

A Universal Bijection for Catalan Magmas

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Abstract

A Catalan magma is a unique factorisation normed magma with only one irreducible element. The norm partitions the base set into subsets enumerated by Catalan numbers. The primary theorem characterises the conditions which a set a with product map must satisfy in order to be a free magma generated by the irreducible elements. This theorem can be used to prove a set of objects (with a product map) is a Catalan magma. The isomorphism between Catalan magmas gives a “universal” bijection – essentially one bijection algorithm for all pairs of families. The morphism property ensures the bijection is recursive. The universal bijection allows us to give some rigour to the idea of an “embedding” bijection between Catalan objects which, in many cases, shows how to embed an element of one Catalan family into one of a different family. Multiplication on the right (respectively left) by the generator gives rise to the right (respectively left) Narayana statistic. The statistic is invariant under the universal bijection and hence allows us to determine what structures of any Catalan family are associated with this refinement. We discuss the relation between the symbolic method for Catalan families and the magma structure on the base set defined by the symbolic method. This shows which “atomic” elements are also irreducible. The appendix gives the magma structure for 14 Catalan families.

Keywords:

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1 Introduction

The sequence of Catalan numbers [1], are defined by

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (1)$$

for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ with $C_0 = 1$, and are well know to count a large number of different families of objects. Sixty-six such families are given in problem set of chapter 6 of [2, 3]. A list of 214 families has been published by R. Stanley [4]. A history of Catalan numbers by I. Pak can be found in the appendix of [4] and a very extensive biography of Catalan (and Bell numbers) was compiled by H. Gould [5]. New Catalan families are regularly being discovered, for example the Catalan floor plans [6].

Each time a new family of objects are discovered and direct enumeration suggests they form a Catalan family, then proof of such a conjecture can be direct (eg. via generating function or sign reversing involution) or by bijection to a known Catalan family. There is a long history of Catalan bijections. We make no attempt to reference them all, but defer to Chapter 3 (Bijective Solutions) of Stanley’s “Catalan Numbers” book [4]. In this paper we focus on such bijections.

The primary motivation of this paper is to try and bring some structure to the large number of Catalan bijections. Once a new family has been bijected to any one of the known Catalan families, then of course, that proves it is a Catalan family, and a composition of the bijections will define a bijection between it and other Catalan families. Whilst mathematically we thus then have bijections between pairs of families, it is not entirely satisfactory when bijecting between any pair to have to go via several other families. Is it possible to define some ‘direct’ way of bijecting between any pair?

Another motivation is to provide some sort of answer to the following question: given any two sets with the same cardinality, say n , then there exists $n!$ possible bijections between – however most of the known Catalan bijections pick out a few of these bijections as somehow more “natural” – what is natural about these bijections?

The central idea presented in this paper is to replace “bijection” with “isomorphism”, in particular, an isomorphism between certain free magmas. (For a definition of a magma – see Definition 1.) In Theorem 1 we give a simple characterisation of the free magmas we require. In the case of a single irreducible element we get Catalan numbers. This motivates the definition of a Catalan magma – see Definition 5. Using Theorem 1 it is possible to prove a family of objects (with appropriate product) is a Catalan magma *without* direct reference to any other Catalan family.

All Catalan magmas are isomorphic. It is this isomorphism that gives us a bijection between any pair of families. Several additional results also follow from this formulation:

- The isomorphism is essentially the same for any pair of Catalan magmas which leads to the notion of a ‘universal’¹ bijection and hence a universal bijection between any pair of Catalan families. Many of the bijections that are defined by the universal bijection are the same as known bijections in the sense that each Catalan object is mapped to the same corresponding object, however, the algorithm defining the map is different in almost all cases.
- The universal bijection is a morphism, thus it respects the product structure of the magmas which turns out to be the “natural” recursive structure of Catalan objects [7]. Thus the universal bijection is natural in that it respects the recursive of Catalan objects.

¹The bijection is ‘universal’ in the sense that it is the same (meta) algorithm for *any* pair of families.

- For the ‘geometrical’ families, the isomorphism gives rise to the idea of an “embedding” bijection. This is usually a geometrical way of embedding an object from one Catalan family into an object from another Catalan family – in a sense it shows how one object is “hidden” in another and thus shows manifestly why the two families have the same enumeration. For families which are sequences adding a bijection to a geometrical family achieves a similar result.
- Right multiplication by the irreducible element of a Catalan magma has a Narayana statistic which is preserved by the isomorphism. Since the irreducible element usually has a natural interpretation in any Catalan family the magma formulation gives a direct identification of the part of the object in any family carrying the Narayana statistic.

The above ideas can be generalised to give a similar form of universal bijection between Motzkin [8] families and between Schröder families [9]. In [10] it is shown that both families require a unary map in addition to a binary map.

In Section 7 we briefly discuss the relationship between the magma structure discussed in this paper and the symbolic method [11]. This is primarily to clarify the scope of both methods as they both use the recursive structure of the combinatorial objects, but do so in different ways. In the symbolic method the recursive structure manifests itself in the set equation satisfied by the set of combinatorial objects ie. it is used to define (or construct) a set. From this set equation an equation for the generating function is obtained. A magma assumes the base set is known and then defines a product map on this set. The definition of the product map usually uses the same recursive structure but it does so to define a map rather than define a set. In the magma case it is the properties of the product map that are of primary interest whilst for the symbolic method it is the set equation.

The idea that Catalan objects define a magma has been stated for a few families, for example, in the context of binary trees [12], in this paper we give a simple characterisation (see Theorem 1 and Definition 5) of the constraints the magma must satisfy to be associated with Catalan numbers and thus extend the idea to all Catalan families. The emphasis on using the recursive structure to define a bijection between combinatorial structures has been used by Forcey et. al. [13] to define ‘deconstruction-reconstruction’ bijections.

2 Magmas

In this section we summarise the standard definitions associated with magmas (see Bourbaki [14] Chapter 1, §1 and §7) and state some theorems useful for proving when a magma is associated with Catalan numbers. We are only interested in countable sets and hence all magmas discussed are assumed to be defined on such sets.

Definition 1 (Magma). *Let \mathcal{M} be a non-empty countable set called the **base set**. A **magma** defined on \mathcal{M} is a pair (\mathcal{M}, \star) where \star is a **product map**,*

$$\star : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}.$$

*The range of the map \star is called the set of **reducible elements** and usually denoted by \mathcal{M}^+ . The elements of the set $\mathcal{M}^0 = \mathcal{M} \setminus \mathcal{M}^+$ are called **irreducible elements** and the set \mathcal{M}^0 called the **set of irreducibles**. If $\star : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is injective then we will call (\mathcal{M}, \star) a **unique factorisation***

magma. If (\mathcal{N}, \bullet) is magma then a **magma morphism** θ from \mathcal{M} to \mathcal{N} , is a map $\theta : \mathcal{M} \rightarrow \mathcal{N}$ such that for all $m, m' \in \mathcal{M}$, $\theta(m \star m') = \theta(m) \bullet \theta(m')$. A **sub-magma**, (\mathcal{N}, \star) , is a subset $\mathcal{N} \subseteq \mathcal{M}$ closed under \star .

Note, there may be no irreducible elements, also if \mathcal{M} is finite then the product cannot be injective. The above definition places no constraints on the map \star (ie. associative, surjective etc.) however we will only be interested in product maps that satisfy the conditions necessary to be associated with Catalan (or similar) numbers. In order to state the conditions we will use a norm. The norm will partition the magma into subsets but will also give a simple way to prove a magma is a Catalan magma (defined below). This motivates the following definition.

Definition 2 (Norm). Let (\mathcal{M}, \star) be a magma. A **norm** is a super-additive map $\|\cdot\| : \mathcal{M} \rightarrow \mathbb{N}$. If (\mathcal{M}, \star) has a norm it will be called a **normed magma**.

A norm is **super-additive** if it satisfies: for all $m_1, m_2 \in \mathcal{M}$, $\|m_1 \star m_2\| \geq \|m_1\| + \|m_2\|$ and called **additive** if the equality always holds, that is: for all $m_1, m_2 \in \mathcal{M}$ $\|m_1 \star m_2\| = \|m_1\| + \|m_2\|$. An immediate consequence of the existence of a norm is the following property.

Proposition 1. Let (\mathcal{M}, \star) be a normed magma and let $X_{\min} \subset \mathcal{M}$ be the set of elements with minimal norm. Then X_{\min} is non-empty and all elements of X_{\min} are irreducible.

Proof. The base set \mathcal{M} is assumed non-empty and hence the norm has non-empty range $A \subseteq \mathbb{N}$. Since \mathbb{N} is a well ordered set the subset A has a least element, r , and thus the pre-image set of r , X_{\min} is not empty. Assume $m \in \mathcal{M}^+ \cap X_{\min}$. Since $m \in \mathcal{M}^+$ it has at least one pre-image pair (m_1, m_2) ie. $\star : (m_1, m_2) \mapsto m$, and by assumption $m \in X_{\min}$ hence $\|m\|$ is minimal. But $\|m\| \geq \|m_1\| + \|m_2\|$ with $\|m_1\|, \|m_2\| > 0$ so $\|m\|$ is not minimal – a contradiction. Thus $\mathcal{M}^+ \cap X_{\min}$ is empty. \square

Note, the converse is false: a normed magma may have irreducible elements that do not have minimal norm.

We now consider free magmas generated by a subset X of \mathcal{M} . The following is based on the definition in Bourbaki [14].

Definition 3 (Standard Free Magma). Let X be a non-empty finite set. Define the sequence $W_n(X)$ of sets of nested 2-tuples recursively by:

$$W_1(X) = X \tag{2a}$$

$$W_n(X) = \bigcup_{p=1}^{n-1} W_p(X) \times W_{n-p}(X), \quad n > 1. \tag{2b}$$

Let $\mathcal{W}_X = \bigcup_{n \geq 1} W_n(X)$ and $\mathcal{W}_X^+ = \mathcal{W}_X \setminus X$. Define the product map $\diamond : \mathcal{W}_X \times \mathcal{W}_X \rightarrow \mathcal{W}_X$ by

$$m_1 \diamond m_2 \mapsto (m_1, m_2) \tag{3}$$

and the map $\|\cdot\| : \mathcal{W}_X \rightarrow \mathbb{N}$ is defined by $\|m\| = n$ when $m \in W_n(X)$. The pair $(\mathcal{W}_X, \diamond)$ is called the **standard free magma** generated by X .

If (\mathcal{M}, \star) is freely generated by X we will also on occasion use the notation (\mathcal{M}, \star, X) . For the above constructed magma we have the obvious properties following directly from the definition.

Proposition 2. *If $(\mathcal{W}_X, \diamond)$ is the standard free magma of Definition 3, then*

1. $(\mathcal{W}_X, \diamond)$ is a unique factorisation magma and
2. the map $\|\cdot\|$ is a norm and
3. an element has minimal norm iff it is a generator and
4. the generators are irreducible.
5. if $Y \subseteq X$ then $(\mathcal{W}_Y, \diamond)$ is a sub-magma of $(\mathcal{W}_X, \diamond)$.

If $X = \{\epsilon\}$, then the base set of the standard free magma, $(\mathcal{W}_\epsilon, \diamond)$ begins (sorting by norm):

$$\begin{aligned} & \epsilon, \quad (\epsilon, \epsilon), \quad (\epsilon, (\epsilon, \epsilon)), \quad ((\epsilon, \epsilon), \epsilon), \\ & (\epsilon, (\epsilon, (\epsilon, \epsilon))), \quad ((\epsilon, (\epsilon, \epsilon)), \epsilon), \quad (\epsilon, ((\epsilon, \epsilon), \epsilon)), \quad (((\epsilon, \epsilon), \epsilon), \epsilon), \quad ((\epsilon, \epsilon), (\epsilon, \epsilon)) \dots \end{aligned} \quad (4)$$

Note, the norm is the number of occurrences of the generator, ϵ .

If we wish to make the product explicit we can use the left-hand side of (3), that is

$$\epsilon, \quad \epsilon \diamond \epsilon, \quad \epsilon \diamond (\epsilon \diamond \epsilon), \quad (\epsilon \diamond \epsilon) \diamond \epsilon \dots \quad (5)$$

The form (5) is also called infix order. There are two other common notations for binary products: prefix order and postfix order. For prefix and postfix order notation the brackets and commas can be omitted without any ambiguity. For example, using prefix notation (5) becomes

$$\epsilon, \quad \diamond \epsilon \epsilon, \quad \diamond \epsilon \diamond \epsilon \epsilon, \quad \diamond \diamond \epsilon \epsilon \epsilon \dots \quad (6)$$

or using postfix-order they become

$$\epsilon, \quad \epsilon \epsilon \diamond, \quad \epsilon \epsilon \epsilon \diamond \diamond, \quad \epsilon \epsilon \diamond \epsilon \diamond, \quad \dots \quad (7)$$

Thus $\epsilon \epsilon \epsilon \diamond \diamond$, $\diamond \epsilon \diamond \epsilon \epsilon$ and $(\epsilon \diamond (\epsilon \diamond \epsilon))$ are all the same element of \mathcal{W}_ϵ , namely $(\epsilon, (\epsilon, \epsilon))$.

The following theorem is well known and makes the connection with Catalan numbers. Denote the size of a set S by $|S|$.

Proposition 3 ([7]). *Let \mathcal{W}_ϵ be the free magma of Definition 2 generated by the single element ϵ . If*

$$W_\ell = \{m \in \mathcal{W}_\epsilon : \|m\| = \ell\}, \quad \ell \geq 1,$$

then

$$|W_\ell| = C_{\ell-1} = \frac{1}{\ell} \binom{2\ell-2}{\ell-1}. \quad (8)$$

The conventional proof [7] goes via the recurrence

$$|W_n| = \sum_{k=1}^{n-1} |W_k| |W_{n-k}|, \quad n > 1 \quad (9)$$

which follows from equation (2), and then verifying the Catalan number (8) satisfies the same recurrence with $|W_1| = 1$. This is usually done by using the generating function for Catalan numbers.

Note, unlike the conventional form of the Catalan recurrence, when using the norm there is no shift of +1 in the subscripts of (9).

For each Catalan family, the norm is usually a simple function of the conventional Catalan objects “size” parameter eg. for triangulation’s of a regular n sided polygon (see family \mathbb{F}_8 in the Appendix) the size is usually the number of triangles ($= n - 2$) or polygon edges ($= n$), but neither size parameter is additive over the product. To achieve additivity the norm of the triangulation has to be $n - 1$. For Dyck paths with $2n$ steps (see family \mathbb{F}_7 in the Appendix) the norm is $n + 1$.

Factorisation in Magmas

Given a base set \mathbb{F} , a subset $X \subseteq \mathbb{F}$, and a product map $\star : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ how can we show (\mathbb{F}, \star, X) is freely generated by X ? One method is of course to biject \mathbb{F} to \mathcal{W}_X and then show the bijection is also an isomorphism. *We don’t want to do this every time we have a new magma (\mathbb{F}, \star) .* We want to show (\mathbb{F}, \star) is freely generated by some subset without *explicitly* mapping to \mathcal{W}_X . Theorem 1 (stated at the end of this section) will enable us to do this. The reason for choosing this route is that we then want to use the fact that if (\mathbb{F}, \star) is freely generated by X then it must be isomorphic to \mathcal{W}_X . We will then use the existence of an isomorphism to construct a “universal” recursive bijection between *any* pair of free magmas generated by subsets of the same size. In order to get to Theorem 1 we need to find a sufficient set of conditions (\mathbb{F}, \star) has to satisfy to ensure there exists an isomorphism $\pi : \mathbb{F} \rightarrow \mathcal{W}_X$.

Consider an arbitrary magma (\mathcal{M}, \star) . For all reducible elements, $m \in \mathcal{M}^+$ there exists at least one factorisation of m into two parts: $m = m_1 \star m_2$. Call m_1 a **left factor** and m_2 a **right factor**. In general the product map is not injective and hence an element $m \in \mathcal{M}^+$ will have a set of pre-images

$$\varphi(m) = \{(m_1, m_2) \in \mathcal{M} \times \mathcal{M} : m_1 \star m_2 = m\} \quad (10)$$

and hence a set of left-right factor pairs. Note, $\varphi(m)$ may have infinite cardinality.

We now attempt to map \mathcal{M} into \mathcal{W}_X . This will only succeed if the magma satisfies certain conditions. We repeat the pair factorisation process on the left and right factors of each pair in $\varphi(m)$. If a factor m_i is reducible then it can be further factorised into at least one pair of factors, if m_i is irreducible then it cannot be further factorised. Collecting all the possible factorisations into a set motivates the following recursive function: If (\mathcal{M}, \star) is *not* a unique factorisation magma then define $\hat{\pi}$ as

$$\hat{\pi}(m) = \begin{cases} \bigcup_{(m_1, m_2) \in \varphi(m)} \hat{\pi}(m_1) \times \hat{\pi}(m_2) & \text{if } m \in \mathcal{M}^+ \\ \{\sigma(m)\} & \text{if } m \in \mathcal{M}^0 \end{cases} \quad (11a)$$

where $\sigma : \mathcal{M}^0 \rightarrow X$ is any injective map (this assumes $|X| \geq |\mathcal{M}^0|$). If (\mathcal{M}, \star) is a unique factorisation magma then we can dispense with the sets and define $\hat{\pi}$ as

$$\hat{\pi}(m) = \begin{cases} (\hat{\pi}(m_1), \hat{\pi}(m_2)) & \text{if } m \in \mathcal{M}^+ \text{ with } m_1 \star m_2 = m \\ \sigma(m) & \text{if } m \in \mathcal{M}^0. \end{cases} \quad (11b)$$

Of primary interest is when the recursion (11) terminates. This motivates the following definition.

Definition 4 (Finite decomposition). *Let (\mathcal{M}, \star) be a magma. If for all reducible elements, $m \in \mathcal{M}^+$, the recursive definition (11) terminates then (\mathcal{M}, \star) will be called a **finite decomposition magma**. If (\mathcal{M}, \star) is a finite decomposition magma then the image $\hat{\pi}(m)$ will be called the **decomposition** of m .*

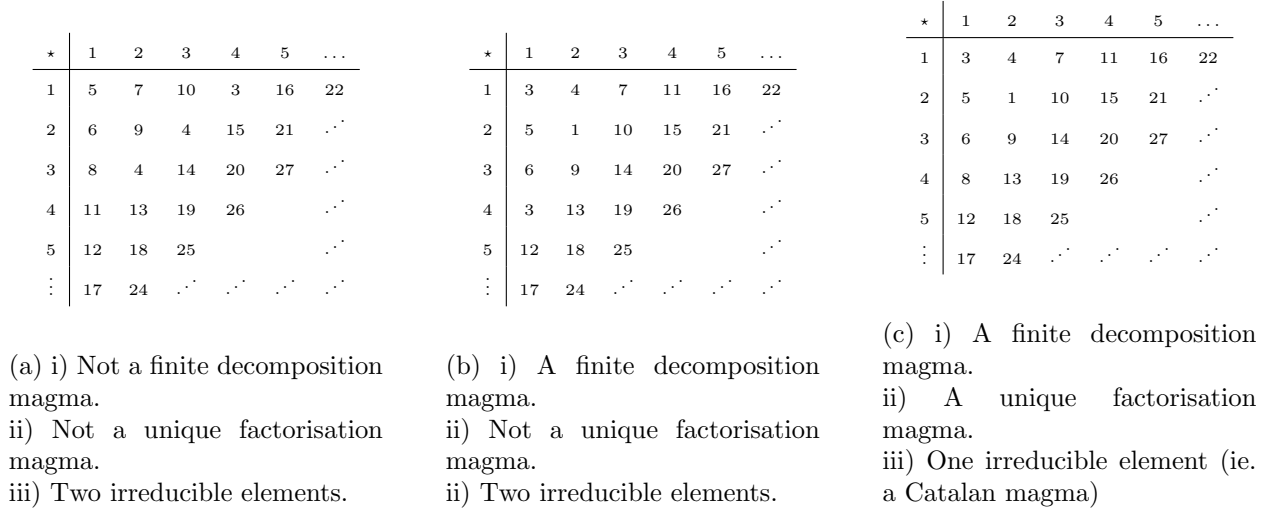


Figure 1: The magmas above have base set \mathbb{N} and product rule defined by the table. Only a portion of the table is shown explicitly. The rest of the table is filled by the remaining elements of \mathbb{N} consecutively in the diagonal pattern illustrated (ie. the north-east diagonals starting with 12, 13, 14, ...).

Note, the decomposition of m is a single element if \star is injective, or a set if \star is not injective. The recursion terminates if all factors always result in an irreducible element after a finite number of iterations of (11). There are several mechanisms that would result in the recursion (11) not terminating. For example, if there are no irreducible elements, or, if at some stage of the recursion a factor of some element m is m itself ie. $\hat{\pi}(m) = \dots(\hat{\pi}(m), \hat{\pi}(m'))\dots$ resulting in an infinitely nested tuple. Three example magmas are given in Figure 1.

If (\mathcal{M}, \star) has a norm then the recursion must terminate as given by the following proposition.

Proposition 4. *Let (\mathcal{M}, \star) be magma. Then (\mathcal{M}, \star) is a normed magma if and only if (\mathcal{M}, \star) is a finite decomposition magma.*

Proof. Forward: If (\mathcal{M}, \star) has a norm, by Proposition 1, the set of irreducibles, \mathcal{M}^0 is non-empty and all elements of minimal norm are in \mathcal{M}^0 . Let R be the range of the norm. Since \mathbb{N} is well ordered the subset R has a minimal element, say r_0 . If $m \in \mathcal{M}^0$ the recursion terminates. If $m \notin \mathcal{M}^0$ then for each product map pre-image of m , $(m_1, m_2) \in \varphi(m)$ we have that $\|m_1 \star m_2\| \geq \|m_1\| + \|m_2\|$ and since the norm takes values in \mathbb{N} we have that $\|m_1\| < \|m_1 \star m_2\|$ and $\|m_2\| < \|m_1 \star m_2\|$. Thus every factor of m has strictly smaller norm than m . This process continues until a factor is an element of \mathcal{M}^0 (in which case the recursion terminates) or the norm of the factor is r_0 . In the latter case, by Proposition 1 the factor must be in \mathcal{M}^0 (and hence the recursion terminates). Thus in all cases the recursion terminates and hence (\mathcal{M}, \star) is a finite decomposition magma.

Converse: If (\mathcal{M}, \star) is a finite decomposition magma each element in the set $\hat{\pi}(m) = \{\pi_1, \pi_2, \dots\}$ has a finite number of occurrences of irreducible elements. Denote the number of occurrences of irreducible elements in $\pi_i \in \hat{\pi}(m)$ by $|\pi_i|$. Define

$$\|m\| = \max \{|\pi| : \pi \in \hat{\pi}(m)\}. \quad (12)$$

Let $m = m_1 \star m_2$ and consider $\|m_1 \star m_2\| = \|m\|$. Since $\hat{\pi}(m_1) \times \hat{\pi}(m_2) \subseteq \hat{\pi}(m)$ we have that

$$\begin{aligned} \max \{|\pi| : \pi \in \hat{\pi}(m)\} &\geq \max \{|\pi| : \pi \in \hat{\pi}(m_1) \times \hat{\pi}(m_2)\} \\ &= \max \{|\mu| : \mu \in \hat{\pi}(m_1)\} + \max \{|\mu| : \mu \in \hat{\pi}(m_2)\} \end{aligned}$$

Thus

$$\|m_1 \star m_2\| \geq \|m_1\| + \|m_2\|.$$

Hence the map defined by (12) is a norm and thus (\mathcal{M}, \star) is a normed magma. □

In the case the product map is injective the norm (12) simplifies.

Corollary 1. *If (\mathcal{M}, \star) is a unique factorisation magma then the norm (12) becomes*

$$\|m\| = |\pi(m)|$$

and satisfies $\|m_1 \star m_2\| = \|m_1\| + \|m_2\|$.

Assuming (\mathcal{M}, \star) is a finite decomposition magma and a unique factorisation magma then (11) defines a map

$$\hat{\pi} : \mathcal{M} \rightarrow \mathcal{W}_X \quad (13)$$

where \mathcal{W}_X is a standard free magma with $|\mathcal{M}^0| \leq |X|$. Then since $(\hat{\pi}(m_1), \hat{\pi}(m_2))$ is the product on \mathcal{W}_X we can write (11) as

$$\hat{\pi}(m_1 \star m_2) = \hat{\pi}(m_1) \blacklozenge \hat{\pi}(m_2). \quad (14)$$

Thus, when \star is injective, $\hat{\pi}$ is a magma morphism.

We can map from \mathcal{W}_X to \mathcal{M} as follows. Let (\mathcal{M}, \star) be an arbitrary magma with a non-empty set of irreducibles, \mathcal{M}^0 . If $0 < |X| \leq |\mathcal{M}^0|$ then consider

$$\mu : \mathcal{W}_X \rightarrow \mathcal{M},$$

which we call **full multiplication** and is defined recursively as follows:

$$\mu(w) = \begin{cases} \mu(w_1) \star \mu(w_2) & \text{if } w = (w_1, w_2) \\ \tau(w) & \text{if } w \in X \end{cases} \quad (15)$$

where

$$\tau : X \rightarrow \mathcal{M}^0,$$

is any injective map. Since (w_1, w_2) is the product of \mathcal{W}_X , the full multiplication definition can be written

$$\mu(w_1 \blacklozenge w_2) = \mu(w_1) \star \mu(w_2) \quad (16)$$

and thus μ is a magma morphism. The map μ is only injective if (\mathcal{M}, \star) is a unique factorisation magma.

If $|\mathcal{M}^0| = |X|$ and (\mathcal{M}, \star) is a unique factorisation magma then all the maps π , σ , μ and τ become bijections. If τ is chosen to be σ^{-1} then π is the inverse of μ and visa versa. The two magma morphisms (14) and (16) then become isomorphisms. Thus we have the following proposition.

Proposition 5. *Let \mathcal{W}_X be the standard free magma of Definition 3 with generating set X and (\mathcal{M}, \star) a magma with non-empty set of irreducible elements \mathcal{M}^0 . If*

1. $|\mathcal{M}^0| = |X|$ and
2. (\mathcal{M}, \star) is a unique factorisation magma and
3. (\mathcal{M}, \star) has a norm

then the maps π and μ defined by (11) and (15) are magma isomorphisms.

Note we get an isomorphism for each choice of σ (or τ). Combining Proposition 4 and Proposition 5 gives the following theorem.

Theorem 1. *Let (\mathcal{M}, \star) be a unique factorisation normed magma. Then (\mathcal{M}, \star) is isomorphic to a standard free magma of Definition 3 generated by the irreducible elements of \mathcal{M} .*

Returning to the original motivation of wanting to prove that a product can be added to a set \mathbb{F} to form a free magma generated by a set X without explicitly constructing an isomorphism to \mathcal{W}_X Theorem 1 shows that the following is sufficient:

- Define a product map $\star : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ (and show it is injective)
- Define a map $\|\cdot\| : \mathbb{F} \rightarrow \mathbb{N}$ (and show $\|\cdot\|$ is additive over \star).
- Select a subset of elements of \mathbb{F} (and show they are the only irreducible elements).

This motivates the following definition.

Definition 5 (Catalan Magma). *A unique factorisation normed magma with only one irreducible element is called a **Catalan magma**.*

3 Catalan Families, Free Magmas and a Universal Bijection

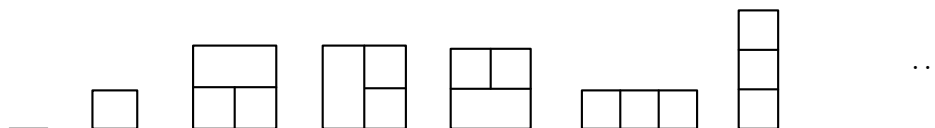
One of the recurring problems associated with Catalan families is that of finding bijections between them. In this section we show how to define a universal bijection $\Upsilon : \mathbb{F}_i \rightarrow \mathbb{F}_j$ between any pair of families. This bijection also has the property of having the same recursive structure (as partially defined by the recurrence relation (9)) of Catalan objects.

The algorithm is constructed by first showing that each family is a free magma with one generator (ie. by showing Theorem 1 holds for each family) and then using the natural magma isomorphism between the two magmas to establish a bijection. Since the map is an isomorphism it automatically satisfies the recursive requirement. In this paper we only consider the case of a single generator.

Thus here we only consider bijections between Catalan Magmas – see Brak and Mahony [10] for Motzkin and Schroöder families.

The first step is to establish the appropriate free magma structure of any given family: Given a set of objects \mathbb{F} , a product rule $\star : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$, we need to show Theorem 1 holds and that there is only single irreducible element.

For example the following are the first few elements of a set a certain type of floor plan (see \mathbb{F}_2 in the Appendix for a definition):



Which element is the generator object? What is the product rule?

Clearly one approach is to biject the set \mathbb{F} to the base set (eg. by ranking the objects of the same size) of the standard free magma and then show its possible to define the product to be (3). We take a more intrinsic approach. Rather than explicitly bijecting to the standard free magma, we define a product map $\star : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and a map $\|\cdot\| : \mathbb{F} \rightarrow \mathbb{N}$, and then show that $(\mathbb{F}, \star, \varepsilon)$ is a Catalan magma ie. is a unique factorisation normed magma with one irreducible element. In detail: Theorem 1 requires that to show $(\mathbb{F}, \star, \varepsilon)$ is a Catalan magma, it is sufficient to:

1. Show the product map is injective.
2. The compliment of the range of the product (ie. the set of generators) contains a single element ε .
3. The map $\|\cdot\| : \mathbb{F} \rightarrow \mathbb{N}$ is an additive norm with $\|\varepsilon\|$ minimal (conventionally² $\|\varepsilon\| = 1$).

We first consider two examples for which the recursive structure is well known: the Dyck path family and the triangulation of regular n -sided polygons.

Dyck Paths: \mathbb{F}_7 . The following is the list of all of Dyck paths of norm four or less: (see \mathbb{F}_7 in the Appendix for a definition):

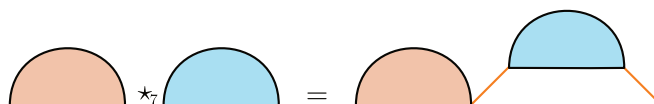


The Dyck path magma generator is

$$\varepsilon_7 = \circ$$

where \circ represents a single vertex (the vertex is shown with a white fill as only a subset of all the vertices of the path correspond to generators). The norm of the generator is defined to be one, and the norm of a path d is defined as $\|d\| = (\text{number of up steps}) + 1$.

The product $d_1 \star_7 d_2$ of two paths d_1 and d_2 is illustrated schematically as



(17)

²Any $k \in \mathbb{N}$ can be used but will result in every value of the norm being a multiple of k

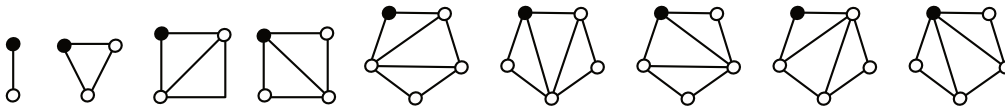
Thus the product takes the left path d_1 , appends an up step on its right, then appends the right path and finally appends a down step. The different colour of the the two added steps is for clarity only. This is the standard (right) factorisation of Dyck paths used to define a magma product ³. For example, the product of two generators is

$$\circ \star_7 \circ = \text{graph with 3 vertices and 2 edges}$$

where the generator vertices are filled white (see \mathbb{F}_7 in the appendix for more examples). Note, the sum of the norms of the left two paths ($= 1 + 1$) which equals that of the product path.

The product is clearly injective: the ‘rightmost’ Dyck path factor is unique. In all cases the norms add: Let $2n_1$ (respec. $2n_2$) be the number of steps in d_1 (respec. d_2). The number of steps in $d_1 \star_2 d_2$ is $2(n_1+n_2)+2$. Thus $\|d_i\| = n_i+1$, $\|d_1 \star_7 d_2\| = (n_1+n_2+1)+1$ and thus $\|d_1 \star_7 d_2\| = \|d_1\| + \|d_2\|$. Thus, by Theorem 1, \mathbb{F}_7 is a Catalan magma.

Triangulation’s: \mathbb{F}_8 . The second Catalan family is the triangulation of regular polygons (see \mathbb{F}_8 in the Appendix). The following is the list of all triangulations of norm four or less:



Note, the polygons are drawn so that the polygon has a marked vertex in a fixed position.

We can construct a magma from triangulations as follows. The generator, ε_8 is a single edge (with the top vertex marked), denoted

$$\varepsilon_8 = \text{graph with 2 vertices and 1 edge}$$

which is defined to have norm one. The norm of a triangulation of a regular n -gon for $n > 2$ is $n - 1$.

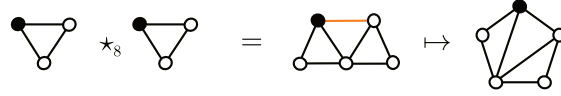
We represent the product⁴, $t_1 \star_8 t_2$, of two triangulation’s t_1 and t_2 schematically as

which represents how the nodes a and d are joined by a new edge and the nodes b and c are combined into a single node. The node a remains the only marked node.

For example, the product of two generators, $\varepsilon_8 \star_8 \varepsilon_8$, is

and the product $(\varepsilon_8 \star_8 \varepsilon_8) \star_8 (\varepsilon_8 \star_8 \varepsilon_8)$ is

³The left factorisation can be used to a related, but different, magma – see Section 4
⁴This product appears in Conway and Coxeter [15] as a way of constructing frieze patterns



where the product representation is shown in more conventional form in the rightmost diagram (see \mathbb{F}_8 in the Appendix for more examples).

The product is clearly injective and the norm adds: if $\|t_1\| = n_1 - 1$, $\|t_2\| = n_2 - 1$ then $t_1 \star_8 t_2$ is an $((n_1 + n_2 - 2) + 1)$ -gon and thus $\|t_1 \star_8 t_2\| = (n_1 + n_2 - 1) - 1 = \|t_1\| + \|t_2\|$. Thus, by Theorem 1, \mathbb{F}_8 is a Catalan magma.

For some families identifying the geometry (or symbols) that represent the generator is simple (eg. the vertex for Dyck paths) whilst for others less so (usually when the generator is the ‘empty’ object). For the latter case the following equivalent process defines the product. Let $\mathbb{F}^+ = \mathbb{F} \setminus \{\varepsilon\}$:

1. Define the object: $\varepsilon \star \varepsilon$
2. For all $f \in \mathbb{F}^+$ define the object: $\varepsilon \star f$
3. For all $f \in \mathbb{F}^+$ define the object: $f \star \varepsilon$
4. For all $f_1, f_2 \in \mathbb{F}^+$ define the object: $f_1 \star f_2$.

An alternative strategy is to add some auxiliary geometry to the objects which is uniquely associated with the generator. This geometry is not strictly part of the the standard definition of the set of objects but used to clarify the magma structure. It can also be used when considering the embedding bijections discussed in Section 5.

Defining the Universal Catalan Bijection. The primary motivation for giving a Catalan family a magma structure is to construct a bijection between the base sets. If we consider the five objects of norm four of two Catalan magmas, by listing (ie. ranking) the two sets we have defined $5! = 120$ possible bijections - the magma isomorphism corresponds to one of these (which is recursive since it is a morphism).

Any two Catalan magmas $(\mathbb{F}_i, \star_i, \varepsilon_i)$ and $(\mathbb{F}_j, \star_j, \varepsilon_j)$ are isomorphic under the morphism,

$$\Upsilon_{i,j} : \mathbb{F}_i \rightarrow \mathbb{F}_j \quad (19)$$

induced by mapping to and from standard free magmas using the decomposition map, (11) and the full multiplication map (14), that is,

$$\pi : \mathbb{F}_i \rightarrow (W_{\varepsilon_i}, \star_i, \varepsilon_i) \quad (20)$$

$$\mu : (W_{\varepsilon_j}, \star_j, \varepsilon_j) \rightarrow \mathbb{F}_j \quad (21)$$

and using the trivial ‘substitution’ map

$$\theta_{i,j} : W_{\varepsilon_i} \rightarrow W_{\varepsilon_j}; \quad \varepsilon_i \mapsto \varepsilon_j, \star_i \mapsto \star_j, \quad (22)$$

then $\Upsilon_{i,j}$ is defined by requiring the diagram,

$$\begin{array}{ccc} \mathbb{F}_i & \xrightarrow{\pi} & W_{\varepsilon_i} \\ \Upsilon_{i,j} \downarrow & & \downarrow \theta_{i,j} \\ \mathbb{F}_i & \xleftarrow{\mu} & W_{\varepsilon_j} \end{array} \quad (23)$$

commute. This gives $\Upsilon_{i,j}$ as

$$\Upsilon_{i,j} = \mu \circ \theta_{i,j} \circ \pi \tag{24}$$

which, since all of π , μ and $\theta_{i,j}$ are isomorphisms (24) shows $\Upsilon_{i,j}$ is also an isomorphism. Note, strictly $\Upsilon_{i,j}$ is different for every pair of families however, the difference is only in the trivial map (22). For this reason we define

$$\Upsilon = \mu \circ \theta \circ \pi \tag{25}$$

where θ is a symbolic “substitution” of the product symbol and generator symbol.

Thus we have the following theorem.

Theorem 2. *Let $(\mathbb{F}_i, \star_i, \varepsilon_i)$ and $(\mathbb{F}_j, \star_j, \varepsilon_j)$ be Catalan magmas. Define the map $\Upsilon : \mathbb{F}_i \rightarrow \mathbb{F}_j$ as follows: for all $f \in \mathbb{F}_i \setminus \{\varepsilon_i\}$*

1. *Decompose f into a product of its generators ε_i .*
2. *In the decomposition of f replace every occurrence of ε_i with ε_j and every occurrence of \star_i with \star_j to give an expression w .*
3. *The value of $\Upsilon(f)$ is defined to be the full multiplication of w .*

The map Υ is then a Catalan magma isomorphism.

The action of Υ can written as

$$f \xrightarrow{\text{decompose}} \begin{matrix} \varepsilon_i \rightarrow \varepsilon_j \\ \text{substitute} \\ \star_i \rightarrow \star_j \end{matrix} \xrightarrow{\text{multiply}} g. \tag{26}$$

Since Υ is an isomorphism we get two outcomes: firstly, it gives us a bijection between the base sets of \mathbb{F}_i and \mathbb{F}_j , and secondly, that the bijection is recursive, that is if $f = f_1 \star_i f_2 \in \mathbb{F}_i$ then $\Upsilon(f) = \Upsilon(f_1) \star_j \Upsilon(f_2)$. ie. if the images under Υ are already known for the two factors then decomposition is not necessary – only a single factorisation is required (and a single multiplication).

We illustrate the bijection/isomorphism Υ using an example from the two Catalan families discussed above: Dyck paths and triangulations.

First decompose the Dyck path down to a product of generators

$$\begin{aligned} \text{Dyck path} &= \text{Dyck path} \star_7 \text{Dyck path} \\ &= (\circ \star_7 \circ) \star_7 (\circ \star_7 \circ) \end{aligned}$$

then change generators

$$\circ \rightarrow \bullet$$

and product rule $\star_7 \rightarrow \star_8$, that is,

$$(\circ \star_7 \circ) \star_7 (\circ \star_7 \circ) \mapsto \left(\begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \right) \star_8 \left(\begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \right)$$

then fully multiplying

$$\begin{aligned} \left(\begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \right) \star_8 \left(\begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \right) &= \begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \\ &= \begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \end{aligned}$$

which gives the bijection

$$\begin{array}{c} \circ \\ \circ \end{array} \star_7 \begin{array}{c} \circ \\ \circ \end{array} \mapsto \begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array}$$

Similarly, if we perform the same multiplications for matching brackets (see family \mathbb{F}_1 in the appendix), we get

$$(00 \star_1 00) \star_1 (00 \star_1 00) = \{\} \star_1 \{\} = \{\}\{\}\{\}$$

or for Catalan floor plans (family \mathbb{F}_{13} in the appendix)

$$(\text{---} \star_{13} \text{---}) \star_{13} (\text{---} \star_{13} \text{---}) = \square \star_{13} \square = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

or for nested matchings (family \mathbb{F}_5 in the appendix),

$$(\bullet \circ \star_{13} \bullet \circ) \star_{13} (\bullet \circ \star_{13} \bullet \circ) = \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ = \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ$$

Thus we have the bijections:

$$\begin{array}{c} \circ \\ \circ \end{array} \star_7 \begin{array}{c} \circ \\ \circ \end{array} \leftrightarrow \begin{array}{c} \bullet \\ \circ \end{array} \star_8 \begin{array}{c} \bullet \\ \circ \end{array} \leftrightarrow \{\}\{\}\{\} \leftrightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \leftrightarrow \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ \star_{13} \bullet \circ$$

Some well known Catalan bijections are the same map as defined by Υ but the algorithm stated to define the codomain element given a domain element can be dramatically different. For example, the bijection from Dyck paths to Staircase polygons (family \mathbb{F}_{10} in the appendix) defined by Delest and Viennot [16] (where staircase polygons are called polyominoes) used the heights of the peaks and valleys of a path to define the heights of the staircase columns (ie. number of cells) and their

overlaps (ie. number of adjacent cells), for example:

$$\begin{array}{c}
 \text{Dyck Path} \Rightarrow 1\ 1\ 2\ 2\ 2\ 1\ 1 \Rightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline \square & & \square & \square & \square & & \square \\ \hline \square & & \square & \square & \square & & \square \\ \hline \end{array} \\
 \Rightarrow \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}
 \end{array} \tag{27}$$

whilst the image path obtained by using Υ is

$$\begin{array}{c}
 \text{Dyck Path} = (((\circ *_{\gamma} \circ) *_{\gamma} ((\circ *_{\gamma} \circ) *_{\gamma} \circ)) *_{\gamma} \circ) \\
 \\
 ((\circ *_{\gamma} \circ) *_{\gamma} ((\circ *_{\gamma} \circ) *_{\gamma} \circ)) *_{\gamma} \circ \Rightarrow (((\text{---} *_{10} \text{---}) *_{10} ((\text{---} *_{10} \text{---}) *_{10} \text{---})) *_{10} \text{---}) \\
 \\
 = ((\square *_{10} (\square *_{10} \text{---})) *_{10} \text{---}) \\
 \\
 = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}
 \end{array}$$

that is, the same image. The proof of the equivalence of the definitions can be seen combinatorially by comparing the embedding bijections discussed in Section 5 (in particular that of a complete binary tree into a path and into a staircase polygon where the correspondence between the peak and valley heights and the tree substructures become clear).

Convention: In order to avoid a profusion of symbols in the remainder of the paper we will adopt the convention of generally using the same symbols, \star and ε , for the product and generators of *different* Catalan magmas. Only when it is necessary to make an explicit distinction will \star and ε be embellished with subscripts and superscripts.

4 Opposite, Reverse and Reflected Families

We now consider new products defined by combining a given product $\star : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ with a bijection $\gamma : \mathcal{M}^+ \rightarrow \mathcal{M}^+$ acting only on the set of reducible elements, \mathcal{M}^+ , of \star . Thus we define the product \star^γ as

$$\star^\gamma : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \quad ; \quad u \star^\gamma v := \gamma(u \star v) \tag{28}$$

If \star is injective then since γ is a bijection, \star^γ will also be injective and thus $(\mathcal{M}, \star^\gamma)$ also a unique factorisation magma.

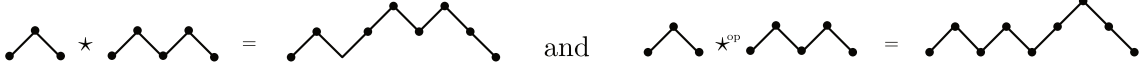
Choosing γ appropriately can show how certain products defined on the same base set (eg. left and right factorisation of Dyck paths) are related. They will also be useful in understanding embedding bijections discussed in Section 5.

We are primarily interested in three examples of γ :

1. The **opposite** map. This map is defined by $\text{opp}(u \star v) = v \star u$. As a product we denote it as \star^{op} , that is

$$\star^{\text{op}} : (u, v) \mapsto \text{opp}(u \star v) = v \star u.$$

called the **opposite product**. Since the opposite map is defined for every magma, every Catalan family, \mathbb{F}_i has an **opposite family**, denoted \mathbb{F}_i^{op} . It is the magma with the same base set but with the opposite product. For example, for Dyck paths



Note, the opposite map, as defined above, is not recursive: $\text{opp}(u \star v) \neq \text{opp}(u) \star \text{opp}(v)$ and not an isomorphism: $\text{opp}(u \star v) \neq \text{opp}(u) \star^{\text{op}} \text{opp}(v)$. The identity map $\text{id} : \mathbb{F} \rightarrow \mathbb{F}^{\text{op}}; u \mapsto u$ is however an anti-isomorphism: $\text{id}(u \star v) = \text{id}(v) \star^{\text{op}} \text{id}(u)$.

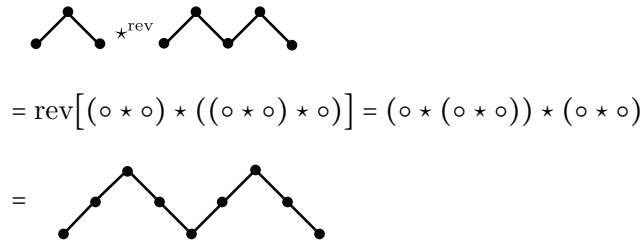
2. The **reverse** map. If w is a word, then $\text{rev}(w)$ is w written in reverse order (ie. read right to left, but written left to right). If (\mathcal{M}, \star) is freely generated, then for $m \in \mathcal{M}^+$ we define $\text{rev} = \mu \circ \text{rev} \circ \pi$ where π and μ are the maps of (11) and (15). As a product we denote it as \star^{rev} , that is

$$\star^{\text{rev}} : (u, v) \mapsto \mu(\text{rev}(\pi(v \star u)))$$

This map is the map induced by mapping the element into the standard magma (ie. fully factorisation), then ‘reversing the word’, then mapping the resulting element back into the magma (ie. full multiplication). This definition is equivalent to the recursive definition

$$\text{rev}(u \star v) = \text{rev}(v) \star \text{rev}(u). \tag{29}$$

Every Catalan family, \mathbb{F}_i has an **reverse family**, denoted $\mathbb{F}_i^{\text{rev}}$. It is the magma with the same base set but with the reverse product. For example, for Dyck paths



3. **Reflection** maps. A reflection map, ref , is any non-trivial map γ which is also an involution: $\gamma^2 = \text{id}$, where id is the identity map. Thus both the opposite and reverse maps are examples of reflection maps, however the definition of ref is motivated by Catalan families that have some sort of geometrical structure for which a vertical or horizontal geometrical reflection can be well defined. For families constructed on circles the reflected structure is (if well defined)

the original structure read clockwise, but drawn counter clockwise. As a product it is denoted by \star^{ref} , that is

$$\star^{\text{ref}} : (u, v) \mapsto \text{ref}(u \star v).$$

Examples of the reflection maps are



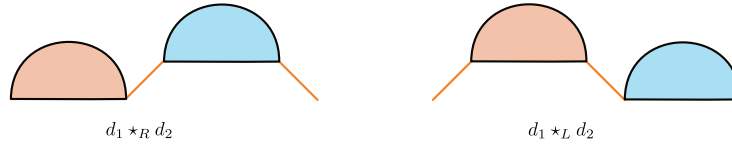
defined by a geometrical reflection across a vertical line. If the Catalan family \mathbb{F}_i has a reflection map defined then the family with the same base set but with the reflection product will be called the **reflection family**, denoted $\mathbb{F}_i^{\text{ref}}$.

Of primary interest is the relationship between the above families and any other product that might be defined on the same base set. For example, many Catalan families have a left and right version of the product. If we start from, say the right product, is it or its opposite/reverse/reflected version related to the left product?

We will consider two examples:

Dyck Paths. It is well know that Dyck paths can be factored in *two* ways giving rise to two different magma products (defined on the same base set), that is, a left-product, which in this section we will denote \star_L and a right product, \star_R .

Thus, schematically, if d_1 and d_2 are two Dyck paths, then the two products are



Clearly these two products define two different free magmas (with the same base set), denote these \mathbb{F}_7^L and \mathbb{F}_7^R .

It would appear the the left product and right products are related by a vertical reflection, however it is not quite as simple. In fact the relationship between the left and right Dyck path products is

$$\star_L^{\text{op}} = \star_R^{\text{ref}} \tag{30}$$

or equivalently, $\star_L = (\star_R^{\text{ref}})^{\text{op}}$. This is because a reflection swaps d_1 and d_2 which the opposite product swaps back: $\text{ref}(d_1 \star_R d_2) = \text{ref}(d_2) \star_L \text{ref}(d_1) = \text{opp}(\text{ref}(d_1) \star_L \text{ref}(d_2))$.

Complete Binary Tree. For complete binary trees there is no natural left and right products but there is an obvious reflection map obtained by ‘reflecting’ the tree across a vertical axis through the root. This is equivalent to recursively swapping left and right sub-trees, that is, we can define the reflection map recursively by

$$\text{ref}(t_1 \star t_2) = \text{ref}(t_2) \star \text{ref}(t_1) \quad \text{and} \quad \text{ref}(\circ) = \circ \tag{31}$$

where \circ is the generator and \star the product for \mathbb{F}_3 . From this definition its clear that for the complete binary tree product

$$\star^{\text{ref}} = \star^{\text{rev}} \neq \star^{\text{op}}. \tag{32}$$

5 Factorisation and Embedding bijections

In order to use the universal bijection one has to first factorise an object. Factorisation of a Catalan object is clearly a recursive algorithm. Whilst such an algorithm can always be programmed it is also desirable to implement the process ‘by hand’. For some Catalan families this is substantially easier than others. For example, the factorisation of a binary tree is trivial whilst for other families, such as frieze diagrams, it is not.

However, it may be that a family with a complicated factorisation algorithm may have a simple bijection to a family where the factorisation is a lot simpler.

In this section we address the problem of factorisation in a particular way which is to embed the objects of one family (whose factorisation is simple) into the objects of another (whose factorisation is more difficult). We will define a process which can always be implemented for any pair of families. Even though the embedding algorithm is defined for any pair of families, currently the embedding gives a simpler factorisation process only in some cases.⁵

Embedding bijections have previously been defined (for example p58 of [4] and [17]) however, the primary aim is to prove that the given set of objects is indeed Catalan rather than determine its factorisation. Here the primary motivation is factorisation: we assume we know the product rules but now want, if possible, a geometrical representation (eg. to a complete binary tree) of the inverse operation. Simple tree traversal will give any of the three decomposition representations (5), (6) and (7) as illustrated in Figure 2.

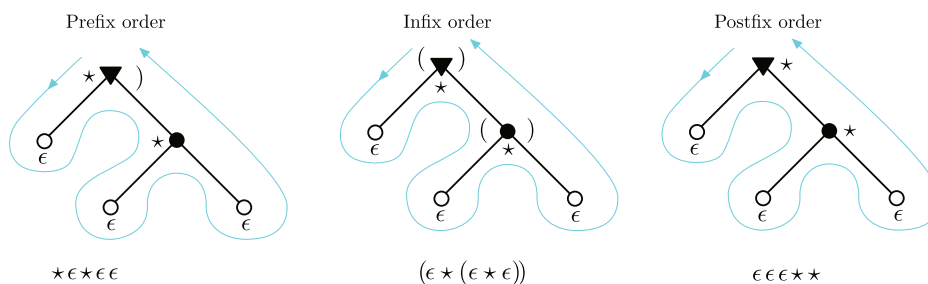


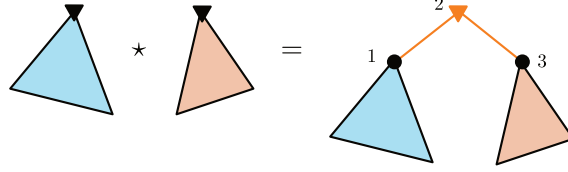
Figure 2: Counter-clockwise traversal of complete binary trees and the three product forms.

The case where the Catalan family is not geometrical (eg. a sequence) will be discussed further below.

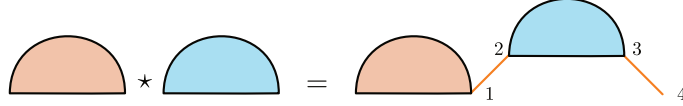
Geometrical Families

If the set of objects is geometrical, knowing the product gives us two important sets of information: 1) we know exactly which parts of the geometry correspond to the generators and 2) which parts of the geometry are added to create the product object each time two objects are multiplied together. We will call this ‘new’ geometry the **product geometry** and the geometry associated with the generators the **generator geometry**. The generator and product geometry is known for a complete binary tree (CBT): the leaves are the generator geometry and the arcs and internal nodes the product geometry. We can represent the product geometry of the CBT,

⁵There are at least 22791 pairs - not all of which have been checked.



by a triple $\alpha = (1, 2, 3)$, where $(1, 2)$ and $(2, 3)$ are the left and right arcs leaving the node 2. For the Dyck path product,



the product geometry is the 2-tuple of up and down steps $\alpha = ((1, 2), (3, 4))$.

Let $w = w_1 \star w_2$ be an element of some Catalan family \mathbb{F} with $\|w\| = \ell > 1$ and generator ε . We can define the **product geometry sequence** $\mathbb{P}_{\mathbb{F}}(w)$ recursively as follows:

$$\mathbb{P}_{\mathbb{F}}(w_1 \star w_2) = \alpha_1, \dots, \alpha_{\ell-1} = \begin{cases} \alpha_1, \mathbb{P}_{\mathbb{F}}(w_1), \mathbb{P}_{\mathbb{F}}(w_2) & \text{if } w_1, w_2 \neq \varepsilon \\ \alpha_1, \mathbb{P}_{\mathbb{F}}(w_1) & \text{if } w_1 \neq \varepsilon, w_2 = \varepsilon \\ \alpha_1, \mathbb{P}_{\mathbb{F}}(w_2) & \text{if } w_1 = \varepsilon, w_2 \neq \varepsilon \\ \alpha_1 & \text{if } w_1 = w_2 = \varepsilon. \end{cases} \quad (33)$$

where α_1 is any chosen representation (eg. vertices, edges, steps etc.) of the product geometry of $w_1 \star w_2$. The **generator geometry sequence** $\mathbb{G}_{\mathbb{F}}(w)$ is similarly defined

$$\mathbb{G}_{\mathbb{F}}(w) = \beta_1, \dots, \beta_{\ell} = \begin{cases} \mathbb{G}_{\mathbb{F}}(w_1), \mathbb{G}_{\mathbb{F}}(w_2) & \text{if } w = w_1 \star w_2 \\ \beta & \text{if } w = \varepsilon. \end{cases} \quad (34)$$

where β is any chosen representation of the generator geometry (eg. point, vertex etc.) associated with ε .

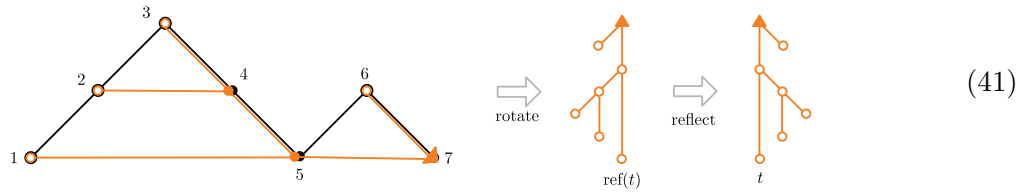
As an example consider the following norm four CBT $t \in \mathbb{F}_3$:



which has decomposition $t = (\circ \star (\circ \star \circ)) \star \circ$ where \circ is a leaf. The product geometry sequence is then computed as

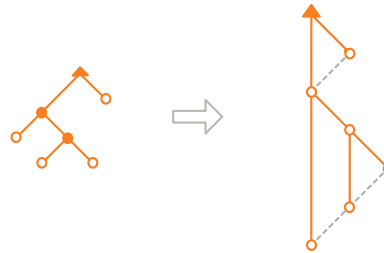
$$\begin{aligned} \mathbb{P}_3(t) &= (2, 1, 7), \mathbb{P}_3(\varepsilon \star (\varepsilon \star \varepsilon)) \\ &= (2, 1, 7), (3, 2, 4), \mathbb{P}_3(\varepsilon \star \varepsilon) \\ &= (2, 1, 7), (3, 2, 4), (5, 4, 6) \end{aligned} \quad (36)$$

However, it is clearer if the elements of the tree product elements can be mapped (injectively) to particular elements of the geometry of the path. There are clearly several possibilities, one of which is (shown left):

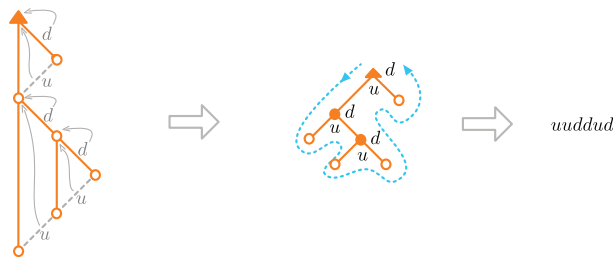


Note, if the embedded tree is rotated to the more conventional form (centre above) it is *not* t , but the reflected (or reverse) tree, $\text{ref}(t)$ as defined by (31). Reflecting this tree gives the correct tree (shown above right). This can be checked by the standard counter clockwise traversal of t which gives the correct Dyck path decomposition $p = (\circ \star (\circ \star \circ)) \star \circ$.

We can consider the converse problem of embedding the path in the tree. This can be done by an “appropriate” stretching of the tree as illustrated left and centre below:

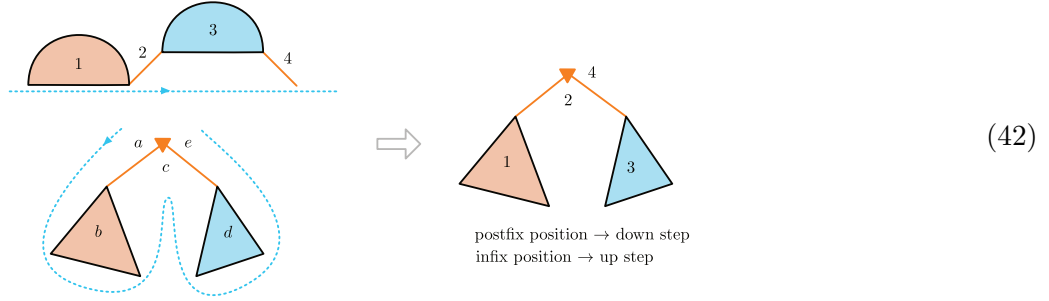


however for a large tree this becomes cumbersome and it is simpler to obtain the Dyck word (and hence trivially the path) by a counter clockwise traversal of t . To do this we map the steps injectively to the geometry of the path. One possibility is to map the up steps to the in-fix node (ie. placed below internal nodes) and the down steps to post-fix node (ie. right side of the internal node and root).



For the above example this mapping (above illustration right) gives the Dyck word $uuddud$ and thus the path $\Upsilon(t)$. This (injective) association can also be obtained by comparing the product

diagrams for the tree and the path:



With a counter-clockwise traversal of the tree: $abcde$ corresponds to

prefix position \rightarrow left sub-tree \rightarrow infix position \rightarrow right sub-tree \rightarrow postfix position

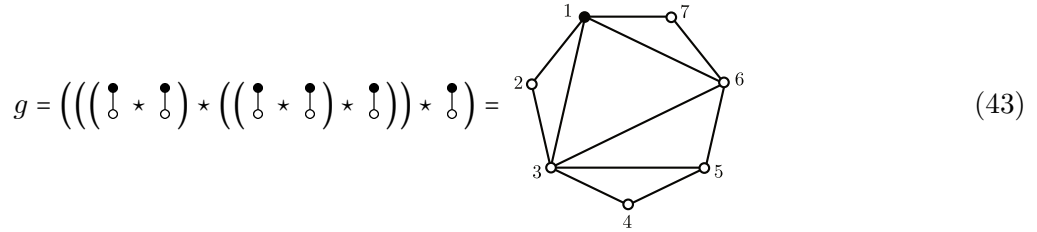
Traversing the path from left to right: 1234 shows that if we map

$$1 \mapsto b \quad 2 \mapsto c \quad 3 \mapsto d \quad 4 \mapsto c$$

we obtain the same embedding as illustrated previously.

This is a variation on the standard ‘labelled CBT tree traversal’ bijection from a CBT to a Dyck path and thus shows that the bijection can be obtained from the magma structure and appropriate mapping of product and generator geometry.

The same process for the triangulated n -gon gives a CBT embedding. For example, consider the triangulation



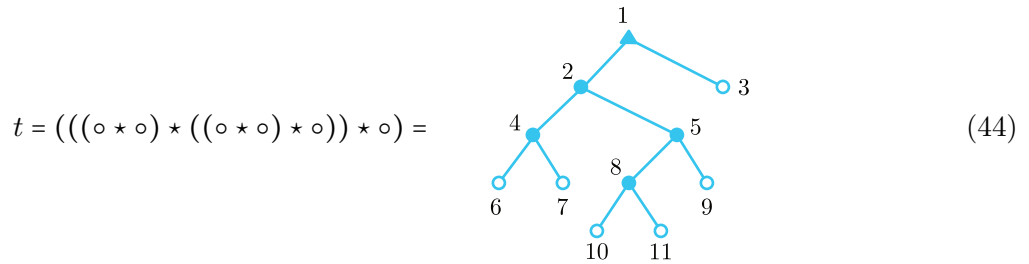
The generator geometry is a sequence of six edges

$$\mathbb{G}_8(g) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)$$

and the product geometry is the sequence of an edge and four chords,

$$\mathbb{P}_8(g) = (1, 7), (1, 6), (1, 3), (3, 6), (3, 5).$$

The CBT $t = \Upsilon(g)$ is

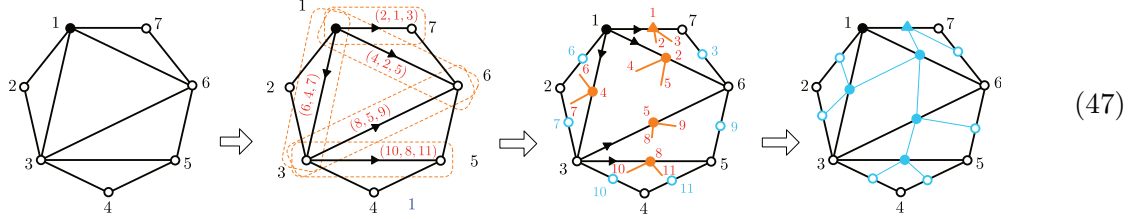


which has product and generator sequences

$$\mathbb{G}_3(g) = 6, 7, 10, 11, 9, 3, \tag{45}$$

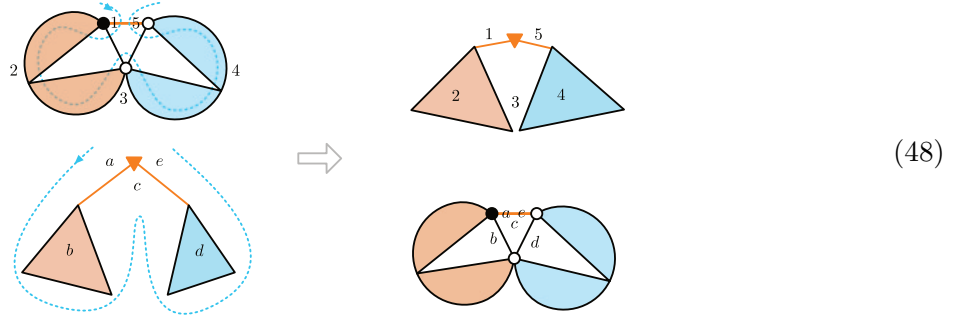
$$\mathbb{P}_3(g) = (2, 1, 3), (4, 2, 5), (6, 4, 7), (8, 5, 9), (10, 8, 11) . \tag{46}$$

Using the map (40) results in the embedding



The chords and edge (1, 7) have been oriented by assigning arrows such that the arrow points away from the node that is first encountered in a counter clockwise traversal around the outside of the polygon. This is necessary as the tree embedded product geometry is added on the right side of the arc (ie. use the right-hand rule). By choosing an injective mapping of associated geometry (one possibility is show above right) gives essentially the same embedding as the example on p58 of [4] or [17] which is now seen as derivable from the magma structure.

This embedding can also be deduced by comparing the two product definitions:

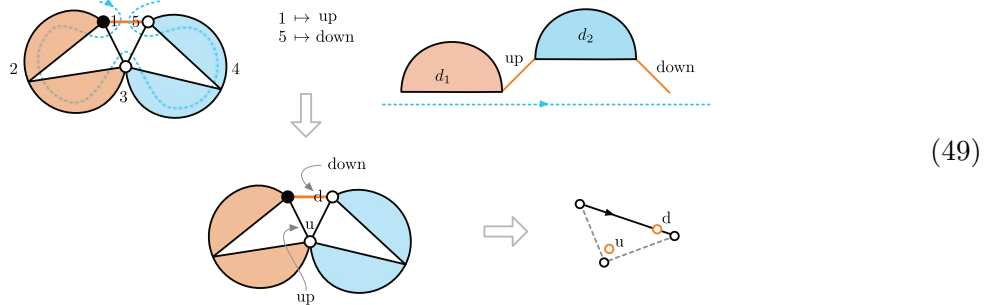


The motivation for embedding a CBT was decomposition, however the process can be used to embed any family into any other family. Thus, we can map the a Dyck path p into a triangulation g using

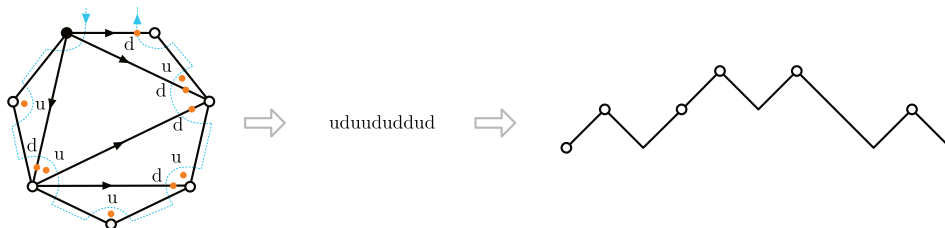
$$\lambda_{p,g} : \mathbb{G}_7(p) \rightarrow \mathbb{G}_8(g)$$

$$\rho_{p,g} : \mathbb{P}_7(p) \rightarrow \mathbb{P}_8(g)$$

or, equivalently by composing the maps defined in(48) and (42) to give



where the bottom right diagram shows the general case (with edge orientations as defined for (47)): End of each arc is a down step, angle subtended by arc (on right) is an up step. For example



This is another well know bijection from triangulations to Dyck paths which we see can be obtained by embedding of the generators and product geometry.

Sequence Families

If the Catalan object is a sequence, for example frieze patterns \mathbb{F}_{14} , then associating a CBT with the elements of the sequence may not be of much utility as the factorisation process may be sufficiently simple to perform on the sequence itself. For example, for matching brackets (or Dyck words) finding the rightmost bracket matching the last bracket (to the right) is straightforward. However, for particular sequence families there may exist a simple version of the universal bijection to some geometric family and then for the geometric family one can construct the CBT embedding and hence obtain the factorisation. Clearly this is the case for matching brackets: biject to Dyck paths and then use the above reflected tree embedding.

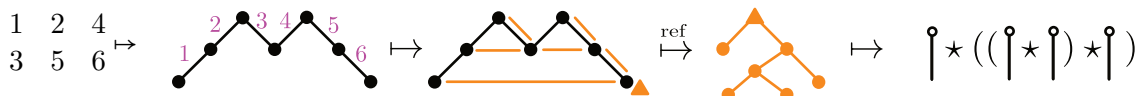
A less trivial example is two row standard tableaux family (\mathbb{F}_{12} in the appendix), considered as a pair of sequences

$$(\text{top row, bottom row}) = (t_1 \dots t_n, b_1 \dots b_n).$$

In this case the universal bijection to Dyck paths takes a simple, form: the top row indexes the up steps and the bottom row indexes the down steps. Thus the (reflected) CBT embedding (and hence decomposition) of

$$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}$$

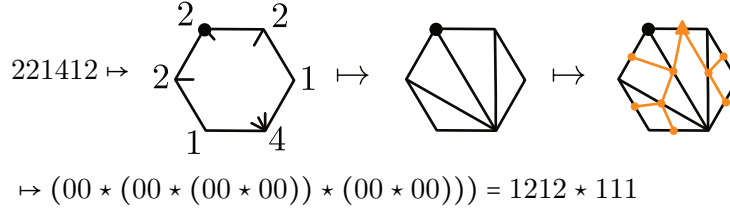
becomes



If only a single level of factorisation is required then the Dyck path can be factored and each factor mapped back to a tableau.

In the case of the frieze sequences the bijection to n -gon triangulation's is an obvious choice.

For example:



Similarly to the tableau, if only a single level of factorisation is required then the triangulation can be factored and each factor mapped back to a frieze sequence.

6 Narayana Connection

The distribution of the number of right (or left) multiplications by the generator in the full decomposition of any element of some Catalan family \mathbb{F}_i is of interest because, as will be shown below, it is given by the Narayana distribution. Let $N_{n,k}$ be the number of elements in \mathcal{W}_ε with norm n and k right multiplications by the generator. The $N_{n,k}$ distribution can also be considered to arise from a map

$$N_r : \mathcal{W}_\varepsilon \rightarrow \mathbb{N}; w \mapsto N_r(w),$$

where $N_r(w)$ is the number of right multiplications by the generator in the infix form of w , then $N_{n,k} = |\{w \in \mathcal{W}_\varepsilon : \|w\| = n, N_r(w) = k\}|$. In the following example the right multiplications by the generator are boxed and the image under N_r is shown:

$$\begin{aligned} \varepsilon * (\varepsilon * (\varepsilon * \boxed{\varepsilon})) &\mapsto 1 \\ (\varepsilon * (\varepsilon * \boxed{\varepsilon})) * \boxed{\varepsilon} &\mapsto 2 \\ (\varepsilon * \boxed{\varepsilon}) * (\varepsilon * \boxed{\varepsilon}) &\mapsto 2 \\ \varepsilon * ((\varepsilon * \boxed{\varepsilon}) * \boxed{\varepsilon}) &\mapsto 2 \\ ((\varepsilon * \boxed{\varepsilon}) * \boxed{\varepsilon}) * \boxed{\varepsilon} &\mapsto 3. \end{aligned}$$

Thus $N_{4,1} = 1$, $N_{4,2} = 3$ and $N_{4,3} = 1$. The N_r map on \mathcal{W}_ε is clearly additive over the product, that is, it is an additive morphism:

$$N_r(w_1 * w_2) = N_r(w_1) + N_r(w_2). \quad (50)$$

From the counter-clockwise traversal of a complete binary tree – see Figure 2 – it is clear that right leaves correspond to right multiplications by the generator. Furthermore, from the embedding bijection of complete binary trees into Dyck paths (see (41)) it is clear that peaks in the path (ie. an up step immediately followed by a down step) correspond to right leaves in the tree and hence the number of peaks in a $2n$ step path is also given by $N_{n,k}$. Finally, Narayana showed [18] that the number of $2n$ step paths with k peaks is given by

$$N_{n,k} = \frac{1}{n-1} \binom{n-1}{k} \binom{n-1}{k-1}, \quad 1 \leq k \leq n-1. \quad (51)$$

which tells us that $N_{n,k}$ is the Narayana distribution.

Since the universal map $\Upsilon : \mathbb{F}_i \rightarrow \mathbb{F}_j$ does not change the number of right multiplications the map, N_r is invariant under the action of Υ , that is, for any pair $w \in \mathbb{F}_i$ and $u \in \mathbb{F}_j$ with $w = \Upsilon(u)$ we have that

$$N_r(w) = N_r(\Upsilon(w)).$$

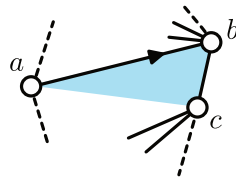
Thus we have the following result.

Proposition 6. *Let $(\mathbb{F}_i, \star_i, \varepsilon_i)$ be a Catalan magma and $\mathbb{F}_{i,n}$ the subset of elements with norm n . Then the number of elements, $N_{n,k}$, in $\mathbb{F}_{i,n}$ whose decomposition has k right multiplications by the generator, is given by the Narayana distribution (51).*

What is of interest is, is there any structure or pattern in the Catalan object that is directly associated with right multiplication and hence will have a Narayana distribution? The previous examples: the right leaves of the complete binary tree and the number of peaks in a Dyck path [19, 20] are two such structures.

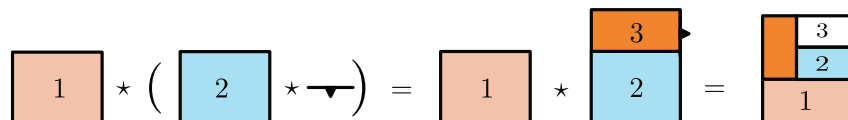
The previous argument connecting peaks in Dyck paths to right multiplications is a specific instance of the following method: if the universal bijection embeds w into $u = \Upsilon(w)$ and we can identify the Narayana structures in w , then their image under the embedding form of Υ will give the corresponding structure in the image object.

Using the embedding of the complete binary tree into a triangulation – illustrated in (47) – shows that the Narayana structure is (schematically),



that is, the polygon edge cb which is defined by the existence of a chord ab that is incident on a node b and adjacent to an earlier (traversing the polygon counter-clockwise from the marked node) node a .

A related approach is to consider the schematic form of the magma product and to track the effect of a right multiplication. For example, for the Catalan floor plans (\mathbb{F}_{13} in Appendix) a right multiplication by the generator (an edge) results in a rectangle spanning the floor plan (ie. a new row), which upon further multiplication on the left by an arbitrary floor plan, moves the ‘row’ (rectangle 3) to the right hand side of the floor plan, that is,



Multiplication of the right by an arbitrary plan does not change this right alignment. Furthermore, the only way to add a new row is by right multiplication by the generator. Thus the right multiplication rectangle always attaches to the right side of the floor plan at any stage of multiplication which shows that the Narayana structure corresponds to the number of ‘rows’ on the right side of the floor plan. The structure having a Narayana distribution is given for all the families in the Appendix.

Other properties of the distribution can be seen in the magma structure. There is an obvious symmetry $k \rightarrow n - k$ obtained algebraically from the binomial identity $\binom{a}{b} = \binom{a}{a-b}$ showing that $N_{n,k} = N_{n,n-k}$. The map $k \rightarrow n - k$ corresponds to reversing the decomposition word (ie. the map defined by (29)) thus replacing right multiplication by a generator by left multiplication by a generator. Thus the number of left generators N_ℓ takes takes on the ‘complimentary’ value $N_\ell(w) = n - N_r(w)$ but has the same distribution $N_{n,\ell}$ where $n = \|w\|$.

7 The Symbolic Enumeration Method and Magmatisation

In this section we briefly discuss the relation between Catalan families considered as magmas and the symbolic enumeration method for counting Catalan families. The latter method is explained and illustrated in the book by Flajolet and Sedgewick [11] and used in Goulden and Jackson [21]. Bracketing words are discussed in Comtet [17] and their representation of complete binary trees and polygon triangulations.

The symbolic method is primarily a technique for finding an equation satisfied by the generating function of a combinatorial family and thereby solving the enumeration problem. The generating function equation is obtained by a simple translation of a “set equation” satisfied by the set of combinatorial objects, \mathbb{S} (or a set in bijection with the objects of the family).

The set \mathbb{S} is defined as a solution to a set equation which is an equation containing \mathbb{S} , sets of ‘atomic’ objects $\{z_1, z_2, \dots\}$, disjoint unions, $\dot{+}$, Cartesian products, \times , markings etc. As will be seen in the examples below, in some sense the atoms “generate” the elements of the combinatorial set but they do not always correspond to the magma generator(s). The symbolic method is well suited to finding the generating function of a *single family* as a large amount of detail can be placed in the elements of \mathbb{S} which then translate directly to a multi-variable generating function.

Different families (with the same generating functions) can have different defining set equations. For example, the set equation for complete binary trees [11] can be written with atoms, $z_1 = \square$ (representing leaves) and $z_2 = \bullet$ (representing internal nodes). The set T , in simple bijection with the trees (assuming trees are defined graphically), satisfies the set equation

$$T = \{\square\} \dot{+} \{\bullet\} \times T \times T \quad (52)$$

and begins

$$T = \{\square, (\bullet, \square, \square), (\bullet, \square, (\bullet, \square, \square)), \dots\} . \quad (53)$$

For Dyck paths there are three atoms $z_1 = \circ$ (representing ‘left’ vertices), $z_2 = /$ (representing up steps) and $z_3 = \backslash$ (representing down steps). The set D , in simple bijection with the set of Dyck paths (assuming paths are defined graphically), satisfies the set equation

$$D = \{\circ\} \dot{+} D \times \{/ \} \times D \times \{\backslash\} \quad (54)$$

and begins

$$D = \{\circ, (\circ, /, \circ, \backslash), (\circ, /, (\circ, /, \circ, \backslash), \backslash), \dots\} . \quad (55)$$

A given family can be associated with several different set equations, each equation defining a slightly different combinatorial set. For example, complete binary trees can equally well be associated with the set equation

$$T' = \{\square\} \dot{+} T' \times \{\bullet\} \times T' \quad (56)$$

with

$$T' = \{\square, (\square, \bullet, \square), (\square, \bullet, (\square, \bullet, \square)), \dots\} \quad (57)$$

with a simple bijection relating T and T' . Because of this type of flexibility, the symbolic enumeration method can result in many different (but similar) symbolic expressions representing the *same* combinatorial family and thus bijecting one Catalan family to another using the elements defined by the associated set equation becomes sensitive to the precise details of the set equation.

The situation with a magma is different – it is algebraic structure. It *assumes* there exists a base set and then defines a product map on top of the base set. Furthermore, there is a natural notion of maps *between* magmas (the morphisms). The magma product map then determines the properties (is it free?, what are the irreducible elements? etc.). The morphisms gives us relations between different families ie. bijections. In the Catalan case, each family is an isomorphic concrete representation of the set of elements of an algebraic structure (a free magma generated by a single element of the base set). The standard form of the symbolic method lacks the notion of irreducible elements (the ‘atoms’ may or may not be magma generators) and of morphisms (constraining the maps *between* families with the same generating functions).

It is however constructive to think of the set equations arising in the symbolic method as equations that define the elements of the base set. One can then place a product structure on top of this base set. For example, for complete binary trees, rather than starting with a graphical definition of the base set (and hence define the product as with \mathbb{F}_3), we can start with the base set defined by the set equation, (53), then define a product $\star_1 : T \times T \rightarrow T$ as

$$t_1 \star_1 t_2 = (\bullet, t_1, t_2) \quad \text{for all } t_1, t_2 \in T$$

and define the norm as the number of \square 's in the element. Then the pair (T, \star_1) is a Catalan magma (proved by showing Theorem 1 is satisfied) with generator \square (the only irreducible element in T). Thus in this case one of the atoms is the generator whilst the other is part of the construction defining the elements of the base set.

Similarly, for the Dyck path example with base set D given by (55) a possible product is

$$d_1 \star_2 d_2 = (d_1, \swarrow, d_2, \searrow)$$

with norm defined as the number of \circ 's in the element. Then the pair (D, \star_2) is a Catalan magma with generator \circ (the only irreducible element in D), again, the two other atoms \swarrow and \searrow are not generators but only part of the definition of the base set. Clearly in both these examples the definition of the products is closely related to the Cartesian product in the set equations as both are related to the recursive structure of the family.

By adding products to the base sets defined by the symbolic method (and proving they are Catalan magmas) they have been effectively “magmatized”. The notion of an isomorphism between them is now well defined and can be used to define bijections between the base sets. For example, using T and D the universal bijection, Theorem 2, gives,

$$(\bullet, \square, (\bullet, \square, \square)) = \square \star_1 (\square \star_1 \square) \quad \mapsto \quad \circ \star_2 (\circ \star_2 \circ) = (\circ, \swarrow, (\circ, \swarrow, \circ, \searrow), \searrow) \quad (58)$$

thus $(\bullet, \square, (\bullet, \square, \square))$ bijects to $(\circ, \swarrow, (\circ, \swarrow, \circ, \searrow), \searrow)$ as expected.

Note, changing the details of the complete binary tree set equation eg. (52) to (56), changes the product definition in a minor way – it becomes $t_1 \star_1 t_2 = (t_1, \bullet, t_2)$ – and similarly changes the details of the bijection (58) in a minor way (note, the form of the left element changes to $(\square, \bullet, (\square, \bullet, \square))$).

In the set equations (52) and (54) the base set only appears twice in each Cartesian product. In other applications of the symbolic method higher order Cartesian products occur as well as other set constructions. It is of interest to see how the ideas associated with free magmas carry across to these cases. Some progress with this generalisation has been made in [10] where a unary map is added in addition to a product map to study Motzkin and Schröder families. For these situations if we have two families \mathbb{M}_1 and \mathbb{M}_2 and four maps (unary and binary)

$$\begin{array}{ll} \alpha_1 : \mathbb{M}_1 \rightarrow \mathbb{M}_1 & \alpha_2 : \mathbb{M}_2 \rightarrow \mathbb{M}_2 \\ \star_1 : \mathbb{M}_1 \times \mathbb{M}_1 \rightarrow \mathbb{M}_1 & \star_2 : \mathbb{M}_2 \times \mathbb{M}_2 \rightarrow \mathbb{M}_2 \end{array}$$

then a map $\Gamma : \mathbb{M}_1 \rightarrow \mathbb{M}_2$ is considered a morphism between the two families if

$$\begin{array}{ll} \Gamma(\alpha_1(m)) = \alpha_2(\Gamma(m)), & m \in \mathbb{M}_1 \\ \Gamma(m_1 \star_1 m_2) = \Gamma(m_1) \star_2 \Gamma(m_2), & m_1, m_2 \in \mathbb{M}_1. \end{array}$$

A map Γ satisfying these equations leads to a universal bijection induced by decomposing an element into a composition of the unary and binary maps acting on the irreducible elements (ie. elements that cannot be written as a composition of the maps).

8 Acknowledgement

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9 Appendix

In this appendix we list fourteen Catalan families and a list of information defining their magma structure. We use the convention of Stanley [2, 4] of giving a brief description of the family followed by the list of the five objects with norm four. If the problem (or trivially equivalent) occurs in [4] then it is referenced in the section header.

For each family the following information is itemised:

- The generator of the magma.
 - It is not strictly necessary to define exactly what ε maps to, so long as the left and right multiplication by ε is defined (and the element $\varepsilon \star \varepsilon$).
- A schematic definition of the product.
 - We will use the same symbol, \star for the product for every family, but of course, the product is different in each case, so strictly a different product symbol should be used in each case.
- Norm.
 - In all cases the norm of the generator is one: $\|\varepsilon\| = 1$. This, together with the additive property (see Definition 2) defines the norm of every object. However, the norm will map to a conventional or convenient parameter associated with the objects in each particular family. This correspondence is stated here.
- Examples of the product.
 - Products where the LHS or RHS is a generator can be poorly defined when using only the schematic diagram. A alternative way to define the product is to consider each case separately as illustrated in the with binary tree family \mathbb{F}_3 . However, doing this in every family can obscure the simplicity of the general case product, thus we clarify the product when one side is a generator by giving examples. For several of the examples the geometry (if appropriate) associated with the generator is emphasises as is the product geometry.

Given the above data for any family it is then possible to biject any object of one family to its image in another family using the universal bijection (26).

Catalan Families

- \mathbb{F}_1 – Matching brackets and Dyck words
- \mathbb{F}_2 – Non-crossing chords ? the circular form of nested matchings
- \mathbb{F}_3 – Complete Binary trees and Binary trees
- \mathbb{F}_8 – Planar Trees
- \mathbb{F}_5 – Nested matchings or Link Diagrams
- \mathbb{F}_6 – Non-crossing partitions
- \mathbb{F}_7 – Dyck paths
- \mathbb{F}_8 – Polygon triangulations
- \mathbb{F}_9 – 321-avoiding permutations
- \mathbb{F}_{10} – Staircase polygons
- \mathbb{F}_{11} – Pyramid of heaps of segments
- \mathbb{F}_{12} – Two row standard tableau
- \mathbb{F}_{13} – Floor Plans
- \mathbb{F}_{14} – Frieze Patterns

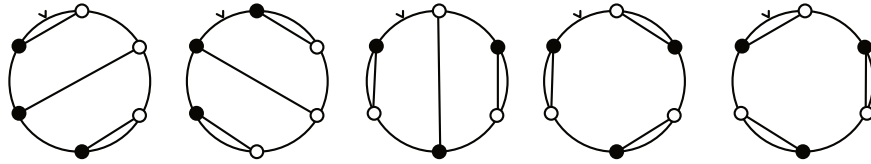
\mathbb{F}_1 : **Matching brackets and Dyck words (Ex77, 211 of [4])** A language in the alphabet containing the two symbols $\{$ and $\}$ (or x and \bar{x} for Dyck words), such that there is an equal number of left as right brackets and the brackets match “pairwise” (ie. every prefix of the word contains no more right brackets than left).

$$\{\{\{\}\}\} \quad \{\{\}\}\} \quad \{\{\}\}\} \quad \{\{\}\}\} \quad \{\{\}\}\}$$

- Generator: $\varepsilon = \emptyset$ (ie. the empty word)
- Product: $b_1 \star b_2 = b_1\{b_2\}$ ie. word concatenation (and add two brackets).
- Norm = (Half the number of brackets) + 1.
- Examples:

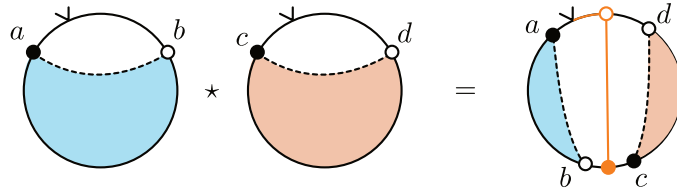
$$\begin{aligned} \emptyset \star \emptyset &= \{\} \\ \{\} \star \emptyset &= \{\}\{\} \\ \emptyset \star \{\} &= \{\}\{\} \\ \{\} \star \{\} &= \{\}\{\}\{\} \end{aligned}$$

\mathbb{F}_2 : **Non-crossing chords – the circular form of nested matchings (Ex59 of [4])** A circle with $2n$ nodes and one marked segment of the circumference. Each node is joined to exactly one other node by a chord such that the chords do not intersect.

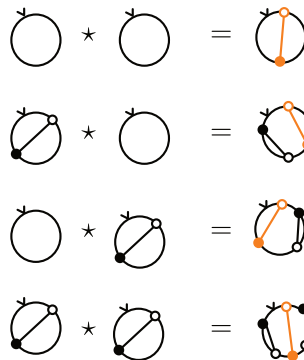


- Generator and product:

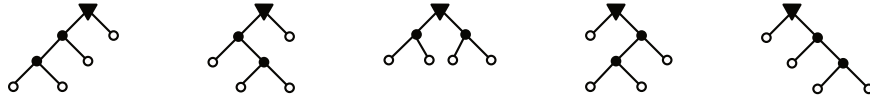
$$\epsilon = \text{circle with a marked segment}$$



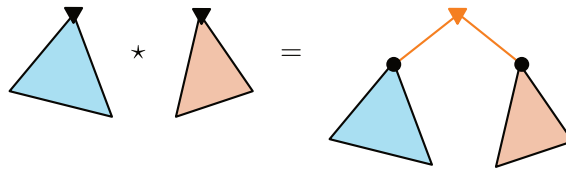
- Norm = (Number of chords) + 1
- Examples:



\mathbb{F}_3 : **Complete Binary trees and Binary trees (Ex 4, 5 of [4])** Binary trees are trivially related to complete binary trees: just delete the leaves. The root of a complete binary tree is denoted by a triangle, the internal nodes with solid circles and the leaves with white circles.



- Generator: $\varepsilon = \circ$ (a leaf).
- Product:⁶



- Norm = Number of leaves.
- Examples:

$$\circ * \circ = \text{tree with root triangle and two children (leaves)}$$

$$\circ * \text{tree with root triangle and two children (leaves)} = \text{tree with root triangle, left child leaf, right child tree with root triangle and two children (leaves)}$$

$$\text{tree with root triangle and two children (leaves)} * \circ = \text{tree with root triangle, left child tree with root triangle and two children (leaves), right child leaf}$$

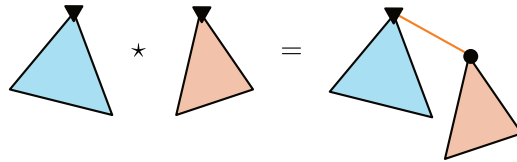
$$\text{tree with root triangle and two children (leaves)} * \text{tree with root triangle and two children (leaves)} = \text{tree with root triangle, left child tree with root triangle and two children (leaves), right child tree with root triangle and two children (leaves)}$$

⁶This is the original magma product – see Appendix C of [12]

\mathbb{F}_4 : **Planar Trees (Ex6, of [4])** A graph with a marked vertex (the root) and no cycles (ie. a unique path from the root to every leaf (degree one vertex)). The planar tree is conventionally drawn with the root at the top.



- Generator: $\varepsilon = \bullet$ (ie. a node).
- Product:



- Norm = Number of nodes.
- Examples:

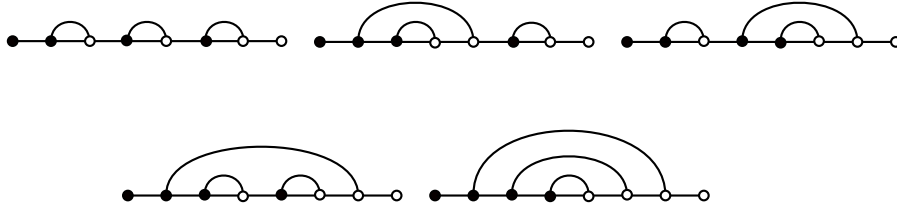
$$\bullet * \bullet = \begin{array}{c} \blacktriangledown \\ \bullet \end{array}$$

$$\bullet * \begin{array}{c} \blacktriangledown \\ \bullet \end{array} = \begin{array}{c} \blacktriangledown \\ \bullet \\ \bullet \end{array}$$

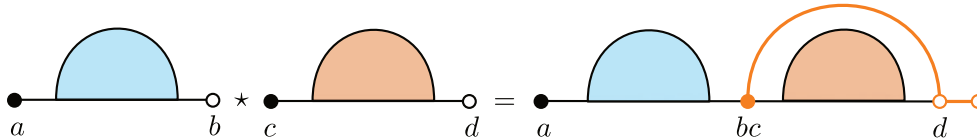
$$\begin{array}{c} \blacktriangledown \\ \bullet \end{array} * \bullet = \begin{array}{c} \blacktriangledown \\ \bullet \\ \bullet \end{array}$$

$$\begin{array}{c} \blacktriangledown \\ \bullet \end{array} * \begin{array}{c} \blacktriangledown \\ \bullet \end{array} = \begin{array}{c} \blacktriangledown \\ \bullet \\ \bullet \\ \bullet \end{array}$$

\mathbb{F}_5 : **Nested matchings or Link Diagrams (Ex61 of [4])** A line of $2n$ nodes with all but the first and last nodes⁷ connected pairwise by $n - 1$ links such that the arcs representing the links (always drawn above the line) do not intersect.

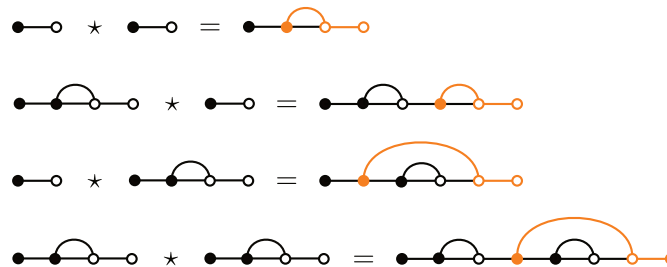


- Generator: $\varepsilon = \bullet \text{---} \circ$
- Product:



Nodes b and c become a single node.

- Norm = (Number of links) + 1 = Half the number of nodes.
- Example:



⁷Conventionally these two nodes are not drawn, but keeping them simplifies the product definition.

\mathbb{F}_6 : **Non-crossing partitions (Ex159 of [4])** A partition of $\{1, 2, \dots, n\}$ is non-crossing if whenever four elements, $1 \leq a < b < c < d \leq n$, are such that a, c are in the same block and b, d are in the same block, then the two blocks coincide [22]. We will use the marked circular representation of the partitions. The five non-crossing partitions with for $n = 3$ are:

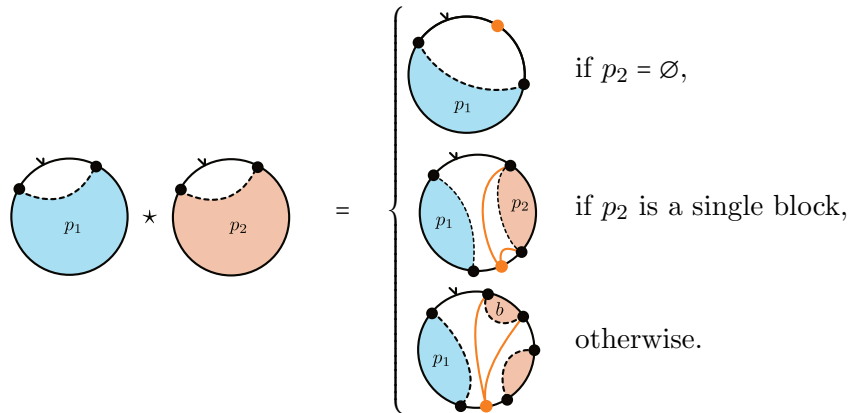


To convert to a partition of $\{1, 2, \dots, n\}$ assume the nodes are enumerated *anti-clockwise* from the mark. Nodes joined by a chord are in the same block of the partition.

- Generator: $\varepsilon = \curvearrowright$ (arc of circle).

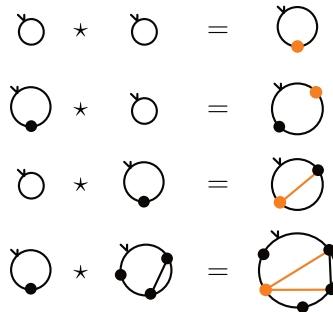
In a general non-crossing partition the i^{th} generator is the arc between node $i-1$ and i (nodes labelled anti-clockwise). The arc between the last node and the mark is not a generator.

- Product:

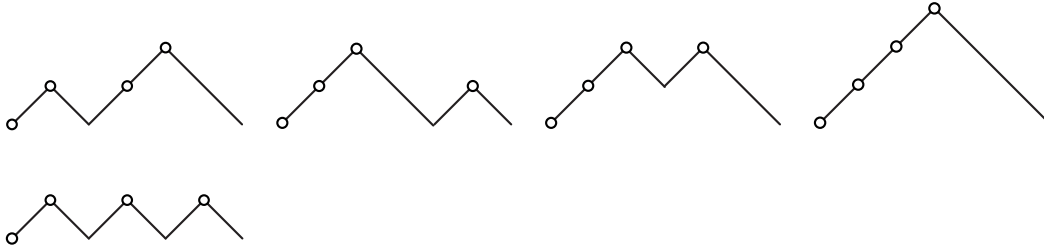


In the last case b is the block in p_2 containing the last node.

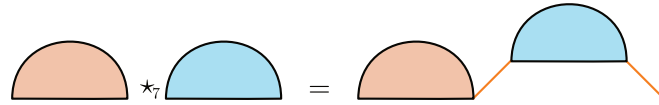
- Norm = (Number of nodes) + 1
- Examples:



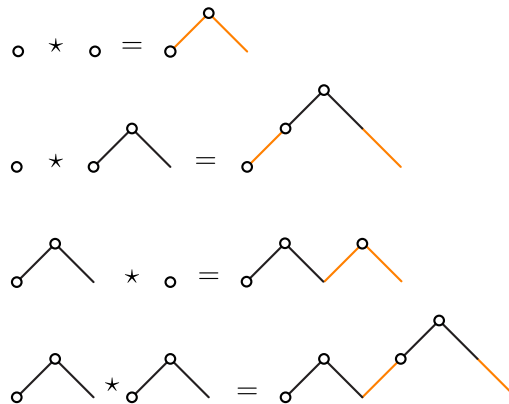
\mathbb{F}_7 : **Dyck paths (Ex30 of [4])** A Dyck path is a sequence of $2n$ steps: n ‘up’ steps and n ‘down’ steps such that the path starts and ends at the same height and does not step below the height of the leftmost vertex. There are five paths with six steps.



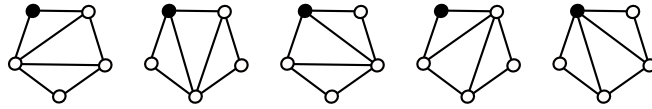
- Generator: $\varepsilon = \circ$ (a vertex).
- Product:



- Norm = (Number of up steps) + 1.
- Examples:



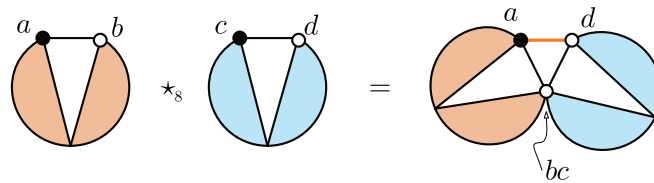
\mathbb{F}_8 : **Polygon triangulations (Ex1 of [4]), [8]** . A triangulated n -gon is a partition of an n sided polygon into $n - 2$ triangles by means of $n - 3$ non-crossing chords. There are five triangulation's of a 5-gon:



The marked node is used in the product definition.

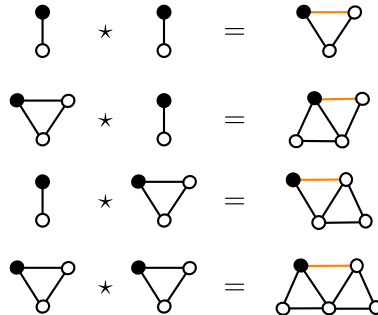
- Generator $\epsilon =$

- Product:



where nodes b and c are merged⁸.

- Norm = (Number of triangles) + 1
- Example:

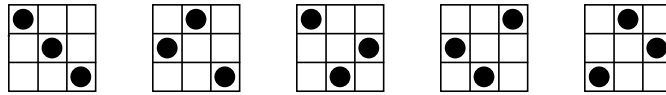


⁸This concatenation of triangulations appears in [15] in connection with frieze patterns – F_{14} .

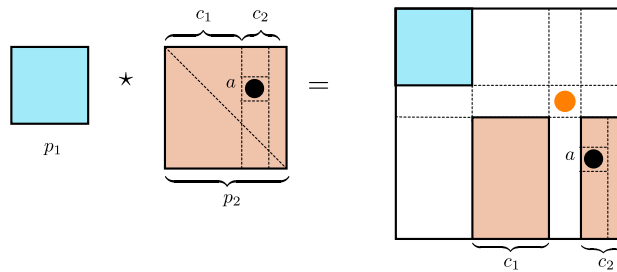
\mathbb{F}_9 : **321-avoiding permutations (Ex115 of [4])** A permutation $\sigma = \sigma_1 \dots \sigma_n$ of $\{1, \dots, n\}$ is called a 321-avoiding permutation if it does not contain the triple $\sigma_k < \sigma_j < \sigma_i$ when $i < j < k$. The five 321-avoiding permutations of 123 are

$$123, \quad 213, \quad 132, \quad 312, \quad 231.$$

and as Rothe diagrams [23] (row i has a dot in column σ_i – rows are labelled from the top),

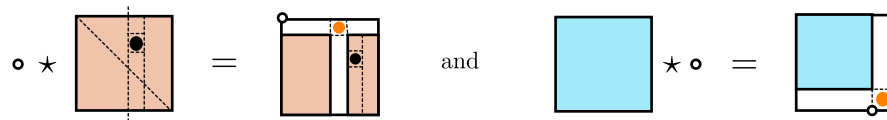


- Generator: $\varepsilon = \cdot$ (a point or the empty permutation). On occasion the generator will be represented by a \circ for clarity.
- Product:



The permutation p_2 is partitioned immediately to the left of the dot a into columns c_1 and c_2 . The dot a is the leftmost dot in p_2 wholly above the diagonal.

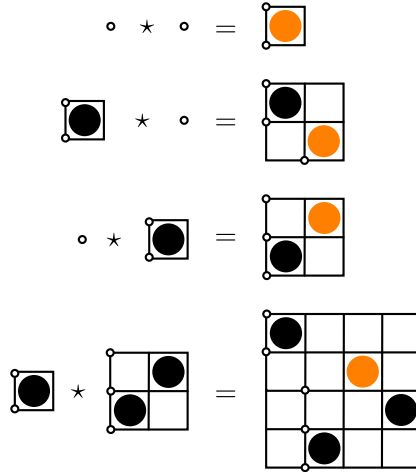
If there are no dots wholly above the diagonal then c_2 is empty. If p_1 or p_2 are generators then the product can be represented as



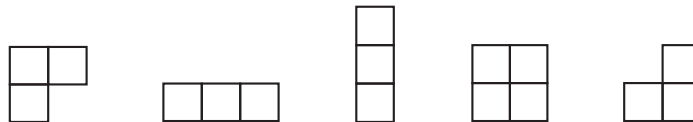
where the generator points are enlarged for clarity.

- Norm = (number of dots) + 1

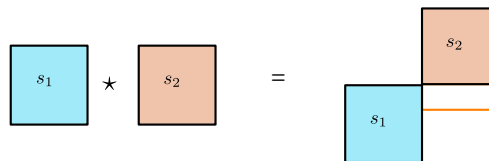
- Examples:



\mathbb{F}_{10} : **Staircase polygons (Ex57 of [4])** A pair of n step binomial paths⁹ (a sequence of vertical and horizontal steps) which: 1) do not intersect (ie. no vertices in common) except for the first and last vertex and 2) start and end at the same position.



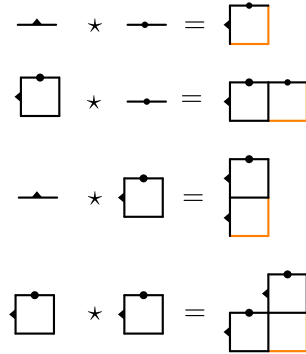
- Generator $\varepsilon = \text{—}$ (an edge).
- Product:



with the convention that if $s_1 = \varepsilon$ then the generator edge is rotated vertically (illustrated with a triangle in the examples below). The orange edge signifies that a single cell is added below each column of s_2 .

⁹Called binomial paths as they are enumerated by binomial coefficients

- Norm = Number of steps in either binomial path
- Examples (the circle and triangle are added to the generators only to clarify how they contribute to the geometry):



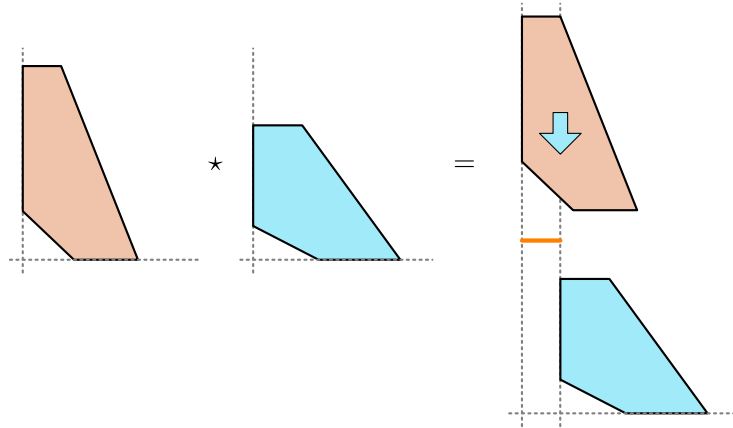
\mathbb{F}_{11} : **Pyramid of heaps of segments** [24]. Heaps of segments is an example of a graphical representation of a certain commutation monoid [25]. We will define then in physical terms. The segments can be considered as identical ‘coins’ which are added to pegs (through a hole in the centre) attached to a board at the base. The pegs are placed in a line an equal distance apart (the pegs are between the dashed lines illustrated below). The diameter of the coin is slightly larger than the spacing between the pegs (thus preventing it from falling below any coins on the peg to the immediate right or left). The heap of coins is a ‘pyramid’ if it has i) exactly one maximal coin and ii) the maximal coin is on the leftmost peg. A coin/segment is maximal if when pushed downwards *all* the coins are pushed downwards.

In the example below the ‘coins’ (a.k.a. segments) are separated vertically for clarity. There are five pyramids of heaps of segments containing three segments.



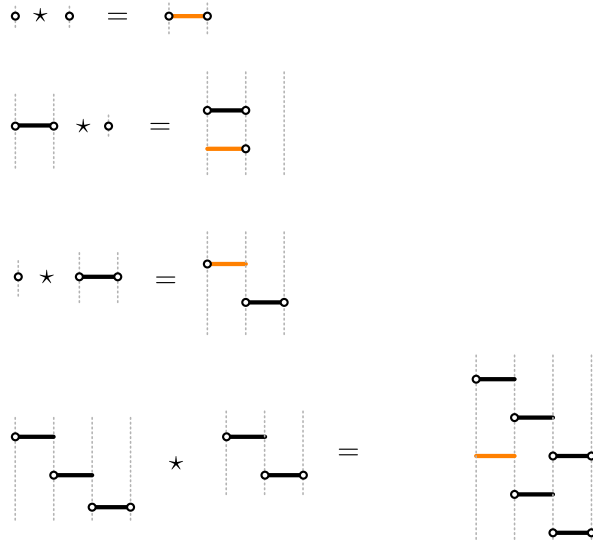
- Generator: $\varepsilon = \circ$ (a point - shown large for clarity).

- Product:



The down arrow shows the segments of the upper heap must be “dropped” onto the lower heap – see the last example below. When multiplying on the left by a generator (an empty heap or point) the generator (denoted orange) is placed at the left end of the new segment (denoted orange) and at the right end when right multiplying.

- Norm = (Number of segments) + 1
- Examples:



\mathbb{F}_{12} : **Two row standard tableau (Ex168 of [4])** Two rows of n square cells. The cells contain the integers 1 to $2n$. Each integer occurs exactly once and the integers increase left to right along each row and increase down each column.

1	3	5
2	4	6

1	2	5
3	4	6

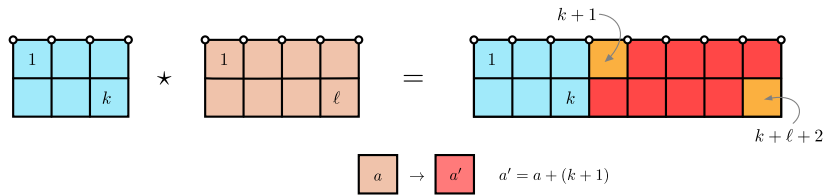
1	2	3
4	5	6

1	2	4
3	5	6

1	3	4
2	5	6

- Generator: $\varepsilon = \begin{array}{|c} \circ \\ \hline \end{array}$

- Product:



- Norm = (Number of columns) + 1

- Examples:

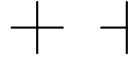
$$\begin{array}{|c} \circ \\ \hline \end{array} \star \begin{array}{|c} \circ \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 1 & \\ \hline 2 & \\ \hline \end{array} \star \begin{array}{|c} \circ \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

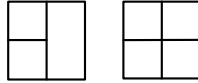
$$\begin{array}{|c} \circ \\ \hline \end{array} \star \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \star \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}$$

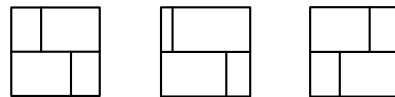
\mathbb{F}_{13} : **Catalan Floor Plans** [6] Catalan floor plans are equivalence classes of rectangles containing rectangles. A size n floor plan rectangle contains n non-nested rectangles – called the ‘rooms’ of the plan. The rooms are constrained by the junctions between the interior walls. Junctions of the two forms



are forbidden. For example, the following are forbidden floor plans

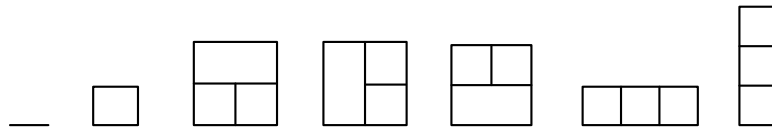


Any two floor plans are equivalent if one can be obtained from another by sequences of wall “slidings”: a horizontal (respec. vertical) wall can be moved up or down (respec. left or right) so long as it does not pass over or coincide with any other wall (or result in one of the forbidden junctions). For example the following are all equivalent.

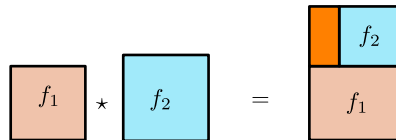


The rectangular shape of the outer wall rectangle is irrelevant.

Thus the five Catalan floor plans with three rooms are:



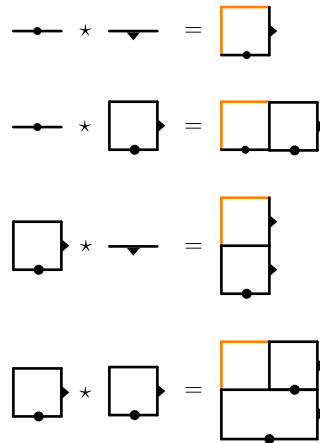
- Generator: $\varepsilon = \text{—}$ (an edge).
- Product:



Thus, first add a new column (the product geometry) spanning the left side of f_2 , then scale f_1 to fit beneath. If f_1 or f_2 are generators then we use the convention that if $f_1 = \varepsilon$ then the edge is scaled to fit the width of f_2 above and if $f_2 = \varepsilon$ then the edge is rotated vertically.

- Norm = (Number of rectangles) + 1

- Examples (the circle and triangle are added to the generators only to clarify how they contribute to the geometry):



\mathbb{F}_{14} : **Frieze Patterns (Ex197 of [4]) [15, 26]** Positive integer sequences a_1, a_2, \dots, a_n which generate an infinite array by periodically translating the $n - 1$ row rhombus (first and last row all 1's),

$$\begin{array}{cccccccccccc}
 \dots & 1 & & 1 & & \dots & & 1 & & \dots & & & & \\
 & & \dots & a_1 & & a_2 & & \dots & & a_n & & \dots & & \\
 & & & \dots & b_1 & & b_2 & & \dots & & b_n & & \dots & \\
 & & & & & \ddots & & & & & & \ddots & & \\
 & & & & & \dots & r_1 & & r_2 & & \dots & & r_n & & \dots \\
 & & & & & & \dots & 1 & & 1 & & \dots & & 1 & & \dots
 \end{array} \tag{59}$$

to the left and right . The values of the sequences in rows three to the second last row are fixed by requiring that any quadrangle

$$\begin{array}{ccc}
 & r & \\
 s & & t \\
 & u &
 \end{array} \tag{60}$$

of four neighbouring entries satisfies the ‘unimodular’ property: $st - ru = 1$. The unimodular quadrangles can always be iterated downwards, the constraint on the a_i ’s is that row $n - 1$ must all be 1’s.

For $n = 5$ there are only five sequences which satisfy the above constraint:

$$12213, \quad 22131, \quad 21312, \quad 13122, \quad 31221. \tag{61}$$

The sequence 12213 defines the four row frieze pattern:

$$\begin{array}{cccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & \dots & 1 & & 2 & & 2 & & 1 & & 3 & & \dots \\
 & & & & \dots & 1 & & 3 & & 1 & & 2 & & 2 & & \dots \\
 & & & & & & \dots & 1 & & 1 & & 1 & & 1 & & 1
 \end{array} \tag{62}$$

- Generator: $\varepsilon = 00$.
- Product¹⁰

$$a_1, a_2, \dots, a_n \star b_1, b_2, \dots, b_m = c_1, c_2, \dots, c_{n+m-1}$$

where

$$c_i = \begin{cases} a_1 + 1 & i = 1 \\ a_i & 1 < i < n \\ a_n + b_1 + 1 & i = n \\ b_i & n < i < n + m - 1 \\ b_m + 1 & i = n + m - 1 \end{cases} \quad (63)$$

- Norm = (Length of sequence) - 1
- Examples:

$$\begin{aligned} 00 \star 00 &= 111 \\ 00 \star 111 &= 1212 \\ 111 \star 00 &= 2121 \\ 111 \star 111 &= 21312 \end{aligned}$$

¹⁰This result appears in [15] as a way of generating new patterns from old and is a consequence of the bijection to triangulations of n -gons - \mathbb{F}_8 .

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