathcal{P}_{13}$ and $w \in \mathcal{W}^5_{\mathcal{P}_{13}}$,

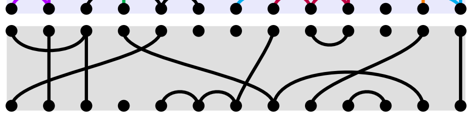
(3.3)

$d =$ 

$w =$ 

$=$ 

$= d \circ w \circ d^T.$

$d^T =$ 

We order the m propagating blocks of a symmetric m-diagram according to their maximum entry. So, for example, we order the blocks in $\pi(w) = \{1,2 \mid 4 \mid 8,9,10 \mid 12 \mid 7,13\}$ as follows: $\{1,2\} < \{4\} < \{8,9,10\} < \{12\} < \{7,13\}$. We refer to this as *max-entry order*. Furthermore, by convention, *we always draw the propagating edges in a symmetric m diagram as identity edges connecting the maximum entries in the blocks.* Upon conjugating a symmetric m-diagram w by $d \in \mathcal{P}_k$, if $\mathsf{rank}(d \circ w \circ d^T) = m$, then the propagating blocks of w have been permuted, and we let $\sigma_{d,w} \in \mathbf{S}_m$ be the permutation of the fixed blocks, so that (in max-entry order),

(3.4) *the ith propagating block in w gets sent to the $\sigma_{d,w}(i)$th propagating block in $d \circ w \circ d^T$.*

We refer to $\sigma_{d,w}$ as the *twist* of the conjugation of w by d. For example in (3.3) $\sigma_{d,w}$ is the three-cycle $(4,3,2)$.

Remark 3.5. The following properties can be verified through simple diagram calculus for $d \in \mathcal{P}_k$.

1. If w is a symmetric m-diagram, then $d \circ w \circ d^T$ is a symmetric m'-diagram with $m' = \mathsf{rank}(d \circ w \circ d^T) \leq \mathsf{rank}(w) = m$.
2. If $d = d^T$ then $d \circ d \circ d^T = d$.

3.2. **Irreducible modules $\mathbf{A}_k^\lambda$.** For any of the diagram algebras $\mathbf{A}_k$, let

$$\mathbf{W}^m_{\mathcal{A}_k} := \mathbb{C}\mathcal{W}^m_{\mathcal{A}_k} = \mathbb{C}\text{-span}\left\{d \in \mathcal{A}_k \mid d \text{ is a symmetric } m\text{-diagram}\right\}, \tag{3.6}$$

and for $d \in \mathcal{A}_k$ and $w \in \mathcal{W}^m_{\mathcal{A}_k}$ define

$$d \cdot w = \begin{cases} n^{\ell(d,w)} d \circ w \circ d^T & \text{if } \mathsf{rank}(d \circ w \circ d^T) = m, \\ 0 & \text{if } \mathsf{rank}(d \circ w \circ d^T) < m, \end{cases} \tag{3.7}$$

where $\ell(d,w)$ is the number of blocks removed from the middle row during the diagram concatenation $d \circ w$. For example, for the diagrams in (3.3), we have $d \cdot w = n(d \circ w \circ d^T)$, since we remove one block in the concatenation $d \circ w$. By Remark 3.5(1), if $d \in \mathcal{A}_k$ and $w \in \mathcal{W}^m_{\mathcal{A}_k}$ then $d \cdot w \in \mathbf{W}^m_{\mathcal{A}_k}$.

Proposition 3.8. *The conjugation action defined in* (3.7) *makes $\mathbf{W}^m_{\mathcal{A}_k}$ an $\mathbf{A}_k$-module.*

Proof. We show that for any two diagrams $d_1, d_2 \in \mathcal{A}_k$, we have $(d_1 d_2) \cdot w = d_1 \cdot (d_2 \cdot w)$. When $\mathsf{rank}\left((d_1 d_2) \circ w \circ (d_1 d_2)^T\right) < m$, then by the associativity of diagram multiplication either $\mathsf{rank}(d_2 \circ w \circ d_2^T) < m$ or $\mathsf{rank}(d_1 \circ (d_2 \circ w \circ d_2^T) \circ d_1^T) < m$. The action gives zero in either case. Now assume $\mathsf{rank}\left((d_1 d_2) \circ w \circ (d_1 d_2)^T\right) = m$. For ease of notation let $d_1 \circ d_2 = d_3$ and let $d_2 \circ w \circ d_2^T = w' \in \mathcal{W}^m_{\mathcal{A}_k}$. Then we have

$$\begin{aligned} d_1 \cdot (d_2 \cdot w) = n^{\ell(d_2,w)} d_1 \cdot \left(d_2 \circ w \circ d_2^T\right) &= n^{\ell(d_1,w')} n^{\ell(d_2,w)} \left(d_1 \circ (d_2 \circ w \circ d_2^T) \circ d_1^T\right) \\ &= n^{\ell(d_1,w')} n^{\ell(d_2,w)} \left(d_3 \circ w \circ d_3^T\right). \end{aligned}$$

On the other hand, we have

$$(d_1 d_2) \cdot w = n^{\ell(d_1,d_2)} d_3 \cdot w = n^{\ell(d_1,d_2)} n^{\ell(d_3,w)} \left(d_3 \circ w \circ d_3^T\right),$$

and from simple diagram calculus, it is clear that for any $x, y, z, u \in \mathcal{A}_k$, $\ell(x \circ y, z \circ u) = \ell(y, z)$, and hence $\ell(d_1, w') = \ell(d_1, d_2)$ and $\ell(d_2, w) = \ell(d_3, w)$. Finally, the linear extension of (3.7) to the full algebra $\mathbf{A}_k$ makes $\mathbf{W}^m_{\mathcal{A}_k}$ into an $\mathbf{A}_k$-module. □

For each partition $\mu \vdash m$ there is an irreducible module $\mathbf{S}^\mu_m$ for the symmetric group $\mathbf{S}_m$ labeled by μ. The dimension of $\mathbf{S}^\mu_m$ is given by the number f^μ of standard Young tableaux of shape μ, where a standard Young tableaux of shape μ is a filling of the boxes of the diagram of μ with the numbers $1, 2, \ldots, m$ such that the rows increase from left to right and the columns increase from top to bottom. We let $\mathcal{SYT}(\mu)$ denote the set of standard Young tableaux of shape μ. For example, there are five standard Young tableaux of shape $\mu = [3,2]$:

$$\mathcal{SYT}([3,2]) = \left\{ t_1 = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array},\ t_2 = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array},\ t_3 = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array},\ t_4 = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array},\ t_5 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \right\}.$$

We let $\{v_t \mid t \in \mathcal{SYT}(\mu)\}$ denote a basis of $\mathbf{S}_m^\mu$. For example, this basis could be one of Young's natural, seminormal, or orthogonal bases (see Section 3.3).

For $\lambda \in \Lambda_n^{\mathbf{A}_k}$ with $|\lambda^*| = m$, let $\mathbf{A}_k^\lambda$ be the vector space

$$\mathbf{A}_k^\lambda := \mathbf{W}_{\mathcal{A}_k}^m \otimes \mathbf{S}_m^{\lambda^*} = \mathbb{C}\text{-span}\left\{ w \otimes v_t \;\middle|\; w \in \mathcal{W}_{\mathcal{A}_k}^m,\ t \in \mathcal{SYT}(\lambda^*) \right\}. \tag{3.9}$$

If $w \in \mathcal{W}_{\mathcal{A}_k}^m$ and $t \in \mathcal{SYT}(\lambda^*)$, we define the action of $d \in \mathcal{A}_k$ on the basis element $w \otimes v_t$ to be

$$d \cdot (w \otimes v_t) = \begin{cases} n^{\ell(d,w)} (d \circ w \circ d^T) \otimes \sigma_{d,w} \cdot v_t & \text{if } \mathsf{rank}(d \circ w \circ d^T) = m, \\ 0 & \text{if } \mathsf{rank}(d \circ w \circ d^T) < m, \end{cases} \tag{3.10}$$

where as in (3.7) $\ell(d,w)$ is the number of connected components removed from the middle row during the diagram concatenation $d \circ w$ and $\sigma_{d,w} \in \mathbf{S}_m$ is the twist of the conjugation of w by d defined in (3.4); that is, $\sigma_{d,w}$ is the permutation on the propagating blocks of w induced by d.

Proposition 3.11. *The twisted conjugation action defined in* (3.10) *makes* $\mathbf{A}_k^\lambda$ *an* $\mathbf{A}_k$*-module.*

Proof. We show that for any two diagrams $d_1, d_2 \in \mathcal{A}_k$, we have $(d_1 d_2) \cdot (w \otimes v_t) = d_1 \cdot (d_2 \cdot (w \otimes v_t))$. We only need to consider the case that $\mathsf{rank}\left((d_1 d_2) \circ w \circ (d_1 d_2)^T\right) = \mathsf{rank}\left(d_1 \circ (d_2 \circ w \circ d_2^T) \circ d_1^T\right) = m$. Letting $d_1 \circ d_2 = d_3$ and let $d_2 \circ w \circ d_2^T = w'$, we have (from the proof of 3.8),

$$d_1 \cdot (d_2 \cdot (w \otimes v_t)) = n^{\ell(d_1,d_2)} n^{\ell(d_3,w)} \left(d_3 \circ w \circ d_3^T \otimes \sigma_{d_1,w'} \sigma_{d_2,w} \cdot v_t\right),$$

On the other hand, we have

$$(d_1 d_2) \cdot (w \otimes v_t) = n^{\ell(d_1,d_2)} n^{\ell(d_3,w)} (d_3 \circ w \circ d_3^T \otimes \sigma_{d_3,w} \cdot v_t),$$

so we need to show that

$$\sigma_{d_1,w'} \sigma_{d_2,w} \cdot v_t = \sigma_{d_3,w} \cdot v_t.$$

The equality of the permutations $\sigma_{d_1,w'}\sigma_{d_2,w}$ and $\sigma_{d_3,w}$ follows from the composition of twists, and $\sigma_{d_1,w'}\sigma_{d_2,w} \cdot v_t = \sigma_{d_3,w} \cdot v_t$ because the action on the symmetric group module $\mathbf{S}_m^{\lambda^*}$ is associative. Finally, the linear extension of (3.10) to the full algebra $\mathbf{A}_k$ makes $\mathbf{A}_k^\lambda$ into an $\mathbf{A}_k$-module. ☐

The above construction of $\mathbf{A}_k^\lambda$ does not depend on the parameter n, though for exceptional n-values the algebra $\mathbf{A}_k$ ceases to be semisimple. The structure of $\mathbf{P}_k(n)$ for the non-semisimple cases is investigated in [DW], [ME], and [MS2]. The following result holds when $n \geq 2k$, which we assume for the remainder of the paper.

Theorem 3.12. *When* $n \geq 2k$ *and* $\lambda \in \Lambda_{k,n}$, $\mathbf{A}_k^\lambda$ *is an irreducible* $\mathbf{A}_k$*-module.*

Proof. First we note that the modules $\mathbf{A}_k^\lambda$ and $\mathbf{A}_k^\mu$ are inequivalent if $\lambda \neq \mu$. For, if $|\lambda^*| = m$ and $|\mu^*| < m$, one can easily find a diagram with propagating number $|\mu^*|$ which necessarily acts as zero on $\mathbf{A}_k^\lambda$ but not on $\mathbf{A}_k^\mu$. Namely, for any basis element $w \otimes v_t$ of $\mathbf{A}_k^\mu$, we have, from Remark 3.5 (2), that $w \cdot (w \otimes v_t) = n^{\ell(w,w)} w \otimes v_t$, whereas w acts as zero on $\mathbf{A}_k^\lambda$. Hence the modules cannot be related by a change of basis. Now suppose $|\lambda^*| = |\mu^*| = m$ but $\lambda \neq \mu$. This does not happen in the planar case as only the trivial $\mathbf{S}_m$-module appears. In the non-planar cases, for any $\sigma \in \mathbf{S}_m$ we can choose a $d \in \mathbf{A}_k$ such that $d \cdot (w \otimes v_t) = w \otimes \sigma \cdot v_t$. Hence, the modules $\mathbf{A}_k^\lambda$ and $\mathbf{A}_k^\mu$ are inequivalent because $\mathbf{S}_m^{\lambda^*} \not\cong \mathbf{S}_m^{\mu^*}$.

Finally, the set $\{\mathbf{A}_k^\lambda \mid \lambda \in \Lambda_n^{\mathbf{A}_k}\}$ accounts for the full dimension of $\mathbf{A}_k$. For $\lambda \in \Lambda_n^{\mathbf{A}_k}$, the dimension of $\mathbf{A}_k^\lambda$ is given by

$$\dim \mathbf{A}_k^\lambda = \left|\mathcal{W}_{\mathcal{A}_k}^{|\lambda^*|}\right| f^{\lambda^*}.$$

We now consider the planar and non-planar cases separately. For the non-planar algebras, we sum the squares of the dimensions of the inequivalent modules we have constructed,

$$\sum_{\lambda\in\Lambda_n^{\mathbf{A}_k}} \left(\left|\mathcal{W}_{\mathcal{A}_k}^{|\lambda^*|}\right| f^{\lambda^*}\right)^2 = \sum_{m\in\Gamma_{\mathbf{A}_k}} \sum_{\mu\vdash m} |\mathcal{W}_{\mathcal{A}_k}^m|^2 (f^\mu)^2 = \sum_{m\in\Gamma_{\mathbf{A}_k}} |\mathcal{W}_{\mathcal{A}_k}^m|^2 m! = \dim \mathbf{A}_k,$$

where we have used the well-known symmetric group identity $m! = \sum_{\mu\vdash m}(f^\mu)^2$. The first equality uses the bijection between (2.17) and (2.13). The last equality is justified as follows: $|\mathcal{W}_{\mathcal{A}_k}^m|$ counts the number of possible top (resp., bottom) rows of diagrams in $\mathcal{A}_k$ with m blocks distinguished as propagating blocks, so that $|\mathcal{W}_{\mathcal{A}_k}^m|^2$ counts the number of top and bottom rows with m blocks chosen from each to be propagating blocks. The distinguished blocks can be matched up in $m!$ ways, and summing over the possible ranks gives the number of basis diagrams for $\mathbf{A}_k$. The planar case is similar, except we have only the trivial partition $[m]$ for each $m \in \Gamma^{\mathbf{A}_k}$, so that

$$\sum_{\lambda\in\Lambda_n^{\mathbf{A}_k}} \left(\left|\mathcal{W}_{\mathcal{A}_k}^{|\lambda^*|}\right| f^{\lambda^*}\right)^2 = \sum_{m\in\Gamma^{\mathbf{A}_k}} |\mathcal{W}_{\mathcal{A}_k}^m|^2 = \dim \mathbf{A}_k.$$

In the planar case there is no $m!$ because propagating edges cannot cross. □

Remark 3.13. When $|\lambda^*| = k$, the only diagrams that do not act as zero on $\mathbf{A}_k^\lambda$ are the permutation diagrams in $\mathbf{S}_k \subseteq \mathcal{P}_k$. Then the action (3.10) is exactly the action of $\mathbf{S}_k$ on the irreducible module $\mathbf{S}_k^{\lambda^*}$, and there is an isomorphism $\mathbf{A}_k^\lambda \cong \mathbf{S}_k^{\lambda^*}$ as $\mathbf{S}_k$ modules.

3.3. **Choice of basis for $\mathbf{S}_m^{\lambda^*}$.** For $\lambda \in \Lambda_n^{\mathbf{A}_k}$ with $|\lambda^*| = m$, the construction (3.9) of the irreducible $\mathbf{A}_k$-module $\mathbf{A}_k^\lambda = \mathbf{W}_{\mathcal{A}_k}^m \otimes \mathbf{S}_m^{\lambda^*}$ allows for any choice of basis of the symmetric group module $\mathbf{S}_m^{\lambda^*}$. Young's [You] natural, seminormal, or othogonal bases are obvious choices but any choice will work.

In Young's natural basis $\{\mathsf{n}_t \mid t \in \mathcal{SYT}(\lambda^*)\}$, the defining feature is that for any permutation $\sigma \in \mathbf{S}_m$ we have

$$\sigma \cdot \mathsf{n}_t = \mathsf{n}_{\sigma(t)}, \tag{3.14}$$

where $\sigma(t)$ is the tableau given by permuting the entries of t according to σ and where we view $\mathsf{n}_{\sigma(t)}$ as an element of $\mathbf{S}_m^\mu$ regardless of whether $\sigma(t)$ is standard or not. When $\sigma(t)$ is nonstandard, the vector $\mathsf{n}_{\sigma(t)}$ can be re-expressed as an *integer* linear combination of basis elements n_t for $t \in \mathcal{SYT}(\lambda^*)$ by applying a recursive "straightening" algorithm using (for example) *Garnir relations*, which are identities in the group algebra $\mathbb{C}\mathbf{S}_m$. See [Sag], [CLL] for details. One can also directly compute the matrix of σ in the natural basis using a combinatorial method called tableaux intersection [GM].

The action on the seminormal basis $\{\mathsf{v}_t \mid t \in \mathcal{SYT}(\lambda^*)\}$ is defined only on the adjacent transpositions $\mathfrak{s}_i = (i, i+1) \in \mathbf{S}_m$,

$$\mathfrak{s}_i \cdot \mathsf{v}_t = \frac{1}{\Delta_i(t)}\mathsf{v}_t + \left(1 + \frac{1}{\Delta_i(t)}\right)\mathsf{v}_{\mathfrak{s}_i(t)}, \tag{3.15}$$

where $\mathsf{v}_{\mathfrak{s}_i(t)} = 0$ if $\mathfrak{s}_i(t)$ is nonstandard, and $\Delta_i(t)$ is the *axial distance* from i to $i+1$ in t, defined to be

$$\Delta_i(t) = (c_i - r_i) - (c_{i+1} - r_{i+1}), \tag{3.16}$$

if i is in the position (r_i, c_i) and $(i+1)$ is in the position (r_{i+1}, c_{i+1}) in t. The action on the orthogonal basis $\{\mathsf{u}_t \mid t \in \mathcal{SYT}(\lambda^*)\}$ is identical to that of the seminormal basis but with the off-diagonal entry $1 + \frac{1}{\Delta_i(t)}$ replaced by $\sqrt{1 - (\frac{1}{\Delta_i(t)})^2}$.

Example 3.17. Let $k = 12$ and $\lambda = [n-5, 3, 2]$. There are five standard Young tableaux of shape $\lambda^* = [3, 2]$ as shown in (3.2). Let $w \in \mathcal{W}_5^{\mathcal{P}_{13}}$ be the symmetric 5-diagram shown in (3.3) and consider the basis element $w \otimes v_{t_4}$ of $\mathbf{P}_k^\lambda$. There is one block removed during the diagram concatenation $d \circ w$, and the five fixed blocks of w are twisted by the permutation $\sigma_{d,w} = (4, 3, 2)$. Hence $d \cdot (w \otimes v_{t_4}) = n(w' \otimes \sigma_{d,w} \cdot v_{t_4})$, where $w' = d \circ w \circ d^T$. We now exemplify how $\sigma_{d,w}$ acts when we take v_t to be the natural and seminormal bases.

Let n_{t_4} be a basis element of Young's natural representation. Then $\sigma_{d,w} \cdot \mathsf{n}_{t_4} = \mathsf{n}_{\sigma_{d,w}(t_4)}$, where

$$\sigma_{d,w}(t_4) = \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

The second equality above comes from the Garnir relations (see [Sag]), and it follows that

$$d \cdot (w \otimes \mathsf{n}_{t_4}) = nw' \otimes \mathsf{n}_{\sigma_{d,w}(t_4)} = nw' \otimes \mathsf{n}_{t_2} - nw' \otimes \mathsf{n}_{t_1}.$$

Now let v_{t_4} be a basis element of Young's seminormal representation. The permutation $\sigma_{d,w}$ must be re-written in terms of adjacent transpositions, $\sigma_{d,w} = \mathfrak{s}_4\mathfrak{s}_3$. Then we have

$$\begin{aligned}\sigma_{d,w} \cdot \mathsf{v}_{t_4} = \mathfrak{s}_4 \cdot (\mathfrak{s}_3 \cdot \mathsf{v}_{t_4}) &= \mathfrak{s}_4 \cdot \left(\frac{1}{3}\mathsf{v}_{t_4} + (1 + \frac{1}{3})\mathsf{v}_{\mathfrak{s}_3(t_4)}\right) \\ &= \frac{1}{3}\left(-\frac{1}{2}\mathsf{v}_{t_4} + (1 - \frac{1}{2})\mathsf{v}_{\mathfrak{s}_4(t_4)}\right) + \frac{4}{3}\left(\mathsf{v}_{\mathfrak{s}_3(t_4)} + (1 + \frac{1}{1})\mathsf{v}_{\mathfrak{s}_4\mathfrak{s}_3(t_4)}\right).\end{aligned}$$

Because $\mathfrak{s}_4\mathfrak{s}_3(t_4)$ is nonstandard, $\mathsf{v}_{\mathfrak{s}_4\mathfrak{s}_3(t_4)} = 0$, and $\mathfrak{s}_4 \cdot (\mathfrak{s}_3 \cdot \mathsf{v}_{t_4}) = -\frac{1}{6}\mathsf{v}_{t_4} + \frac{1}{6}\mathsf{v}_{t_3} + \frac{4}{3}\mathsf{v}_{t_5}$. It follows that

$$d \cdot (w \otimes \mathsf{v}_{t_4}) = -\frac{n}{6}w' \otimes \mathsf{v}_{t_4} + \frac{n}{6}w' \otimes \mathsf{v}_{t_3} + \frac{4n}{3}w' \otimes \mathsf{v}_{t_5}.$$

4. Set-Partition Tableaux

In this section we describe the irreducible modules of the algebras $\mathbf{A}_k$ on a basis indexed by set-partition tableaux. These tableaux first appear for the partition algebra implicitly in [BHH, Sec. 5.3] and explicitly in [BH3, Def. 3.14]. They also appear (independently) as multiset tableaux in [OZ]. In Section 4.3 we restrict the definition of set-partition tableaux to work for each of the diagram subalgebras $\mathbf{A}_k$.

Definition 4.1. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ be an integer partition of n into ℓ parts, with $\lambda^* = [\lambda_2, \ldots, \lambda_\ell]$, and let π be a set partition of $\{1, \ldots, k\}$ into t blocks with $|\lambda^*| \leq t \leq n$. A *set-partition tableau* T *of shape* λ *and content* π is a filling of the boxes of the skew shape $\lambda/[n-t]$ with the blocks of π so that each box of $\lambda/[n-t]$ contains a unique block of π. The blocks below the first row of T are called *propagating* blocks, while the blocks in the first row are called *non-propagating.* A set-partition tableau is *standard* if all of the entries of T increase across the rows from left to right and down the columns using max-entry order on the blocks of π.

Example 4.2. Below is a set-partition tableau T of shape $\lambda = [8, 4, 3, 1] \vdash 16$ and content

$$\pi = \{3 \mid 5 \mid 6 \mid 8 \mid 2, 9 \mid 12 \mid 4, 7, 10, 14 \mid 13, 15 \mid 1, 16 \mid 11, 17\}$$

which has $t = 10$ blocks. The blocks are increasing across the rows and down the columns, so T is standard. We have emphasized max-entry order by underlining the maximal elements in each block of π.

						$\underline{12}$	$1, \underline{16}$
$\underline{3}$	$\underline{6}$	$\underline{8}$	$11, \underline{17}$				
$\underline{5}$	$4, 7, 10, \underline{14}$	$13, \underline{15}$					
$2, \underline{9}$							

Remark 4.3. Let $\lambda \vdash n$ and π be a set partition of $\{1, \ldots, k\}$, and let T be a set-partition tableau of shape λ and content π. When $n \geq 2k$ (which we assume for the semisimplicity of $\mathbf{P}_k(n)$) there is no column of T with both propagating and non-propagating blocks. To simplify our figures we omit unnecessary boxes from the first row, and let a single box with "$\cdots$" denote the correct number of boxes. For instance, consider the same tableau as in the example above, but where $\lambda \in \Lambda_{17,n}$ is the partition $[n-8, 4, 3, 1]$ for some $n \geq 34$,

$n-10$ boxes (over the first row)

				$\cdots$	$\underline{12}$	$1, \underline{16}$
$\underline{3}$	$\underline{6}$	$\underline{8}$	$11, \underline{17}$			
$\underline{5}$	$4, 7, 10, \underline{14}$	$13, \underline{15}$				
$2, \underline{9}$						

For $k \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \Lambda_{k,n}$, define $\mathcal{SPT}(\lambda, k)$ to be the set of set-partition tableaux of shape λ whose content is a set partition of $\{1, \ldots, k\}$, and define $\overline{\mathcal{SPT}}(\lambda, k)$ to be the subset of these tableaux whose first row is increasing from left to right. Finally, define $\mathcal{SSPT}(\lambda, k)$ to be the subset of standard set-partition tableaux. For a fixed λ and k, $\mathcal{SSPT}(\lambda, k) \subseteq \overline{\mathcal{SPT}}(\lambda, k) \subseteq \mathcal{SPT}(\lambda, k)$, and the sizes of these sets (when $n \geq 2k$) are given by

$$|\mathcal{SPT}(\lambda, k)| = \sum_t \left\{ {k \atop t} \right\} \binom{t}{m} t!, \tag{4.4a}$$

$$|\overline{\mathcal{SPT}}(\lambda, k)| = \sum_t \left\{ {k \atop t} \right\} \binom{t}{m} m! = |\mathcal{W}_{\mathcal{P}_k}^m| m!, \tag{4.4b}$$

$$|\mathcal{SSPT}(\lambda, k)| = \sum_t \left\{ {k \atop t} \right\} \binom{t}{m} f^{\lambda/[n-t]} = \dim \mathbf{P}_k^\lambda = \sum_t \left\{ {k \atop t} \right\} \binom{t}{m} f^{\lambda^*}, \tag{4.4c}$$

which are justified as follows: first partition the set $\{1, \ldots, k\}$ into at $t \geq m$ blocks, and choose m of these blocks to propagate. For (4.4a), we can arrange the blocks of the tableau in $t!$ ways, for (4.4b) the first row is fixed and we are free to arrange the propagating blocks in $m!$ ways, and for (4.4c) the number of standard arrangements of the blocks is equal to $f^{\lambda/[n-t]}$. The second equality in (4.4c) also holds when $n < 2k$ [BH2, BH3, BHH].

Recall that $\mathcal{W}_{\mathcal{P}_k}^m$ is the set of symmetric m-diagrams in $\mathcal{P}_k$, and let $\mathcal{YT}(\mu)$ be the set of Young tableaux of shape μ. For each $\lambda \in \Lambda_{k,n}$, there is an easy-to-verify bijection,

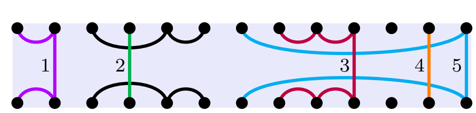
$$\overline{\mathcal{SPT}}(\lambda, k) \longleftrightarrow \mathcal{W}_{\mathcal{P}_k}^{|\lambda^*|} \times \mathcal{YT}(\lambda^*), \tag{4.5}$$

which is given by associating $\mathsf{T} \in \overline{\mathcal{SPT}}(\lambda, k)$ with the pair $(w, t) \in \mathcal{W}_{\mathcal{P}_k}^{|\lambda^*|} \times \mathcal{YT}(\lambda^*)$ where w is the symmetric $|\lambda^*|$-diagram whose propagating and non-propagating blocks are those of T and where t is the standard tableau with entries $\{1, \ldots, |\lambda^*|\}$ such that i is placed in the same position the ith propagating block of w occupies in T. See Example 4.6.

Example 4.6. If $w \otimes v_{t_4}$ is the basis element from (3.3), then the bijection in (4.5) gives the pairing,

$n-7$ blocks (over the first row); rows 2–3 form λ^*

			$\cdots$	$3, 5, \underline{6}$	$\underline{11}$
$1, \underline{2}$	$\underline{4}$	$\underline{12}$			
$8, 9, \underline{10}$	$7, \underline{13}$				

$\longleftrightarrow$ ( ,

1	2	4
3	5	

).

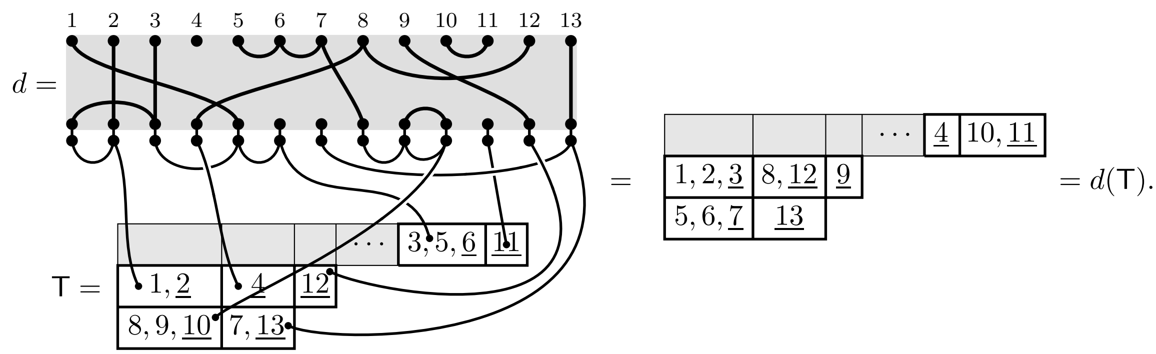
4.1. **Action of diagrams on set-partition tableaux.** Now we define an action of diagrams in $\mathcal{A}_k$ on set-partition tableaux that generalizes the permutation action of the symmetric group on Young tableaux. For $d \in \mathcal{A}_k$ let $\mathsf{top}(d)$ be the partition of $\{1, \dots, k\}$ induced on the top row of d.

Definition 4.7. For a diagram $d \in \mathcal{A}_k$ and a set partition π of $\{1, \dots, k\}$, let $d \circ \pi$ denote the diagram concatenation of d with π, where π is viewed as a one-line set-partition diagram. Given a set-partition tableau T of shape $\lambda \vdash n$ and content π, define the action of d on T, denoted $d(\mathsf{T})$, to be the set-partition tableau of shape λ where:

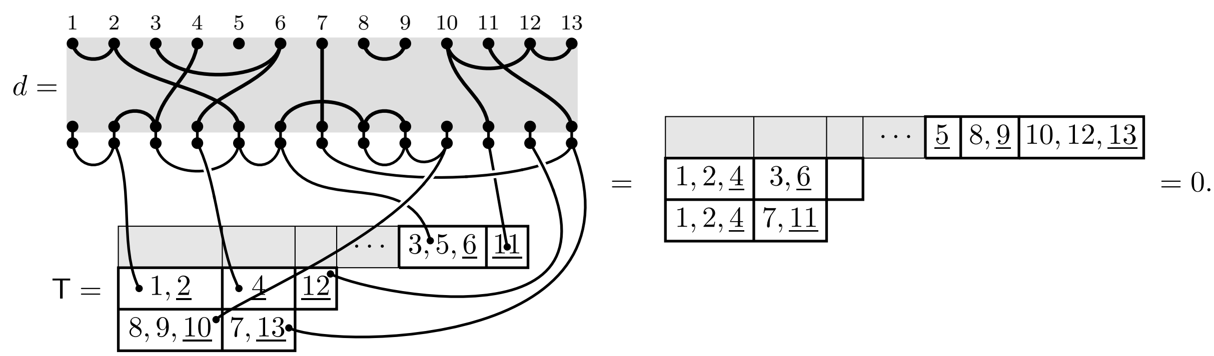
(a) the propagating blocks in $d(\mathsf{T})$ are obtained by replacing each propagating block of T with the block it is connected to in $\mathsf{top}(d \circ \pi)$,
(b) the non-propagating blocks in $d(\mathsf{T})$ are
   (i) the non-propagating blocks of $\mathsf{top}(d \circ \pi)$,
   (ii) blocks of $\mathsf{top}(d \circ \pi)$ which are connected only to non-propagating blocks of T,
(c) the non-propagating blocks increase from left to right in the first row of $d(\mathsf{T})$,
(d) if the results of the above steps do not produce a set-partition tableau, then $d(\mathsf{T}) = 0$.

The action of a diagram d on a tableau T is easily obtained by placing d above T, drawing edges from the blocks of T to the corresponding blocks on the bottom row of d, and performing diagram multiplication. For instance, see Examples 4.8 and 4.19.

Example 4.8. Let T be the set-partition tableau from Example 4.6. Acting with the diagram d from (3.3), we find



The following diagram acts as zero on T, since the result is not a set-partition tableau.



Remark 4.9. A diagram d acts as zero on a set-partition tableau T if

(a) two propagating blocks of T become connected when constructing $d(\mathsf{T})$, or
(b) a propagating block of T does not propagate to the top of d when constructing $d(\mathsf{T})$.

4.2. **Natural basis.** For $\lambda \in \Lambda_{k,n}$, let $\{\mathsf{N}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SSPT}(\lambda, k)\}$ be a set of vectors indexed by the standard set-partition tableaux of shape λ. Define

$$\mathsf{P}_k^\lambda = \mathbb{C}\text{-span}\left\{\mathsf{N}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SSPT}(\lambda, k)\right\}, \tag{4.10}$$

and for $d \in \mathcal{P}_k$ and $\mathsf{T} \in \mathcal{SSPT}(\lambda, k)$ define

$$d \cdot \mathsf{N}_\mathsf{T} = \begin{cases} n^{\ell(d,\mathsf{T})}\mathsf{N}_{d(\mathsf{T})} & \text{if } d(\mathsf{T}) \text{ is a set-partition tableau,} \\ 0 & \text{if } d(\mathsf{T}) \text{ is not a set-partition tableau,} \end{cases} \tag{4.11}$$

where $d(\mathsf{T})$ is defined in Definition 4.7 and $\ell(d, \mathsf{T})$ is the number of connected components removed in the construction of $d(\mathsf{T})$. If $d(\mathsf{T})$ is not standard, then $\mathsf{N}_{d(\mathsf{T})}$ can be expressed as an integer linear combination of basis elements using Garnir relations (see Section 3.3).

Example 4.12. Let d and T be defined as in the first example from Example 4.8. In the construction of $d(\mathsf{T})$ there is one connected component removed, so that

$$d \cdot \mathsf{N}_\mathsf{T} = n\mathsf{N}_{d(\mathsf{T})}, \text{ where } d(\mathsf{T}) = \begin{array}{|c|c|c|c|c|c|} \hline & & & \cdots & \underline{4} & 10,\underline{11} \\ \hline 1,2,\underline{3} & 8,\underline{12} & \underline{9} \\ \cline{1-3} 5,6,\underline{7} & \underline{13} \\ \cline{1-2} \end{array}.$$

The result is nonstandard, with a descent in the first row. The Garnir relation for straightening $\mathsf{N}_{d(\mathsf{T})}$ is:

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & \cdots & \underline{4} & 10,\underline{11} \\ \hline 1,2,\underline{3} & 8,\underline{11} & \underline{9} \\ \cline{1-3} 5,6,\underline{7} & \underline{12} \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline & & & \cdots & \underline{4} & 10,\underline{11} \\ \hline 1,2,\underline{3} & \underline{9} & 8,\underline{11} \\ \cline{1-3} 5,6,\underline{7} & \underline{12} \\ \cline{1-2} \end{array} - \begin{array}{|c|c|c|c|c|c|} \hline & & & \cdots & \underline{4} & 10,\underline{11} \\ \hline 1,2,\underline{3} & \underline{9} & \underline{12} \\ \cline{1-3} 5,6,\underline{7} & 8,\underline{11} \\ \cline{1-2} \end{array},$$

and hence

$$d \cdot \mathsf{N}_\mathsf{T} = n\mathsf{N}_{\mathsf{T}_1} - n\mathsf{N}_{\mathsf{T}_2},$$

where T_1 and T_2 are the two standard set-partition tableaux appearing above. This can be compared to Example 3.17 (a), which gives the analogous action on the diagram basis.

Theorem 4.13. *The action defined in* (4.11) *makes* P_k^λ *into a* $\mathbf{P}_k(n)$*-module, and* $\mathsf{P}_k^\lambda \cong \mathbf{P}_k^\lambda$.

Proof. We show that the action defined on set-partition tableaux is simply the result of applying the bijection (4.5) to the action defined in (3.10) when the basis for $\mathbf{S}_m^{\lambda^*}$ is Young's natural basis $v_t = \mathsf{n}_t$. Let T be a standard set-partition tableau of shape λ and content π, and let $w \otimes \mathsf{n}_t$, be the basis element associated to T via the bijection (4.5). Assuming $\mathsf{rank}(d \circ w \circ d^T) = m$, we have $d \cdot (w \otimes \mathsf{n}_t) = n^{\ell(d,w)} d \circ w \circ d^T \otimes \mathsf{n}_{\sigma_{d,w}(t)}$, where the ith propagating block of w gets sent to the $\sigma_{d,w}(i)$th propagating block of $d \circ w \circ d^T$. To obtain $\sigma_{d,w}(t)$, we replace i with $\sigma_{d,w}(i)$. If T' is the set-partition tableau associated to $d \circ w \circ d^T \otimes \mathsf{n}_{\sigma_{d,w}(t)}$ by (4.5), then the propagating blocks of T' are obtained by replacing the ith propagating block of T with the $\sigma_{d,w}(i)$th propagating block of $d \circ w \circ d^T$ for each i, and the non-propagating blocks of T' are the non-propagating blocks of $d \circ w \circ d^T \otimes \mathsf{n}_{\sigma_{d,w}(t)}$, which are either the non-propagating blocks of d or the blocks of d connected only to non-propagating blocks of w. Hence $\mathsf{T}' = d(\mathsf{T})$. One can easily confirm that the connected components removed in the construction of $d(\mathsf{T})$ are connected only to non-propagating blocks of T, otherwise the action gives zero. Hence $\ell(d, \mathsf{T}) = \ell(d, w)$. Finally, by Remark 4.9 the criteria for $d(\mathsf{T}) = 0$ are equivalent to the criteria for $\mathsf{rank}(d \circ w \circ d^T) < m$. □

Remark 4.14. The construction defined in (4.10) and (4.11) is a partition algebra analogue of Young's natural basis for the irreducible modules of the symmetric group, and we refer to $\{\mathsf{N}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SSPT}(\lambda, k)\}$ as the *natural basis* for $\mathbf{P}_k^\lambda$. Analogous modules can be constructed when the basis for $\mathbf{S}_m^{\lambda^*}$ is seminormal v_t or orthogonal u_t. However, the action on these modules, though isomorphic to the one defined above, lacks the "naturalness" evident in (4.11).

The generators $\mathfrak{s}_i, \mathfrak{p}_i, \mathfrak{b}_i$, have particularly nice actions on set-partition tableaux, which we describe in the following theorem. The actions of the generators $\mathfrak{e}_i, \mathfrak{l}_i$, and $\mathfrak{r}_i$ of the subalgebras are omitted for brevity but can easily be obtained from $\mathfrak{s}_i, \mathfrak{b}_i$ and $\mathfrak{p}_i$.

Theorem 4.15. *Let $\lambda \in \Lambda_{k,n}$ and $\mathsf{T} \in \mathcal{SSPT}(\lambda, k)$, so that N_T is an element of the natural basis for P_k^λ. Then the action of $\mathfrak{s}_i$, $\mathfrak{p}_i$, $\mathfrak{b}_i$, and $\mathfrak{e}_i$ on N_T are given by:*

(a) $\mathfrak{s}_i \cdot \mathsf{N}_\mathsf{T} = \mathsf{N}_{\mathfrak{s}_i(\mathsf{T})}$, *where $\mathfrak{s}_i(\mathsf{T})$ is the set-partition tableau in $\overline{\mathcal{SPT}}(\lambda, k)$ obtained from T by swapping i and $i+1$, and standardizing the first row.*

(b) $$\mathfrak{p}_i \cdot \mathsf{N}_\mathsf{T} = \begin{cases} n\mathsf{N}_\mathsf{T} & \text{if } \{i\} \text{ is a non-propagating singleton block in } \mathsf{T}, \\ 0 & \text{if } \{i\} \text{ is a propagating singleton block in } \mathsf{T}, \\ \mathsf{N}_{\mathfrak{p}_i(\mathsf{T})} & \text{otherwise,} \end{cases}$$
where $\mathfrak{p}_i(\mathsf{T})$ is the set-partition tableau in $\overline{\mathcal{SPT}}(\lambda, k)$ obtained from T by removing i from its block, placing the singleton block $\{i\}$ into the first row, and standardizing the first row.

(c) $$\mathfrak{b}_i \cdot \mathsf{N}_\mathsf{T} = \begin{cases} \mathsf{N}_\mathsf{T} & \text{if } i \text{ and } i+1 \text{ are in the same block in } \mathsf{T}, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in different propagating blocks in } \mathsf{T}, \\ \mathsf{N}_{\mathfrak{b}_i(\mathsf{T})} & \text{otherwise,} \end{cases}$$
where $\mathfrak{b}_i(\mathsf{T})$ is the set-partition tableau in $\overline{\mathcal{SPT}}(\lambda, k)$ obtained from T by joining the block containing i with the block containing $i+1$, and standardizing the first row. The resulting block becomes propagating if one of the original blocks was propagating, and otherwise stays non-propagating.

If $\mathfrak{s}_i(\mathsf{T})$, $\mathfrak{p}_i(\mathsf{T})$, $\mathfrak{b}_i(\mathsf{T})$ is a nonstandard set-partition tableau then $\mathsf{N}_{\mathfrak{s}_i(\mathsf{T})}$, $\mathsf{N}_{\mathfrak{p}_i(\mathsf{T})}$, $\mathsf{N}_{\mathfrak{b}_i(\mathsf{T})}$ can be expressed as an integer linear combination of basis elements using Garnir relations (see Section 3.3).

Proof. The action is easily obtained from (4.11) through diagram calculus as in Example 4.8. □

Example 4.16. We give examples of the explicit action of $\mathfrak{p}_i$ and $\mathfrak{b}_i$ described above.

(a) *Action of $\mathfrak{p}_i$.* Consider the following standard set-partition tableau T of shape $[n-4, 3, 1]$,

$$\mathfrak{p}_5\left(\begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & \underline{1} & 5,\underline{6} \\ \hline \underline{4} & 2,3,\underline{8} & \underline{9} \\ \cline{1-3} \underline{7} \\ \cline{1-1} \end{array}\right) = \begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & \underline{1} & \underline{5} & \underline{6} \\ \hline \underline{4} & 2,3,\underline{8} & \underline{9} \\ \cline{1-3} \underline{7} \\ \cline{1-1} \end{array},$$

$$\mathfrak{p}_8\left(\begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & \underline{1} & 5,\underline{6} \\ \hline \underline{4} & 2,3,\underline{8} & \underline{9} \\ \cline{1-3} \underline{7} \\ \cline{1-1} \end{array}\right) = \begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & \underline{1} & 5,\underline{6} & \underline{8} \\ \hline \underline{4} & 2,\underline{3} & \underline{9} \\ \cline{1-3} \underline{7} \\ \cline{1-1} \end{array}.$$

Since 1 is a non-propagating singleton, $\mathfrak{p}_1 \cdot \mathsf{N}_\mathsf{T} = n\mathsf{N}_\mathsf{T}$. Since 4 is a propagating singleton, $\mathfrak{p}_4$ acts as zero on T. When $\mathfrak{p}_5$ acts on T, it separates 5 and 6. When $\mathfrak{p}_8$ acts on T, it moves 8 to its own block on the first row, and the result is nonstandard. We then have $\mathfrak{p}_5 \cdot \mathsf{N}_\mathsf{T} = \mathsf{N}_{\mathfrak{p}_5(\mathsf{T})}$ and $\mathfrak{p}_8 \cdot \mathsf{N}_\mathsf{T} = \mathsf{N}_{\mathfrak{p}_8(\mathsf{T})}$.

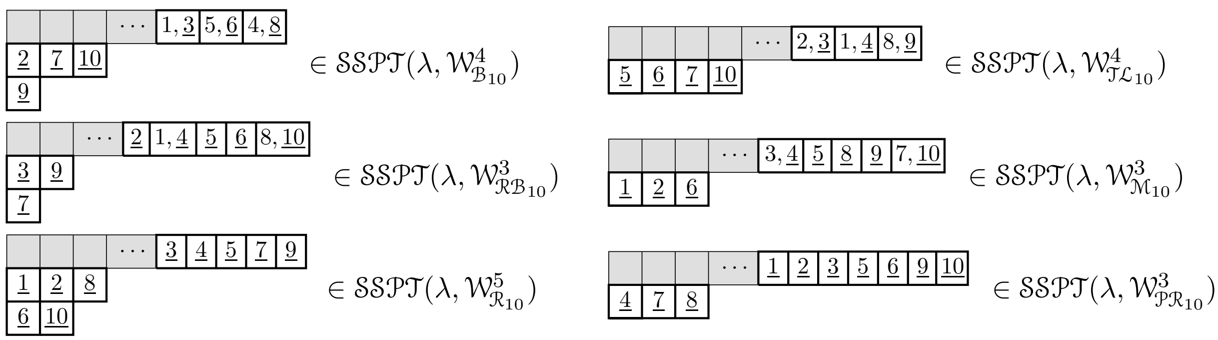
(b) *Action of $\mathfrak{b}_i$.* Consider the following set-partition tableau T of shape $[n-4, 3, 1]$,

$$\mathfrak{b}_2\left(\begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & 6,\underline{8} & 1,2,\underline{9} \\ \hline \underline{3} & \underline{7} & \underline{10} \\ \cline{1-3} 4,\underline{5} \\ \cline{1-1} \end{array}\right) = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \cdots & 6,\underline{8} \\ \hline 1,2,3,\underline{9} & \underline{7} & \underline{10} \\ \cline{1-3} 4,\underline{5} \\ \cline{1-1} \end{array},$$

$$\mathfrak{b}_8\left(\begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & 6,\underline{8} & 1,2,\underline{9} \\ \hline \underline{3} & \underline{7} & \underline{10} \\ \cline{1-3} 4,\underline{5} \\ \cline{1-1} \end{array}\right) = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \cdots & 1,2,6,8,\underline{9} \\ \hline \underline{3} & \underline{7} & \underline{10} \\ \cline{1-3} 4,\underline{5} \\ \cline{1-1} \end{array}.$$

Since 1 and 2, and 4 and 5, are in the same block, both $\mathfrak{b}_1$ and $\mathfrak{b}_4$ fix T. Since 3 and 4 are in different propagating blocks, $\mathfrak{b}_3$ acts as zero on T. When $\mathfrak{b}_2$ acts on T, the contents of the block containing 2 are appended to the block containing 3, and the result is nonstandard. Finally, $\mathfrak{b}_8$ acts by joining the blocks containing 8 and 9. Thus $\mathfrak{b}_2 \cdot \mathsf{N}_{\mathsf{T}} = \mathsf{N}_{\mathfrak{b}_2(\mathsf{T})}$ and $\mathfrak{b}_8 \cdot \mathsf{N}_{\mathsf{T}} = \mathsf{N}_{\mathfrak{b}_8(\mathsf{T})}$.

4.3. **Subalgebras.** When $\mathbf{A}_k$ is a subalgebra of the partition algebra, applying the bijection (4.5) to basis elements of $\mathbf{A}_k^\lambda$ yields restricted types of standard set-partition tableaux of shape $\lambda \in \Lambda_n^{\mathbf{A}_k}$. In particular, for all of the proper subalgebras the propagating blocks are singletons. For the Brauer and Temperley-Lieb algebras the non-propagating blocks are pairs, for the rook-Brauer and Motzkin algebras the non-propagating blocks are pairs or singletons, and for the rook monoid and planar rook monoid algebras the non-propagating blocks are singletons. Below are example set-partition tableaux for these subalgebras.



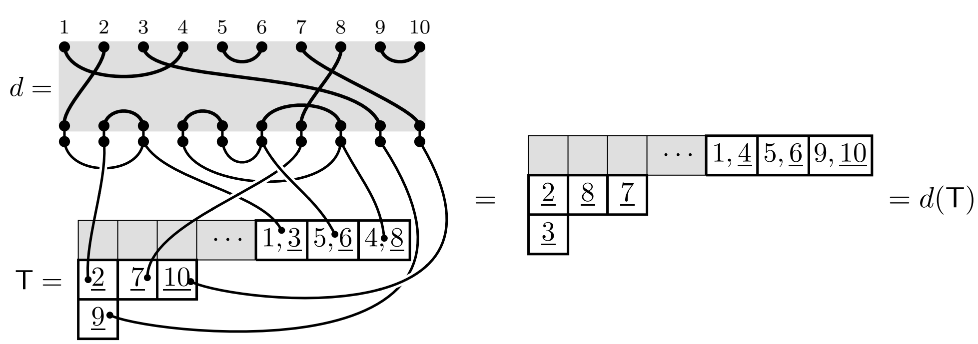
When restricted to the subalgebra $\mathbf{A}_k$, Definition 4.7 defines an action of $\mathbf{A}_k$ on set-partition tableaux $\mathsf{T} \in \mathcal{SSPT}(\lambda, \mathcal{W}^m_{\mathcal{A}_k})$. This leads to the following theorem, whose proof is identical to that of Theorem 4.13.

Theorem 4.17. *When restricted to any of the subalgebras* $\mathbf{A}_k$*, the action* (4.11) *defines an analogue of Young's natural representation for* $\mathbf{A}_k$*.*

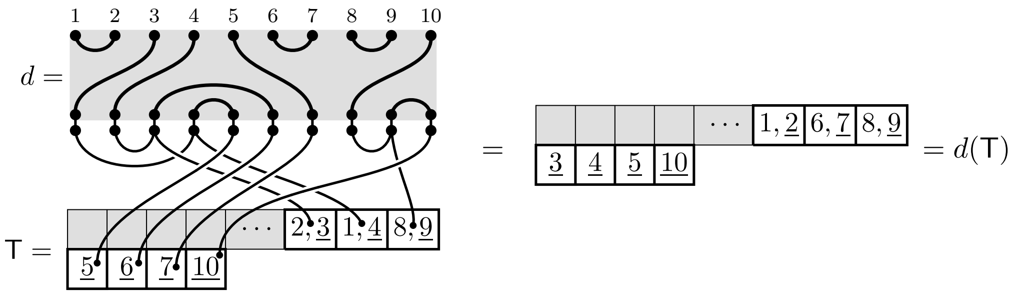
Remark 4.18. When $|\lambda^*| = k$, the standard set-partition tableaux of shape λ^* have k propagating singletons and no non-propagating blocks, and thus are standard Young tableaux. Furthermore, the only diagrams which are nonzero on $\mathbf{A}_k^\lambda$ are permutation diagrams. Upon restriction to the subalgebra $\mathbb{C}\mathbf{S}_k$, the module $\mathbf{A}_k^\lambda$ corresponds exactly to Young's natural representation.

Example 4.19. We give examples in the Brauer, Temperley-Lieb, and Rook monoid algebras.

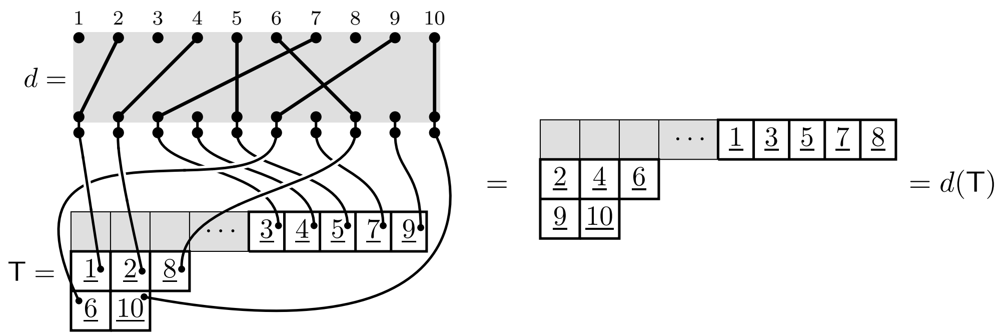
(a) *Brauer algebra.* In the example below $d \cdot \mathsf{N}_{\mathsf{T}} = n\mathsf{N}_{d(\mathsf{T})}$, where $\mathsf{N}_{d(\mathsf{T})}$ can be re-expressed in the basis of standard tableaux using Garnir relations as in Section 3.3 (in this particular case, the Garnir relation is simple: $\mathsf{N}_{d(\mathsf{T})} = \mathsf{N}_{\mathsf{T}'}$, where T' has 7 and 8 switched).



(b) *Temperley-Lieb algebra.* In the example below $d \cdot \mathsf{N}_{\mathsf{T}} = \mathsf{N}_{d(\mathsf{T})}$.



(c) *Rook monoid algebra.* In the example below $d \cdot \mathsf{N}_{\mathsf{T}} = n^3 \mathsf{N}_{d(\mathsf{T})}$.

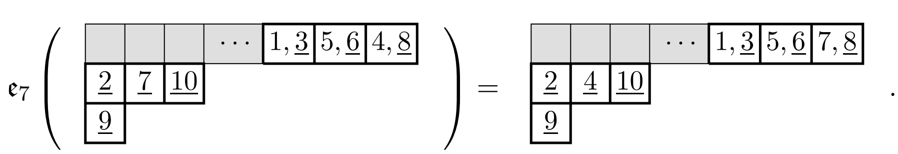


The action of the generator $\mathfrak{e}_i$ on set-partition tableaux takes a nice form that can be verified using Theorem 4.15 and the relation $\mathfrak{e}_i = \mathfrak{b}_i \mathfrak{p}_i \mathfrak{p}_{i+1} \mathfrak{b}_i$,

$$\mathfrak{e}_i \cdot \mathsf{N}_{\mathsf{T}} = \begin{cases} n\mathsf{N}_{\mathsf{T}} & \text{if } \{i, i+1\} \text{ is a non-propagating block in } \mathsf{T}, \\ 0 & \begin{array}{l}\text{if } i \text{ and } i+1 \text{ are in propagating blocks in } \mathsf{T}, \text{ or if } \{i\} \text{ and} \\ \{i+1\} \text{ are singleton blocks in } \mathsf{T} \text{ with one propagating,}\end{array} \\ n\mathsf{N}_{\mathfrak{e}_i(\mathsf{T})} & \text{if } \{i\} \text{ and } \{i+1\} \text{ are non-propagating singleton blocks in } \mathsf{T}, \\ \mathsf{N}_{\mathfrak{e}_i(\mathsf{T})} & \text{otherwise,} \end{cases} \tag{4.20}$$

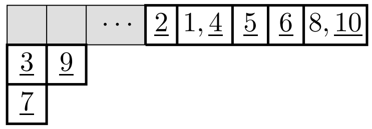
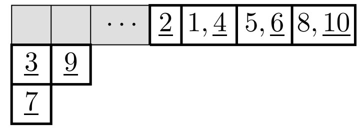
where $\mathfrak{e}_i(\mathsf{T})$ is the set-partition tableau in $\overline{\mathcal{SPT}}(\lambda, k)$ obtained from T by removing i and $i+1$ from their blocks, making $\{i, i+1\}$ into a non-propagating block, joining the remaining elements from the blocks which contained i and $i+1$, and standardizing the first row. The resulting block becomes propagating if one of the original blocks was propagating, and otherwise stays non-propagating.

Example 4.21. Below are examples of the action of $\mathfrak{e}_i$ on set-partition tableaux of Brauer and rook-Brauer type. Consider the following set-partition tableau T in $\mathcal{SSPT}(\lambda, \mathcal{W}^4_{\mathcal{B}_{10}})$, where $\lambda = [n-4, 3, 1]$,



Since 9 and 10 are distinct propagating singletons, $\mathfrak{e}_9$ acts as zero on T. Since $\{5, 6\}$ is a non-propagating block, $\mathfrak{e}_5 \cdot \mathsf{N}_{\mathsf{T}} = n\mathsf{N}_{\mathsf{T}}$. When $\mathfrak{e}_7$ acts on T, $\{7, 8\}$ becomes a non-propagating block and 4 becomes a propagating singleton, so that $\mathfrak{e}_7 \cdot \mathsf{N}_{\mathsf{T}} = \mathsf{N}_{\mathfrak{e}_7(\mathsf{T})}$.

Consider the following set-partition tableau in $\mathcal{SSPT}(\lambda, \mathcal{W}^3_{\mathcal{RB}_{10}})$, where $\lambda = [n-3, 2, 1]$,

$\mathfrak{e}_5$ () =  .

Since 2 is a non-propagating singleton and 3 is a propagating singleton, $\mathfrak{e}_2$ acts as zero on T. The same is true for $\mathfrak{e}_6$. When $\mathfrak{e}_5$ acts on T, $\{5, 6\}$ becomes a non-propagating block, and $\mathfrak{e}_5 \cdot \mathsf{N}_{\mathsf{T}} = n\mathsf{N}_{\mathfrak{e}_5(\mathsf{T})}$.

5. Characters

As an application of the explicit construction of the simple module $\mathbf{A}_k^\lambda$, we provide a closed form for the irreducible characters of the partition algebra and its diagram subalgebras. If $d \in \mathcal{A}_k$, then taking the trace in the diagram basis, with the action defined in (3.10), gives the following result.

Theorem 5.1. *Let $d \in \mathcal{A}_k$ be a basis diagram for $\mathbf{A}_k$ and let $\lambda \in \Lambda_n^{\mathbf{A}_k}$ with $|\lambda^*| = m$. The value of the irreducible character $\chi^\lambda_{\mathbf{A}_k}$ on the diagram $d \in \mathcal{A}_k$ is given by*

$$\chi^\lambda_{\mathbf{A}_k}(d) = \sum_{w \in \mathcal{F}^m_{\mathcal{A}_k}(d)} n^{\ell(d,w)} \chi^{\lambda^*}_{\mathbf{S}_m}(\sigma_{d,w}), \tag{5.1a}$$

where $n^{\ell(d,w)}$ is the number of connected components removed in the concatenation of d and w, $\sigma_{d,w}$ is the twist of $d \circ w \circ d^T$, and $\mathcal{F}^m_{\mathcal{A}_k}(d)$ is the set of diagrams in $\mathcal{W}^m_{\mathcal{A}_k}$ fixed under conjugation by d,

$$\mathcal{F}^m_{\mathcal{A}_k}(d) := \left\{ w \in \mathcal{W}^m_{\mathcal{A}_k} \;\middle|\; d \circ w \circ d^T = w \right\}. \tag{5.1b}$$

Let γ_r be the r-cycle $(r, r-1, \ldots, 1)$ in $\mathbf{S}_r \subseteq \mathbf{P}_r(n)$, and for a partition $\kappa = [\kappa_1, \kappa_2, \ldots, \kappa_\ell]$ define

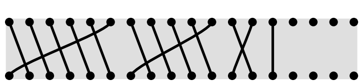
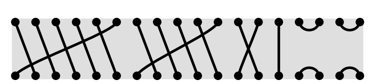
$$\gamma_\kappa = \gamma_{\kappa_1} \otimes \gamma_{\kappa_2} \otimes \cdots \otimes \gamma_{\kappa_\ell}, \tag{5.2}$$

where here the tensor product denotes the juxtaposition of diagrams. It follows from the basic construction (see Section 2.4 and [HR1, Lem. 2.8]) that the irreducible characters of $\mathbf{A}_k$ are completely determined by their values on diagrams of the form $\gamma_\kappa \otimes \mathsf{e}^{\otimes s}$, where

(5.3a) $\mathsf{e} = \frac{1}{n}\mathfrak{e}_1 = \frac{1}{n}$  and $|\kappa| + 2s = k$ for $\mathbf{B}_k(n)$ and $\mathbf{TL}_k(n)$,

(5.3b) $\mathsf{e} = \frac{1}{n}\mathfrak{p}_1 = \frac{1}{n}$ and $|\kappa| + s = k$ for $\mathbf{P}_k(n)$ and its other diagram subalgebras.

Thus, the diagrams $\gamma_\kappa \otimes \mathsf{e}^{\otimes s}$ are conjugacy class analogs for $\mathbf{A}_k$. For example, if $k = 18$ and $\kappa = [6, 5, 2, 1] \vdash 14$, then

$\gamma_\kappa \otimes \mathfrak{p}_1^{\otimes 4} = \frac{1}{n^4}$  and $\gamma_\kappa \otimes \mathfrak{e}_1^{\otimes 2} = \frac{1}{n^2}$  .

are conjugacy class representatives in $\mathbf{P}_{18}(n)$ and $\mathbf{B}_{18}(n)$, respectively. If the algebra $\mathbf{A}_k$ is planar, then the only partition κ used is $\kappa = [1, \ldots, 1]$ so that γ_κ is the identity diagram. Furthermore from [HR1, Eq. 2.17, Eq. 2.22] and [Hal1, Cor. 4.2.3], the irreducible characters satisfy

$$\chi^\lambda_{\mathbf{A}_k}(\gamma_\kappa \otimes \mathsf{e}^{\otimes s}) = \begin{cases} 0 & \text{if } |\kappa| < |\lambda^*|, \\ \chi^\lambda_{\mathbf{A}_{|\kappa|}}(\gamma_\kappa) & \text{if } |\kappa| \geq |\lambda^*|. \end{cases} \tag{5.4}$$

It follows from 5.4 that characters of $\mathbf{A}_k$ are determined by the characters of $\mathbf{A}_{k-1}$ and the values $\chi^\lambda_{\mathbf{A}_k}(\gamma_\kappa)$ for $\kappa \vdash k$. When $\kappa \vdash k$, Theorem 5.1 simplifies to the following.

Corollary 5.5. *For $\lambda \in \Lambda_n^{\mathbf{A}_k}$ such that $|\lambda^*| = m$ and $\kappa \vdash k$, we have*

$$\chi^{\lambda}_{\mathbf{A}_k}(\gamma_\kappa) = \sum_{\mu \vdash m} \mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k} \chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu), \tag{5.5a}$$

where $\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k} := |\mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa)|$ is the cardinality of the following set,

$$\mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa) := \left\{ w \in \mathcal{W}^{\mathcal{A}_k}_m \;\middle|\; \gamma_\kappa \circ w \circ \gamma_\kappa^T = w,\ \sigma_{\gamma_\kappa, w} \in \mathbf{S}_m \text{ has cycle type } \mu \right\} \subseteq \mathcal{F}^{m}_{\mathcal{A}_k}(\gamma_\kappa). \tag{5.5b}$$

Proof. Clearly $n^{\ell(\gamma_\kappa, w)} = 1$ for all γ_κ and w, and on these special elements the sum (5.1a) becomes

$$\chi^{\lambda}_{\mathbf{A}_k}(\gamma_\kappa) = \sum_{w \in \mathcal{F}^m_{\mathcal{A}_k}(\gamma_\kappa)} \chi^{\lambda^*}_{\mathbf{S}_m}(\sigma_{\gamma_\kappa, w}) = \sum_{\mu \vdash m} \sum_{w \in \mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa)} \chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu) = \sum_{\mu \vdash m} \mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k} \chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu),$$

where in the third equality we use the fact that characters are a class function. □

5.1. **Fixed points $\mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa)$.** We now characterize the fixed diagrams $\mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa)$ defined in (5.5b). Many of the statements in this section are straightforward generalizations of the $m = 0$ case to $m \geq 0$, and the proofs of Lemma 5.6, Proposition 5.9, and Lemma 5.14 are nearly identical to the proofs of Lemma 2, Proposition 4, and Lemma 6 in [FH].

A symmetric m-diagram w in $\mathcal{W}^m_{\mathcal{A}_k}$ is determined uniquely by the set partition $\mathsf{top}(w)$ making up its top row, and the m blocks of $\mathsf{top}(w)$ distinguished as propagating. The bottom row $\mathsf{bot}(w)$ is the mirror image of $\mathsf{top}(w)$, so we use $\mathsf{top}(w)$ to denote the set partition of both the top and bottom rows of w.

Lemma 5.6. *The k-cycle γ_k fixes $w \in \mathcal{W}^m_{\mathcal{A}_k}$ if and only if the following conditions hold:*

- (a) *all of the blocks of w propagate if $m > 0$,*
- (b) *none of the blocks of w propagate if $m = 0$, and*
- (c) *$i \overset{w}{\sim} j$ if and only if $(i + r) \overset{w}{\sim} (j + r)$, for all $r \in \mathbb{Z}$, where $i + r$ and $j + r$ are computed* mod k

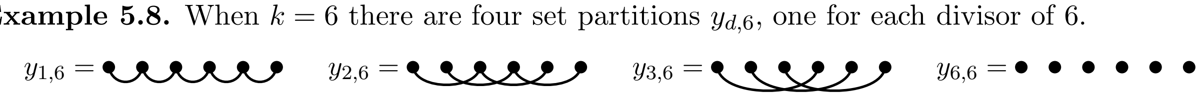
Proof. The action of γ_k on w is to shift each vertex one step to the left, mod k. Thus, if $i \overset{w}{\sim} j$ then $(i-1) \overset{\gamma_k \cdot w}{\sim} (j-1)$, viewing the subtraction mod k. Now, if $w \in \mathcal{F}^{\mathcal{A}_k}_m(\gamma_k)$, then $w = \gamma_k^r \cdot w$ for any $r \in \mathbb{Z}$. Thus $i \overset{w}{\sim} j$ implies $(i - r) \overset{w}{\sim} (j - r)$. If $w \in \mathcal{F}^{\mathcal{A}_k}_m(\gamma_k)$, then the blocks of w either all propagate or all do not propagate, for if this were not the case, γ_k would send a propagating block to a non-propagating block and visa versa. □

Definition 5.7. [FH, Def. 3] For each divisor d of k, define the set partition $y_{d,k}$ of $\{1, \dots, k\}$ by the rule

$$a \overset{y_{d,k}}{\sim} b \quad \text{if and only if} \quad a \equiv b \mod d.$$

The set partition $y_{d,k}$ has d connected components each of size k/d. We refer to the connected components of $y_{d,k}$ as *d-components.*

Example 5.8. When $k = 6$ there are four set partitions $y_{d,6}$, one for each divisor of 6.



Proposition 5.9. *A diagram $w \in \mathcal{W}^m_{\mathcal{A}_k}$ is fixed by γ_k if and only if $\mathsf{top}(w) = y_{d,k}$ for $d|k$, so that*

$$\mathcal{F}^m_{\mathcal{A}_k}(\gamma_\kappa) = \begin{cases} \left\{ w \in \mathcal{W}^m_{\mathcal{A}_k} \mid \mathsf{top}(w) = y_{d,k}, \text{ where } d|k \right\} & \text{if } m = 0, \\ \left\{ w \in \mathcal{W}^m_{\mathcal{A}_k} \mid \mathsf{top}(w) = y_{m,k} \right\} & \text{if } m > 0 \text{ and } m \mid k, \\ \emptyset & \text{if } m > 0 \text{ and } m \nmid k. \end{cases}$$

Proof. If $d|k$ and $\mathsf{top}(w) = y_{d,k}$ then w satisfies the conditions of Lemma 5.6 and w is fixed by γ_k. If $m = 0$, none of the blocks propagate and we can construct w from $y_{d,k}$ for any $d|k$. If $m > 0$, then by Lemma 5.6 w must have m blocks, all of which propagate, and so $\mathsf{top}(w) = y_{m,k}$. Conversely, let $w \in \mathcal{F}^m_{\mathcal{A}_k}(\gamma_\kappa)$, and let d be the *minimum* horizontal distance between two vertices that are connected by an edge in w. That is,

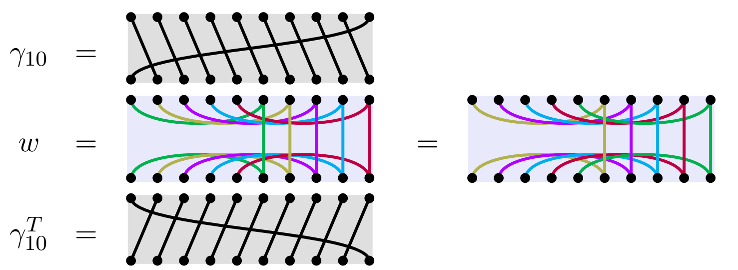
$$d = \begin{cases} k, & \text{if } w \text{ has no horizontal connections,} \\ \min\left\{(i-j) \mod k \mid i \overset{w}{\sim} j, i \neq j\right\}, & \text{otherwise.} \end{cases}$$

Choose i and j so that $i \overset{w}{\sim} j$ with $(i-j) \mod k = d$. Then by Lemma 5.6, we have $(i+r) \overset{w}{\sim} (j+r)$ for $0 \leq r \leq k$. Now, d must divide k, otherwise all of the vertices of w are connected implying $d = 1$, which divides k. If there were a connection in $\mathsf{top}(w)$ not in $y_{d,k}$, then $\mathsf{top}(w)$ would connect two vertices which are closer together than d, contradicting the minimality of d. Thus $\mathsf{top}(w) = y_{d,k}$. ☐

Lemma 5.10. *If $m > 0$ divides k and $w \in \mathcal{W}^m_{\mathcal{A}_k}$ such that $\mathsf{top}(w) = y_{m,k}$, then the permutation induced when γ_k conjugates w is $\sigma_{\gamma_k,w} = \gamma_m$, where γ_m is the m-cycle $(m, m-1, \ldots, 1) \in \mathbf{S}_m$.*

Proof. If $m > 0$ and $w \in \mathcal{W}^m_{\mathcal{A}_k}$, then all m connected components in w are fixed blocks. Using max-entry order, label these fixed blocks in increasing order mod m, so that $w_i < w_j$ if $i < j$. The action of γ_k on w is to shift the fixed blocks one step to the left, which shifts w_1 to w_m, w_m to w_{m-1}, and so on, down to w_2 being sent to w_1. ☐

Example 5.11. Let $k = 10$ and $m = 5$. In the example below, we conjugate a 5-diagram w, whose propagating blocks are $y_{5,10}$, by the cycle γ_{10}. The induced permutation on the fixed blocks of w is $\sigma_{\gamma_{10},w} = (5, 4, 3, 2, 1) = \gamma_5$.

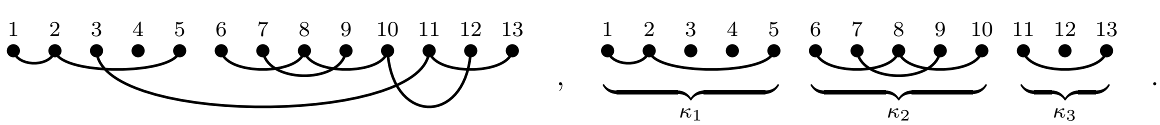


Definition 5.12. [FH, Def. 5] For a partition $\kappa = [\kappa_1, \ldots, \kappa_\ell]$ of k and a set partition π of $\{1, \ldots, k\}$, we say that the κ-blocks of π are the ℓ sub-set partitions given by grouping the elements of $\{1, \ldots, k\}$ into the subsets

$$\{1, \ldots, \kappa_1\}, \{\kappa_1 + 1, \ldots, \kappa_1 + \kappa_2\}, \ldots, \{\kappa_1 + \cdots + \kappa_{\ell-1} + 1, \ldots, k\}$$

where within a κ-block we inherit any connection from π, but ignore any connections between different κ-blocks. A κ-block is said to be *of type d* if it has d connected components.

Example 5.13. Below is a set partition of $\{1, \ldots, 13\}$. If $\kappa = [5, 5, 3] \vdash 13$, the κ-blocks of the set partition are of type 3, 2, and 2, respectively.



Lemma 5.14. *Let $\kappa = [\kappa_1, \ldots, \kappa_\ell] \vdash k$ and $\mu = [\mu_1, \ldots, \mu_s] \vdash m \leq k$. Then $w \in \mathcal{W}^m_{\mathcal{A}_k}$ is in $\mathcal{F}^\mu_{\mathcal{A}_k}(\kappa)$ if and only if the following conditions hold:*

(a) *for each i, the κ_i-block of $\mathsf{top}(w)$ is of the form y_{d_i,κ_i} for some divisor d_i of κ_i,*
(b) *if a κ_i-block of type d_i and a κ_j-block of type d_j have connections between them in w, then*

(i) $d_i = d_j$,
(ii) *each* d_i*-component of the* κ_i*-block is connected to a unique* d_i*-component of the* κ_j*-block,*
(iii) *there are no further connections between these two blocks,*

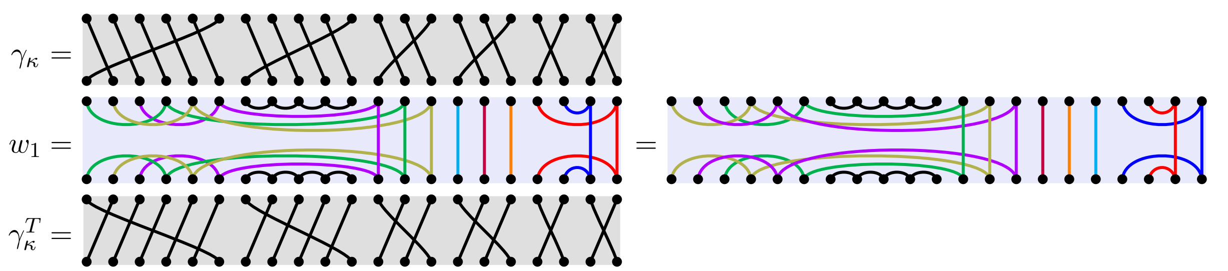
(c) *for each* i*, there are* $\mathsf{m}_i(\mu)$ *sets of connected* κ*-blocks of type* i *that propagate, where* $\mathsf{m}_i(\mu)$ *is the multiplicity of* i *in* μ*.*

Proof. (a) When γ_κ acts on w, the cycle γ_{κ_i} acts on the κ_i-block, so by Proposition 5.9 the κ_i-block must be of the form y_{d_i,κ_i} for some divisor d_i of κ_i.

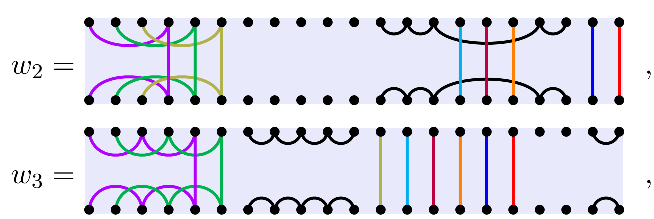
(b) If a d_i-component in the κ_i-block is connected to two d_j-components d_j^1 and d_j^2 in the κ_j-block, then by transitivity $d_j^1 \sim d_j^2$. Thus, each d_i-component is connected to a unique d_j-component. When γ_κ acts on w it permutes the d_i-components in the κ_i-block and the d_j-components in the κ_j-block. If $d_j > d_i$ then γ_κ sends a d_j-component that is not connected to the κ_i-block to a d_j-component that is connected to the κ_i-block, which cannot happen. The same is true when $d_j < d_i$, and thus $d_i = d_j$. There can be no further connections between blocks because that would force two components in one to be connected to a single component in the other.

(c) Now suppose that a set of κ-blocks of type i are connected to each other and all propagate. Then there are i propagating edges from the rightmost κ-block in the set, and by Lemma 5.10, when γ_κ acts on w the i edges are permuted according to the i-cycle γ_i. Hence, if for each i there are $\mathsf{m}_i(\mu)$ sets of connected κ-blocks of type i that propagate then there are $\sum_{i=1}^m \mathsf{m}_i(\mu) = m$ propagating blocks, which are permuted by $\sigma_{\gamma_\kappa,w} = \gamma_{i_1} \otimes \gamma_{i_2} \otimes \cdots \otimes \gamma_{i_s}$, where $i_j \in \mu$. Clearly $\sigma_{\gamma_\kappa,w}$ has cycle type μ, and any other choices for the number and type of propagating blocks gives a different cycle type, so that $w \in \mathcal{F}^{\nu}_{\mathcal{A}_k}(\kappa)$ for $\nu \neq \mu$. □

Example 5.15. Let $\kappa = [6,5,3,3,2,2] \vdash 21$ and $\mu = [3,3,2] \vdash 8$. The following diagram is fixed under conjugation by γ_κ, and the permutation of the fixed blocks is $\sigma_{\gamma_\kappa,w_1} = (3,2,1)(6,5,4)(7,8) \in \mathbf{S}_8$.



The following diagrams are also in $\mathcal{F}^{\mu}_{\mathcal{P}_{21}}(\kappa)$,

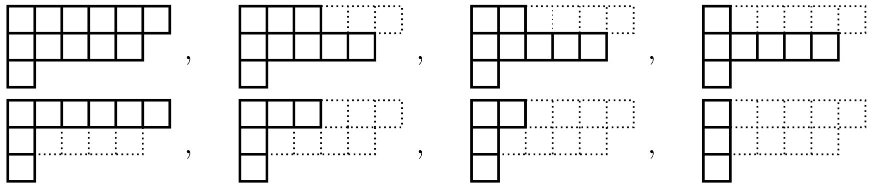


with permutations $\sigma_{\gamma_\kappa,w_2} = (3,2,1)(6,5,4)(7,8)$ and $\sigma_{\gamma_\kappa,w_3} = (1,2)(5,4,3)(8,7,6)$, respectively, which both have cycle type $[3,3,2]$. It is easy to verify that these three diagrams satisfy the properties of Lemma 5.14.

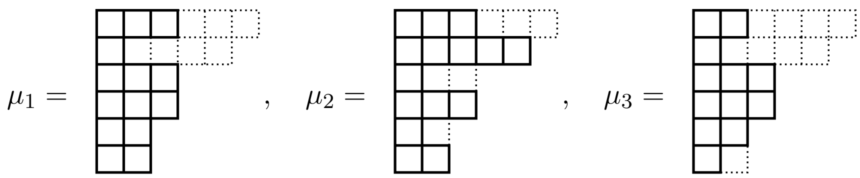
5.2. **The partition algebra.** We now count the number of symmetric m-diagrams in $\mathcal{W}^m_{\mathcal{P}_k}$ that satisfy the conditions of Lemma 5.14.

Definition 5.16. Let $\kappa = [\kappa_1, \ldots, \kappa_\ell]$ be an integer partition of k into ℓ parts. We say that a *divisor* of κ is a composition $\nu = [\nu_1, \ldots, \nu_\ell]$ such that $\nu_i | \kappa_i$ for all $i = 1, \ldots, \ell$, and we let $\nu | \kappa$ indicate that ν is a divisor of κ.

Example 5.17. The following diagrams depict the eight divisors of $\kappa = [6,5,1] \vdash 12$.



Example 5.18. Each diagram in $\mathcal{F}^{\mu}_{\mathcal{P}_k}(\kappa)$ determines a divisor of κ: by Lemma 5.14 (a) the blocks of w are of the form y_{d_i,κ_i} where $d_i | \kappa_i$, and the collection $\nu = [d_1, \ldots, d_\ell]$ is a divisor of κ. For the three diagrams shown in Example 5.15, the corresponding divisors of κ are



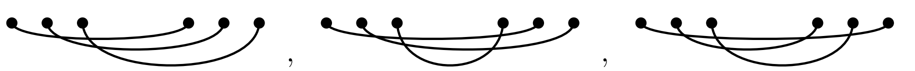
Proposition 5.19. *Let $\kappa \vdash k$ and $\mu \vdash m$. The number of diagrams in $\mathcal{F}^{\mu}_{\mathcal{P}_k}(\kappa)$ is given by*

$$\mathsf{F}^{\mu,\kappa}_{\mathbf{P}_k(n)} = \sum_{\nu|\kappa} \prod_i \sum_t \left\{ {\mathsf{m}_i(\nu) \atop t} \right\} \binom{t}{\mathsf{m}_i(\mu)} i^{\mathsf{m}_i(\nu)-t},$$

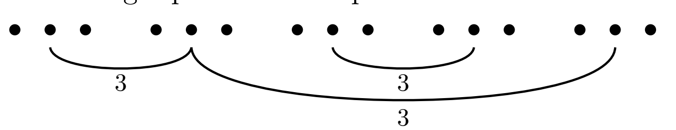
where the outer sum is over divisors of κ and $\mathsf{m}_i(\nu)$ is the number of parts of size i in ν.

Proof. Given a divisor $\nu|\kappa$ consider the symmetric diagram w whose κ_i block is of the form y_{ν_i,κ_i}. We count the number of ways of making w into a symmetric m-diagram in $\mathcal{F}^{\mu}_{\mathcal{P}_k}(\kappa)$. By Lemma 5.14 (c), for some i and $w \in \mathcal{F}^{\mu}_{\mathcal{P}_k}(\kappa)$ there must be $\mathsf{m}_i(\mu)$ sets of connected κ-blocks of type i that propagate. The number of available κ-blocks of type i in w is given by $\mathsf{m}_i(\nu)$. If $\mathsf{m}_i(\nu) < \mathsf{m}_i(\mu)$, there are not enough κ-blocks of type i to propagate and the sum gives zero. Suppose $\mathsf{m}_i(\nu) \geq \mathsf{m}_i(\mu)$. To construct w we choose a set partition of these $\mathsf{m}_i(\nu)$ κ-blocks of type i into t blocks, where $\mathsf{m}_i(\mu) \leq t \leq \mathsf{m}_i(\nu)$. There are $\left\{ {\mathsf{m}_i(\nu) \atop t} \right\}$ ways to do this. Then we choose $\mathsf{m}_i(\mu)$ of these blocks to propagate in $\binom{t}{\mathsf{m}_i(\mu)}$ ways. The remaining blocks do not propagate.

We now count the number of ways of connecting the individual κ-blocks. There are i ways of connecting two κ-blocks of type i. For instance, there are three ways of connecting two κ-blocks of type 3:



A set partition of $\mathsf{m}_i(\nu)$ κ-blocks of type i into t blocks can be depicted as a one-line set-partition diagram where each edge is labelled by the i ways to connect two κ-blocks. For instance, if $i = 3$ and $\mathsf{m}_i(\nu) = 5$, then the following represents a set partition of 5 κ-blocks of type 3 into two blocks:

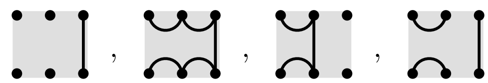


Thus, the number of ways of connecting $\mathsf{m}_i(\nu)$ κ-blocks of type i into t blocks is given by $i^{\mathsf{m}_i(\nu)-t}$. The inner sum is over $\mathsf{m}_i(\mu) \leq t \leq \mathsf{m}_i(\nu)$, but $\left\{ {\mathsf{m}_i(\nu) \atop t} \right\} \binom{t}{\mathsf{m}_i(\mu)}$ is identically zero outside this interval, so we can sum over all t. The connections between (and propagation of) blocks of each type are independent, so taking the product over all i and summing over the divisors ν of κ completes the proof. □

Example 5.20. Let $\kappa = [2,1]$ and $\mu = [1]$. The two divisors of κ are κ itself and the trivial divisor $\nu = [1,1]$. The number of symmetric 1-diagrams in $\mathcal{F}^{\mu}_{\mathcal{P}_3}(\kappa)$ is

$$\left(\sum_{t=1}^{1}\left\{{1 \atop t}\right\}\binom{t}{1}1^{1-t}\right)\left(\sum_{t=0}^{1}\left\{{1 \atop t}\right\}\binom{t}{0}2^{1-t}\right)+\sum_{t=1}^{2}\left\{{2 \atop t}\right\}\binom{t}{1}1^{2-t}=(1)(0+1)+(1+2)=4.$$

Indeed,

$$\mathcal{F}^{\mu}_{\mathcal{P}_3}(\kappa) = \Bigg\{$$

$$\Bigg\}.$$

The first diagram corresponds to the divisor κ while the others correspond to the divisor ν. This coefficient appears in the factorization of the character table for $\mathbf{P}_3(n)$ in Example 5.25 (a).

Combining Proposition 5.19 with (5.4) gives our main result.

Theorem 5.21. *If λ is a partition of n such that $|\lambda^*| = m$, and κ is an integer partition such that $|\kappa| \leq k$, then*

$$\chi^{\lambda}_{\mathbf{P}_k(n)}(\gamma_\kappa \otimes \mathsf{e}^{\otimes s}) = \sum_{\mu \vdash m}\sum_{\nu|\kappa}\prod_i \sum_t \left\{{\mathsf{m}_i(\nu) \atop t}\right\}\binom{t}{\mathsf{m}_i(\mu)} i^{\mathsf{m}_i(\nu)-t}\,\chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu),$$

where the first sum is over partitions μ of m, the second sum is over divisors ν of κ, and $\mathsf{m}_i(\nu)$ is the number of parts of ν equal to i.

Remarks 5.22.

(R1) A recursive Murnaghan-Nakayama rule for computing $\chi^{\lambda}_{\mathbf{P}_k(n)}(\gamma_\kappa \otimes \mathsf{e}^{\otimes s})$ is given in [Hal1]. The closed formula in Theorem 5.21 in terms of the symmetric group character $\chi^{\lambda^*}_{\mathbf{S}_m}$ is new.

(R2) When $|\lambda^*| = k$, the only divisor of κ that contributes to the sum is κ itself, and the only partition $\mu \vdash k$ that contributes to the sum is $\mu = \kappa$. Hence

$$\chi^{\lambda}_{\mathbf{P}_k(n)}(\gamma_\kappa) = \prod_i \left\{{\mathsf{m}_i(\kappa) \atop \mathsf{m}_i(\kappa)}\right\}\binom{\mathsf{m}_i(\kappa)}{\mathsf{m}_i(\kappa)}\chi^{\lambda^*}_{\mathbf{S}_k}(\gamma_\kappa) = \chi^{\lambda^*}_{\mathbf{S}_k}(\gamma_\kappa).$$

(R3) When $|\lambda^*| = 0$, we have $\mu = \emptyset$ and $\chi^{\emptyset}_{\mathbf{S}_0}(\gamma_\emptyset) = 1$, so the character formula specializes to

$$\chi^{\lambda}_{\mathbf{P}_k(n)}(\gamma_\kappa \otimes \mathsf{e}^{\otimes s}) = \sum_{\nu|\kappa}\prod_i \sum_{t \geq 0}\left\{{\mathsf{m}_i(\nu) \atop t}\right\} i^{\mathsf{m}_i(\nu)-t}.$$

This is a new formula for this character value, which is studied in [FH, Thm. 9] and used there to prove a "second orthogonality relation" for the characters of $\mathbf{P}_k(n)$ and compute the joint mixed moments of the number of fixed points of σ^i for $\sigma \in \mathbf{S}_n$.

5.3. **Subalgebras.** We now count the number of symmetric m-diagrams in $\mathcal{W}^m_{\mathcal{A}_k}$ that satisfy the conditions of Lemma 5.14, where $\mathbf{A}_k$ is one of the subalgebras of $\mathbf{P}_k(n)$. We first consider the non-planar algebras, giving new character formulas for the Brauer and rook-Brauer algebras, and the known character formula obtained in [Sol, Prop. 3.5] for the rook monoid algebra.

Theorem 5.22. *If $\mathbf{A}_k$ is one of the non-planar subalgebras of $\mathbf{P}_k(n)$ and $\lambda \in \Lambda^{\mathbf{A}_k}_n$ with $|\lambda^*| = m$, then*

$$\chi^{\lambda}_{\mathbf{A}_k}(\gamma_\kappa \otimes \mathsf{e}^{\otimes s}) = \sum_{\mu \vdash m}\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k}\,\chi^{\lambda^*}_{\mathbf{S}_m}(\gamma_\mu),$$

where κ is a partition such that $|\kappa| + 2s = k$ for the Brauer algebra and $|\kappa| + s = k$ for the others. The coefficients $\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k}$ for the Brauer, rook-Brauer, and rook monoid algebras, respectively, are given by,

(a) $\mathsf{F}^{\mu,\kappa}_{\mathbf{B}_k(n)} = \prod_i \binom{\mathsf{m}_i(\kappa)}{\mathsf{m}_i(\mu)} \sum_t \binom{\mathsf{d}_i(\kappa,\mu)}{2t} (2t-1)!!\ i^t \left(\frac{1+(-1)^i}{2}\right)^{\mathsf{d}_i(\kappa,\mu)-2t}$,

(b) $\mathsf{F}^{\mu,\kappa}_{\mathbf{RB}_k(n)} = \prod_i \binom{\mathsf{m}_i(\kappa)}{\mathsf{m}_i(\mu)} \sum_t \binom{\mathsf{d}_i(\kappa,\mu)}{2t} (2t-1)!!\ i^t \left(\frac{3+(-1)^i}{2}\right)^{\mathsf{d}_i(\kappa,\mu)-2t}$,

(c) $\mathsf{F}^{\mu,\kappa}_{\mathbf{R}_k} = \prod_i \binom{\mathsf{m}_i(\kappa)}{\mathsf{m}_i(\mu)}$,

where $\mathsf{d}_i(\kappa,\mu) = \mathsf{m}_i(\kappa) - \mathsf{m}_i(\mu)$ *and for* (a) *we adopt the convention* $0^0 = 1$ *as in* [Knu].

Proof. We count the number of symmetric m-diagrams in $\mathcal{F}^{\mu}_{\mathcal{A}_k}(\kappa)$ for each algebra. For each of the proper subalgebras of $\mathbf{P}_k(n)$, the propagating blocks of a symmetric m-diagram are identity edges, so the propagating κ-blocks are of the form y_{κ_i,κ_i}. Consider first the Brauer algebra. There are $\mathsf{m}_i(\kappa)$ blocks which can become propagating blocks of type i. If $\mathsf{m}_i(\kappa) < \mathsf{m}_i(\mu)$, there are not enough blocks of type i to propagate and the coefficient is zero. If $\mathsf{m}_i(\kappa) \geq \mathsf{m}_i(\mu)$, then we choose $\mathsf{m}_i(\mu)$ of these to propagate. There are $\mathsf{d}_i(\kappa,\mu) = \mathsf{m}_i(\kappa) - \mathsf{m}_i(\mu)$ blocks of type i remaining. The non-propagating blocks in Brauer diagrams have size two, and if i is even, we can choose $2t$ of the remaining blocks to pair up in $(2t-1)!!$ ways, where a given pair can be connected in i^t ways. The remaining $\mathsf{d}_i(\kappa,\mu) - 2t$ blocks are not paired up, and are made into blocks of type $i/2$ by pairing up vertices within each block. If i is odd and $\mathsf{d}_i(\kappa,\mu)$ is even we pair all $\mathsf{d}_i(\kappa,\mu)$ blocks together, which can happen in $(\mathsf{d}_i(\kappa,\mu)-1)!!\ i^{\mathsf{d}_i(\kappa,\mu)/2}$ ways. If both i and $\mathsf{d}_i(\kappa,\mu)$ are odd, there are zero ways of pairing the non-propagating vertices. Taking the product over all values of i gives the result.

For the rook-Brauer algebra it is possible to have non-propagating blocks of size one. If i is even, each of the $\mathsf{d}_i(\kappa,\mu) - 2t$ blocks designated as non-propagating can either consist of singletons or pairs, so there are $2^{\mathsf{d}_i(\kappa,\mu)-2t}$ configurations for the non-propagating blocks. If i is odd, all of the non-propagating blocks must be singletons, so there is only one choice.

Finally, for the rook monoid algebra, all of the non-propagating blocks are singletons, so for each i we simply choose $\mathsf{m}_i(\mu)$ blocks of type i to propagate. □

The characters of the planar subalgebras are determined by their values on the identity diagram $\mathbf{1}_r$, for $r \leq k$ (see (5.4)). It follows that the set of fixed points equals the set of symmetric m-diagrams. This gives the known character formulas obtained in [HR1, Sec. 2] for the Temperley-Lieb algebra, in [BH, Sec. 4.3] for the Motzkin algebra, and in [FHH, Sec. 5] for the planar rook monoid algebra.

Theorem 5.23. *If* $\mathbf{A}_k$ *is one of the planar subalgebras of* $\mathbf{P}_k(n)$ *and* $\lambda \in \Lambda_n^{\mathbf{A}_k}$ *with* $|\lambda^*| = m$, *then*

$$\chi^{\lambda}_{\mathbf{A}_k}(\mathbf{1}_r \otimes \mathsf{e}^{\otimes s}) = \mathsf{F}^{m,r}_{\mathbf{A}_k},$$

where $r + 2s = k$ *for the Temperley-Lieb algebra and* $r + s = k$ *for the others. The coefficients* $\mathsf{F}^{m,r}_{\mathbf{A}_k}$ *for the Temperley-Lieb, Motzkin, and planar rook monoid algebras, respectively, are given by,*

(a) $\mathsf{F}^{m,r}_{\mathbf{TL}_k(n)} = \binom{r}{\frac{r-m}{2}} - \binom{r}{\frac{r-m}{2} - 1}$,

(b) $\mathsf{F}^{m,r}_{\mathbf{M}_k(n)} = \sum_t \binom{r}{m+2t} \left(\binom{m+2t}{t} - \binom{m+2t}{t-1} \right)$,

(c) $\mathsf{F}^{m,r}_{\mathbf{PR}_k} = \binom{r}{m}$.

Proof. The proof is by counting symmetric m-diagrams in the planar algebras, which is done in [HR3, Sec. 5.5–5.7]. □

5.4. **Character tables.** When viewed as a matrix, the character table of $\mathbf{A}_k$, denoted $\Xi_{\mathbf{A}_k}$, can be expressed as the product of a direct sum of character tables $\Xi_{\mathbf{S}_m}$ for $0 \leq m \leq k$ and the matrix $\mathsf{F}_{\mathbf{A}_k}$, whose μ, κ entry is $\mathsf{F}^{\mu,\kappa}_{\mathbf{A}_k}$. It is clear from the definitions above that, in all cases, $\mathsf{F}_{\mathbf{A}_k}$ is unitriangular (with respect to lexicographic order on partitions) with entries in $\mathbb{Z}_{\geq 0}$ and has determinant equal to one. As a result, the absolute value of the determinant of the character table $\Xi_{\mathbf{A}_k}$ is equal to the absolute value of the product of determinants of symmetric group character tables $\Xi_{\mathbf{S}_m}$. In [Jam] and [SS] it is shown that the absolute value of the determinant of $\Xi_{\mathbf{S}_m}$ is equal to the product of all parts of all partitions of m: $|\det \Xi_{\mathbf{S}_m}| = \prod_{\mu \vdash m} \prod_i i^{\mathsf{m}_i(\mu)}$. This leads to the following result.

Proposition 5.24. *Let $\mathbf{A}_k$ be any of the diagram algebras above, and let $\Xi_{\mathbf{A}_k}$ denote the character table of $\mathbf{A}_k$ viewed as a matrix with integer entries. Then*

$$|\det \Xi_{\mathbf{A}_k}| = \begin{cases} \displaystyle\prod_{\lambda} \prod_{i} i^{\mathsf{m}_i(\lambda^*)} & \text{if } \mathbf{A}_k \text{ is non-planar,} \\ 1 & \text{if } \mathbf{A}_k \text{ is planar,} \end{cases}$$

where the product is over partitions $\lambda \in \Lambda_n^{\mathbf{A}_k}$.

We conclude the section by providing examples of character tables for the non-planar algebras.

Example 5.25. In the following examples, the rows of $\Xi_{\mathbf{A}_k}$ are indexed by the irreducible $\mathbf{A}_k$-modules, which are labelled by partitions $\lambda \in \Lambda_n^{\mathbf{A}_k}$, and the columns are indexed by conjugacy class analogs, which are labelled by partitions of $0, \ldots, k$. Both are arranged in lexicographic order. For example, the rows of $\Xi_{\mathbf{P}_3(n)}$ are indexed by $\{[n], [n-1,1], [n-2,2][n-2,1,1], [n-3,3], [n-3,2,1], [n-3,1,1,1]\}$ and the columns are indexed by $\{\emptyset, [1], [2], [1,1], [3], [2,1], [1,1,1]\}$.

(a) *The partition algebra, $\mathbf{P}_3(n)$. Note that the entry $\mathsf{F}^{[1],[2,1]}_{\mathbf{P}_3(n)} = 4$ is computed in Example 5.20:*

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 5 \\ \cdot & 1 & 1 & 3 & 1 & 4 & 10 \\ \cdot & \cdot & 1 & 1 & 0 & 2 & 6 \\ \cdot & \cdot & -1 & 1 & 0 & 0 & 6 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{P}_3(n)}} = \underbrace{\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{S}_0} \oplus \Xi_{\mathbf{S}_1} \oplus \Xi_{\mathbf{S}_2} \oplus \Xi_{\mathbf{S}_3}} \underbrace{\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 5 \\ \cdot & 1 & 1 & 3 & 1 & \mathbf{4} & 10 \\ \cdot & \cdot & 1 & 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & 1 & 0 & 1 & 6 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}}_{\mathsf{F}_{\mathbf{P}_3(n)}}$$

(b) *The rook-Brauer algebra, $\mathbf{RB}_3(n)$:*

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 4 \\ \cdot & 1 & 0 & 2 & 0 & 2 & 6 \\ \cdot & \cdot & 1 & 1 & 0 & 1 & 3 \\ \cdot & \cdot & -1 & 1 & 0 & -1 & 3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{RB}_3(n)}} = \underbrace{\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{S}_0} \oplus \Xi_{\mathbf{S}_1} \oplus \Xi_{\mathbf{S}_2} \oplus \Xi_{\mathbf{S}_3}} \underbrace{\begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 4 \\ \cdot & 1 & 0 & 2 & 0 & 2 & 6 \\ \cdot & \cdot & 1 & 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & 1 & 0 & 0 & 3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}}_{\mathsf{F}_{\mathbf{RB}_3(n)}}$$

(c) *The rook monoid algebra,* $\mathbf{R}_3$*:*

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 0 & 2 & 0 & 1 & 3 \\ \cdot & \cdot & 1 & 1 & 0 & 1 & 3 \\ \cdot & \cdot & -1 & 1 & 0 & -1 & 3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{R}_3(n)}} = \underbrace{\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{S}_0} \oplus \Xi_{\mathbf{S}_1} \oplus \Xi_{\mathbf{S}_2} \oplus \Xi_{\mathbf{S}_3}} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 0 & 2 & 0 & 1 & 3 \\ \cdot & \cdot & 1 & 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & 1 & 0 & 0 & 3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}}_{\mathsf{F}_{\mathbf{R}_3}}$$

(d) *The Brauer algebra,* $\mathbf{B}_4(n)$*:*

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 3 & 1 & 3 \\ \cdot & 1 & 1 & 0 & 0 & 2 & 2 & 6 \\ \cdot & -1 & 1 & 0 & 0 & -2 & 0 & 6 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & -1 & 0 & -1 & 1 & 3 \\ \cdot & \cdot & \cdot & 0 & -1 & 2 & 0 & 2 \\ \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 3 \\ \cdot & \cdot & \cdot & -1 & 1 & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{B}_4(n)}} = \underbrace{\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & -1 & 0 & -1 & 1 & 3 \\ \cdot & \cdot & \cdot & 0 & -1 & 2 & 0 & 2 \\ \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 3 \\ \cdot & \cdot & \cdot & -1 & 1 & 1 & -1 & 1 \end{pmatrix}}_{\Xi_{\mathbf{S}_0} \oplus \Xi_{\mathbf{S}_2} \oplus \Xi_{\mathbf{S}_4}} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 3 & 1 & 3 \\ \cdot & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ \cdot & \cdot & 1 & 0 & 0 & 0 & 1 & 6 \\ \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}}_{\mathsf{F}_{\mathbf{B}_4(n)}}$$

References

[BH] G. Benkart and T. Halverson. Motzkin algebras. *European J. Combin.* **36**(2014), 473–502.

[BH2] G. Benkart and T. Halverson. Partition algebras $P_k(n)$ with $2k > n$ and the fundamental theorems of invariant theory for the symmetric group S_n. *ArXiv e-prints* (2017).

[BH3] G. Benkart and T. Halverson. Partition algebras and the invariant theory of the symmetric group. *J. London Math. Soc* (forthcoming).

[BHH] G. Benkart, T. Halverson, and N. Harman. Dimensions of irreducible modules for partition algebras and tensor power multiplicities for symmetric and alternating groups. *J. Algebraic Combin.* **46**(2017), 77–108.

[Bra] R. Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)* **38**(1937), 857–872.

[CLL] C. Carré, A. Lascoux, and B. Leclerc. Turbo-straightening for decomposition into standard bases. *Int. J. Alg. Comp.* **2**(1992), 275–290.

[DW] W. Doran and D. Wales. The partition algebra revisited. *J. Algebra* **231**(2000), 265–330.

[Eny] J. Enyang. A seminormal form for partition algebras. *J. Combin. Theory Ser. A* **120**(2013), 1737–1785.

[FH] J. Farina and T. Halverson. Character orthogonality for the partition algebra and fixed points of permutations. *Adv. Appl. Math.* **31**(2003), 113–131.

[FHH] D. Flath, T. Halverson, and K. Herbig. The planar rook algebra and Pascal's triangle. *Enseign. Math.* **55**(2009), 77–92.

[GM] A.M. Garsia and T.J. McLarnan. Relations between Young's natural and the Kazhdan–Lusztig representations of S_n. *Adv. in Math.* **69**(1988), 32–92.

[Hal1] T. Halverson. Characters of the partition algebras. *J. Algebra* **238**(2001), 502–533.

[Hal2] T. Halverson. Representations of the q-rook monoid. *J. Algebra* **273**(2004), 227–251.

[Hd] T. Halverson and E. delMas. Representations of the rook-Brauer algebra. *Comm. Algebra* **42**(2014), 423–443.

[HMR] T. Halverson, M. Mazzocco, and A. Ram. Commuting families in Hecke and Temperley–Lieb algebras. *Nagoya Math. J.* **195**(2009), 125–152.

[HR1] T. Halverson and A. Ram. Characters of algebras containing a Jones basic construction: The Temperley–Lieb, Okada, Brauer, and Birman-Wenzl algebras. *Adv. Math.* **116**(1995), 263–321.

[HR2] T. Halverson and A. Ram. Partition algebras. *European J. Combin.* **26**(2005), 869–921.

[HR3] T. Halverson and M. Reeks. Gelfand models for diagram algebras. *J. Algebraic Combin.* **41**(2015), 229–255.

[HW] P. Hanlon and D. Wales. On the decomposition of Brauer's centralizer algebras. *J. Algebra* **121**(1989), 409–445.

[Jam] G.D. James. *The Representation Theory of the Symmetric Groups.* Lecture Notes in Mathematics. Springer-Verlag, 1978.

[Jon] V.F.R. Jones. The Potts model and the symmetric group. In H. Araki, Kawahigashi Y., and Kosaki H., editors, *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras (Kyuzeso, 1993)*, pages 259–267. World Scientific, 1994.

[Knu] D.E. Knuth. Two notes on notation. *The American Mathematical Monthly* **99**(1992), 403–422.

[KM] G. Kudryavtseva and V. Mazorchuk. Combinatorial Gelfand models for some semigroups and q-rook monoid algebras. *Proc. Edinb. Math. Soc. (2)* **52**(2009), 707–718.

[Mar1] P.P. Martin. Representations of graph Temperley–Lieb Algebras. *Publ. Res. Inst. Math. Sci.* **26**(1990), 485–503.

[Mar2] P.P. Martin. *Potts Models and Related Problems in Statistical Mechanics*. Series on advances in statistical mechanics. World Scientific, 1991.

[Mar3] P.P. Martin. Temperley–Lieb algebras for non-planar statistical mechanics – the partition algebra construction. *Journal of Knot Theory and Its Ramifications* **03**(1994), 51–82.

[Mar4] P.P. Martin. The partition algebra and the Potts model transfer matrix spectrum in high dimensions. *J. Phys. A* **33**(2000), 3669.

[ME] P.P. Martin and A. Elgamal. The structure of the partition algebras. *J. Algebra* **183**(1996), 319–358.

[MM] P.P. Martin and V. Mazorchuk. On the representation theory of partial Brauer algebras. *Q. J. Math.* **65**(2014), 225–247.

[MS1] P.P. Martin and H. Saleur. On an algebraic approach to higher-dimensional statistical mechanics. *Comm. Math. Phys.* **158**(1993), 155–190.

[MS2] P.P. Martin and H. Saleur. Algebras in higher-dimensional statistical mechanics – the exceptional partition (mean field) algebras. *Lett. Math. Phys.* **30**(1994), 179–185.

[Naz] M. Nazarov. Young's orthogonal form for Brauer's centralizer algebra. *J. Algebra* **182**(1996), 664–693.

[OEIS] N.J.A. Sloane. The On-Line Encyclopedia of Integer Sequences. Published electronically at `http://oeis.org`.

[OZ] R. Orellana and M. Zabrocki. Symmetric group characters as symmetric functions. *ArXiv e-prints* (2016).

[Sag] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer, New York, 2001.

[SS] F.W. Schmidt and R. Simion. On a partition identity. *J. Combin. Theory Ser. A* **36**(1984), 249–252.

[Sol] L. Solomon. Representations of the rook monoid. *J. Algebra* **256**(2002), 309–342.

[TL] H.N.V. Temperley and E.H. Lieb. Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A* **322**(1971), 251–280.

[Wes] B. Westbury. The representation theory of the Temperley–Lieb algebras. *Math. Z.* **219**(1995), 539–565.

[You] A. Young. On quantitative substitutional analysis (second paper). *Proc. Lond. Math. Soc.* **34**(1902), 361–397.

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