

# Contributions to the Problems of Recognizing and Coloring Gammoids

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*“That may be impossible, sir.”*

— *Data.*

*“Things are only impossible until they’re not!”*

— *Jean-Luc Picard.*



## Abstract

This work provides a thorough introduction to the field of gammoids and presents new results that are considered helpful for solving the problems of recognizing and coloring gammoids.

Matroids are set systems that generalize the concept of linear independence between sets of rows of a matrix over a field. Gammoids are those matroids that may be represented by directed graphs where the corresponding independence is modeled as the existence of certain families of pair-wise vertex disjoint paths. The seminal papers in gammoid theory have been written by J.H. Mason [Mas72], A.W. Ingleton and M.J. Piff [IP73]. Natural applications of gammoids can be found within the realms of connectivity of both directed and undirected graphs.

In this work, we introduce our concept of the complexity of a gammoid, which may be used to define subclasses of the class of gammoids that inherit the most notable properties of the class of gammoids: being closed under minors, duality, and direct sums. Furthermore, we provide a comprehensive method for deciding whether a given matroid is a gammoid. We give a new procedure for

## Zusammenfassung

Diese Arbeit gibt eine gründliche Einführung in das Gebiet der Gammoiden, und legt neue Ergebnisse dar, die für die Probleme des Erkennens und des Färbens von Gammoiden dienlich sind.

Matroide sind Mengensysteme, welchen Begriff der linearen Unabhängigkeit zwischen Mengen von Zeilen einer Matrix über einem Körper verallgemeinern. Gammoiden sind jene Matroide, welche so durch gerichtete Graphen dargestellt werden können, dass ihre zugehörige Unabhängigkeit durch die Existenz gewisser Familien von paarweise knotendisjunkten Pfaden beschreibbar ist. Die grundlegenden Arbeiten zur Theorie der Gammoiden wurden von J.H. Mason [Mas72], A.W. Ingleton und M.J. Piff [IP73] verfasst. Natürliche Anwendung finden Gammoiden im Bereich des Zusammenhangs sowohl von gerichteten als auch von ungerichteten Graphen.

In dieser Arbeit führen wir unseren Begriff der Komplexität eines Gammoids ein, welcher verwendet werden kann, um Unterklassen der Klasse der Gammoiden zu definieren. Diese Unterklassen erben die bedeutendsten Eigenschaften der Klasse der Gammoiden, nämlich die Abgeschlossenheit unter Minoren, unter Dualität sowie unter direkten Summen. Des Weiteren stellen wir eine umfassende Methode bereit, mit der entschieden wer-

obtaining an  $\mathbb{R}$ -matrix, that represents a gammoid given by the means of a directed graph, which avoids using power series. We present the first purely combinatorial way of obtaining orientations of gammoids. We prove that every lattice path matroid is 3-colorable.

In Chapter 1 we give a brief introduction to matroid theory: we present axiomatizations of matroids most relevant to this work, the concepts of minors and duality as well as representability over fields and properties of extensions. The same chapter also contains a brief introduction to the theory of transversals, including the Theorems of Hall, Rado, Ore, and Perfect, and an introduction to transversal matroids. Also, we provide a short introduction to directed graphs, we introduce the concept of a routing in a directed graph and we close the chapter with Menger's Theorem and its consequences.

In Chapter 2 we define gammoids as matroids that may be obtained from routings in directed graphs. We explore the properties of their directed graph representations and along that we define our notion of a duality respecting representation which correlates the duality-like notion of op-

den kann, ob ein gegebenes Matroid ein Gammoid ist. Wir geben eine neue Vorgehensweise an, die eine  $\mathbb{R}$ -Matrix-Darstellung eines Gammoids, welches mittels eines gerichteten Graphen gegeben ist, findet, ohne auf Potenzreihen zurückzugreifen. Wir stellen das erste rein kombinatorische Verfahren vor, das Orientierungen eines Gammoids liefert. Wir zeigen, dass alle Lattice-Path-Matroiden 3-färbbar sind.

Im ersten Kapitel geben wir eine kurze Einführung in die Matroidtheorie: Wir stellen die Axiomatisierungen von Matroiden, welche am besten zu dieser Arbeit passen, den Minorenbegriff, die Dualität sowie die Darstellbarkeit über Körpern und Eigenschaften von Erweiterungen vor. Das Kapitel enthält außerdem eine Einführung in die Transversaltheorie, welche die Sätze von Hall, Rado, Ore und Perfect sowie eine Einführung der Transversalmatroiden umfasst. Außerdem stellen wir gerichtete Graphen kurz vor, erläutern den Begriff des Routings in gerichteten Graphen und beenden das Kapitel mit dem Satz von Menger sowie Schlussfolgerungen aus diesem.

Im zweiten Kapitel definieren wir Gammoiden als Matroiden, die durch Routings in gerichteten Graphen beschrieben werden können. Wir untersuchen die Eigenschaften ihrer Darstellungen mit gerichteten Graphen und definieren dabei unseren Begriff einer dualitätsachtenden Darstellung, welche den dualitätsnahen

posite directed graphs with the notion of duality with respect to gammoids. Furthermore, we introduce our three complexity measures for gammoids that yield subclasses of gammoids which are closed under minors and duality. We present Mason's  $\alpha$ -criterion for strict gammoids, and we examine the properties of strict gammoids and transversal matroids. We analyze the problem of recognizing gammoids, we develop the notion of an  $\alpha$ -violation, and we present our best approach for deciding instances of the recognition problem. At the end of Chapter 2, we present our method for determining an  $\mathbb{R}$ -matrix representing a gammoid from a given representation in terms of a directed graph.

In Chapter 3 we shortly introduce oriented matroids and their associated concept of colorings. We show that all orientations of lattice path matroids have 3-colorings. Then we introduce our concept of a heavy arc orientation of a gammoid that yields a purely combinatorial way to obtain representable orientations of gammoids. In Chapter 4 we summarize our new results and give an overview of new and old open problems.

Begriff gegenläufig gerichteter Graphen und den Dualitätsbegriff von Gammoiden in Wechselbeziehung stellt. Weiterhin stellen wir drei Komplexitätsmaße für Gammoide vor, welche Unterklassen von Gammoiden liefern, die unter Minoren und Dualität abgeschlossen sind. Wir stellen Mason's  $\alpha$ -Kriterium für strikte Gammoide vor, und wir untersuchen die Eigenschaften von strikten Gammoiden und von Transversalmatroiden. Wir analysieren das Problem des Erkennens von Gammoiden, wir entwickeln den Begriff der  $\alpha$ -Verletzung und wir präsentieren unseren besten Ansatz zum Entscheiden, ob ein Matroid ein Gammoid ist. Zum Schluß des zweiten Kapitels stellen wir unsere Methode vor, eine  $\mathbb{R}$ -Matrix-Darstellung eines Gammoids aus einer Darstellung vermöge gerichteter Graphen zu erhalten.

Im dritten Kapitel geben wir eine kurze Einführung in orientierte Matroide und den damit verbundenen Begriff der Färbung. Wir zeigen, dass alle Orientierungen von Lattice-Path-Matroiden eine 3-Färbung besitzen. Danach stellen wir unseren Begriff der Heavy-Arc-Orientierung eines Gammoids vor, welcher eine rein kombinatorische Vorgehensweise, eine repräsentierbare Orientierung eines Gammoids zu finden, liefert. Im vierten Kapitel resümieren wir unsere neuen Resultate und geben einen Überblick über neue und alte offene Fragestellungen.





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# Chapter 1

## Preliminaries

In this chapter, we introduce those aspects of matroid theory that are most important to the comprehension of the later chapters. For a thorough introduction to matroid theory, we would like to redirect the reader to the following books, in no particular order.

- *Matroid Theory* by J.G. Oxley [Oxl11] is a comprehensive resource on matroid theory covering most of the current state of the art. Matroids are introduced using a variety of cryptomorphic axiom systems starting from independence axioms and base axioms. This book is the authoritative standard reference for matroid theory and we guarantee that all definitions made in this work are compatible with those found in J.G. Oxley's book.
- *Matroid Theory* by D.J.A. Welsh [Wel76] is an introduction to matroid theory that also covers the greedy algorithm, transversal theory, Menger's Theorem and gammoids, polymatroids, and infinite generalizations of matroids. Although this book is not the most recent one on this topic, it is the book that we would like to recommend to anyone who wants to read *only one* book on matroid theory, as it presents the theory in remarkable clarity.
- *On the Foundations of Combinatorial Theory: Combinatorial Geometries* by H.H. Crapo and G.-C. Rota [CR70] is a remarkably well structured introduction to matroid theory with lattice theory as a starting point. Unfortunately, a regular edition never followed the preliminary edition.

## Notation

All notation used in this work is either standard mathematical notation, or declared in the corresponding definitions. We would like to point out one less common notational detail: If we denote a set  $X = \{a, b, c\}$  we are stating that the set  $X$  consists of the elements  $a$ ,  $b$ , and  $c$ ; but we do not require any two or all three of  $a, b, c$  to be distinct elements. Thus  $|X| = 1$ ,  $|X| = 2$ , and  $|X| = 3$  are possibly true assertions with this notation. But if we denote a set  $Y = \{a, b, c\}_{\neq}$ , then we are stating that  $Y$  consists of the elements  $a$ ,  $b$ , and  $c$ ; and that no two of these elements are equal, therefore  $|Y| = 3$  is the only possibility here.

We will denote the set of non-negative integers by  $\mathbb{N} = \{0, 1, 2, \dots\}$ , the set of integers by  $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ , the field of the rational numbers by  $\mathbb{Q}$ , and the field of the real numbers by  $\mathbb{R}$ . The cardinality of a set  $X$  is denoted by  $|X|$ , the power set of  $X$  is denoted by  $2^X$ . The set of subsets of  $X$  with cardinality  $n$  is denoted by  $\binom{X}{n}$ . The set of all maps  $f: X \rightarrow Y$  is denoted by  $Y^X$ .

If  $f: X \rightarrow Y$  is a map and  $X' \subseteq X$ , then we denote the set of images of  $x' \in X'$  under  $f$  by  $f[X'] = \{f(x') \mid x' \in X'\}$ . We denote the restriction of  $f$  to  $X'$  by  $f|_{X'}$ .

Whenever  $\mathcal{A} \subseteq 2^X$  is a family of sets, we denote the union of all those sets by  $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ . If  $\mathcal{A} \neq \emptyset$ , we denote the intersection of all sets in  $\mathcal{A}$  by  $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$ . For  $\mathcal{A} = \emptyset$ , we set  $\bigcap \mathcal{A} = \bigcap_X \emptyset = X$ .

We use the  $O$ -notation in the usual way: If  $f, g, h: \mathbb{N} \rightarrow \mathbb{R}$  are maps, we write  $f = O(g)$  in order to denote that  $\limsup_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$ . We write  $O(g) = O(h)$  if the implication  $f = O(g) \Rightarrow f = O(h)$  holds for all  $f \in \mathbb{R}^{\mathbb{N}}$ . Please keep in mind that  $O(g) = O(h)$  is not equivalent to  $O(h) = O(g)$ . (!) Instead, the  $O$ -notation is asymmetric and has to be read from left-to-right. We also use the straight-forward generalization of the  $O$ -notation to several non-negative integer variables in an informal way, for instance we would write  $O(x^2 y^3) = O(2^x y^4)$ . Similarly, we write  $f = \Omega(g)$  in order to denote that  $\limsup_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| > 0$ .

### 1.1 Canonical Preliminaries

This section contains canonical definitions, which are most unrelated to matroid theory. The authors know that it is quite uncommon to have a canonical preliminaries section within the preliminaries of a work. We are certain that any person who did study mathematics to some extent knows the contents of this section by heart, yet we include



it in order to maintain a higher level of self-sufficiency of this work as well as to fix certain formal aspects of the common basic definitions.

**Definition 1.1.1.** *Let  $X$  be any set. The **multi-sets over  $X$**  are the elements of the set*

$$\mathbb{N}^X = \{f: X \longrightarrow \mathbb{N}\}.$$

*The **finite multi-sets over  $X$**  are defined to be*

$$\mathbb{N}^{(X)} = \left\{ f \in \mathbb{N}^X \mid |\{x \in X \mid f(x) \neq 0\}| < \infty \right\}. \quad \blacksquare$$

**Notation 1.1.2.** *Let  $X$  be a set,  $\mathbb{K}$  be a field. The **vectors** of the  $X$ -dimensional vector space  $\mathbb{K}^X$  over  $\mathbb{K}$  are identified with the maps  $v: X \longrightarrow \mathbb{K}$ . If  $X$  is finite, then the canonical basis of  $\mathbb{K}^X$  is the set  $\{e_i \mid i \in X\}$  where*

$$e_i: X \longrightarrow \mathbb{K}, \quad x \mapsto \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{otherwise.} \end{cases}$$

*For  $\alpha \in \mathbb{K}$  and  $v \in \mathbb{K}^X$ , we shall denote the scalar multiplication of  $\alpha$  and  $v$  both by  $\alpha \cdot v$  and by*

$$\alpha v: X \longrightarrow \mathbb{K}, \quad x \mapsto \alpha \cdot v(x).$$

*For  $X$  finite and  $\alpha, \beta \in \mathbb{K}^X$  we denote the scalar product of  $\alpha$  and  $\beta$  by*

$$\langle \alpha, \beta \rangle = \sum_{x \in X} \alpha(x) \cdot \beta(x). \quad \blacksquare$$

**Definition 1.1.3.** *Let  $K$ ,  $R$ , and  $C$  be any sets. An  **$R \times C$ -matrix over  $K$**  is a map  $\mu: R \times C \longrightarrow K$ . Every  $r \in R$  is a **row-index of  $\mu$** , and every  $c \in C$  is a **column-index of  $\mu$** . For every  $r \in R$ , the map*

$$\mu_r: C \longrightarrow K, \quad c \mapsto \mu(r, c)$$

*is the  **$r$ -th row of  $\mu$** . Analogously, for every  $c \in C$ , the map*

$$\mu_c^\top: R \longrightarrow K, \quad r \mapsto \mu(r, c)$$

*is the  **$c$ -th column of  $\mu$** . The class of  $R \times C$ -matrices over  $K$  shall be denoted by  $K^{R \times C}$ . If  $R = \{1, 2, \dots, n\} \subseteq \mathbb{N}$  and  $C = \{1, 2, \dots, m\} \subseteq \mathbb{N}$ , then we also write  $K^{n \times m}$*

for  $K^{R \times C}$ . For every matrix  $\mu \in K^{R \times C}$ , we define the **transposed matrix**  $\mu^\top$  to be the map  $\mu^\top: C \times R \rightarrow K, (c, r) \mapsto \mu(r, c)$ . ■

**Definition 1.1.4.** Let  $X$  be any set,  $\mathbb{K}$  be a field or ring with zero and one. The **identity matrix for  $X$  over  $\mathbb{K}$**  is the map

$$\text{id}_{\mathbb{K}}(X): X \times X \rightarrow \mathbb{K}, (r, c) \mapsto \begin{cases} 1 & \text{if } r = c, \\ 0 & \text{otherwise.} \end{cases}$$

■

**Definition 1.1.5.** Let  $X, Y, Z$  be sets,  $Y$  finite,  $\mathbb{R}$  a ring. Let further  $\mu \in \mathbb{R}^{X \times Y}$  and  $\nu \in \mathbb{R}^{Y \times Z}$  be matrices. Then the **matrix multiplication of  $\mu$  with  $\nu$**  shall be the matrix

$$\mu * \nu: X \times Z \rightarrow \mathbb{R}, (x, z) \mapsto \sum_{y \in Y} \mu(x, y) \cdot \nu(y, z).$$

Let  $\alpha \in \mathbb{R}^Y$ . Analogously, the **vector-matrix multiplication of  $\alpha$  with  $\nu$**  shall be the vector

$$\alpha * \nu: Z \rightarrow \mathbb{R}, z \mapsto \sum_{y \in Y} \alpha(y) \cdot \nu(y, z),$$

and the **matrix-vector multiplication of  $\mu$  with  $\alpha$**  shall be

$$\mu * \alpha: X \rightarrow \mathbb{R}, x \mapsto \sum_{y \in Y} \mu(x, y) \cdot \alpha(y). \quad \blacksquare$$

**Definition 1.1.6.** Let  $\mu \in K^{R \times C}$  be an  $R \times C$ -matrix over  $K$ ,  $R_0 \subseteq R$ , and  $C_0 \subseteq C$ . The **restriction of  $\mu$  to  $R_0$**  is defined to be the map

$$\mu|_{R_0}: R_0 \times C \rightarrow K, (r, c) \mapsto \mu(r, c).$$

The **restriction of  $\mu$  to  $R_0 \times C_0$**  is defined to be the map

$$\mu|_{R_0 \times C_0}: R_0 \times C_0 \rightarrow K, (r, c) \mapsto \mu(r, c). \quad \blacksquare$$

**Definition 1.1.7.** Let  $\mathbb{K}$  be a field or a commutative ring,  $X = \{x_1, x_2, \dots, x_m\}_{\neq}$  and  $Y = \{y_1, y_2, \dots, y_m\}_{\neq}$  be finite sets of equal cardinality that have implicit linear orders given by the indexes, and let  $\mu \in \mathbb{K}^{X \times Y}$  be a square matrix over  $\mathbb{K}$ . The **determinant of  $\mu$**  is defined to be

$$\det \mu = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \mu(x_i, y_{\sigma(i)})$$

where  $\mathfrak{S}_m$  consists of all permutations  $\sigma: \{1, 2, \dots, m\} \longrightarrow \{1, 2, \dots, m\}$ . ■

**Definition 1.1.8.** Let  $R$  and  $C$  be finite sets,  $\mu \in \mathbb{K}^{R \times C}$ , and  $n = \min\{|R|, |C|\}$ . The **determinant-indicator of  $\mu$**  is defined to be

$$\operatorname{idet} \mu = \begin{cases} 1 & \text{if } n = 0, \\ 1 & \text{if for some } R_n \in \binom{R}{n}, C_n \in \binom{C}{n}: \det(\mu|_{R_n \times C_n}) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

■

**Notation 1.1.9.** Let  $\mathbb{R}$  be a commutative ring,  $X$  be a set. The **polynomial ring over  $\mathbb{R}$  with variables  $X$**  shall be denoted by  $\mathbb{R}[X]$ . The **unit monomials of  $\mathbb{R}[X]$** , i.e. polynomials of the form  $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$  where  $\{x_1, x_2, \dots, x_k\}_{\neq} \subseteq X$ , may be identified with the finite multi-sets  $\mathbb{N}^{(X)}$  and thus they shall be denoted by  $\mathbb{N}^{(X)}$ , too. It is also customary to identify the polynomial ring  $\mathbb{R}[\emptyset]$  with the ring  $\mathbb{R}$  itself, and to write  $\mathbb{R}[x_1, x_2, \dots, x_k]$  for  $\mathbb{R}[\{x_1, x_2, \dots, x_k\}]$ . Furthermore, for every polynomial  $p \in \mathbb{R}[X]$ ,  $Y \subseteq X$ , and every  $\eta \in \mathbb{R}^Y$ , we obtain a polynomial  $p[Y = \eta] \in \mathbb{R}[X \setminus Y]$  by setting  $y = \eta(y)$  in  $p$  for every  $y \in Y$ . For  $Y = \{x_1, x_2, \dots, x_i\}_{\neq}$ , we also write  $p[x_1 = \eta(x_1), x_2 = \eta(x_2), \dots, x_i = \eta(x_i)]$  in order to denote  $p[Y = \eta]$ . For  $p \in \mathbb{R}[x]$  and  $r \in \mathbb{R}$ , we denote  $p[x = r]$  by  $p(r)$ . ■

**Definition 1.1.10.** Let  $X \subseteq \mathbb{R}$  be a set of reals. Then  $X$  shall be called  **$\mathbb{Z}$ -independent**, if for the injection  $\xi: X \longrightarrow \mathbb{R}$  with  $\xi(x) = x$  and for all  $p \in \mathbb{Z}[X]$  the equivalency

$$p[X = \xi] = 0 \iff p = 0$$

holds. ■

**Lemma 1.1.11.** Let  $n \in \mathbb{N}$ . There is a set  $X = \{x_1, x_2, \dots, x_n\}_{\neq} \subseteq \mathbb{R}$  such that  $X$  is  $\mathbb{Z}$ -independent, where  $\mathbb{R}$  denotes the set of reals.

*Proof.* By induction on  $\mathbb{N}$ . The base case is clear. For the induction step, let  $X' = \{x_1, x_2, \dots, x_{n-1}\}_{\neq} \subseteq \mathbb{R}$  be  $\mathbb{Z}$ -independent. Then for  $x \in \mathbb{R}$ ,  $X' \cup \{x\}$  is not

$\mathbb{Z}$ -independent, if and only if there is a non-zero polynomial  $p \in \mathbb{Z}[x_1, x_2, \dots, x_{n-1}, x]$  such that the polynomial  $p_0$  has the root  $p_0(x) = 0$ , where  $p_0 \in \mathbb{R}[x]$  arises from  $p$  — which can be interpreted as a polynomial over  $\mathbb{R}$  — by setting

$$p_0 = p[x_1 = x_1, x_2 = x_2, \dots, x_{n-1} = x_{n-1}] \in \mathbb{R}[x].$$

In other words,  $p_0$  arises from  $p$  by identification of the monomials  $x' \in X'$  with their natural real value. Since  $X'$  is  $\mathbb{Z}$ -independent, we obtain  $p_0 \neq 0$  unless  $p = 0$ . Thus each polynomial  $p_0$  obtained in this way has only finitely many roots. Furthermore, the set  $\mathbb{Z}[x_1, x_2, \dots, x_{n-1}, x]$  is countable, therefore there are only countably many real numbers  $x \in \mathbb{R}$  such that the set  $X' \cup \{x\}$  is not  $\mathbb{Z}$ -independent. But  $\mathbb{R}$  is uncountably infinite, so there is some  $x \in \mathbb{R} \setminus X'$ , such that  $X' \cup \{x\}$  is  $\mathbb{Z}$ -independent.  $\square$

**Definition 1.1.12** ([Bir67], p.1). *Let  $(P, \leq)$  be a pair, where  $P$  is any set – called the **support set of  $(P, \leq)$**  – and  $\leq$  is a binary relation on  $P$ . Then  $(P, \leq)$  is a **poset**, if the following properties hold for all  $p, q, r \in P$ :*

(i)  $p \leq p$ ;

(ii) if  $p \leq q$  and  $q \leq p$  holds, then  $p = q$ ; and

(iii) if  $p \leq q$  and  $q \leq r$  holds, then  $p \leq r$  holds, too.

If the poset  $(P, \leq)$  is clear from the context, we also denote  $(P, \leq)$  by its support set  $P$ , or by its binary relation symbol  $\leq$ . Furthermore, we shall write  $p < q$  – where we may use an analogue symbol corresponding to the symbol used to denote the binary relation of the poset in question – whenever  $p \leq q$  and  $p \neq q$  holds. A poset  $(P, \leq)$  is called **finite**, if  $P$  is finite. For every poset  $(P, \leq)$  and every  $y \in P$ , the  **$(P, \leq)$ -down-set of  $y$**  shall be the set

$$\downarrow_{(P, \leq)} y = \{x \in P \mid x \leq y\}. \quad \blacksquare$$

**Example 1.1.13.** Let  $X$  be a finite set, and  $P \subseteq 2^X$ . Then  $(P, \subseteq)$  is a poset, where  $\subseteq$  denotes the usual set-inclusion.  $\blacktriangle$

**Definition 1.1.14** ([Bir67], pp.101f). Let  $(P, \leq)$  be a finite poset. The **zeta-matrix of  $(P, \leq)$**  shall be the map

$$\zeta_{(P, \leq)}: P \times P \longrightarrow \mathbb{Z}, (p, q) \mapsto \begin{cases} 1 & \text{if } p \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

If the poset is clear from the context, we shall denote  $\zeta_{(P, \leq)}$  by  $\zeta_P$  or  $\zeta$ . The **Möbius-function of  $(P, \leq)$**  is defined as

$$\mu_{(P, \leq)}: P \times P \longrightarrow \mathbb{Z}, (p, q) \mapsto \begin{cases} 0 & \text{if } p \not\leq q, \\ 1 & \text{if } p = q, \\ - \sum_{q' \in P, p \leq q' < q} \mu(p, q') & \text{otherwise.} \end{cases}$$

Again, if the poset is clear from the context, we shall denote  $\mu_{(P, \leq)}$  by  $\mu_P$  or  $\mu$ . ■

**Lemma 1.1.15** ([Rot64], Proposition 1). Let  $(P, \leq)$  be a finite poset. Then

$$\mu_P * \zeta_P = \text{id}_{\mathbb{Z}(P)}.$$

In other words, the Möbius-function of a poset is the inverse matrix of the zeta-matrix of that poset, and thus all  $\zeta_P$  are invertible in the ring of integer matrices.

*Proof.* Let  $(P, \leq)$  be a finite poset, and let  $\mu = \mu_P$  and  $\zeta = \zeta_P$  be defined as in Definition 1.1.14. Let  $p, r \in P$ , then we have

$$(\mu * \zeta)(p, r) = \sum_{q \in P} \mu(p, q) \cdot \zeta(q, r) = \sum_{q \in P, p \leq q \leq r} \mu(p, q) \cdot \zeta(q, r)$$

because if  $p \not\leq q$ , then  $\mu(p, q) = 0$ , and if  $q \not\leq r$ , then  $\zeta(q, r) = 0$ . Therefore we obtain that for all  $p \in P$ ,

$$(\mu * \zeta)(p, p) = \sum_{q \in P, p \leq q \leq p} \mu(p, q) \cdot \zeta(q, p) = \mu(p, p) \cdot \zeta(p, p) = 1 \cdot 1 = 1 = \text{id}_{\mathbb{Z}(P)}(p, p).$$

Now let  $p, r \in P$  with  $p \neq r$ . Since  $\zeta(p, q) = 1$  whenever  $p \leq q$ , we have

$$\begin{aligned}
 \sum_{q \in P, p \leq q \leq r} \mu(p, q) \cdot \zeta(q, r) &= \left( \sum_{q \in P, p \leq q < r} \mu(p, q) \cdot \zeta(q, r) \right) + \mu(p, r) \cdot \zeta(r, r) \\
 &= \left( \sum_{q \in P, p \leq q < r} \mu(p, q) \right) + \mu(p, r) \\
 &= \left( \sum_{q \in P, p \leq q < r} \mu(p, q) \right) - \left( \sum_{q \in P, p \leq q < r} \mu(p, q) \right) \\
 &= 0 = \text{id}_{\mathbb{Z}}(P)(p, r).
 \end{aligned}$$

Therefore  $\mu * \zeta = \text{id}_{\mathbb{Z}}(P)$ . □

**Lemma 1.1.16** (Principle of Inclusion-Exclusion, [Rot64]). *Let  $X$  be a finite set. Then for all  $A, B \subseteq X$*

$$\mu_{(2^X, \subseteq)}(A, B) = \begin{cases} (-1)^{|B|-|A|} & \text{if } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases}$$

## 1.2 Matroid Basics

In this section, we give a quick and incomplete review of some axiomatizations of matroids. A more complete picture as well as some proofs<sup>1</sup> of cryptomorphy can be obtained from J.G. Oxley's book [Oxl11].

### 1.2.1 Independence Axioms

All definitions, lemmas, theorems, and proofs in this subsection are canonical and can be found in [Oxl11]. Readers familiar with matroid theory may safely skip this section.

**Definition 1.2.1.** Let  $E$  be a finite set,  $\mathcal{I} \subseteq 2^E$ . Then the pair  $(E, \mathcal{I})$  is an **independence matroid**, or shorter **matroid**, if the following properties hold:

(I1)  $\emptyset \in \mathcal{I}$ ,

(I2) for  $I \in \mathcal{I}$  and every  $J \subseteq I$ , we have  $J \in \mathcal{I}$ .

(I3) If  $J, I \in \mathcal{I}$  and  $|J| < |I|$ , then there is some  $i \in I \setminus J$ , such that  $J \cup \{i\} \in \mathcal{I}$ .

Let  $X \subseteq E$ , we say that  $X$  is **independent** in the matroid  $M = (E, \mathcal{I})$ , if  $X \in \mathcal{I}$ . Otherwise, we say that  $X$  is **dependent** in  $M$ . ■

**Example 1.2.2.** Let  $E$  be any finite set, then the **free matroid** on the ground set  $E$  shall be the matroid  $M = (E, \mathcal{I})$  where all subsets of  $E$  are independent, i.e. where  $\mathcal{I} = 2^E$ . ▲

Matroids have the natural concept of isomorphism.

**Definition 1.2.3.** Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids. A bijective map

$$\varphi: E \longrightarrow E'$$

is called **matroid isomorphism between  $M$  and  $N$** , if for all  $X \subseteq E$

$$X \in \mathcal{I} \iff \varphi[X] \in \mathcal{I}'$$

holds. As usual, an  **$M$ -automorphism** is a matroid isomorphism between  $M$  and itself. ■

<sup>1</sup>Some axiomatizations can be found in the exercise sections, where, of course, the proofs are left for the reader.

For now, we will stick to the independence axioms of matroids and define the typical matroid concepts in terms of their independence systems.

**Definition 1.2.4.** Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids such that  $E \cap E' = \emptyset$ . Then the **direct sum of  $M$  and  $N$**  is the matroid  $M \oplus N = (E \cup E', \mathcal{I}_\oplus)$  where

$$\mathcal{I}_\oplus = \{X \cup X' \mid X \in \mathcal{I}, X' \in \mathcal{I}'\}. \quad \blacksquare$$

**Lemma 1.2.5.** Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids such that  $E \cap E' = \emptyset$ . Then  $M \oplus N$  is indeed a matroid.

*Proof.* Each matroid axiom may be easily deduced from the fact that every summand satisfies that axiom:  $\emptyset \in \mathcal{I}_\oplus$  since  $\emptyset \in \mathcal{I}$  and  $\emptyset \in \mathcal{I}'$ , (I1) holds. Let  $X \cup X' \in \mathcal{I}_\oplus$  for some  $X \in \mathcal{I}$  and  $X' \in \mathcal{I}'$ . Let  $Y \subseteq X \cup X'$ , then  $Y = (Y \cap X) \cup (Y \cap X')$ , and since  $(Y \cap X) \subseteq X$  and  $(Y \cap X') \subseteq X'$ , we have  $(Y \cap X) \in \mathcal{I}$  and  $(Y \cap X') \in \mathcal{I}'$ , therefore  $Y \in \mathcal{I}_\oplus$ , (I2) holds. Let  $X \cup X' \in \mathcal{I}_\oplus$  and  $Y \cup Y' \in \mathcal{I}_\oplus$  with  $|X \cup X'| < |Y \cup Y'|$ , i.e.  $X, Y \in \mathcal{I}$  and  $X', Y' \in \mathcal{I}'$ , and  $|X| + |X'| < |Y| + |Y'|$ . By symmetry we may assume without loss of generality that  $|X| < |Y|$ . Then there is some  $y \in Y \setminus X$  such that  $X \cup \{y\} \in \mathcal{I}$ , therefore  $X \cup \{y\} \cup X' \in \mathcal{I}_\oplus$ , thus (I3) holds.  $\square$

**Definition 1.2.6.** Let  $M = (E, \mathcal{I})$  be a matroid. Every maximal element of  $\mathcal{I}$  is called a **base** of  $M$ . For  $F \subseteq E$ , every maximal element of  $\{I \in \mathcal{I} \mid I \subseteq F\}$  is called a **base of  $F$**  in  $M$ . The family of all bases of  $M$  shall be denoted by  $\mathcal{B}(M)$ , and the family of all bases of  $F$  in  $M$  shall be denoted by  $\mathcal{B}_M(F)$ .  $\blacksquare$

It is an important property of matroids, that for every  $F \subseteq E$ , the bases of  $F$  have the same cardinality; and that every independent subset of  $F$  can be augmented to a base of  $F$ . Likewise, any set independent in a matroid  $M$  can be augmented to a base of  $M$ .

**Lemma 1.2.7.** Let  $M = (E, \mathcal{I})$  be a matroid, and let  $F \subseteq H \subseteq E$  with  $F \in \mathcal{I}$ . Then there is a subset  $G \in \mathcal{I}$  with  $F \subseteq G \subseteq H$ , such that  $|G| = \max\{|I| \mid I \in \mathcal{I}, I \subseteq H\}$ .

*Proof.* Let  $\mathcal{I}' = \{I \in \mathcal{I} \mid F \subseteq I \subseteq H\}$ . Clearly,  $F \in \mathcal{I}'$  and  $\mathcal{I}'$  is finite, therefore there is an element  $G \in \mathcal{I}'$  which is maximal with respect to set-inclusion  $\subseteq$ . Now assume that  $|G| < |I|$  for some  $I \in \mathcal{I}$  with  $I \subseteq H$ . By (I3) there is an element  $i \in I \setminus G$  such that  $G \cup \{i\} \in \mathcal{I}$ . But  $i \in I \subseteq H$ , therefore  $G \cup \{i\} \in \mathcal{I}'$ , which contradicts the choice of  $G$  as  $\subseteq$ -maximal element of  $\mathcal{I}'$ . Thus  $|G| = \max\{|I| \mid I \in \mathcal{I}, I \subseteq H\}$ .  $\square$



**Corollary 1.2.8.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $H \subseteq E$ . Let  $F, G$  be maximal elements in  $\{X \in \mathcal{I} \mid X \subseteq H\}$  with respect to set-inclusion. Then  $|F| = |G|$ .*

*Proof.* If, without loss of generality,  $|F| < |G|$ , then  $F$  cannot be maximal with respect to set-inclusion, because then Lemma 1.2.7 gives a proper independent superset of  $F$  in  $H$ .  $\square$

**Corollary 1.2.9.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $F \subseteq E$  and  $B_1, B_2 \subseteq F$  be bases of  $F$  in  $M$ . Then the following property is satisfied:*

(B3') *For every element  $x \in B_1 \setminus B_2$  there is an element  $y \in B_2 \setminus B_1$ , such that  $(B_1 \setminus \{x\}) \cup \{y\}$  is a base of  $F$  in  $M$ .*

*Proof.* Since  $|B_1| = |(B_1 \setminus \{x\}) \cup \{y\}|$  for any  $x \in B_1 \setminus B_2$  and  $y \in B_2 \setminus B_1$ , it suffices to show, that for each such  $x$ , there is a corresponding  $y$  with  $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{I}$ . We give an indirect argument. Assume that for  $x \in B_1 \setminus B_2$ , there is no  $y \in B_2 \setminus B_1$  with  $(B_1 \setminus \{x\}) \cup \{y\}$  independent in  $M$ . Then  $B_1 \setminus \{x\}$  is a base of  $B' = (B_1 \setminus \{x\}) \cup (B_2 \setminus B_1)$ . Clearly,  $B' = (B_1 \cup B_2) \setminus \{x\}$ , but  $x \notin B_2$ , therefore  $B_2 \subseteq B'$ . Now  $B_2 \in \mathcal{I}$  together with  $|B_2| > |B_1 \setminus \{x\}|$  contradicts that  $B_1 \setminus \{x\}$  is a base of  $B'$ . Therefore, there is some  $y \in B_2 \setminus B_1$  such that  $(B_1 \setminus \{x\}) \cup \{y\}$  is a base of  $F$  in  $M$ .  $\square$

**Lemma 1.2.10.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $F \subseteq E$  and  $B_1, B_2 \subseteq F$  be bases of  $F$  in  $M$ . For every element  $y \in B_2 \setminus B_1$  there is an element  $x \in B_1 \setminus B_2$ , such that  $(B_1 \setminus \{x\}) \cup \{y\}$  is a base of  $F$  in  $M$ .*

D.J.A. Welsh gives the following nice and short proof of this lemma in [Wel76].

*Proof.* Let  $y \in B_2 \setminus B_1$ , thus  $\{y\} \in \mathcal{I}$ . From Lemma 1.2.7 we obtain that there is a basis  $B'$  of  $F' = B_1 \cup \{y\}$  with  $\{y\} \subseteq B'$ . Since  $B_1$  is a base of  $F$  and a proper subset of  $F' \subseteq F$ ,  $F'$  is dependent. Thus  $B'$  is a proper subset of  $F'$  and therefore there is an element  $x \in B_1 \setminus B'$ . Since  $B_1$  and  $B'$  are bases of  $F' = B_1 \cup \{y\} = B' \cup \{x\}$  in  $M$ , and  $B_1$  and  $B_2$  are bases of  $F$  in  $M$ , we have  $|B'| = |B_1| = |B_2|$ , so  $B' = (B_1 \setminus \{x\}) \cup \{y\}$  is a base of  $F$  in  $M$ , too.  $\square$

**Definition 1.2.11.** *Let  $M = (E, \mathcal{I})$  be a matroid. A set  $C \subseteq E$  is called **circuit** of  $M$ , if  $C$  is dependent, yet any proper subset of  $C$  is independent in  $M$ . The set of circuits of  $M$  is denoted by*

$$\mathcal{C}(M) = \{C \subseteq E \mid C \notin \mathcal{I}, \forall c \in C: C \setminus \{c\} \in \mathcal{I}\}. \quad \blacksquare$$

Obviously, we may restore  $\mathcal{I}$  from  $\mathcal{C}(M)$  since the independent sets of  $M$  are those subsets of  $E$ , which do not contain a circuit. The following property of  $\mathcal{C}(M)$  is called **strong circuit elimination** and also plays a role in axiomatizing matroids using axioms governing its family of circuits.

**Lemma 1.2.12** ([Oxl11], Proposition 1.4.12). *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C_1, C_2 \in \mathcal{C}(M)$  be circuits of  $M$ . Furthermore, let  $e \in C_1 \cap C_2$  and  $f \in C_1 \setminus C_2$ . Then there is a circuit  $C' \in \mathcal{C}(M)$  such that  $f \in C'$  and  $C' \subseteq (C_1 \cup C_2) \setminus \{e\}$ .*

For a proof, see [Oxl11], p.29.

**Definition 1.2.13.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $l \in E$ . Then  $l$  is called a **loop in  $M$** , if the singleton  $\{l\}$  is a circuit of  $M$ . Let  $p_1, p_2 \in E$  such that  $p_1 \neq p_2$ . Then  $p_1$  and  $p_2$  are called **parallel edges in  $M$** , if  $\{p_1, p_2\}$  is a circuit of  $M$ . Let  $c \in E$  such that for all bases  $B$  of  $M$ ,  $c \in B$ . Then  $c$  is called a **coloop in  $M$** . ■*

**Definition 1.2.14.** *Let  $M = (E, \mathcal{I})$  be a matroid. The **rank function** of  $M$  shall be the map*

$$\text{rk}_M: 2^E \longrightarrow \mathbb{N}, X \mapsto \max \{ |Y| \mid Y \subseteq X, Y \in \mathcal{I} \}.$$

*If the matroid  $M$  is clear from the context, we denote  $\text{rk}_M$  by  $\text{rk}$ . ■*

Again,  $\mathcal{I}$  may be retrieved from  $\text{rk}_M$  since the independent sets are precisely those elements of the domain  $2^E$  of  $\text{rk}_M$ , for which the cardinality and the image under the rank function coincide.

**Lemma 1.2.15.** *Let  $M = (E, \mathcal{I})$  be a matroid, and  $X \subseteq Y \subseteq E$ . Then  $\text{rk}(X) \leq \text{rk}(Y)$ .*

*Proof.* Since  $\{I \in \mathcal{I} \mid I \subseteq X\} \subseteq \{I \in \mathcal{I} \mid I \subseteq Y\}$  the maximum expression for  $\text{rk}(Y)$  ranges over a superset of the expression for  $\text{rk}(X)$  and therefore cannot be smaller. □

**Definition 1.2.16.** *Let  $M = (E, \mathcal{I})$  be a matroid. A set  $F \subseteq E$  is called **flat** of  $M$ , if for all  $x \in E \setminus F$ , the equality  $\text{rk}(F \cup \{x\}) = \text{rk}(F) + 1$  holds. The family of all flats of  $M$  is denoted by*

$$\mathcal{F}(M) = \{X \subseteq E \mid \forall y \in E \setminus X: \text{rk}_M(X) < \text{rk}_M(X \cup \{y\})\}.$$

*The **closure operator** of  $M$  is defined to be the map*

$$\text{cl}_M: 2^E \longrightarrow 2^E, X \mapsto \bigcap \{F \in \mathcal{F}(M) \mid X \subseteq F\}.$$

*If the matroid  $M$  is clear from the context, we denote  $\text{cl}_M$  by  $\text{cl}$ . ■*

Clearly, for every matroid  $M = (E, \mathcal{I})$ , the ground set  $E \in \mathcal{F}(M)$  is a flat, and therefore the defining expression of  $\text{cl}(X)$  is well-defined, as it is never an intersection of an empty family. The following properties are easy consequences from the definition of the closure operator.

**Lemma 1.2.17.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq Y \subseteq E$ . Then  $X \subseteq \text{cl}(X) \subseteq \text{cl}(Y)$ .*

*Proof.* Since  $\emptyset \neq \{F \in \mathcal{F}(M) \mid Y \subseteq F\} \subseteq \{F \in \mathcal{F}(M) \mid X \subseteq F\}$ , we have

$$X \subseteq \text{cl}(X) = \bigcap \{F \in \mathcal{F}(M) \mid X \subseteq F\} \subseteq \bigcap \{F \in \mathcal{F}(M) \mid Y \subseteq F\} = \text{cl}(Y). \quad \square$$

**Lemma 1.2.18.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$ . Then  $\text{rk}(X) = \text{rk}(\text{cl}(X))$ .*

*Proof.* By Lemma 1.2.17 we have  $X \subseteq \text{cl}(X)$  and by Lemma 1.2.15 we obtain that  $\text{rk}(X) \leq \text{rk}(\text{cl}(X))$ . Now consider the family  $\mathcal{E} = \{Y \subseteq E \mid X \subseteq Y \text{ and } \text{rk}(X) = \text{rk}(Y)\}$ . Since  $X \in \mathcal{E}$  and  $E$  is finite, there is a maximal element  $F \in \mathcal{E}$  with respect to set-inclusion. Since  $F$  is maximal, we have that  $F \in \mathcal{F}(M)$ . Thus  $\text{cl}(X) \subseteq F$  and so  $\text{rk}(\text{cl}(X)) \leq \text{rk}(F) = \text{rk}(X)$  holds, and consequently  $\text{rk}(X) = \text{rk}(\text{cl}(X))$ .  $\square$

**Lemma 1.2.19.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$ . Then for every  $\mathcal{F}' \subseteq \mathcal{F}(M)$ ,  $\bigcap_E \mathcal{F}' \in \mathcal{F}(M)$ . Furthermore, for all  $X \subseteq E$*

$$\text{cl}(X) \in \mathcal{F}(M) \quad \text{and} \quad \text{cl}(\text{cl}(X)) = \text{cl}(X).$$

*Proof.* Let  $\mathcal{F}' \subseteq \mathcal{F}(M)$ , and let  $F' = \bigcap_E \mathcal{F}' = \{x \in E \mid \forall F \in \mathcal{F}': x \in F\}$ . Let  $e \in E \setminus F'$ , then there is some  $F \in \mathcal{F}'$  with  $e \notin F$ . Since  $\text{rk}(F \cup \{e\}) > \text{rk}(F)$  holds, for every base  $B$  of  $F$ , we must have  $B \cup \{e\} \in \mathcal{I}$ . Now let  $B' \subseteq F'$  be a base of  $F'$ , then by Lemma 1.2.7, there is a base  $B$  of  $F$  with  $B' \subseteq B$ . Since  $B' \cup \{e\} \subseteq B \cup \{e\}$ , we obtain that  $\text{rk}(F' \cup \{e\}) \geq |B' \cup \{e\}| > |B'| = \text{rk}(F')$ . Thus  $F' \in \mathcal{F}(M)$ .

Let  $X \subseteq E$ , since the closure operator  $\text{cl}$  is defined to be the intersection of a family of flats of  $M$ , we have  $\text{cl}(X) \in \mathcal{F}(M)$ . Therefore  $\text{cl}(X)$  is the unique minimal element of  $\{F \in \mathcal{F}(M) \mid X \subseteq F\}$  with respect to set-inclusion  $\subseteq$ . Thus we have the following equality between subfamilies of  $\mathcal{F}(M)$

$$\{F \in \mathcal{F}(M) \mid X \subseteq F\} = \{F \in \mathcal{F}(M) \mid \text{cl}(X) \subseteq F\},$$

which yields  $\text{cl}(\text{cl}(X)) = \text{cl}(X)$ .  $\square$

**Lemma 1.2.20.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq Y \subseteq E$ . Then  $\text{cl}(X) = \text{cl}(Y)$  if and only if there is a base  $B$  of  $Y$  with  $B \subseteq X$ .*

*Proof.* Assume that  $\text{cl}(X) = \text{cl}(Y)$ , then  $\text{rk}(X) = \text{rk}(\text{cl}(X)) = \text{rk}(\text{cl}(Y)) = \text{rk}(Y)$  by Lemma 1.2.18. Let  $B$  be a base of  $X$ , then  $\text{rk}(B) = \text{rk}(Y)$ , so  $B \subseteq X \subseteq Y$  is also a base of  $Y$ . Now assume that  $\text{cl}(X) \neq \text{cl}(Y)$ , thus there is some  $y \in \text{cl}(Y) \setminus \text{cl}(X)$  such that for some base  $B$  of  $\text{cl}(X)$  in  $M$ ,  $B \cup \{y\} \in \mathcal{I}$  is independent. Thus  $\text{rk}(Y) = \text{rk}(\text{cl}(Y)) > \text{rk}(X)$  and therefore no base  $B'$  of  $Y$  is a subset of  $X$ .  $\square$

## 1.2.2 Rank Axioms

There are at least two natural ways to axiomatize matroids through their corresponding rank functions.

**Theorem 1.2.21.** *Let  $E$  be a finite set,  $\rho: 2^E \rightarrow \mathbb{N}$  a map. The following are equivalent:*

- (i) *There is a matroid  $M = (E, \mathcal{I})$  with  $\text{rk}_M = \rho$ ,*
- (ii)  *$\rho$  satisfies the properties (R1') – (R3'), and*
- (iii)  *$\rho$  satisfies the properties (R1) – (R3);*

where

$$(R1') \quad \rho(\emptyset) = 0,$$

$$(R2') \quad \rho(X) \leq \rho(X \cup \{y\}) \leq \rho(X) + 1 \text{ for all } X \subseteq E \text{ and all } y \in E,$$

$$(R3') \quad \text{if } \rho(X) = \rho(X \cup \{y\}) = \rho(X \cup \{z\}), \text{ then } \rho(X) = \rho(X \cup \{y, z\}), \text{ for all } X \subseteq E \text{ and all } y, z \in E;$$

$$(R1) \quad 0 \leq \rho(X) \leq |X| \text{ for all } X \subseteq E,$$

$$(R2) \quad \text{if } X \subseteq Y, \text{ then } \rho(X) \leq \rho(Y) \text{ for all } X, Y \subseteq E,$$

$$(R3) \quad \rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y) \text{ for all } X, Y \subseteq E.$$

We named the rank axioms coherent with J.G. Oxley's book [Oxl11]; D.J.A. Welsh's *Matroid Theory* [Wel76] denotes (R1)–(R3) with (R1')–(R3'), and vice-versa, yet the proof is more along the lines of section 1.6 in D.J.A. Welsh's book [Wel76].

*Proof.* The implication (i)  $\Rightarrow$  (ii).

- By (I1) we obtain  $\text{rk}(\emptyset) = |\emptyset| = 0$ , thus (R1') holds for  $\text{rk}$ .
- Let  $X' \in \mathcal{I}$  with  $X' \subseteq X \cup \{y\}$  such that  $\text{rk}(X \cup \{y\}) = |X'|$ . By (I2)  $X' \setminus \{y\} \in \mathcal{I}$ , therefore  $\text{rk}(X \cup \{y\}) \leq \text{rk}(X) + 1$ . Since every subset of  $X$  is a subset of  $X \cup \{y\}$ , too, we obtain (R2') for  $\text{rk}$ :  $\text{rk}(X) \leq \text{rk}(X \cup \{y\}) \leq \text{rk}(X) + 1$ .
- We prove (R3') via contraposition and show that  $\rho(X) \neq \rho(X \cup \{x, y\})$  implies that  $\rho(X) \neq \rho(X \cup \{x\})$  or  $\rho(X) \neq \rho(X \cup \{y\})$ . We may assume the non-trivial case  $y, z \notin X$ . If  $\text{rk}(X \cup \{y, z\}) > \text{rk}(X)$ , then every  $X' \subseteq X \cup \{y, z\}$ , which has maximal cardinality such that  $X' \in \mathcal{I}$ , must have a non-empty intersection  $X' \cap \{y, z\} \neq \emptyset$ , because  $X' \not\subseteq X$ . Without loss of generality we may assume that  $y \in X'$ . If  $y = z$  or  $z \notin X'$  or  $\text{rk}(X) = \text{rk}(X \cup \{y, z\}) - 2$ , we obtain that  $\text{rk}(X \cup \{y\}) = \text{rk}(X) + 1$ . The remaining case is that  $\{y, z\} \not\subseteq X'$  and  $\text{rk}(X) = |X'| - 1$ . Let  $\tilde{X} \subseteq X$  be a subset with maximal cardinality such that it is still independent, i.e.  $\tilde{X} \in \mathcal{I}$ . Since  $X' \setminus \{y, z\} \in \mathcal{I}$ , (I3) yields that there is an  $x \in \tilde{X} \setminus X'$  such that  $(X' \setminus \{y, z\}) \cup \{x\} \in \mathcal{I}$ . Applying (I3) again yields that either  $(X' \setminus \{y\}) \cup \{x\} \in \mathcal{I}$  or  $(X' \setminus \{z\}) \cup \{x\} \in \mathcal{I}$ , therefore either  $\text{rk}(X) < \text{rk}(X \cup \{y\})$  or  $\text{rk}(X) < \text{rk}(X \cup \{z\})$ . This establishes (R3').

The implication (ii)  $\Rightarrow$  (iii):

- We show (R1) by induction on  $|X|$ . From (R1') we obtain  $0 \leq \rho(\emptyset) = 0 \leq |\emptyset|$ . Now, let  $X \subseteq E$  and  $x \in X$ . By induction hypothesis, we have  $0 \leq \rho(X \setminus \{x\}) \leq |X \setminus \{x\}| = |X| - 1$ . (R2') yields  $\rho(X \setminus \{x\}) \leq \rho(X) \leq \rho(X \setminus \{x\}) + 1$ , which combines with the previous inequality to the desired  $0 \leq \rho(X \setminus \{x\}) \leq \rho(X) \leq (|X| - 1) + 1 = |X|$ .
- In order to show (R2) it suffices to consider  $X \subseteq Y \subseteq E$ . We prove  $\rho(X) \leq \rho(Y)$  by induction on  $|Y \setminus X|$ . The base case implies  $X = Y$  thus  $\rho(X) \leq \rho(Y)$  holds trivially. Now let  $y \in Y \setminus X$ . By induction hypothesis,  $\rho(X) \leq \rho(Y \setminus \{y\})$  holds. From (R2') we obtain  $\rho(Y \setminus \{y\}) \leq \rho(Y)$ , and thus  $\rho(X) \leq \rho(Y \setminus \{y\}) \leq \rho(Y)$  holds.
- We prove that the following auxiliary property ...

(R2'') If  $\rho(X \cup \{y\}) = \rho(X) + 1$  and  $X' \subseteq X$ , then  $\rho(X' \cup \{y\}) = \rho(X') + 1$ ;  
for all  $X \subseteq E$ ,  $y \in E$ .

... follows from (ii) by induction on  $|X \setminus X'|$ . The base case  $X = X'$  is trivial. For the induction step, let  $x \in X \setminus X'$ , and assume that the implication is not vacuously true. By induction hypothesis  $\rho(X' \cup \{x, y\}) = \rho(X' \cup \{x\}) + 1$ . Using (R2') we obtain the

inequalities  $\rho(X') \leq \rho(X' \cup \{x\}) \leq \rho(X') + 1$ , similarly  $\rho(X') \leq \rho(X' \cup \{y\}) \leq \rho(X') + 1$ , and furthermore  $\rho(X' \cup \{y\}) \leq \rho(X' \cup \{x, y\}) \leq \rho(X' \cup \{y\}) + 1$ . We establish  $(R2'')$  by the following case analysis:

- (a)  $\rho(X' \cup \{x\}) = \rho(X') + 1$ , by induction hypothesis  $\rho(X' \cup \{x, y\}) = \rho(X') + 2$  and as a consequence of the last inequality  $\rho(X' \cup \{y\}) = \rho(X') + 1$ .
- (b)  $\rho(X' \cup \{x\}) = \rho(X')$ . If we assume that  $\rho(X' \cup \{y\}) = \rho(X')$ , we could use  $(R3')$  in order to deduce  $\rho(X' \cup \{x, y\}) = \rho(X')$ , which would contradict the induction hypothesis. Therefore,  $\rho(X' \cup \{y\}) = \rho(X') + 1$ .

— In order to show that  $(R3)$  holds for all  $X, Y \subseteq E$ , we may use an inductive argument over  $(|X \setminus Y|, |Y \setminus X|)$  with respect to the well-founded natural coordinate-wise partial order. The base case  $|X \setminus Y| = 0 = |Y \setminus X|$  implies that  $X = Y$  and therefore  $\rho(X \cap Y) + \rho(X \cup Y) = 2\rho(X) = \rho(X) + \rho(Y)$  holds. Due to the commutativity of the operations  $\cap$ ,  $\cup$ , and  $+$ , it suffices to proof the induction step from  $(X \setminus \{x\}, Y)$  to  $(X, Y)$  for  $x \in X \setminus Y$ , as the step from  $(X, Y \setminus \{y\})$  to  $(X, Y)$  for  $y \in Y \setminus X$  follows symmetrically. By induction hypothesis, we may assume that  $\rho((X \setminus \{x\}) \cup Y) + \rho((X \setminus \{x\}) \cap Y) \leq \rho(X \setminus \{x\}) + \rho(Y)$  holds. Since  $x \in X \setminus Y$ , we see that  $x \notin Y$  and thus  $(X \setminus \{x\}) \cap Y = X \cap Y$  as well as  $(X \setminus \{x\}) \cup Y = (X \cup Y) \setminus \{x\}$ , so we may write the induction hypothesis as  $\rho((X \cup Y) \setminus \{x\}) + \rho(X \cap Y) \leq \rho(X \setminus \{x\}) + \rho(Y)$ . Property  $(R2')$  implies that  $\rho(X \cup Y) = \rho((X \cup Y) \setminus \{x\}) + \alpha$  and  $\rho(X) = \rho(X \setminus \{x\}) + \beta$  for some  $\alpha, \beta \in \{0, 1\}$ . The desired inequality  $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$  follows from the fact that  $\alpha \leq \beta$ , which is a consequence of  $(R2'')$  where  $X \setminus \{x\}$  takes the role of  $X'$ ,  $(X \setminus \{x\}) \cup Y$  takes the role of  $X$  and  $x$  takes the role of  $y$ .

The implication  $(iii) \Rightarrow (i)$ :

— First, we prove that  $(iii)$  implies property  $(R2')$  that  $\rho$  is unit-increasing, let  $X \subseteq E$  and  $y \in E$ . If  $y \in X$  the property holds trivially, let  $y \notin X$ . The first inequality  $\rho(X) \leq \rho(X \cup \{y\})$  holds due to  $(R2)$ . With  $(R3)$  we obtain  $\rho(X \cup \{y\}) + \rho(X \cap \{y\}) \leq \rho(X) + \rho(\{y\})$ , and since  $X \cap \{y\} = \emptyset$  we may use  $(R1)$  twice to obtain  $\rho(\{y\}) \leq 1$  and  $\rho(\emptyset) = 0$ , from which we may infer the second inequality of  $(R2')$ , namely  $\rho(X \cup \{y\}) \leq \rho(X) + 1$ .

— We prove that  $(iii)$  implies property

$$(R4) \quad (\forall y \in Y: \rho(X \cup \{y\}) = \rho(X)) \Rightarrow \rho(X \cup Y) = \rho(X) \text{ for all } X, Y \subseteq E.$$

By induction on  $|Y \setminus X|$ . The base cases  $|Y \setminus X| \in \{0, 1\}$  are trivial. Now let  $v, w \in Y \setminus X$ . By induction hypothesis,  $\rho(X) = \rho(X \cup Y \setminus \{v\}) = \rho(X \cup Y \setminus \{w\}) = \rho(X \cup Y \setminus \{v, w\})$ . Using (R3) we obtain  $\rho(X \cup Y \setminus \{v, w\}) + \rho(X \cup Y) \leq \rho(X \cup Y \setminus \{v\}) + \rho(X \cup Y \setminus \{w\})$ . Together with the induction hypothesis we get  $\rho(X \cup Y) \leq \rho(X)$  and the property (R2) that  $\rho$  is isotone yields  $\rho(X \cup Y) = \rho(X)$ .

— Next, we prove that (iii) also implies the following property:

(R5) For every  $X \subseteq E$  there is a subset  $X' \subseteq X$ , such that  $|X'| = \rho(X') = \rho(X)$ .

By induction on  $|X|$ . The base case  $\rho(\emptyset) = 0 = |\emptyset|$  is clear. Now let  $x \in X$  and by induction hypothesis, there is a subset  $X' \subseteq X \setminus \{x\}$  such that  $|X'| = \rho(X') = \rho(X \setminus \{x\})$ . From (R2') we conclude that  $\rho(X) = \rho(X \setminus \{x\}) + \alpha$  for some  $\alpha \in \{0, 1\}$ . The case  $\alpha = 0$  is trivial. For the case  $\alpha = 1$  we give an indirect argument: Assume that  $\rho(X' \cup \{x\}) = \rho(X') = \rho(X \setminus \{x\})$ . Then  $\rho(X) = \rho(X \setminus \{x\})$  follows from (R4), because for every  $y \in X \setminus X'$  we have  $\rho(X' \cup \{y\}) = \rho(X)$ . Yet, this is a contradiction to  $\rho(X) = \rho(X \setminus \{x\}) + 1$ , therefore  $\rho(X' \cup \{x\}) = \rho(X') + 1$  follows from (R2'), thus  $|X' \cup \{x\}| = \rho(X' \cup \{x\}) = \rho(X)$ .

— From  $\rho$ , we define the set system  $\mathcal{I} = \{X \subseteq E \mid \rho(X) = |X|\}$ . For now, let us assume that  $M = (E, \mathcal{I})$  is indeed a matroid. An immediate consequence of property (R5) is that  $\rho(X) \leq \text{rk}_M(X)$  for all  $X \subseteq E$ . By definition of  $\text{rk}_M$ , there is a subset  $X' \subseteq X$  such that  $\text{rk}_M(X) = |X'| = \rho(X') \leq \rho(X)$  due to (R2). Thus  $\rho = \text{rk}_M$ .

— By (R1) we have  $\rho(\emptyset) = 0 = |\emptyset|$ , thus  $\emptyset \in \mathcal{I}$ , so (I1) holds.

— Let  $X \in \mathcal{I}$ . We show that  $X' \in \mathcal{I}$  for all  $X' \subseteq X$  by induction on  $|X \setminus X'|$ . The base case  $X' = X$  is trivial. Now let  $x \in X \setminus X'$ . By induction hypothesis,  $X' \cup \{x\} \in \mathcal{I}$ , therefore  $\rho(X' \cup \{x\}) = |X'| + 1$ . From (R1) we get the inequality  $\rho(X') \leq |X'|$ , and from (R2') we get the inequality  $\rho(X' \cup \{x\}) \leq \rho(X') + 1$ . Thus  $\rho(X') = |X'|$  follows, consequently  $X' \in \mathcal{I}$ , so (I2) holds.

— We give an indirect argument for (I3). Let  $X, Y \in \mathcal{I}$  with  $|X| < |Y|$ , and assume that for all  $y \in Y$ ,  $X \cup \{y\} \notin \mathcal{I}$ . Since  $|X| = \rho(X)$  and by (R2)  $\rho$  is isotone, we can infer that  $\rho(X \cup \{y\}) = \rho(X)$  for all  $y \in Y$ . With (R4) we see that  $\rho(X \cup Y) = \rho(X)$ , and together with (R2) we obtain  $\rho(Y) \leq \rho(X \cup Y) = \rho(X) = |X| < |Y|$ , a contradiction to  $Y \in \mathcal{I}$ . We may now conclude that  $M = (E, \mathcal{I})$  is a matroid.  $\square$



### 1.2.3 Matroids Induced From Submodular Functions

**Definition 1.2.22.** Let  $E$  be any set,  $R \subseteq \mathbb{R}$ , and let  $f: 2^E \rightarrow R$ . We call the map  $f$  **non-decreasing**, if for every  $X \subseteq Y \subseteq E$ , the inequality  $f(X) \leq f(Y)$  holds. We call  $f$  **submodular**, if for all  $X, Y \subseteq E$  the inequality  $f(X \cap Y) + f(X \cup Y) \leq f(X) + f(Y)$  holds.  $\blacksquare$

**Example 1.2.23.** Let  $M = (E, \mathcal{I})$  be a matroid. Then  $\text{rk}_M: 2^E \rightarrow \mathbb{N}$  is a non-decreasing and submodular function.  $\blacktriangle$

The following theorem is the independent-sets version of Proposition 11.1.1 in [Oxl11], which is attributed to J. Edmonds and G.C. Rota.

**Theorem 1.2.24.** Let  $E$  be a finite set, and let  $f: 2^E \rightarrow \mathbb{Z}$  be a non-decreasing, submodular function. Then  $M = (E, \mathcal{I})$  where

$$\mathcal{I} = \left\{ X \subseteq E \mid \forall X' \subseteq X: X' \neq \emptyset \Rightarrow f(X') \geq |X'| \right\}$$

is a matroid.

*Proof.* From the definition, it is clear that  $\emptyset \in \mathcal{I}$  (I1) as well as that for every  $X \in \mathcal{I}$  and every  $Y \subseteq X$ ,  $Y \in \mathcal{I}$  (I2). We have to show that (I3) holds for  $\mathcal{I}$ , too. Let  $X, Y \in \mathcal{I}$  with  $|X| < |Y|$ . We give an indirect argument. Assume that for all  $y \in Y \setminus X$ ,  $X \cup \{y\} \notin \mathcal{I}$ . Since  $|X \setminus Y| + |X \cap Y| = |X| < |Y| = |Y \setminus X| + |X \cap Y|$ , we have  $|Y \setminus X| > |X \setminus Y|$ . For every  $y \in Y \setminus X$ , there is a subset  $X_y \subseteq X$  with minimal cardinality such that  $f(X_y \cup \{y\}) < |X_y| + 1$ . Since  $Y \in \mathcal{I}$  we obtain that  $X_y \cap (X \setminus Y) \neq \emptyset$ . Therefore by a simple counting argument, there are  $y_1, y_2 \in Y \setminus X$  such that there is some  $x \in X \setminus Y$  with  $x \in X_{y_1} \cap X_{y_2}$ . Below, we first use that  $f$  is non-decreasing, then that  $f$  is submodular, and then the fact that  $X_{y_1} \cap X_{y_2} \in \mathcal{I}$  from (I2) and  $X \in \mathcal{I}$ ; finally, we use the fact that neither  $X \cup \{y_1\} \in \mathcal{I}$  nor  $X \cup \{y_2\} \in \mathcal{I}$  and that  $f$  is integer-valued:

$$\begin{aligned} f((X_{y_1} \cup X_{y_2} \cup \{y_1, y_2\}) \setminus \{x\}) &\leq f(X_{y_1} \cup X_{y_2} \cup \{y_1, y_2\}) \\ &\leq f(X_{y_1} \cup \{y_1\}) + f(X_{y_2} \cup \{y_2\}) - f(X_{y_1} \cap X_{y_2}) \\ &\leq f(X_{y_1} \cup \{y_1\}) + f(X_{y_2} \cup \{y_2\}) - |X_{y_1} \cap X_{y_2}| \\ &\leq |X_{y_1}| + |X_{y_2}| - |X_{y_1} \cap X_{y_2}| \\ &= |X_{y_1} \cup X_{y_2}| \\ &= |(X_{y_1} \cup X_{y_2} \cup \{y_1, y_2\}) \setminus \{x\}| - 1. \end{aligned}$$

Thus there must be a subset of minimal cardinality  $C \subseteq (X_{y_1} \cup X_{y_2} \cup \{y_1, y_2\}) \setminus \{x\}$  such that  $f(C) < |C|$ . Then  $C \cap \{y_1, y_2\} = \emptyset$  because otherwise  $C$  would contradict the



minimality of the cardinalities of  $X_{y_1}$  and  $X_{y_2}$ , respectively. But then the fact that  $C \subseteq X_{y_1} \cup X_{y_2} \subseteq X$  would contradict  $X \in \mathcal{I}$ . Therefore there must be some  $y \in Y \setminus X$  such that  $X \cup \{y\} \in \mathcal{I}$ .  $\square$

If we restrict  $f$  to be a map into the non-negative integers, we may simplify the expression that gives  $\mathcal{I}$  analogously to Corollary 8.1 [Wel76].

**Theorem 1.2.25.** *Let  $E$  be a finite set, and let  $f: 2^E \rightarrow \mathbb{N}$  be a non-decreasing, submodular function. Then  $M = (E, \mathcal{I})$  where*

$$\mathcal{I} = \{X \subseteq E \mid \forall X' \subseteq X: f(X') \geq |X'|\}$$

is a matroid. If furthermore  $f(\emptyset) = 0$ , then its rank function is given by

$$\text{rk}(X) = \min \{f(Y) + |X \setminus Y| \mid Y \subseteq X\}.$$

*Proof.* Let  $\mathcal{I}' = \{X \subseteq E \mid \forall X' \subseteq X: X' \neq \emptyset \Rightarrow f(X') \geq |X'|\}$  corresponding to Theorem 1.2.24. From the definitions, it is clear that  $\mathcal{I} \subseteq \mathcal{I}'$ . From inspection we obtain that if  $X \in \mathcal{I}' \setminus \mathcal{I}$ , then  $f(\emptyset) < |\emptyset| = 0$ . But this is impossible for  $f(\emptyset) \in \mathbb{N}$ . Thus  $\mathcal{I} = \mathcal{I}'$ .

The second part of the proof follows the ideas from [Dun76] where a more general statement is proved.<sup>2</sup> Now let us assume that we have the further property  $f(\emptyset) = 0$ , we shall now prove the rank formula. We will denote the formula given in the statement of the theorem by  $\text{rk}$ , whereas we are denoting the rank formula from Definition 1.2.14 by  $\text{rk}_M$ . First, we want to show that  $\text{rk}$  is non-decreasing. Let  $X' \subseteq X \subseteq E$ , we do induction on  $|X \setminus X'|$ . The base case is trivial. Now, let  $X \subseteq E$ , and  $x \in X$ , the induction hypothesis yields that  $\text{rk}(X') \leq \text{rk}(X \setminus \{x\})$ . If there is a subset  $Y \subseteq X$  with  $x \in Y$ , such that  $\text{rk}(X) = f(Y) + |X \setminus Y|$ , then since  $f(Y \setminus \{x\}) \leq f(Y)$  we obtain that

$$\text{rk}(X \setminus \{x\}) \leq f(Y \setminus \{x\}) + |(X \setminus \{x\}) \setminus (Y \setminus \{x\})| \leq f(Y) + |X \setminus Y| = \text{rk}(X).$$

Otherwise let  $Y \subseteq X \setminus \{x\}$  be a subset such that  $\text{rk}(X) = f(Y) + |X \setminus Y|$ . Then

$$\text{rk}(X \setminus \{x\}) \leq f(Y) + |(X \setminus \{x\}) \setminus Y| < f(Y) + |X \setminus Y| = \text{rk}(X),$$

thus in any case  $\text{rk}(X \setminus \{x\}) \leq \text{rk}(X)$ , so  $\text{rk}$  is non-decreasing. Now, in order to show that  $\text{rk}_M(X) \leq \text{rk}(X)$  for all  $X \subseteq E$ , it suffices to show that  $\text{rk}(X) = |X|$  for all

<sup>2</sup>Both D.J.A. Welsh and F.D.J. Dunstan cite a conference abstract of the *Waterloo Conference on Combinatorics 1968* by J. Edmonds and G.C. Rota who proved that for submodular, non-decreasing, integer-valued  $f$  the rank function is given by  $\text{rk}(X) = \min \{|X|, f(Y) + |X \setminus Y| \mid Y \subseteq X\}$ . Unfortunately, we were not able to get a copy of that abstract.

independent  $X \subseteq E$ . Let  $X \in \mathcal{I}$ . By definition of  $\mathcal{I}$ , for all  $Y \subseteq X$ ,  $|Y| \leq f(Y)$  holds. Thus for any  $Y \subseteq X$ , we have

$$|X| = |Y| + |X \setminus Y| \leq f(Y) + |X \setminus Y|.$$

Therefore the minimum in the expression for  $\text{rk}(X)$  is attained for  $Y = \emptyset$ , i.e.

$$\text{rk}(X) = \min \{f(Y) + |X \setminus Y| \mid Y \subseteq X\} = f(\emptyset) + |X \setminus \emptyset| = |X|.$$

To complete the proof that  $\text{rk} = \text{rk}_M$ , we have to show that for every  $X \subseteq E$ , there is a subset  $Y \subseteq X$  such that  $Y \in \mathcal{I}$  and  $\text{rk}(X) = |Y|$ . Let  $Z \subseteq X$  such that  $Z$  has maximal cardinality with  $Z \in \mathcal{I}$ . We give an indirect argument and assume that  $\text{rk}_M(X) = |Z| < \text{rk}(X) \leq |X|$ . Since  $Z$  is maximally independent in  $X$ , for every  $x \in X \setminus Z$  there must be a subset  $Z_x \subseteq Z$  such that  $f(Z_x \cup \{x\}) < |Z_x \cup \{x\}| = |Z_x| + 1$ . From  $Z \in \mathcal{I}$  we may infer that  $f(Z_x) \geq |Z_x|$ , thus we have  $f(Z_x \cup \{x\}) = |Z_x| = f(Z_x)$  due to  $f$  being non-decreasing and integer-valued. We show the auxiliary claim that for all  $X' \subseteq X \setminus Z$ ,  $f(Z) = f(Z \cup X')$ , by induction on  $|X'|$ . The base case is trivial. For the induction step, let  $x' \in X'$ , and by induction hypothesis we may assume that  $f(Z) = f(Z \cup (X' \setminus \{x'\}))$ . Let  $Z_{x'} \subseteq Z$  such that  $f(Z_{x'} \cup \{x'\}) = |Z_{x'}| = f(Z_{x'})$  as above. Since  $f$  is submodular, we obtain that  $f(Z \cup (X' \setminus \{x'\})) + f(Z_{x'} \cup \{x'\}) \geq f(Z_{x'}) + f(Z \cup X')$ , and along with the previous equation this yields  $f(Z \cup (X' \setminus \{x'\})) \geq f(Z \cup X')$ . So, together with the property that  $f$  is non-decreasing and with the induction hypothesis, we obtain the desired equation  $f(Z \cup X') = f(Z \cup (X' \setminus \{x'\})) = f(Z)$ . But now, let  $X' = X \setminus Z$ , then  $Z \cup X' = X$ . We obtain from the auxiliary claim above, that  $f(Z) = f(X)$ , so that, by construction as a minimum,  $\text{rk}(X) \leq f(X) + |X \setminus X| = f(Z) = |Z|$ . Yet this contradicts  $|Z| < \text{rk}(X)$ . Thus there is an independent subset of  $X$  with cardinality  $\text{rk}(X)$  for every  $X \subseteq E$ , and therefore  $\text{rk} = \text{rk}_M$ .  $\square$

### 1.2.4 Dual Matroids

**Definition 1.2.26.** Let  $M = (E, \mathcal{I})$  be a matroid. We call  $X \subseteq E$  **spanning** in  $M$ , if there is a base  $B$  of  $M$ , such that  $B \subseteq X$ .  $\blacksquare$

**Lemma 1.2.27.** Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$ . Then  $X$  is a base if and only if  $X$  is spanning in  $M$ , yet for all  $x \in X$ ,  $X \setminus \{x\}$  is not spanning in  $M$ .

*Proof.* Let  $B \in \mathcal{I}$  be a base of  $M$ , then  $\text{rk}(B) = |B|$  is maximal, so  $\text{cl}(B) = E$ . On the other hand, for every  $b \in B$ ,  $\text{rk}(B \setminus \{b\}) < |B|$ , thus  $b \notin \text{cl}(B \setminus \{b\})$ , so  $\text{cl}(B \setminus \{b\}) \neq E$ . Let  $X \subseteq E$  such that  $X \notin \mathcal{B}(M)$ . If further  $\text{rk}(X) < \text{rk}(E)$ , then  $X$  clearly is not a

spanning set in  $M$ . Now assume that  $\text{rk}(X) = \text{rk}(E)$ , so  $X$  is spanning in  $M$ , and because it is not a base,  $X \notin \mathcal{I}$ . But then there is a base  $B \subsetneq X$  with  $\text{cl}(B) = \text{cl}(X)$  (Lemma 1.2.20). So there is some  $x \in X \setminus B$ , such that  $X \setminus \{x\}$  still contains the base  $B$  and therefore  $X \setminus \{x\}$  is spanning in  $M$ .  $\square$

Matroids allow to be axiomatized cryptomorphically by characterizing the set of bases of  $M$ . For full disclosure on this topic we would like to refer the reader the first chapters in [Wel76] and [Oxl11].

**Theorem 1.2.28.** *Let  $E$  be a finite set,  $\mathcal{I} \subseteq 2^E$ . Let further*

$$\mathcal{B} = \{X \in \mathcal{I} \mid \nexists Y \in \mathcal{I}: X \subsetneq Y\}$$

be the family of maximal elements of  $\mathcal{I}$  with regard to set-inclusion. If

$$(B1) \quad \mathcal{B} \neq \emptyset,$$

$$(B2) \quad \forall X, Y \in \mathcal{B}: |X| = |Y|, \text{ and}$$

$$(B3) \quad \text{for all } X, Y \in \mathcal{B} \text{ and all } x \in X \setminus Y, \text{ there is an element } y \in Y \setminus X, \text{ such that } (X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$$

holds, and if  $\mathcal{I} = \{X \subseteq E \mid \exists B \in \mathcal{B}: X \subseteq B\}$ , then  $M = (E, \mathcal{I})$  is a matroid.

*Proof.* From (B1) we obtain  $B \in \mathcal{B}$ , and clearly  $\emptyset \subseteq B$ , so  $\emptyset \in \mathcal{I}$  (I1). Let  $X \in \mathcal{I}$ , then there is some  $B \in \mathcal{B}$  with  $X \subseteq B$ . For  $Y \subseteq X$  we have  $Y \subseteq B$  and therefore  $Y \in \mathcal{I}$  (I2). Let  $X, Y \in \mathcal{I}$  with  $|X| < |Y|$ . There are  $B_X \supseteq X$  and  $B_Y \supseteq Y$  with  $B_X, B_Y \in \mathcal{B}$ . If  $Y' = B_X \cap (Y \setminus X) \neq \emptyset$ , then let  $y \in Y'$  be an arbitrary choice, and we obtain  $X \cup \{y\} \subseteq B_X$  therefore  $X \cup \{y\} \in \mathcal{I}$ . If  $Y' = \emptyset$ , then let  $\alpha(B_X) = |(B_Y \setminus B_X) \setminus (Y \setminus X)|$ , we prove that we can augment  $X$  by induction on  $\alpha(B_X)$ . Since  $|X| < |Y| \leq |B_Y| = |B_X|$ , there is an element  $x' \in B_X \setminus X$ . We may use (B3) in order to obtain the base  $B'_X = (B_X \setminus \{x'\}) \cup \{y\} \in \mathcal{B}$  where  $y \in B_Y \setminus B_X$ . If  $y \in Y \setminus X$ , then  $X \cup \{y\} \subseteq B'_X$  and therefore  $X \cup \{y\} \in \mathcal{I}$ . Otherwise  $y \in (B_Y \setminus B_X) \setminus (Y \setminus X)$ , then  $\alpha(B'_X) = \alpha(B_X) - 1$  and thus there is some  $y \in Y \setminus X$  with  $X \cup \{y\} \in \mathcal{I}$  by the induction hypothesis. Thus  $\mathcal{I}$  has the property (I3).  $\square$

**Definition 1.2.29.** *Let  $M = (E, \mathcal{I})$  be a matroid. The **dual matroid** of  $M$  shall be the pair  $M^* = (E, \mathcal{I}^*)$  where*

$$\mathcal{I}^* = \{E \setminus X \mid X \subseteq E, \text{ such that } X \text{ is spanning in } M\}. \quad \blacksquare$$

**Lemma 1.2.30.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M^* = (E, \mathcal{I}^*)$  is indeed a matroid.*

*Proof.* First, observe that for  $\mathcal{B}^* = \{E \setminus B \mid B \text{ is a base of } M\}$  we have the set equation

$$\mathcal{I}^* = \{X \subseteq E \mid \exists B' \in \mathcal{B}^*: X \subseteq B'\},$$

because the minimal spanning sets of  $M$  are precisely the bases of  $M$ , which in turn have complements in  $E$  with maximal cardinality. Since  $\emptyset \in \mathcal{I}$  implies that  $M$  has at least one base, we have  $\mathcal{B}^* \neq \emptyset$  (B1). From Corollary 1.2.8 we obtain that for any two  $B, B' \in \mathcal{B}^*$ , we have  $B_0, B'_0$  that are bases of  $M$  with  $B = E \setminus B_0$  and  $B' = E \setminus B'_0$ , therefore  $|B| = |E| - |B_0| = |E| - |B'_0| = |B'|$ , so (B2) holds. Now let  $x \in B \setminus B' = (E \setminus B_0) \setminus (E \setminus B'_0) = B'_0 \setminus B_0$ , then there is a  $y \in B_0 \setminus B'_0 = (E \setminus B'_0) \setminus (E \setminus B_0) = B' \setminus B$  such that  $(B_0 \setminus \{y\}) \cup \{x\}$  is a base of  $M$  (Lemma 1.2.10). But then

$$\begin{aligned} E \setminus ((B_0 \setminus \{y\}) \cup \{x\}) &= E \setminus ((B_0 \cup \{x\}) \setminus \{y\}) \\ &= (E \setminus (B_0 \cup \{x\})) \cup \{y\} \\ &= (B \setminus \{x\}) \cup \{y\} \in \mathcal{B}^*. \end{aligned}$$

So (B3) holds for  $\mathcal{B}^*$ , too, and from Theorem 1.2.28 we obtain that  $M^* = (E, \mathcal{I}^*)$  is a matroid.  $\square$

**Corollary 1.2.31.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $B \subseteq E$ . Then  $B$  is a base of  $M$  if and only if  $E \setminus B$  is a base of  $M^*$ .*

*Proof.* Let  $(E, \mathcal{I}') = M^*$ . If  $B$  is a base of  $M$ , then for all  $b \in B$ ,  $B \setminus \{b\}$  is not spanning  $M$  (Lemma 1.2.27), therefore  $E \setminus B \in \mathcal{I}'$ , yet  $(E \setminus B) \cup \{b\} \notin \mathcal{I}'$ , therefore  $E \setminus B$  is maximally independent with respect to set-inclusion, and thus it is an independent set of  $M^*$  with maximal cardinality (Corollary 1.2.8), so  $E \setminus B$  is a base of  $M^*$ . Conversely, if  $E \setminus B$  is a base of  $M^*$ , then  $E \setminus (E \setminus B) = B$  must be minimally spanning in  $M$ , since otherwise  $E \setminus (B \setminus \{x\}) \in \mathcal{I}'$  for some  $x \in B$  contradicting the maximality of  $E \setminus B$  in  $\mathcal{I}'$ . Thus  $B$  is a base of  $M$  (Lemma 1.2.27).  $\square$

**Corollary 1.2.32.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M = (M^*)^*$ .*

*Proof.* By property (I2), the family of independent sets of a matroid is determined by its maximal elements, which are the bases of  $M$ . By Corollary 1.2.31,  $B$  is base of  $M$ , if and only if  $E \setminus B$  is a base of  $M^*$ , if and only if  $E \setminus (E \setminus B) = B$  is a base of  $(M^*)^*$ . Thus  $M = (M^*)^*$ .  $\square$

The next two lemmas can be found in J.G. Oxley's book ([Oxl11], p.67) and yield an elegant way to characterize the rank function of the dual matroid in terms of the rank function of the primal matroid.

**Lemma 1.2.33.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X, Y \subseteq E$  with  $X \cap Y = \emptyset$  such that  $X \in \mathcal{I}$  is independent in  $M$  and  $Y \in \mathcal{I}^*$  is independent in  $M^*$ . Then there is a base  $B \subseteq E$  of  $M$  such that  $X \subseteq B$  and  $Y \subseteq E \setminus B$ .*

*Proof.* Let  $B$  be a base of  $E \setminus Y$  in  $M$  such that  $X \subseteq B$  (Lemma 1.2.7). Then  $Y \subseteq E \setminus B$ . It remains to show that  $B$  is a base of  $M$ . Assume that  $B$  is not a base of  $M$ , then  $\text{rk}_M(E \setminus Y) < \text{rk}_M(E)$ . But  $Y \in \mathcal{I}^*$ , therefore  $E \setminus Y$  is spanning in  $M$  – a contradiction. Thus  $B$  is the desired base of  $M$ .  $\square$

**Lemma 1.2.34.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $X \subseteq E$ . Then*

$$\text{rk}_{M^*}(X) = |X| + \text{rk}_M(E \setminus X) - \text{rk}_M(E).$$

*Proof.* Let  $B'_X \subseteq X$  be a base of  $X$  in  $M^*$ , and  $B_{E \setminus X} \subseteq E \setminus X$  be a base of  $E \setminus X$  in  $M$ . Then  $\text{rk}_{M^*}(X) = |B'_X|$  and  $\text{rk}_M(E \setminus X) = |B_{E \setminus X}|$ . Clearly  $B'_X \cap B_{E \setminus X} = \emptyset$ , therefore there is a base  $B$  of  $M$  such that  $B_{E \setminus X} \subseteq B$  and  $B'_X \subseteq E \setminus B$  (Lemma 1.2.33). Since  $B_{E \setminus X}$  is a base of  $E \setminus X$  in  $M$ , we have that  $B \cap (E \setminus X) = B_{E \setminus X}$ , and analogously,  $(E \setminus B) \cap X = B'_X$ . We obtain  $B \cap X = X \setminus B'_X$  and therefore  $B = B_{E \setminus X} \dot{\cup} (X \setminus B'_X)$ , so

$$\text{rk}_M(E) = |B| = |B_{E \setminus X}| + |X| - |B'_X| = \text{rk}_M(E \setminus X) + |X| - \text{rk}_{M^*}(X),$$

and as a consequence,  $\text{rk}_{M^*}(X) = |X| + \text{rk}_M(E \setminus X) - \text{rk}_M(E)$ .  $\square$

The following fact will be of interest for oriented matroids in Chapter 3. It can be found as Proposition 2.1.11 in J.G. Oxley's book ([Oxl11], p.68), together with the proof we present here.

**Lemma 1.2.35.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $M^* = (E, \mathcal{I}^*)$  be its dual matroid. Then for every  $C \in \mathcal{C}(M)$  and  $D \in \mathcal{C}(M^*)$ , we have  $|C \cap D| \neq 1$ .*

*Proof.* We give an indirect proof and assume that  $\{x\} = C \cap D$  for some  $C \in \mathcal{C}(M)$  and  $D \in \mathcal{C}(M^*)$ . Since  $D \in \mathcal{C}(M^*)$ , we have  $\text{rk}_{M^*}(D) = |D| - 1$ . We set  $H = E \setminus D$ , then by Lemma 1.2.34, we get

$$\text{rk}_{M^*}(D) = |D| - 1 = |D| + \text{rk}_M(H) - \text{rk}_M(E),$$

and therefore  $\text{rk}_M(H) = \text{rk}(E) - 1$  follows. Clearly,  $\text{cl}_M(H) = H$ , since otherwise there would be an element  $d \in D$  such that  $d \in \text{cl}_M(H) \setminus H$ , which would imply that

$$\text{rk}_{M^*}(D \setminus \{d\}) = |D \setminus \{d\}| + \text{rk}_M(H \cup \{d\}) - \text{rk}_M(E) = |D \setminus \{d\}| - 1,$$

contradicting that  $D \in \mathcal{C}(M^*)$  is a minimally dependent set of  $M^*$  with respect to set-inclusion. But now we arrive at another contradiction: We have  $x \in C \cap D$ ,  $x \notin H = E \setminus D$ , and thus  $C \not\subseteq H$ , yet  $|C \cap H| = |C| - |C \cap D| = |C| - 1$ , and therefore  $\text{cl}_M(C \cap H) = C$ , so we obtain the contradiction  $C \subseteq \text{cl}_M(C \cap H) \subseteq \text{cl}_M(H) = H$  (Lemma 1.2.17). Therefore  $|C \cap D| \neq 1$  must be the case.  $\square$

**Lemma 1.2.36.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $M^* = (E, \mathcal{I}^*)$  be its dual matroid. Let further  $C \in \mathcal{C}(M)$  be a circuit and  $c, d \in C$  with  $c \neq d$ . There there is some  $D \in \mathcal{C}(M^*)$  such that  $C \cap D = \{c, d\}$ .*

The proof presented here is along the lines of the proof of Lemma 2.2.3 in [BV78].

*Proof.* Since  $C \in \mathcal{C}(M)$ , we have  $C \setminus \{c\} \in \mathcal{I}$ . There is a base  $B_c$  of  $M$  with  $C \setminus \{c\} \subseteq B_c$  (Lemma 1.2.7), and since  $C \notin \mathcal{I}$ ,  $c \notin B_c$ . Then  $B'_c = E \setminus B_c$  is a base of  $M^*$  with  $c \in B'_c$  (Corollary 1.2.31). Let  $D' = B'_c \cup \{d\}$ , then  $\text{rk}_{M^*}(D') = \text{rk}_{M^*}(E) = |D'| - 1$ , and therefore there is a unique circuit  $D \subseteq D'$ . Clearly,  $d \in D$  is an element of that circuit. Therefore  $d \in C \cap D$ . Furthermore  $C \subseteq B_c \cup \{c\}$  and  $D \subseteq B'_c \cup \{d\} = (E \setminus B_c) \cup \{d\}$  yield  $C \cap D \subseteq \{c, d\}$ . Since  $|C \cap D| \neq 1$  (Lemma 1.2.35), we obtain that  $C \cap D = \{c, d\}$ .  $\square$

## 1.2.5 Minors

In this section, we introduce the natural substructures for matroids.

**Definition 1.2.37.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $R \subseteq E$ . The **restriction of  $M$  to  $R$**  is the pair  $M|R = (R, \mathcal{I}')$  where*

$$\mathcal{I}' = \{X \in \mathcal{I} \mid X \subseteq R\}. \quad \blacksquare$$

**Lemma 1.2.38.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $R \subseteq E$ . Then  $M|R = (R, \mathcal{I}')$  is a matroid.*

*Proof.*  $\emptyset \subseteq R$  and  $\emptyset \in \mathcal{I}$  thus  $\emptyset \in \mathcal{I}'$  (I1). Let  $X \subseteq Y \in \mathcal{I}'$ , then  $Y \subseteq R$  and  $Y \in \mathcal{I}$ , therefore  $X \subseteq R$  and  $X \in \mathcal{I}$ , so  $X \in \mathcal{I}'$  (I2). Let  $X, Y \in \mathcal{I}'$  with  $|X| < |Y|$ . There is some  $y \in Y \setminus X$  with  $X \cup \{y\} \in \mathcal{I}$ , and since  $X \cup \{y\} \subseteq R$ ,  $X \cup \{y\} \in \mathcal{I}'$  (I3).  $\square$

**Corollary 1.2.39.** *Let  $M = (E, \mathcal{I})$  and  $R \subseteq E$ . Then for all  $X \subseteq R$  we have*

$$\text{rk}_{M|R}(X) = \text{rk}_M(X).$$

*Proof.* Clear from Definition 1.2.14.  $\square$

**Lemma 1.2.40.** *Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids with  $E \cap E' = \emptyset$ . Let  $X \subseteq E \cup E'$ , then*

$$(M \oplus N)|X = (M|X \cap E) \oplus (N|X \cap E').$$

*Proof.* Clear from Definitions 1.2.4 and 1.2.37: the independent sets of the direct sum  $\mathcal{I}_\oplus$  are disjoint unions of independent sets of its parts, therefore the restriction of the family  $\mathcal{I}_\oplus$  to subsets of  $X$  consists of those disjoint unions of the subsets of  $X$ , that are independent with respect to its parts.  $\square$

**Definition 1.2.41.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C \subseteq E$ . The **contraction of  $M$  to  $C$**  is the pair  $M.C = (C, \mathcal{I}')$  where*

$$\mathcal{I}' = \{X \subseteq C \mid \forall B \subseteq E \setminus C: B \in \mathcal{I} \Rightarrow B \cup X \in \mathcal{I}\}. \quad \blacksquare$$

**Lemma 1.2.42.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $C \subseteq E$ , and let  $B$  be a base of  $E \setminus C$  in  $M$ . If further*

$$\begin{aligned} \mathcal{I}_B &= \{X \subseteq C \mid B \cup X \in \mathcal{I}\} \text{ and} \\ \mathcal{I}' &= \{X \subseteq C \mid \forall B' \subseteq E \setminus C: B' \in \mathcal{I} \Rightarrow B' \cup X \in \mathcal{I}\}, \end{aligned}$$

then  $\mathcal{I}' = \mathcal{I}_B$ .

*Proof.* From the definition it is clear that  $\mathcal{I}' \subseteq \mathcal{I}_B$ . First, we show that  $\mathcal{I}_B$  does not depend on the choice of the base of  $E \setminus C$  in  $M$ . Let  $B, B' \subseteq E$  be any two bases of  $E \setminus C$  in  $M$ , and let  $\mathcal{I}_B$  be defined as in the lemma, and let  $\mathcal{I}_{B'} = \{X \subseteq C \mid B' \cup X \in \mathcal{I}\}$ . If  $X \in \mathcal{I}_B$  then  $B \cup X \in \mathcal{I}$ . Let  $F = B \cup B' \cup X$ , then there is a base  $B_X$  of  $F$  with  $B \cup X \subseteq B_X$  (Lemma 1.2.7). Furthermore, we already have  $B_X = B \cup X$ , because both  $B$  and  $B'$  are independent subsets of  $E \setminus C$  with maximal cardinality, so any  $|B| + 1$  elementary subset of  $B \cup B'$  must be dependent and therefore cannot be a subset of  $B_X$ . Again by Lemma 1.2.7, we obtain a base  $B'_X$  of  $F$  with  $B' \subseteq B'_X$ . Since  $|B_X| = |B'_X|$  (Corollary 1.2.8) and the previous argument about subsets of  $B \cup B'$ , we have  $B'_X = B' \cup X$ , therefore  $X \in \mathcal{I}_{B'}$ . This proves  $\mathcal{I}_B \subseteq \mathcal{I}_{B'}$  for any two bases  $B$  and  $B'$  of  $E \setminus C$  in  $M$ , and therefore  $\mathcal{I}_B = \mathcal{I}_{B'}$  for any two such bases.

Let  $X \subseteq E$  and let  $I \subseteq E \setminus C$  such that  $I \in \mathcal{I}$ . Then there is a base  $B'$  of  $E \setminus C$  in  $M$  with  $I \subseteq B'$ . If  $X \cup I \notin \mathcal{I}$ , then clearly  $X \cup B' \notin \mathcal{I}$ . Therefore we may write

$$\begin{aligned} \mathcal{I}' &= \bigcap_{B' \subseteq E \setminus C, B' \in \mathcal{I}} \{X \subseteq C \mid X \cup B' \in \mathcal{I}\} \\ &= \bigcap_{B' \in \mathcal{B}_M(E \setminus C)} \{X \subseteq C \mid X \cup B' \in \mathcal{I}\} = \mathcal{I}_B \end{aligned}$$

where  $B$  is any fixed base of  $E \setminus C$  in  $M$ .  $\square$

**Lemma 1.2.43.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C \subseteq E$ . Then  $M.C = (C, \mathcal{I}')$  is a matroid.*

*Proof.* Let  $B$  be an arbitrarily fixed base of  $E \setminus C$  in  $M$ , then  $\mathcal{I}' = \{X \subseteq C \mid X \cup B \in \mathcal{I}\}$  (Lemma 1.2.42). Clearly  $B \cup \emptyset = B \in \mathcal{I}$ , thus  $\emptyset \in \mathcal{I}'$  (I1). Furthermore, if  $X \in \mathcal{I}'$ , then  $B \cup X \in \mathcal{I}$ , therefore for any  $Y \subseteq X$ , we have  $B \cup Y \in \mathcal{I}$  (I2). Now let  $X, Y \in \mathcal{I}'$  with  $|X| < |Y|$ . Thus  $B \cup X \in \mathcal{I}$  and  $B \cup Y \in \mathcal{I}$  with  $|B \cup X| = |B| + |X| < |B| + |Y| = |B \cup Y|$ . There is  $y \in (B \cup Y) \setminus (B \cup X) = Y \setminus X$  such that  $B \cup X \cup \{y\} \in \mathcal{I}$ , and therefore  $X \cup \{y\} \in \mathcal{I}'$  (I3). Thus  $M.C$  is a matroid.  $\square$

**Corollary 1.2.44.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $C \subseteq E$ . Then for all  $X \subseteq C$*

$$\text{rk}_{M.C}(X) = \text{rk}_M(X \cup (E \setminus C)) - \text{rk}_M(E \setminus C).$$

*Proof.* Immediate consequence from Lemma 1.2.42 and Definition 1.2.14.  $\square$

**Lemma 1.2.45.** *Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids with  $E \cap E' = \emptyset$ . Let  $C \subseteq E \cup E'$ . Then*

$$(M \oplus N).C = (M.C \cap E) \oplus (N.C \cap E').$$

*Proof.* Direct consequence of Definition 1.2.4 and Corollary 1.2.44: Since the independent sets of  $M \oplus N$  are the disjoint unions of the independent sets of  $M$  and  $N$ , it is clear that  $\text{rk}_{M \oplus N}(X) = \text{rk}_M(X \cap E) + \text{rk}_N(X \cap E')$  holds for all  $X \subseteq E \cup E'$  (Definition 1.2.14). Thus

$$\begin{aligned} \text{rk}_{(M \oplus N).C}(X) &= \text{rk}_M((X \cap E) \cup (E \setminus C)) + \text{rk}_N((X \cap E') \cup (E' \setminus C)) \\ &\quad - \text{rk}_M(E \setminus C) - \text{rk}_N(E' \setminus C) \\ &= \text{rk}_{M.C \cap E}(X \cap E) + \text{rk}_{N.C \cap E'}(X \cap E'). \end{aligned} \quad \square$$



The operations of restriction and contraction are related by duality, if you do one of these operations on the dual of  $M$  and then dualize the result, you get the matroid you would have obtained from the other operation on  $M$ .

**Lemma 1.2.46.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C \subseteq E$ . Then  $M.C = (M^*|C)^*$ .*

*Proof.* Clearly,  $M.C = (M^*|C)^*$  holds if and only if  $(M.C)^* = M^*|C$  holds (Corollary 1.2.32). Since the family of independent sets of a matroid can be reconstructed from the values of its rank function, it suffices to show that for any  $X \subseteq C$  the equation

$$\text{rk}_{(M.C)^*}(X) = \text{rk}_{M^*|C}(X)$$

holds. First observe that for  $X \subseteq C \subseteq E$  the set equation  $(E \setminus C) \cup (C \setminus X) = E \setminus X$  holds. Now from Lemma 1.2.34, and the Corollaries 1.2.44 and 1.2.39 we obtain

$$\begin{aligned} \text{rk}_{(M.C)^*}(X) &= |X| + \text{rk}_{M.C}(C \setminus X) - \text{rk}_{M.C}(C) \\ &= |X| + \text{rk}_M((E \setminus C) \cup (C \setminus X)) - \text{rk}_M(E \setminus C) - \text{rk}_M(E) + \text{rk}_M(E \setminus C) \\ &= |X| + \text{rk}_M(E \setminus X) - \text{rk}_M(E) = \text{rk}_{M^*}(X) = \text{rk}_{M^*|C}(X). \quad \square \end{aligned}$$

**Lemma 1.2.47.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C \subseteq E$  and  $R \subseteq E$  such that  $(E \setminus C) \cap (E \setminus R) = \emptyset$ . Then*

$$(M|R).(C \cap R) = (M.C)|(C \cap R).$$

*Proof.* First, we want to establish the fact that  $R \setminus C = E \setminus C$ . Since  $R \subseteq E$ , it remains to show that  $(E \setminus C) \setminus (R \setminus C) = \emptyset$ . For all  $x \in (E \setminus C) \setminus (R \setminus C)$  we have  $x \in E$ ,  $x \notin C$  and  $x \notin R$ , thus  $x \in E \setminus C$  and  $x \in E \setminus R$ . Since  $(E \setminus C) \cap (E \setminus R) = \emptyset$ , we conclude  $(E \setminus C) \setminus (E \setminus R) = \emptyset$ , so  $E \setminus C = R \setminus C$ . Furthermore, it is clear that  $R \setminus (C \cap R) = R \setminus C$  for all sets  $C$  and  $R$ . We give a proof of the statement of the lemma using the rank formulae from Corollaries 1.2.39 and 1.2.44. Let  $X \subseteq C \cap R$ , then

$$\begin{aligned} \text{rk}_{(M|R).(C \cap R)} &= \text{rk}_{M|R}(X \cup (R \setminus C)) - \text{rk}_{M|R}(R \setminus C) \\ &= \text{rk}_M(X \cup (R \setminus C)) - \text{rk}_M(R \setminus C) \\ &= \text{rk}_M(X \cup (E \setminus C)) - \text{rk}_M(E \setminus C) \\ &= \text{rk}_{M.C}(X) = \text{rk}_{(M.C)|(C \cap R)}(X). \quad \square \end{aligned}$$

**Definition 1.2.48.** Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be matroids. We shall call  $N$  a **minor of  $M$** , if there are sets  $X \subseteq Y \subseteq E$  such that

$$N = (M.Y)|X$$

holds. ■

**Remark 1.2.49.** For  $M = (E, \mathcal{I})$  and  $X \subseteq Y \subseteq E$  we have  $Y \cap (E \setminus (Y \setminus X)) = X$  and  $(E \setminus Y) \cap (Y \setminus X) = \emptyset$ , so Lemma 1.2.47 yields that

$$(M.Y)|X = (M|E \setminus (Y \setminus X)).X \quad \text{and} \quad (M|Y).X = (M.E \setminus (Y \setminus X))|X. \quad \bullet$$

**Definition 1.2.50.** Let  $\mathcal{M}$  be a class of matroids. Then  $\mathcal{M}$  shall be called a **minor-closed class**, if for every  $M = (E, \mathcal{I}) \in \mathcal{M}$  and every  $X \subseteq E$ , also  $M|X \in \mathcal{M}$  and  $M.X \in \mathcal{M}$  holds. ■

**Example 1.2.51.** Let  $\mathcal{M}$  be the class where  $M \in \mathcal{M}$  if and only if  $M = (E, 2^E)$  for any set  $E$ , i.e.  $\mathcal{M}$  is the class of all free matroids. Clearly, for every  $M = (E, 2^E)$  and every  $X \subseteq E$  we have  $M|X = M.X = (X, 2^X) \in \mathcal{M}$ . ▲

**Definition 1.2.52.** Let  $\mathcal{M}$  be a minor-closed class of matroids. A matroid  $M = (E, \mathcal{I})$  is called **excluded minor for  $\mathcal{M}$**  if  $M \notin \mathcal{M}$  and if for every  $X \subsetneq E$  we have both  $M|X \in \mathcal{M}$  and  $M.X \in \mathcal{M}$ . Furthermore, a minor-closed class of matroids  $\mathcal{M}$  is called **characterized by finitely many excluded minors** if there are only finitely many pair-wise non-isomorphic excluded minors for  $\mathcal{M}$ . ■

**Example 1.2.53.** A matroid is representable over the 2-elementary field  $\mathbb{F}_2$  (Definition 1.2.59) if and only if it has no minor isomorphic to the rank-2 uniform matroid  $(\{a, b, c, d\}, \{X \subseteq \{a, b, c, d\} \mid |X| \leq 2\})$ . (Theorem 6.5.4 [Oxl11], p.193). Thus the class of all matroids representable over  $\mathbb{F}_2$  is characterized by finitely many, or in this case, a single excluded minor. ▲

**Remark 1.2.54.** If  $\mathcal{M}$  is a minor-closed class of matroids with the property that  $M \in \mathcal{M} \Leftrightarrow M^* \in \mathcal{M}$  holds for all matroids  $M$ , i.e.  $\mathcal{M}$  is closed under duality; then  $N$  is an excluded minor of  $\mathcal{M}$  if and only if  $N^*$  is an excluded minor of  $\mathcal{M}$  (see also Lemma 1.2.47). •

The excluded minors for matroids representable over fields with 2, 3, and 4 elements are known ([Oxl11], p.193), and the famous *Rota's Conjecture* states that for every

finite field  $\mathbb{F}$ , the class of matroids representable over  $\mathbb{F}$  is characterized by finitely many excluded minors. J. Geelen, B. Gerards and G. Whittle claim to have proven Rota's Conjecture and published an overview of their proof in [GGW14]. Furthermore, it has been shown that both the class of matroids representable over the field of the reals  $\mathbb{R}$  and the class of gammoids have the property, that every matroid  $M$  in each respective class is a minor of an excluded minor of that class, therefore those classes cannot be characterized by finitely many excluded minors, because both classes are non-empty and closed under direct sums, thus they are infinite. The result for the class of matroids representable over  $\mathbb{R}$  has been proven by D. Mayhew, M. Newman, and G. Whittle in [MNW09], and the result for the class of gammoids can be found in a paper by D. Mayhew [May16], where the excluded minor constructed for an arbitrary gammoid is also an excluded minor for the class of matroids representable over  $\mathbb{R}$ .

### 1.2.6 Matroids Representable Over a Field

A quite natural class of matroids arises from the notion of linear independence. We only give a short introduction here. Those readers, who are interested in the classes of matroids representable over some given field  $\mathbb{F}$ , shall hereby be referred to J.G. Oxley's book [Oxl11].

**Definition 1.2.55.** *Let  $\mathbb{K}$  be a field,  $E$  and  $C$  be finite sets. Let  $\mu \in \mathbb{K}^{E \times C}$  be an  $E \times C$ -matrix over  $\mathbb{K}$ . The **matroid represented by  $\mu$  over  $\mathbb{K}$**  is the pair  $M(\mu) = (E, \mathcal{I})$  where*

$$\mathcal{I} = \{X \subseteq E \mid \text{idet}(\mu|_X) = 1\}. \quad \blacksquare$$

**Lemma 1.2.56.** *Let  $\mathbb{K}$  be a field,  $E$  and  $C$  be finite sets. Let  $\mu \in \mathbb{K}^{E \times C}$ . Then  $M(\mu)$  is a matroid.*

The proof is essentially elementary linear algebra.

*Proof.* Let  $(E, \mathcal{I}) = M(\mu)$ . It is clear from Definition 1.1.8 that for  $X \subseteq E$ , the equality  $\text{idet}(\mu|_X) = 1$  holds if and only if the set  $V_X = \{\mu_x \mid x \in X\}$  is linear independent in the vector space  $\mathbb{K}^C$  with the further property  $|V_X| = |X|$ . Thus  $\emptyset \in \mathcal{I}$  (I1). For every  $Y \subseteq X \in \mathcal{I}$ , we have that  $V_Y = \{\mu_y \mid y \in Y\}$  is linear independent in  $\mathbb{K}^C$  with  $|V_Y| = |Y|$ , thus  $Y \in \mathcal{I}$  (I2). Let  $X, Y \in \mathcal{I}$  with  $|X| < |Y|$ , and let  $V_X, V_Y$  be defined as above. Since  $|V_Y| > |V_X|$  and  $V_Y$  is linear independent in  $\mathbb{K}^C$ , we have that  $\text{span}_{\mathbb{K}^C}(V_X) \subsetneq \text{span}_{\mathbb{K}^C}(V_X \cup V_Y)$ . Therefore, there is some  $\mu_y \in V_Y$  with  $\mu_y \notin \text{span}_{\mathbb{K}^C}(V_X)$ , and consequently,  $V' = V_X \cup \{\mu_y\}$  is linear independent in  $\mathbb{K}^C$  with  $|V'| = |X| + 1$ , thus  $X \cup \{y\} \in \mathcal{I}$  (I3).  $\square$

**Corollary 1.2.57.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $\mathbb{K}$  be a field,  $E$  and  $C$  be finite sets, and  $\mu \in \mathbb{K}^{E \times C}$  be a matrix. For all  $R \subseteq E$ ,*

$$M(\mu)|R = M(\mu|R).$$

**Remark 1.2.58.** It is a well-known fact from linear algebra that the following operations on  $\mu: E \times C \rightarrow \mathbb{K}$  do not change linear dependency between rows, and therefore do not alter the matroid  $M(\mu)$ :

$$(i) \text{ Interchanging two columns } c_1, c_2 \in C, \text{ i.e. if } \nu(e, c) = \begin{cases} \mu(e, c) & \text{if } c \notin \{c_1, c_2\}, \\ \mu(e, c_2) & \text{if } c = c_1, \\ \mu(e, c_1) & \text{if } c = c_2, \end{cases}$$

then  $M(\mu) = M(\nu)$ .

(ii) Adding a multiple of one column to another column, i.e. for  $c_1, c_2 \in C$  with

$$c_1 \neq c_2 \text{ and } \alpha \in \mathbb{K}, \text{ i.e. if } \nu(e, c) = \begin{cases} \mu(e, c) & \text{if } c \neq c_2, \\ \mu(e, c_2) + \alpha \cdot \mu(e, c_1) & \text{if } c = c_2, \end{cases}$$

then  $M(\mu) = M(\nu)$ .

(iii) Multiplying a column  $c_1 \in C$  with  $\alpha \in \mathbb{K} \setminus \{0\}$ , i.e.

$$\text{if } \nu(e, c) = \begin{cases} \mu(e, c) & \text{if } c \neq c_1, \\ \alpha \cdot \mu(e, c_1) & \text{if } c = c_1, \end{cases} \quad \text{then } M(\mu) = M(\nu).$$

Furthermore, if  $B \subseteq E$  is a base of  $M(\mu)$ , then we can use Gauß-Jordan elimination steps<sup>3</sup> in order to obtain an injective map  $\iota: B \rightarrow C$  and a matrix  $\nu$ , which has the

properties  $M(\nu) = M(\mu)$  and for all  $b \in B$  and all  $c \in C$ ,  $\nu(b, c) = \begin{cases} 1 & \text{if } c = \iota(b), \\ 0 & \text{otherwise.} \end{cases}$

From the matrix  $\nu$ , we can easily read some important properties of  $M(\mu)$ . Let  $e \in E \setminus B$ , then the unique circuit contained in  $B \cup \{e\}$  consists of  $e$  and the elements  $b' \in B$  where  $\nu(e, \iota(b')) \neq 0$ . If  $B' \subseteq B$ , then  $\text{cl}_{M(\nu)}(B')$  consists of  $B'$  and all  $e \in E \setminus B$  which have the property that  $\nu(e, c) = 0$  holds for all  $c \in C \setminus (\iota[B'])$ . •

**Definition 1.2.59.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $\mathbb{K}$  be a field. We say that  $M$  is **representable over**  $\mathbb{K}$ , if there is a finite set  $C$  and a matrix  $\mu \in \mathbb{K}^{E \times C}$ , such that  $M = M(\mu)$ . ■*

<sup>3</sup>This procedure is commonly referred to as "pivoting in the ordered basis  $B$ " in the context of linear programming. Careful pivoting is the foundation of the simplex algorithm for solving linear optimization problems.

**Lemma 1.2.60.** *Let  $E, C$  be finite sets, and  $\mu \in \mathbb{K}^{E \times C}$  be a matrix. Let  $e \in E$  and  $c \in C$ , such that  $\mu(e, c) \neq 0$ . Let*

$$\nu: (E \setminus \{e\}) \times (C \setminus \{c\}) \longrightarrow \mathbb{K}, (f, d) \mapsto \mu(f, d) - \frac{\mu(e, d)}{\mu(e, c)} \cdot \mu(f, c)$$

*be the matrix obtained by carrying out a Gauß-Jordan elimination step with the pivot index  $(e, c)$  and then deleting the corresponding row and column. Then*

$$M(\mu).(E \setminus \{e\}) = M(\nu).$$

*Proof.* Let  $\nu' \in \mathbb{K}^{E \times C}$  where for all  $(f, d) \in E \times C$

$$\nu'(f, d) = \begin{cases} \mu(f, d) - \frac{\mu(e, d)}{\mu(e, c)} \cdot \mu(f, c) & \text{if } d \neq c, \\ \frac{\mu(f, c)}{\mu(e, c)} & \text{if } d = c. \end{cases}$$

Since  $\nu'$  arises from  $\mu$  by elementary column operations, we have  $M(\mu) = M(\nu')$  (Remark 1.2.58). Furthermore,  $\nu'(e, c) = 1$  and  $\nu'(e, d) = 0$  for all  $d \in C \setminus \{c\}$ . Let  $E' \subseteq E \setminus \{e\}$  and  $C' \subseteq C \setminus \{c\}$  with  $|E'| = |C'|$ . Then

$$\det(\nu'|_{(E' \cup \{e\}) \times (C' \cup \{c\})}) = \sigma \cdot \det(\nu'|_{E' \times C'}) = \sigma \cdot \det(\nu|_{E' \times C'})$$

for some  $\sigma \in \{-1, 1\}$ . Thus for  $X \subseteq E \setminus \{e\}$

$$\text{idet}(\nu'|_{(X \cup \{e\})}) = \text{idet}(\nu|_X),$$

and consequently  $X$  is independent in  $M(\nu)$ , if and only if  $X \cup \{e\}$  is independent in  $M(\nu') = M(\mu)$ . Therefore  $M(\nu) = M(\mu).(E \setminus \{e\})$ .  $\square$

**Remark 1.2.61.** Let  $M(\mu)$  be a matroid for some  $\mu \in \mathbb{R}^{E \times C}$ . A straightforward consequence of Lemma 1.2.60 is that for  $X \subseteq E$ , we can pivot in a base  $B$  of  $M(\mu)$  with the property that  $B \setminus X$  is a base of  $E \setminus X$  – which exists due to Lemma 1.2.7 – and then restrict the resulting matrix  $\nu$  to  $X \times C_0$  where  $C_0 = \{c \in C \mid \forall b' \in B \setminus X: \nu(b', c) = 0\}$ . Then  $M(\mu).X = M(\nu|_{X \times C_0})$ .  $\bullet$

**Lemma 1.2.62.** *Let  $M = (E, \mathcal{I})$  be a matroid that is representable over  $\mathbb{K}$ , such that for some  $n, r \in \mathbb{N}$ ,  $E = \{e_1, e_2, \dots, e_n\} \neq \emptyset$  and  $B_0 = \{e_1, e_2, \dots, e_r\} \neq \emptyset$  is a base of  $M$ . Then there is a matrix  $\nu \in \mathbb{K}^{E \times B_0}$  such that  $\nu|_{B_0 \times B_0}$  is the identity matrix for  $B_0$  over  $\mathbb{K}$ .*

*Proof.* This is basic linear algebra. Let  $\mu \in \mathbb{K}^{E \times C}$  be a matrix with  $M = M(\mu)$ . Then the row vectors  $\{\mu_b \mid b \in B_0\}$  form a basis of a sub-vector space  $V \subseteq \mathbb{K}^C$ , and since

$B_0$  is a base of  $M$ , we have that  $\mu_e \in V$  for all  $e \in E$ . Thus every  $\mu_e$  has a unique representation as linear combination of vectors from  $\{\mu_b \mid b \in B_0\}$ , and we can set  $\nu(e, b)$  to be the coefficient of  $\mu_b$  with respect to the linear combination representing  $\mu_e$ , for all  $e \in E$  and  $b \in B_0$ . Since a change of basis in a vector space does not affect linear dependency, we have  $M(\mu) = M(\nu)$ .  $\square$

**Remark 1.2.63.** An immediate consequence of Lemma 1.2.62 is, that every matroid  $M = (E, \mathcal{I})$  representable over  $\mathbb{K}$  with  $r = \text{rk}_M(E)$  has a matrix  $\mu \in \mathbb{K}^{E \times \{1, 2, \dots, r\}}$  such that  $M = M(\mu)$  and such that for some base  $B \subseteq E$  of  $M$ , the matrix  $\mu|(B \times \{1, 2, \dots, r\})$  resembles an identity matrix — up to renaming of the rows. Thus we may consider  $\mu^\top$  to be that identity matrix in apposition with a matrix  $A^\top \in \mathbb{K}^{\{1, 2, \dots, r\} \times (E \setminus B)}$ , i.e. that  $\mu = \begin{pmatrix} I_r & A^\top \end{pmatrix}^\top$ . A matrix of this form is called **standard representation**. If  $\mu$  is a standard representation, then  $\nu = \begin{pmatrix} -A & I_{|E|-r} \end{pmatrix}^\top$  has the property that  $M^* = M(\nu)$  and further, that for all  $e, f \in E$ ,  $\langle \mu_e, \nu_f \rangle = 0$  (Corollary 1, [Wel76], p. 143). Thus for every field  $\mathbb{K}$  the family of matroids representable over  $\mathbb{K}$  is closed under duality.  $\bullet$

## 1.3 Single Element Extensions

H.H. Crapo has exhaustively studied extensions of matroids by single elements in [Cra65].

**Definition 1.3.1.** Let  $M = (E, \mathcal{I})$  be a matroid,  $A, B \subseteq E$ . Then  $A$  and  $B$  are a **modular pair in  $M$** , whenever

$$\text{rk}(A \cap B) + \text{rk}(A \cup B) = \text{rk}(A) + \text{rk}(B)$$

holds. A modular pair is called **trivial**, if  $A \subseteq B$  or  $B \subseteq A$ . ■

**Example 1.3.2.** Let  $M = (E, \mathcal{I})$  be a matroid and  $A, B \subseteq E$  such that  $A \cup B \in \mathcal{I}$ . Then  $A$  and  $B$  are a modular pair in  $M$ , since

$$\text{rk}(A \cup B) + \text{rk}(A \cap B) = |A \cup B| + |A \cap B| = |A| + |B| = \text{rk}(A) + \text{rk}(B). \quad \blacktriangle$$

**Lemma 1.3.3.** Let  $M = (E, \mathcal{I})$  be a matroid,  $A, B \subseteq E$ . Then  $A$  and  $B$  are a modular pair in  $M$  if and only if there is a base  $X$  of  $A \cup B$  such that  $X \cap A$  is a base of  $A$  and  $X \cap B$  is a base of  $B$ .

*Proof.* Let  $X$  be a base of  $A \cup B$  such that  $X \cap A$  is a base of  $A$  and  $X \cap B$  is a base of  $B$ . Then  $X \cap A \cap B$  is a base of  $A \cap B$ : Since  $\text{rk}(A) + \text{rk}(B) \geq \text{rk}(A \cup B) + \text{rk}(A \cap B)$  holds by submodularity of the rank function, and since  $|X \cap A \cap B| \leq \text{rk}(A \cap B)$  holds by (I2) and Definition 1.2.14, we obtain

$$\begin{aligned} |X \cap A \cap B| &\leq \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B) - \text{rk}(A \cup B) \\ &= |X \cap A| + |X \cap B| - |X| = |X \cap A \cap B|. \end{aligned}$$

It has been shown in Example 1.3.2 that  $X \cap A$  and  $X \cap B$  are a modular pair in  $M$ . Thus  $A$  and  $B$  are a modular pair in  $M$ , because  $\text{rk}(A) = \text{rk}(X \cap A)$ ,  $\text{rk}(B) = \text{rk}(X \cap B)$ ,  $\text{rk}(A \cap B) = \text{rk}(X \cap A \cap B)$ , and  $\text{rk}(A \cup B) = \text{rk}(X)$  holds. Now let  $A, B \subseteq E$  such that there is no base  $X$  of  $A \cup B$ , for which both  $X \cap A$  is a base of  $A$  and  $X \cap B$  is a base of  $B$ . Then for all bases  $X$  of  $A \cup B$ , for which  $X \cap A$  is a base of  $A$ , and for which  $X \cap A \cap B$  is a base of  $A \cap B$ , there is some  $b \in B \setminus \text{cl}(X \cap B)$ , i.e.  $\text{rk}(B) > \text{rk}(X \cap B)$ . Lemma 1.2.7 guarantees that there is a base  $X$  of  $A \cup B$  with  $\text{rk}(X \cap A) = \text{rk}(A)$  and  $\text{rk}(X \cap A \cap B) = \text{rk}(A \cap B)$ . Thus we obtain that  $\text{rk}(A) + \text{rk}(B) > |X \cap A| + |X \cap B| = |X| + |X \cap A \cap B| = \text{rk}(A \cup B) + \text{rk}(A \cap B)$  holds, which implies that  $A$  and  $B$  are not a modular pair in  $M$ . □

**Definition 1.3.4.** Let  $M = (E, \mathcal{I})$  be a matroid, and let  $C \subseteq \mathcal{F}(M)$  be a set of flats of  $M$ . We call  $C$  a **modular cut of  $M$** , if  $C$  has the properties that

(i) for all  $A, B \in \mathcal{F}(M)$  the implication

$$\text{rk}(A) + \text{rk}(B) = \text{rk}(A \cap B) + \text{rk}(A \cup B) \implies A \cap B \in C$$

holds, and

(ii) for all  $X, Y \in \mathcal{F}(M)$  with  $X \subseteq Y$ , the implication  $X \in C \implies Y \in C$  holds.

The **class of all modular cuts of  $M$**  shall be denoted by

$$\mathcal{M}(M) = \{C \subseteq \mathcal{F}(M) \mid C \text{ is a modular cut of } M\}. \quad \blacksquare$$

**Definition 1.3.5.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $e \notin E$ . The **class of single element extensions of  $M$  by  $e$**  is defined to be

$$\mathcal{X}(M, e) = \{(E \cup \{e\}, \mathcal{I}') \mid \mathcal{I}' \subseteq 2^{E \cup \{e\}}: \mathcal{I}' \cap 2^E = \mathcal{I} \text{ and } (E \cup \{e\}, \mathcal{I}') \text{ is a matroid}\}.$$

Let  $N = (F, \mathcal{J})$  be a matroid.  $N$  shall be called **single element extension of  $M$** , if  $F \setminus E = \{f\}$  and  $N \in \mathcal{X}(M, f)$ . ■

We convince ourselves that  $N$  is indeed an extension of  $M$  in the sense that  $N$  behaves exactly like  $M$  with respect to subsets of  $E$ .

**Lemma 1.3.6.** Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$ , and let  $X \subseteq E$ . Then  $\text{rk}_M(X) = \text{rk}_N(X)$ .

*Proof.* Let  $N = (E \cup \{e\}, \mathcal{I}')$ . Since  $\mathcal{I}' \cap 2^E = \mathcal{I}$ ,  $\{Y \subseteq X \mid Y \in \mathcal{I}\} = \{Y \subseteq X \mid Y \in \mathcal{I}'\}$ . Thus

$$\text{rk}_M(X) = \max\{|Y| \mid Y \subseteq X, Y \in \mathcal{I}\} = \max\{|Y| \mid Y \subseteq X, Y \in \mathcal{I}'\} = \text{rk}_N(X). \quad \square$$

Now we have all the definitions that we need in order to present H.H. Crapo's results [Cra65]:



**Theorem 1.3.7.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $e \notin E$ . Then there is a bijection*

$$\varphi: \mathcal{X}(M, e) \longrightarrow \mathcal{M}(M)$$

*which maps the single element extension  $N$  to the modular cut*

$$\varphi(N) = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}.$$

*Proof.* First, we show that  $\varphi$  is well-defined. Let  $N \in \mathcal{X}(M, e)$  be a single element extension of  $M$ . We have to prove that  $\varphi(N)$  is indeed a modular cut of  $M$ . Let  $F \in \varphi(N)$ , and  $G \in \mathcal{F}(M)$  with  $F \subseteq G$ , then  $e \in \text{cl}_N(F) \subseteq \text{cl}_N(G)$ , thus  $G \in \varphi(N)$ . Now, let  $A, B \in \varphi(N)$  such that

$$\text{rk}_M(A) + \text{rk}_M(B) = \text{rk}_M(A \cap B) + \text{rk}_M(A \cup B).$$

We give an indirect argument for  $A \cap B \in \varphi(N)$ . Assume that  $A \cap B \notin \varphi(N)$ . Then  $e \notin \text{cl}_N(A \cap B)$  thus  $\text{rk}_N((A \cap B) \cup \{e\}) > \text{rk}_N(A \cap B)$ . By Lemma 1.3.6, we have the equation

$$\text{rk}_N(A) + \text{rk}_N(B) = \text{rk}_N(A \cap B) + \text{rk}_N(A \cup B).$$

Furthermore,  $A, B \in \varphi(N)$ , therefore  $e \in \text{cl}_N(A)$ ,  $e \in \text{cl}_N(B)$ , and  $e \in \text{cl}_N(A \cup B)$ , so  $\text{rk}_N(A \cup \{e\}) = \text{rk}_N(A)$ ,  $\text{rk}_N(B \cup \{e\}) = \text{rk}_N(B)$ , and  $\text{rk}_N(A \cup B \cup \{e\}) = \text{rk}_N(A \cup B)$ . This yields

$$\begin{aligned} \text{rk}_N((A \cap B) \cup \{e\}) &> \text{rk}_N(A \cap B) \\ &= \text{rk}_N(A) + \text{rk}_N(B) - \text{rk}_N(A \cup B) \\ &= \text{rk}_N(A \cup \{e\}) + \text{rk}_N(B \cup \{e\}) - \text{rk}_N(A \cup B \cup \{e\}), \end{aligned}$$

which contradicts (R3), the submodularity of  $\text{rk}_N$ , which guarantees

$$\text{rk}_N(A \cup \{e\}) + \text{rk}_N(B \cup \{e\}) \geq \text{rk}_N(A \cup B \cup \{e\}) + \text{rk}_N((A \cap B) \cup \{e\}).$$

Thus  $A \cap B \in \varphi(N)$ , so  $\varphi(N)$  is indeed a modular cut of  $M$ .

— Now, we show that  $\varphi$  is injective. Let  $N, N' \in \mathcal{X}(M, e)$  with  $N \neq N'$ . Without loss of generality we may assume that there is a set  $X \subseteq E \cup \{e\}$  which is independent in  $N$ , yet dependent in  $N'$ . Since  $N$  coincides with  $M$  on  $2^E$ , we obtain that  $e \in X$  and that

$X \setminus \{e\} \in \mathcal{I}$  is independent in  $N$ ,  $N'$  and  $M$ . Let  $F = \text{cl}_M(X \setminus \{e\}) \supseteq X \setminus \{e\}$ . Then  $e \in \text{cl}_{N'}(X \setminus \{e\}) \subseteq \text{cl}_{N'}(F)$  so  $F \in \varphi(N')$ , but  $e \notin \text{cl}_N(F) = \text{cl}_M(F) = F$ , so  $F \notin \varphi(N)$ . Thus  $\varphi(N) \neq \varphi(N')$ .

— It remains to show that  $\varphi$  is surjective. Let  $C$  be a modular cut of  $M$ . We define  $N = (E \cup \{e\}, \mathcal{I}')$  such that

$$\mathcal{I}' = \mathcal{I} \cup \{X \cup \{e\} \mid X \in \mathcal{I}, \text{cl}_M(X) \notin C\}.$$

Assume for a moment that  $N$  is a matroid, then  $\varphi(N) = C$ : Let  $F \in \mathcal{F}(M)$  and let  $X \subseteq F$  be a base of  $F$  in  $M$ . If  $F \in C$ , then  $X \cup \{e\} \notin \mathcal{I}'$ , thus  $e \in \text{cl}_N(F)$ , and therefore  $F \in \varphi(N)$ . If  $F \notin C$ , then  $X \cup \{e\} \in \mathcal{I}'$  and  $e \notin \text{cl}_N(F)$ , therefore  $F \notin \varphi(N)$ .

— We show that  $N$  is indeed a matroid by explicating D.J.A. Welsh's sketch on p. 319 [Wel76]: Observe that the map

$$\rho: 2^{E \cup \{e\}} \longrightarrow \mathbb{N}, X \mapsto \max \{ |Y| \mid Y \subseteq X, Y \in \mathcal{I}' \}$$

satisfies the equation

$$\rho(X) = \begin{cases} \text{rk}_M(X) & \text{if } e \notin X, \\ \text{rk}_M(X \setminus \{e\}) + 1 & \text{if } e \in X, \text{cl}_M(X \setminus \{e\}) \notin C, \\ \text{rk}_M(X \setminus \{e\}) & \text{if } e \in X, \text{cl}_M(X \setminus \{e\}) \in C. \end{cases}$$

Furthermore, we see that

$$\mathcal{I}' = \{ X \subseteq E \cup \{e\} \mid \forall Y \subseteq X: \rho(Y) \geq |Y| \},$$

which is the family of independent sets of a matroid obtained from  $\rho$  by Theorem 1.2.25, whenever  $\rho$  is non-negatively integer-valued, non-decreasing, and submodular. Clearly,  $\rho$  is non-negatively integer-valued. Furthermore,  $\rho$  restricted to  $2^E$  is  $\text{rk}_M$ , thus  $\rho$  is non-decreasing and submodular on  $2^E$ . Let  $X, Y \subseteq E \cup \{e\}$  such that  $e \in X$  and  $e \in Y$ . If  $\text{cl}_M(X \setminus \{e\}) \in C \Leftrightarrow \text{cl}_M(Y \setminus \{e\}) \in C$ , then  $\rho(X) = \text{rk}_M(X \setminus \{e\}) + \alpha$  and  $\rho(Y) = \text{rk}_M(Y \setminus \{e\}) + \alpha$  for the same value of  $\alpha \in \{0, 1\}$ , thus  $\rho$  is non-decreasing because  $\text{rk}_M$  is non-decreasing. Otherwise, if  $X \subseteq Y$  and  $\text{cl}_M(X \setminus \{e\}) \in C \not\Leftrightarrow \text{cl}_M(Y \setminus \{e\}) \in C$ , then  $\text{cl}_M(X) \notin C$  whereas  $\text{cl}_M(Y) \in C$ , because  $C$  is closed under super-flats and  $\text{cl}_M$  preserves set-inclusion (Lemma 1.2.17). But then  $\text{cl}_M(X \setminus \{e\}) \neq \text{cl}_M(Y \setminus \{e\})$ , so  $\text{rk}_M(\text{cl}_M(X)) < \text{rk}_M(\text{cl}_M(Y))$ , thus  $\rho(X) = \text{rk}_M(X \setminus \{e\}) + 1 \leq \text{rk}_M(Y \setminus \{e\}) = \rho(Y)$ .

Therefore  $\rho$  is non-decreasing on its whole domain. Now let  $A, B \subseteq E \cup \{e\}$ , we have to show that the submodular inequality

$$\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B) \quad (1.1)$$

holds. Clearly

$$\rho(A) + \rho(B) = \text{rk}_M(A \setminus \{e\}) + \text{rk}_M(B \setminus \{e\}) + \alpha$$

for some  $\alpha \in \{0, 1, 2\}$  and analogously

$$\rho(A \cap B) + \rho(A \cup B) = \text{rk}_M((A \cap B) \setminus \{e\}) + \text{rk}_M((A \cup B) \setminus \{e\}) + \beta$$

for some  $\beta \in \{0, 1, 2\}$ . Since  $\text{rk}_M$  is a submodular function, we may deduce inequality (1.1) from  $\alpha \geq \beta$  as well as from

$$\beta - \alpha \leq \text{rk}_M(A \setminus \{e\}) + \text{rk}_M(B \setminus \{e\}) - \text{rk}_M((A \cap B) \setminus \{e\}) - \text{rk}_M((A \cup B) \setminus \{e\}).$$

If  $\beta = 2$ , then  $\text{cl}_M((A \cup B) \setminus \{e\}) \notin C$ , therefore  $\text{cl}_M(A \setminus \{e\}) \notin C$  and  $\text{cl}_M(B \setminus \{e\}) \notin C$  because  $C$  is closed under super-flats. So in this case,  $\alpha = 2 \geq \beta$ . If  $\beta = 0$  then clearly  $\alpha \geq \beta$ , too. So the submodular inequality (1.1) holds due to  $\alpha \geq \beta$  unless  $\beta = 1$  and  $\alpha = 0$ . In this case, if

$$\text{rk}_M(A \setminus \{e\}) + \text{rk}_M(B \setminus \{e\}) - \text{rk}_M((A \cap B) \setminus \{e\}) - \text{rk}_M((A \cup B) \setminus \{e\}) \geq 1,$$

then (1.1) follows as mentioned above. We close our argumentation by showing that for  $\beta = 1$  and  $\alpha = 0$ ,

$$\text{rk}_M(A \setminus \{e\}) + \text{rk}_M(B \setminus \{e\}) - \text{rk}_M((A \cap B) \setminus \{e\}) - \text{rk}_M((A \cup B) \setminus \{e\}) = 0$$

can never be the case. There are two ways that lead to  $\beta = 1$ . Assume that  $e \notin A \cap B$ , then  $\text{cl}_M((A \cup B) \setminus \{e\}) \notin C$ . If  $e \in A$ , then  $\text{cl}_M(A \setminus \{e\}) \notin C$  follows, thus  $\alpha \geq 1$ ; similarly if  $e \in B$ . Thus  $e \in A \cap B$  is implied by  $\beta = 1$  and  $\alpha = 0$ . Consequently,  $\text{cl}_M((A \cap B) \setminus \{e\}) \notin C$  is necessary for  $\beta = 1$ . Furthermore, for  $\alpha = 0$  it is necessary that  $\text{cl}_M(A \setminus \{e\}) \in C$  and  $\text{cl}_M(B \setminus \{e\}) \in C$ . But then  $\text{cl}_M(A \setminus \{e\})$  and  $\text{cl}_M(B \setminus \{e\})$  cannot be a modular pair in  $M$ , because  $C$  is closed under intersections of modular pairs yet  $C$  does not contain the intersection of these two flats. This yields

$$\text{rk}_M(A \setminus \{e\}) + \text{rk}_M(B \setminus \{e\}) - \text{rk}_M((A \cap B) \setminus \{e\}) - \text{rk}_M((A \cup B) \setminus \{e\}) \neq 0.$$

So all premises for Theorem 1.2.25 are satisfied,  $N$  is a matroid and therefore  $\varphi$  is surjective.  $\square$

The next lemma summarizes how the family of flats behaves when a matroid is extended.

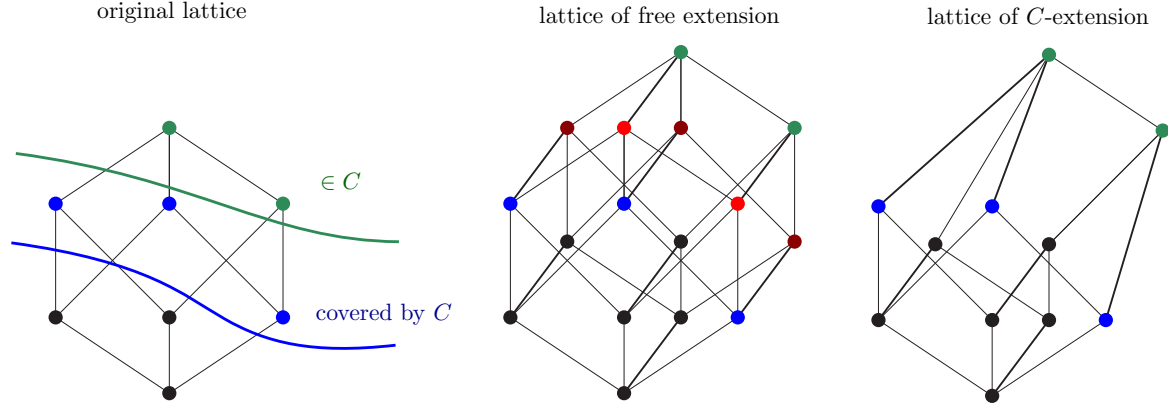


Fig. 1.1 Construction of the lattice of flats of a single element extension from the lattice of flats of the original matroid and the corresponding modular cut (Lemma 1.3.8).

**Lemma 1.3.8.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ , and  $N \in \mathcal{X}(M, e)$ . Furthermore, let  $C \in \mathcal{M}(M)$  be the modular cut of  $M$  where*

$$C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}.$$

Then

$$\begin{aligned} \mathcal{F}(N) &= (\mathcal{F}(M) \setminus C) \cup \{F \cup \{e\} \mid F \in C\} \\ &\quad \cup \{F \cup \{e\} \mid F \in \mathcal{F}(M) \setminus C, \forall x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \notin C\} \\ &= (\mathcal{F}(M) \setminus C) \cup \{F \cup \{e\} \mid F \in \mathcal{F}(M), F \in C \Leftrightarrow \text{cl}_N(F \cup \{e\}) \setminus \{e\} \in C\}. \end{aligned}$$

*Proof.* First, we show that the second equation holds, which is implied by the equation

$$\begin{aligned} &\{F \cup \{e\} \mid F \in C\} \cup \{F \cup \{e\} \mid F \in \mathcal{F}(M) \setminus C, \forall x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \notin C\} \\ &= \{F \cup \{e\} \mid F \in \mathcal{F}(M), F \in C \Leftrightarrow \text{cl}_N(F \cup \{e\}) \setminus \{e\} \in C\}. \end{aligned}$$

If  $F \in C$ , then  $F \in \mathcal{F}(M)$  and  $e \in \text{cl}_N(F)$ , so  $\text{cl}_N(F \cup \{e\}) = F \cup \{e\}$  and therefore  $\text{cl}_N(F \cup \{e\}) \setminus \{e\} = F \in C$ . If  $F \notin C$  and for all  $x \in E \setminus F$  we have  $\text{cl}_M(F \cup \{x\}) \notin C$ , i.e. whenever  $F \notin C$  is not covered by a flat  $G \in C$  in  $\mathcal{F}(M)$ , then  $\text{cl}_N(F \cup \{e\}) =$

$F \cup \{e\}$ , since otherwise  $G = \text{cl}_N(F \cup \{e\}) \setminus \{e\}$  would be a maximal subset of  $E$  with rank  $\text{rk}_N(G) = \text{rk}_N(F) + 1 = \text{rk}_M(F) + 1$ , and therefore we would have  $G \in \mathcal{F}(M)$ . Furthermore, there would have to be some element  $g \in G \setminus F$ , so we would have found a flat  $G \in C$  and with  $\text{cl}_M(F \cup \{g\}) = G$ , contradicting the assumption. Thus we have  $\text{cl}_N(F \cup \{e\}) \setminus \{e\} = (F \cup \{e\}) \setminus \{e\} = F \notin C$ . Therefore we obtain

$$\begin{aligned} & \{F \cup \{e\} \mid F \in C\} \cup \{F \cup \{e\} \mid F \in \mathcal{F}(M) \setminus C, \forall x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \notin C\} \\ & \subseteq \{F \cup \{e\} \mid F \in \mathcal{F}(M), F \in C \Leftrightarrow \text{cl}_N(F \cup \{e\}) \setminus \{e\} \in C\}. \end{aligned}$$

Now let  $F' \in \{F \cup \{e\} \mid F \in \mathcal{F}(M), F \in C \Leftrightarrow \text{cl}_N(F \cup \{e\}) \setminus \{e\} \in C\}$  and  $F = F' \setminus \{e\}$ . If  $F \in C$ , then clearly  $F' \in \{F \cup \{e\} \mid F \in C\}$ . If  $F \notin C$ , we give an indirect argument and assume that

$$F' \notin \{F \cup \{e\} \mid F \in \mathcal{F}(M) \setminus C, \forall x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \notin C\}.$$

So there is some  $x \in E \setminus F$  such that  $\text{cl}_M(F \cup \{x\}) \in C$ . Let  $G = \text{cl}_M(F \cup \{x\})$ , then  $\text{cl}_N(G) = G \cup \{e\}$ , thus  $\text{rk}_N(G \cup \{e\}) = \text{rk}_N(G) = \text{rk}_M(G) = \text{rk}_M(F) + 1 = \text{rk}_N(F) + 1$ . Thus  $\text{cl}_N(F \cup \{e\}) = \text{cl}_N(G) = G \cup \{e\}$ , so  $\text{cl}_N(F \cup \{e\}) \setminus \{e\} = G \in C$ , contradicting the assumption that  $F'$  is a member of the right-hand side set. Consequently, the second equation of the lemma holds.

We show the inclusion of the left-hand side of the first equation in the right-hand side of the first equation. Let  $X \in \mathcal{F}(N)$ , we have to treat two cases. If  $e \notin X$ , then the defining property of  $X \in \mathcal{F}(N)$  is that the strict inequality  $\text{rk}_N(X \cup \{y\}) > \text{rk}_N(X)$  holds for all  $y \in (E \cup \{e\}) \setminus X$ . This together with  $\text{rk}_M = \text{rk}_N|_{2E}$  implies that  $X \in \mathcal{F}(M)$ . Furthermore,  $X = \text{cl}_N(X)$  implies  $e \notin X$  and therefore  $X \notin C$ , so  $X \in \mathcal{F}(M) \setminus C$ . Now assume that  $e \in X$ , and let  $X' = X \setminus \{e\}$ . If  $e \in \text{cl}_N(X')$ , then clearly  $X' \in C$  and so  $X \in \{F \cup \{e\} \mid F \in C\}$ . If otherwise  $e \notin \text{cl}_N(X')$ , then we must have  $\text{rk}_N(X') = \text{rk}_N(X) - 1$ , and  $X' \in \mathcal{F}(N)$  because  $\text{cl}_N(X') \subsetneq \text{cl}_N(X) = X = X' \cup \{e\}$ . As a consequence, we obtain that  $X' \in \mathcal{F}(M)$  and  $X' \notin C$ . Assume that for some  $y \in E \setminus X'$ ,  $\text{cl}_M(X' \cup \{y\}) \in C$ , then  $e \in \text{cl}_N(X' \cup \{y\})$  so  $X = \text{cl}_N(X') \cup \{e\}$  would be a proper subset of the flat  $\text{cl}_N(X' \cup \{y\})$  of rank  $\text{rk}_N(X') + 1$ , but  $\text{rk}_N(X) = \text{rk}_N(X') + 1$ , which is impossible. Therefore, for all  $y \in E \setminus X'$  we have  $\text{cl}_M(X' \cup \{y\}) \notin C$ . Thus we obtain

$$X \in \{F \cup \{e\} \mid F \in \mathcal{F}(M) \setminus C, \forall x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \notin C\}.$$

Finally, we show the inclusion of the right-hand side of the first equation in the left-hand side of the first equation. Let  $X \in \mathcal{F}(M) \setminus C$ , then  $e \notin \text{cl}_N(X)$ , so  $\text{cl}_N(X) =$

$\text{cl}_M(X) = X$ , thus  $X \in \mathcal{F}(N)$ . Let  $X \in C$ , and let  $X' = X \cup \{e\}$ . Then  $\text{cl}_N(X) = \text{cl}_M(X) \cup \{e\} = X \cup \{e\}$ , and therefore  $X \cup \{e\} \in \mathcal{F}(N)$ . Now let  $X \notin C$  and for all  $y \in E \setminus X$ ,  $\text{cl}_M(X \cup \{y\}) \notin C$ . Let  $G = \text{cl}_N(X \cup \{e\})$ . Assume that we have the proper inclusion  $X \cup \{e\} \subsetneq G$ , then  $\text{rk}_N(G) = \text{rk}_N(X) + 1$  yields that there is some  $g \in G \setminus (X \cup \{e\})$  such that  $\text{cl}_N(X \cup \{g\}) = G$ . This leads us to the contradiction  $G \setminus \{e\} = \text{cl}_M(X \cup \{g\}) \in C$ . Therefore we must have  $X \cup \{e\} = G \in \mathcal{F}(N)$ .  $\square$

## 1.4 Theorems of Hall, Rado, Ore, and Perfect

D.J.A. Welsh gives the following very elegant generalization of the theorems of Rado and Hall in [Wel71]. From this generalization, the theorems of Hall, Rado, Ore, and Perfect follow as an easy corollary each. Before we present the theorem, we need some definitions.

**Definition 1.4.1.** Let  $I$  and  $E$  be sets. A **family of subsets** of  $E$  indexed by  $I$  is a map  $A_\bullet: I \rightarrow 2^E$  with domain  $I$ , such that for every  $i \in I$  the image  $A_i$  is a subset of  $E$ . We denote such a family by writing  $(A_i)_{i \in I} \subseteq E$ , or shorter  $(A_i)_{i \in I}$  whenever  $E$  is clear from the context. We call  $(A_i)_{i \in I}$  **finite** if  $I$  is finite. Further, we call  $(A_i)_{i \in I}$  a **family of non-empty subsets**, if for all  $i \in I$ ,  $A_i \neq \emptyset$ . ■

**Definition 1.4.2.** Let  $I, E$  be sets, and let  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of subsets of  $E$ . A **system of representatives** is a map  $x_\bullet: I \rightarrow E$  such that there is a bijection  $\sigma: I \rightarrow I$  with  $x_i \in A_{\sigma(i)}$  for all  $i \in I$ . We will denote such a family by writing  $(x_i)_{i \in I} \in \mathcal{A}$ . A system of representatives is called **system of distinct representatives**, if  $x_\bullet$  is an injective map. A **transversal** of  $\mathcal{A}$  is a subset  $T \subseteq E$  such that there is a bijection  $\sigma: T \rightarrow I$  with  $t \in A_{\sigma(t)}$  for all  $t \in T$ . A **partial transversal** of  $\mathcal{A}$  is a subset  $P \subseteq E$  such that there is an injection  $\iota: P \rightarrow I$  with  $t \in A_{\iota(t)}$  for all  $t \in P$ . If  $P$  is a partial transversal of  $\mathcal{A}$ , we define the **defect** of  $P$  to be  $|I| - |P|$ , i.e. the cardinality of those indices in  $I$  that are not in the image of the corresponding  $\iota$ . ■

**Theorem 1.4.3.** Let  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a finite family of non-empty subsets of  $E$ , and let  $\mu: 2^E \rightarrow \mathbb{N}$  be a map with the properties that

- (i) for all  $X \subseteq Y \subseteq E$ ,  $\mu(X) \leq \mu(Y)$ , and
- (ii) for all  $X, Y \subseteq E$ ,  $\mu(X) + \mu(Y) \geq \mu(X \cap Y) + \mu(X \cup Y)$ .

Then there is a system of representatives  $(x_i)_{i \in I} \in \mathcal{A}$  with the property that

- (1) for all  $J \subseteq I$ ,  $\mu(\{x_i \mid i \in J\}) \geq |J|$

if and only if  $\mathcal{A}$  has the property that

- (2) for all  $J \subseteq I$ ,  $\mu(\bigcup_{i \in J} A_i) \geq |J|$ .

This proof of the theorem follows the course of [Wel71] — a very nice version of which can be found on p.100 of [Wel76] — and it focuses more on details than brevity.

*Proof.* Let  $(x_i)_{i \in I} \in \mathcal{A}$  be such a system of representatives, that (1) holds, and let  $\sigma: I \rightarrow I$  be a permutation that has the property  $x_i \in A_{\sigma(i)}$  for all  $i \in I$ . Let  $J \subseteq I$ , then  $\{x_{\sigma^{-1}(i)} \mid i \in J\} \subseteq \bigcup_{i \in J} A_i$ . By (i)  $\mu$  is non-decreasing, therefore

$$|J| = |\sigma^{-1}[J]| \leq \mu(\{x_i \mid i \in \sigma^{-1}[J]\}) \leq \mu\left(\bigcup_{i \in J} A_i\right).$$

For the converse implication, we employ induction on the integer vector  $v = (|A_i|)_{i \in I}$ . The base case is  $v_i = 1$  for all  $i \in I$  where every  $A_i$  is a singleton set, thus for any system of representatives  $(x_i)_{i \in I} \in \mathcal{A}$ , we have  $A_i = \{x_{\sigma^{-1}(i)}\}$  for all  $i \in I$ . Therefore,  $\{x_i \mid i \in \sigma^{-1}[J]\} = \bigcup_{i \in J} A_i$  and the equivalence is obvious. For the induction step, let  $i' \in I$  such that  $|A_{i'}| > 1$ . In this case, we claim that there is some  $x \in A_{i'}$ , such that the derived family  $\mathcal{A}' = (A'_i)_{i \in I}$  where  $A'_i = A_i$  if  $i \neq i'$ , and  $A'_{i'} = A_{i'} \setminus \{x\}$  still has the property (2). Assume that this claim is false, then for any  $\{x, y\} \neq \subseteq A_{i'}$  there are  $J_x, J_y \subseteq I \setminus \{i'\}$  such that

$$\begin{aligned} \mu\left((A_{i'} \setminus \{x\}) \cup \bigcup_{i \in J_x} A_i\right) &\leq |J_x| < |J_x| + 1, \text{ and} \\ \mu\left((A_{i'} \setminus \{y\}) \cup \bigcup_{i \in J_y} A_i\right) &\leq |J_y| < |J_y| + 1. \end{aligned}$$

We use the submodularity (ii) of  $\mu$  in order to obtain that

$$\mu\left((A_{i'} \setminus \{x\}) \cup \bigcup_{i \in J_x} A_i\right) + \mu\left((A_{i'} \setminus \{y\}) \cup \bigcup_{i \in J_y} A_i\right) \geq \mu(B_\cap) + \mu\left(A_{i'} \cup \bigcup_{i \in J_x \cup J_y} A_i\right)$$

where

$$B_\cap = \left((A_{i'} \setminus \{x\}) \cup \bigcup_{i \in J_x} A_i\right) \cap \left((A_{i'} \setminus \{y\}) \cup \bigcup_{i \in J_y} A_i\right).$$

Clearly,  $\bigcup_{i \in J_x \cap J_y} A_i \subseteq B_\cap$ , and since  $\mu$  is non-decreasing due to property (i), we obtain that

$$\mu(B_\cap) + \mu\left(A_{i'} \cup \bigcup_{i \in J_x \cup J_y} A_i\right) \geq \mu\left(\bigcup_{i \in J_x \cap J_y} A_i\right) + \mu\left(A_{i'} \cup \bigcup_{i \in J_x \cup J_y} A_i\right).$$



We now may use property (2) with  $J = J_x \cup J_y \cup \{i'\}$ , and  $J = J_x \cap J_y$ , respectively. We add the respective inequalities and obtain

$$\mu \left( A_{i'} \cup \bigcup_{i \in J_x \cup J_y} A_i \right) + \mu \left( \bigcup_{i \in J_x \cap J_y} A_i \right) \geq (|J_x \cup J_y| + 1) + |J_x \cap J_y| = |J_x| + |J_y| + 1.$$

Yet, this yields

$$\mu \left( (A_{i'} \setminus \{x\}) \cup \bigcup_{i \in J_x} A_i \right) + \mu \left( (A_{i'} \setminus \{y\}) \cup \bigcup_{i \in J_y} A_i \right) \geq |J_x| + |J_y| + 1$$

which contradicts

$$\mu \left( (A_{i'} \setminus \{x\}) \cup \bigcup_{i \in J_x} A_i \right) + \mu \left( (A_{i'} \setminus \{y\}) \cup \bigcup_{i \in J_y} A_i \right) \leq |J_x| + |J_y|.$$

Thus the claim holds, and since  $|A_{i'}| < v_{i'}$ , we may use the induction hypothesis on  $\mathcal{A}'$  which guarantees the existence of a system of representatives  $(x_i)_{i \in I}$  with property (1). Every such  $(x_i)_{i \in I}$  is also a system of representatives of  $\mathcal{A}$ , therefore  $(x_i)_{i \in I}$  with (1) exists.  $\square$

**Corollary 1.4.4** (Hall). *Let  $\mathcal{A} = (A_i)_{i \in I}$  be a finite family of sets, then  $\mathcal{A}$  has a transversal if and only if for all  $J \subseteq I$ ,*

$$\left| \bigcup_{i \in J} A_i \right| \geq |J|.$$

*Proof.* Apply Theorem 1.4.3 with  $\mu(X) = |X|$  and  $E = \bigcup_{i \in I} A_i$ .  $\square$

**Corollary 1.4.5** (Rado). *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $\mathcal{A} = (A_i)_{i \in I}$  be a finite family of subsets of  $E$ , then  $\mathcal{A}$  has a transversal which is independent in  $M$  if and only if for all  $J \subseteq I$ ,*

$$\text{rk}_M \left( \bigcup_{i \in J} A_i \right) \geq |J|.$$

*Proof.* Apply Theorem 1.4.3 with  $\mu(X) = \text{rk}_M(X)$ .  $\square$

**Corollary 1.4.6** (Ore). *Let  $\mathcal{A} = (A_i)_{i \in I}$  be a finite family of sets, and  $d \in \mathbb{N}$ , then  $\mathcal{A}$  has a partial transversal  $T$  with defect  $\leq d$  if and only if for all  $J \subseteq I$ ,*

$$\left| \bigcup_{i \in J} A_i \right| \geq |J| - d.$$

*Proof.* Apply Theorem 1.4.3 with  $\mu(X) = |X| + d$  and  $E = \bigcup_{i \in I} A_i$ .  $\square$

**Corollary 1.4.7** (Perfect). *Let  $M = (E, \mathcal{I})$  be a matroid,  $d \in \mathbb{N}$ , and let  $\mathcal{A} = (A_i)_{i \in I}$  be a finite family of subsets of  $E$ , then  $\mathcal{A}$  has a partial transversal  $T$  with defect  $\leq d$  which is independent in  $M$  if and only if for all  $J \subseteq I$ ,*

$$\text{rk}_M \left( \bigcup_{i \in J} A_i \right) \geq |J| - d.$$

*Proof.* Apply Theorem 1.4.3 with  $\mu(X) = \text{rk}_M(X) + d$ .  $\square$

### 1.4.1 Matroids Induced by Bipartite Graphs

In this section, we describe how matchings in bipartite graphs can be used to induce a matroid on one color-class from a matroid given on the other color-class. The class of transversal matroids consists of those matroids, which can be obtained from a free matroid by bipartite matroid induction.

**Definition 1.4.8.** *Let  $D = (V, A)$  be a digraph, and let  $M \subseteq \binom{V}{2}$  be a set of unordered pairs of vertices of  $D$ . We call  $M$  a **matching in  $D$** , if the sets in  $M$  are pair-wise disjoint, and if for every  $\{x, y\} \in M$ , there is an arc  $(x, y) \in A$  or  $(y, x) \in A$ .  $\blacksquare$*

**Definition 1.4.9.** *Let  $A, B$  be sets with  $A \cap B = \emptyset$  and  $\Delta \subseteq A \times B$ . We call the digraph  $D = (A \dot{\cup} B, \Delta)$  the **directed bipartite graph** for  $\Delta$  from  $A$  to  $B$ .  $\blacksquare$*

If there are no isolated vertices in the directed bipartite graph  $(A \dot{\cup} B, \Delta)$ , then the partition of its vertices into  $A$  and  $B$  can be deduced from  $\Delta$ . Thus it is reasonable to identify the directed bipartite graph  $(A \dot{\cup} B, \Delta)$  with its arcs  $\Delta$ .

**Definition 1.4.10.** *Let  $A, B$  be finite sets with  $A \cap B = \emptyset$ , and let  $\Delta \subseteq A \times B$ . The **arc system of  $(A \dot{\cup} B, \Delta)$**  shall be denoted by  $\mathcal{A}_\Delta$ . It is defined to be the family  $\mathcal{A}_\Delta = (A_i)_{i \in B} \subseteq A$  where*

$$A_b = \{a \in A \mid (a, b) \in \Delta\}$$

*for every  $b \in B$ .  $\blacksquare$*

**Theorem 1.4.11.** *Let  $D, E$  be finite sets with  $D \cap E = \emptyset$ ,  $M = (E, \mathcal{I})$  be a matroid, and let  $\Delta \subseteq D \times E$ . Furthermore, let  $N = (D, \mathcal{I}')$  be such that  $\mathcal{I}' \subseteq 2^D$  with the defining property that for all  $X \subseteq D$ ,  $X \in \mathcal{I}'$  if and only if  $X$  is a partial transversal of the arc system  $\mathcal{A}_\Delta$  such that there is an injective map  $\iota: X \rightarrow E$  with  $x \in A_{\iota(x)}$  for all*

$x \in X$  with the additional property that  $\{\iota(x) \mid x \in X\}$  is independent in  $M$ . Then  $N$  is a matroid.

The following proof is based on the proof in [Wel76], p.119.

*Proof.* Let  $\mu: 2^D \rightarrow \mathbb{N}$  be the map where

$$\mu(X) = \text{rk}_M(\{e \in E \mid \exists x \in X: x \in A_e\})$$

for every  $X \subseteq D$ . Clearly,  $\mu(\emptyset) = \text{rk}_M(\emptyset) = 0$  and  $\mu$  is non-decreasing and submodular. By Theorem 1.2.25, the set

$$\mathcal{I}'' = \{X \subseteq D \mid \forall X' \subseteq X: \mu(X') \geq |X'|\}$$

defines a matroid  $N' = (D, \mathcal{I}'')$ . We show that  $\mathcal{I}' = \mathcal{I}''$ . Let  $X \in \mathcal{I}'$ , and let  $\iota: X \rightarrow E$  an injective map such that  $\iota[X] \in \mathcal{I}$  and for all  $x \in X$ ,  $x \in A_{\iota(x)}$ . Then, clearly, for all  $X' \subseteq X$ ,  $\iota[X'] \subseteq \{e \in E \mid \exists x' \in X': x' \in A_e\}$ . Therefore for all  $X' \subseteq X$ ,

$$|X'| = \text{rk}_M(\iota[X']) \leq \text{rk}_M(\{e \in E \mid \exists x' \in X': x' \in A_e\}) = \mu(X'),$$

thus  $X \in \mathcal{I}''$ . Conversely, assume that  $X \in \mathcal{I}''$ . We flip around the arc system and consider the following family of subsets of  $E$ : Let  $\mathcal{B}_X = (B_i)_{i \in X} \subseteq E$  be the family of subsets of  $E$  where for  $x \in X$ , the subset  $B_x = \{e \in E \mid x \in A_e\}$  consists of all elements of  $E$  that  $x$  can pair with in  $\Delta$ . By Rado's Theorem (Corollary 1.4.5), the family  $\mathcal{B}_X$  has a transversal  $Y \subseteq E$  that is independent in  $M$ , if and only if for all  $X' \subseteq X$  we have the inequality

$$\text{rk}_M\left(\bigcup_{x' \in X'} B_{x'}\right) \geq |X'|.$$

But  $\bigcup_{x' \in X'} B_{x'} = \bigcup_{x' \in X'} \{e \in E \mid x' \in A_e\} = \{e \in E \mid \exists x' \in X': x' \in A_e\}$ , which gives that  $\text{rk}_M(\bigcup_{x' \in X'} B_{x'}) = \mu(X')$ . By definition,  $X \in \mathcal{I}''$  implies that for all  $X' \subseteq X$ ,  $\mu(X') \geq |X'|$ . Thus we may infer that there is an  $M$ -independent transversal  $Y$  of  $\mathcal{B}_X$ . This gives rise to a bijective map  $\sigma: Y \rightarrow X$  such that for every  $y \in Y$  we have  $y \in B_{\sigma(y)}$ . Yet  $y \in B_{\sigma(y)}$  implies that  $\sigma(y) \in A_y$ . Therefore there is an injective map  $\tilde{\iota}: X \rightarrow E$  with  $\tilde{\iota}(x) = \sigma^{-1}(x)$  which witnesses that  $X$  is a partial transversal of  $\mathcal{A}_\Delta$ , and therefore  $X \in \mathcal{I}'$ .  $\square$

**Remark 1.4.12.** Obviously, the premise  $D \cap E = \emptyset$  in Theorem 1.4.11 may be dropped, since we may give the elements of  $D$  distinct names  $D'$ , apply the construction, and then rename the elements of the so obtained matroid back to  $D$ .  $\bullet$

**Definition 1.4.13.** Let  $D, E$  be finite sets,  $\Delta \subseteq D \times E$ , and let  $M_0 = (E, \mathcal{I})$  be a matroid. The **matroid induced by  $\Delta$  from  $M_0$**  shall be the pair  $M(\Delta, M_0) = (D, \mathcal{I}_{\Delta, M_0})$  where  $\mathcal{I}_{\Delta, M_0} \subseteq 2^D$  consists of all partial transversals  $X$  of the family  $\mathcal{A}_\Delta$  that can be admissibly injected into an independent set of  $M_0$ , i.e. there is an injective map  $\iota: X \rightarrow E$  such that for all  $x \in X$ ,  $x \in A_{\iota(x)}$  and  $\iota[X] \in \mathcal{I}$ . ■

## 1.4.2 Transversal Matroids

In this section, we provide the definition of transversal matroids and forestall those important properties of transversal matroids, that we need in order to develop the theory of routings in directed graphs — which, in turn, is essential for defining gammoids. Further properties of transversal matroids are treated in Section 2.3.

**Definition 1.4.14.** Let  $E, I$  be a finite sets, and  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of subsets. The **transversal matroid presented by  $\mathcal{A}$**  shall be the pair  $M(\mathcal{A}) = (E, \mathcal{I}_\mathcal{A})$  where  $\mathcal{I}_\mathcal{A} \subseteq 2^E$  with the property that for all  $X \subseteq E$ ,  $X \in \mathcal{I}_\mathcal{A}$  if and only if  $X$  is a partial transversal of  $\mathcal{A}$ . ■

**Corollary 1.4.15.** Let  $E, I$  be a finite sets, and  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of subsets. Then  $M(\mathcal{A}) = (E, \mathcal{I}_\mathcal{A})$  is a matroid.

*Proof.* Let  $M_0 = (I, 2^I)$  be the free matroid on  $I$ , and let  $\Delta = \{(e, i) \in E \times I \mid e \in A_i\}$ . Then  $M(\mathcal{A}) = M(\Delta, M_0)$  is the matroid induced by  $\Delta$  from  $M_0$ , which is a matroid by Theorem 1.4.11. □

We just proved that the maximal partial transversals are bases of a matroid  $M(\mathcal{A})$ .

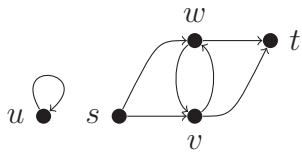
**Corollary 1.4.16.** Let  $E, I$  be finite sets, and let  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of subsets. Two maximal partial transversals  $S, T \subseteq E$  of  $\mathcal{A}$  have  $|S| = |T|$ .

**Definition 1.4.17.** Let  $M = (E, \mathcal{I})$  be a matroid. We call  $M$  **transversal matroid**, if there is a finite family of subsets  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$ , such that  $M = M(\mathcal{A})$ , i.e.  $M$  is the transversal matroid presented by  $\mathcal{A}$ . ■

## 1.5 Directed Graphs

In this section, we present the basic definitions and properties of directed graphs used in the course of this work. We aim to be consistent regarding terminology with the monograph *Digraphs: Theory, Algorithms, and Applications* by J. Bang-Jensen and G. Gutin [BJG09], although we may divert from it in technical details since we do not need the full generality of [BJG09].

**Definition 1.5.1.** A pair  $D = (V, A)$  is called **directed graph**, or shorter **digraph**, whenever  $V$  is a finite set and  $A \subseteq V \times V$ . Every  $v \in V$  is called **vertex** of  $D$  and every  $a = (u, v) \in A$  is called **arc** of  $D$ . Furthermore,  $u$  is called the **tail** of the arc  $a$  and  $v$  is called the **head** of  $a$ . We also say that  $a = (u, v)$  is an arc that **leaves**  $u$  and **enters**  $v$ , or shorter that  $a$  **goes from**  $u$  **to**  $v$ . Furthermore,  $u$  and  $v$  are the **end vertices** of  $a$ , and we say that  $u$  and  $v$  are **incident** with  $a$ . Two vertices that are incident with the same arc are called **adjacent**. An arc  $a = (u, v)$  with  $u = v$  is called a **loop**. ■



**Example 1.5.2.** Consider  $V = \{u, v, w, s, t\}_{\neq}$  and  $A = \{(u, u), (v, w), (w, v), (s, v), (s, w), (w, t), (v, t)\}$ . Then  $D = (V, A)$  is a directed graph. We can represent  $D$  by a figure, where each vertex is represented by a small circle

which may or may not have a name next to it, and where each arc is represented by an arrow which points from the tail vertex circle of the arc to the head vertex circle of the arc. The figure on the left represents  $D$  as above. ▲

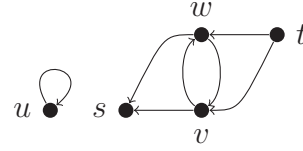
Clearly, we can construct another directed graph  $D^{\text{opp}}$  from any directed graph  $D$  by swapping heads and tails of all arcs of  $D$ , thus effectively reorienting all arcs to their opposite direction.

**Definition 1.5.3.** Let  $D = (V, A)$  be a digraph. The **opposite digraph** is defined to be the unique directed graph  $D^{\text{opp}} = (V, A^{\text{opp}})$  with the property

$$(u, v) \in A^{\text{opp}} \Leftrightarrow (v, u) \in A.$$

It is easy to see that  $(D^{\text{opp}})^{\text{opp}} = D$ . ■

**Example 1.5.4.** Consider the digraph  $D$  from Example 1.5.2. Its opposite digraph  $D^{\text{opp}}$  has the same vertex set as  $D$ , whereas the arcs are reversed to  $A^{\text{opp}} = \{(u, u), (w, v), (v, w), (v, s), (w, s), (t, w), (t, v)\}$ .  $\blacktriangle$



**Definition 1.5.5.** Let  $D = (V, A)$  be a digraph,  $x \in V$ . We call  $x$  a **source** in  $D$  if  $x$  is never the head of an arc in  $D$ . Analogously, we call  $x$  a **sink** in  $D$  if  $x$  is never the tail of an arc in  $D$ .  $\blacksquare$

From this definition it is clear that  $x$  is a source in  $D$  if and only if  $x$  is a sink in  $D^{\text{opp}}$ , and analogously,  $x$  is a sink in  $D$  if and only if  $x$  is a source in  $D^{\text{opp}}$ .

**Example 1.5.6.** Consider the digraph  $D$  from Example 1.5.2. The vertex  $s$  is a source in  $D$  and a sink in  $D^{\text{opp}}$ , the vertex  $t$  is a sink in  $D$  and a source in  $D^{\text{opp}}$ , whereas the vertices  $u, v, w$  are neither sinks nor sources in both  $D$  and  $D^{\text{opp}}$ .  $\blacktriangle$

**Definition 1.5.7.** Let  $D = (V, A)$  be a digraph, the **outer-extension operator in  $D$**  shall be the map

$$\vec{\bullet}^D: 2^V \longrightarrow 2^V, X \mapsto \vec{X}^D$$

where

$$\vec{X}^D = X \cup \{v \in V \mid \exists x \in X: (x, v) \in A\}.$$

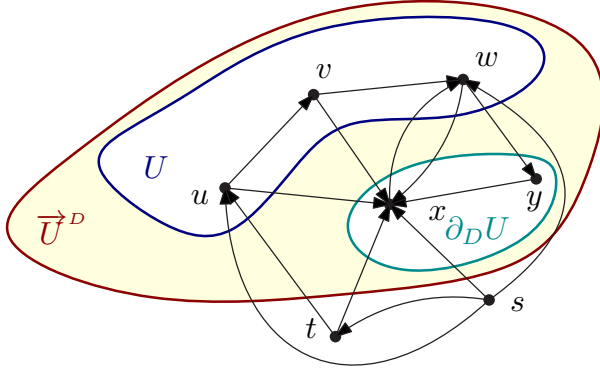
We call  $\vec{X}^D$  the **outer extension of  $X$  in  $D$** . If the digraph is clear from the context, we write  $\vec{\bullet}$  for  $\vec{\bullet}^D$ . The **outer-margin operator in  $D$**  is defined to be the map

$$\partial_D \bullet: 2^V \longrightarrow 2^V, X \mapsto \partial_D X$$

where

$$\partial_D X = \vec{X}^D \setminus X.$$

We call  $\partial_D X$  the **outer margin of  $X$  in  $D$** . Again, if no confusion can occur, we write  $\partial \bullet$  as a shorthand for  $\partial_D \bullet$ .  $\blacksquare$



**Example 1.5.8.** Consider the digraph shown on the left. Let  $U = \{u, v, w\}$ . The outer extension of  $U$  is  $\vec{U}^D = \{u, v, w, x, y\}$  and the outer margin of  $U$  is  $\partial_D U = \{x, y\}$ .  $\blacktriangle$

**Definition 1.5.9.** Let  $D = (V, A)$  be a digraph,  $\mathbb{N} \ni n > 0$ , and  $w = (w_i)_{i=1}^n \in V^n$ . Then  $w$  is a **walk** in  $D$ , if for all  $i \in \{1, 2, \dots, n-1\}$  there is an arc  $(w_i, w_{i+1}) \in A$ . The **start vertex** — or **initial vertex** — of  $w$  is  $w_1$ , and the **end vertex** — or **terminal vertex** — of  $w$  is denoted by  $w_{-1} = w_n$ . The set of vertices **visited by  $w$**  is denoted by  $|w| = \{w_1, w_2, \dots, w_n\}$ . The set of all arcs **traversed by  $w$**  is denoted by  $|w|_A = \{(w_i, w_{i+1}) \mid i = 1, 2, \dots, n-1\}$ . The set of all walks in  $D$  is denoted by

$$\mathbf{W}(D) = \left\{ w \in \bigcup_{n=1}^{\infty} V^n \mid w \text{ is a walk in } D \right\}.$$

The **length** of the walk  $w = (w_i)_{i=1}^n$  is  $n$ . A walk  $w$  is called **trivial**, if its length is 1. We say that a walk  $w$  is a **path**, if no vertex is visited twice by  $w$ , i.e. if  $w_i = w_j$  already implies  $i = j$ . The family of paths in  $D$  is denoted by

$$\mathbf{P}(D) = \{p \in \mathbf{W}(D) \mid p \text{ is a path}\}.$$

Furthermore, for all  $u, v \in V$ , we shall denote the set of all walks from  $u$  to  $v$  in  $D$  by

$$\mathbf{W}(D; u, v) = \{w \in \mathbf{W}(D) \mid w_1 = u \text{ and } w_{-1} = v\}$$

and the set of all paths from  $u$  to  $v$  in  $D$  by

$$\mathbf{P}(D; u, v) = \{p \in \mathbf{P}(D) \mid p_1 = u \text{ and } p_{-1} = v\}. \quad \blacksquare$$

Instead of  $w = (w_i)_{i=1}^n = (w_1, w_2, \dots, w_n)$  we shall also write  $w_1 w_2 \dots w_n$ . Furthermore, we set the convention that  $(w_1 w_2 \dots w_n)^i$  shall denote the walk consisting of  $i$ -iterations of the visited-vertex sequence  $w_1 w_2 \dots w_n$ , i.e.  $(abc)^3$  shall denote the non-path walk  $abcabcabc$ .

**Definition 1.5.10.** Let  $D = (V, A)$  be a digraph, and let  $w = (w_i)_{i=1}^n \in \mathbf{W}(D)$  and  $q = (q_i)_{i=1}^m \in \mathbf{W}(D)$  be walks. Then we say that  $w$  is **compatible with  $q$** , if  $w_n = q_1$ . In that case, we define the **concatenation of  $w$  and  $q$**  to be the walk  $w.q = w_1w_2 \dots w_nq_2q_3 \dots q_m$ . ■

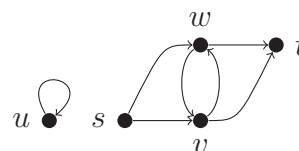
**Example 1.5.11.** Consider again the digraph  $D$  from

Example 1.5.2. The trivial walks in  $D$  are  $u, v, w, s,$

and  $t$ . The paths in  $D$  of length greater than 1 are  $sw,$

$sv, vw, vt, wv, wt, svw, svt, swt, svwt, swvt$ . The non-

path walks in  $D$  are  $u^i$  ( $i \geq 2$ ),  $w(vw)^j, v(wv)^j, w(vw)^jv, v(wv)^jw, sw(vw)^j, sv(wv)^j, sw(vw)^jv, sv(wv)^jw, w(vw)^jt, v(wv)^jt, w(vw)^jvt, v(wv)^jvt, sw(vw)^jt, sv(wv)^jt, sw(vw)^jvt,$  and  $sv(wv)^jvt$  ( $j \geq 1$ ). ▲



**Definition 1.5.12.** Let  $D = (V, A)$  be a digraph. A walk  $w = (w_i)_{i=1}^n \in \mathbf{W}(D)$  is called **cycle walk**, or shorter, **cycle**, if  $w_1 = w_n$  and  $w_2w_3 \dots w_n$  is a path. ■

Observe that there is no “empty walk”, thus the trivial walks are not considered to be cycles.

**Definition 1.5.13.** Let  $D = (V, A)$  be a digraph.  $D$  shall be called **acyclic digraph**, if every walk  $w \in \mathbf{W}(D)$  is a path, i.e. whenever  $\mathbf{W}(D) = \mathbf{P}(D)$ . ■

**Corollary 1.5.14.** Let  $D = (V, A)$  be a digraph. Then  $D$  is acyclic if and only if there is no cycle walk  $w \in \mathbf{W}(D)$ .

## 1.5.1 Routings and Transversals

In this section, we introduce a correspondence between families of pair-wise vertex disjoint paths in digraphs and transversals of certain families of sets, which will be valuable for the study of gammoids.

**Definition 1.5.15.** Let  $D = (V, A)$  be a digraph, and  $X, Y \subseteq V$ . A **routing** from  $X$  to  $Y$  in  $D$  is a family of paths  $R \subseteq \mathbf{P}(D)$  such that

- (i) for each  $x \in X$  there is some  $p \in R$  with  $p_1 = x$ ,
- (ii) for all  $p \in R$  the end vertex  $p_{-1} \in Y$ , and
- (iii) for all  $p, q \in R$ , either  $p = q$  or  $|p \cap q| = \emptyset$ .



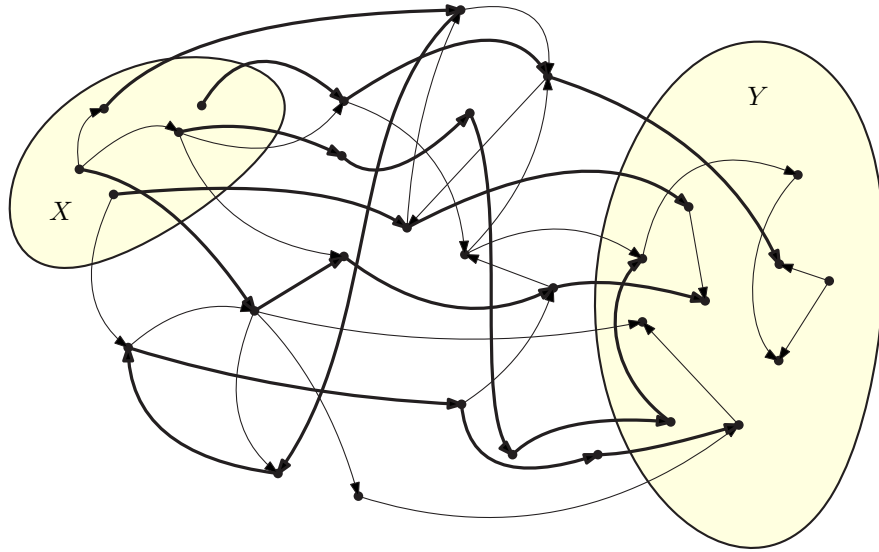


Fig. 1.2 Example of a routing  $R: X \rightrightarrows Y$  in a digraph. The paths that belong to  $R$  are depicted by bold arrows.

We will write  $R: X \rightrightarrows Y$  in  $D$  as a shorthand for “ $R$  is a routing from  $X$  to  $Y$  in  $D$ ”, and if no confusion is possible, we just write  $X \rightrightarrows Y$  instead of  $R$  and  $R: X \rightrightarrows Y$ . A routing  $R$  is called **linking** from  $X$  to  $Y$ , if it is a routing onto  $Y$ , i.e. whenever  $Y = \{p_{-1} \mid p \in R\}$ . ■

**Remark 1.5.16.** We defined a routing to consist of paths only, but for most of what we are concerned with when using the concept of a routing in  $D$ , the property that the walks of  $R$  are indeed paths is not crucial. Let  $R' \subseteq \mathbf{W}(D)$  be a family of walks such that for all  $w, q \in R'$  the implication  $|w| \cap |q| \neq \emptyset \Rightarrow w = q$  holds. Now let  $w \in R' \setminus \mathbf{P}(D)$  be a non-path walk in  $R'$ . Then there is a vertex  $v \in |w|$  such that  $w = w_0 v p' v w_1$  where  $w_0, w_1, p' \in \mathbf{W}(D)$  such that  $v p' v$  is a cycle walk. Clearly  $\hat{w} = w_0 v w_1$  has less such cycle sub-walks than  $w$  and  $|w_0 v w_1| \subseteq |w|$ , thus we may iteratively straighten out any cycles in the walks from  $R'$  without changing the start and the end vertices. The property, that the family of walks consists of pair-wise vertex disjoint walks, remains intact throughout the procedure. The result of this process is a family of paths which is a linking from  $\{w_1 \mid w \in R'\}$  onto  $\{w_{-1} \mid w \in R'\}$  in  $D$ . •

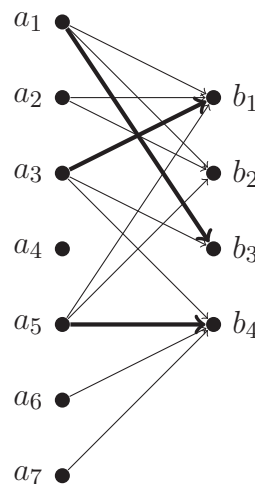
The straightening out of cycle walks in routings is a special case of the following construction.

**Definition 1.5.17.** Let  $D = (V, A)$  be a digraph,  $w \in \mathbf{W}(D)$  be a walk. Then  $w$  is called **essential path in  $D$** , if for all  $w' \in \mathbf{W}(D; w_1, w_{-1})$  with  $|w'| \subseteq |w|$  we have  $|w'| = |w|$ . Let  $R \subseteq \mathbf{P}(D)$  be a routing, then  $R$  is called **essential routing in  $D$** , if  $p$  is an essential path in  $D$  for all  $p \in R$ . ■

**Lemma 1.5.18.** Let  $D = (V, A)$  be a digraph and let  $R \subseteq \mathbf{P}(D)$  be a routing from  $X$  to  $Y$  in  $D$ . Then there is an essential routing from  $X$  to  $Y$  in  $D$ .

*Proof.* We show this by induction on the number of paths in  $R$  that are not essential. In the base case,  $R$  itself is an essential routing from  $X$  to  $Y$ . Now let  $p \in R$  be a path that is not essential in  $D$ . Then there is a path  $p' \in \mathbf{P}(D, p_1, p_{-1})$  with  $|p'| \subsetneq |p|$ , such that  $|p'|$  is  $\subseteq$ -minimal. Such  $p'$  is an essential path. Then  $(R \setminus \{p\}) \cup \{p'\}$  is a routing from  $X$  to  $Y$  in  $D$  with fewer non-essential paths, so by induction hypothesis there is an essential routing from  $X$  to  $Y$  in  $D$ . □

**Example 1.5.19.** Let  $A, B$  be finite disjoint sets, and let  $\Delta \subseteq A \times B$ . Then  $D = (A \dot{\cup} B, \Delta)$  is a directed bipartite graph. Let  $R: X \rightrightarrows Y$  be a linking in  $D$  with  $X \subseteq A$  and  $Y \subseteq B$ . Then the set  $M = \{|p| \mid p \in R\}$  is a matching in  $D$ . Conversely, if  $M'$  is a matching in  $D$ , we can construct an induced linking in  $D$  from  $M'$ :  $R' = \{ab \mid a \in A, b \in B, \{a, b\} \in M'\}$ . Furthermore, let  $\mathcal{A}_\Delta = (A_i)_{i \in B}$  be the family of subsets of  $A$  where  $A_b = \{a \in A \mid (a, b) \in \Delta\}$  for all  $b \in B$ . The linking  $R$  then induces a partial transversal  $P = \{p_1 \mid p \in R\}$  of  $\mathcal{A}_\Delta$ . Conversely, if  $P'$  is a partial transversal of  $\mathcal{A}_\Delta$ , then there is an injective map  $\iota: P' \rightarrow B$  such that for all  $p \in P'$ ,  $p \in A_{\iota(p)}$ . Thus  $R'' = \{ab \mid a \in P', b = \iota(a)\}$  is a linking in  $D$ . ▲



The following connection between the linkings in directed graphs and the transversals of a set system, which define linkings in a bipartite graph that can be deduced from the digraph, has first been pointed out by A.W. Ingleton and M.J. Piff in [IP73]. But first, we need to clarify how to deduce the correct family of sets given a digraph and a set of targets.

**Definition 1.5.20.** Let  $D = (V, A)$  be a digraph, and let  $T \subseteq V$  be a set of vertices. The **linkage system of  $D$  to  $T$**  – denoted by  $\mathcal{A}_{D,T}$  – is defined to be the family

$$\mathcal{A}_{D,T} = \left( A_i^{(D,T)} \right)_{i \in V \setminus T} \subseteq V$$

where for  $v \in V \setminus T$

$$A_v^{(D,T)} = \{w \in V \mid (v, w) \in A\} \cup \{v\}. \quad \blacksquare$$

**Lemma 1.5.21.** Let  $D = (V, A)$  be a digraph,  $T \subseteq V$ . Every maximal partial transversal of  $\mathcal{A}_{D,T}$  is a transversal of  $\mathcal{A}_{D,T}$ .

*Proof.* Clearly,  $V \setminus T$  is a transversal of  $\mathcal{A}_{D,T}$ , and therefore  $\text{rk}_{M(\mathcal{A}_{D,T})} = |V \setminus T|$ . Let  $P$  be a maximal partial transversal of  $\mathcal{A}_{D,T}$ , then  $P$  is a base of  $M(\mathcal{A}_{D,T})$  and thus  $|P| = |V \setminus T|$  due to the equicardinality of bases (B2). Therefore every injective map  $\iota: P \rightarrow V \setminus T$  with  $p \in A_{\iota(p)}^{(D,T)}$  for all  $p \in P$  is a bijection that witnesses that  $P$  is a transversal of  $\mathcal{A}_{D,T}$ .  $\square$

The following lemma has been named *The Fundamental Lemma* by A.W. Ingleton and M.J. Piff [IP73], who used it as the key to proving that strict gammoids are precisely the duals of transversal matroids. We are going to use it in order to show augmentation properties of routings in digraphs, too.

**Lemma 1.5.22.** Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ . Then there is a linking from  $S$  to  $T$  in  $D$ , if and only if  $V \setminus S$  is a transversal of the linkage system  $\mathcal{A}_{D,T}$ .

The proof presented here can be found on p.217 [Wel76], where the lemma is called *The Linkage Lemma*.

*Proof.* Assume that  $R: S \rightrightarrows T$  is a linking in  $D$ . We construct the bijective map  $\sigma: V \setminus S \rightarrow V \setminus T$  such that for  $v \in V \setminus S$ , the image

$$\sigma(v) = \begin{cases} u & (a) \text{ if } \exists p \in R: (u, v) \in |p|_A, \text{ and} \\ v & (b) \text{ otherwise.} \end{cases}$$

The map  $\sigma$  is well-defined because  $R$  consists of pair-wise vertex disjoint paths in  $D$ ; and whenever  $v \in T$ , then either  $v \in S$  in which case  $v$  is not part of the domain of  $\sigma$ , or there is a non-trivial path  $p \in R$  that ends in  $v$ . Then  $\sigma(v) \notin T$  since otherwise  $R$  could not be onto  $T$  as every path has precisely one end vertex. From the definition of  $\mathcal{A}_{D,T}$  and the construction of  $\sigma$  it is clear, that for every  $v \in V \setminus S$ ,  $v \in A_{\sigma(v)}^{(D,T)}$ . Assume that

$\sigma$  is not injective, thus there are  $v, w \in V \setminus S$  with  $v \neq w$ , yet  $\sigma(v) = \sigma(w)$ . This is not possible if  $v$  and  $w$  are in the same case of  $\sigma$ . Thus without loss of generality we may assume that  $\sigma$  maps  $v$  through case (a) and  $w$  through case (b). Thus  $\sigma(v) = \sigma(w) = w$ , and  $(w, v) \in |p|_A$  for some  $p \in R$ . Since for  $w$  case (b) holds, we can infer that  $w = p_1$  is the initial vertex of a path in  $R$ . But then  $w \in S$  which is not part of the domain of  $\sigma$ . Therefore no such  $v, w \in V$  exist and  $\sigma$  is an injective map. Since  $R$  is a linking,  $|S| = |T|$  and  $|V \setminus S| = |V \setminus T| < \infty$ , thus  $\sigma$  is a bijection and  $V \setminus S$  is indeed a transversal of  $\mathcal{A}_{D,T}$ .

Conversely, assume that  $V \setminus S$  is a transversal of  $\mathcal{A}_{D,T}$ . Thus there is a bijection  $\sigma: V \setminus S \rightarrow V \setminus T$  such that for all  $v \in V \setminus S$ ,  $v \in A_{\sigma(v)}^{(D,T)}$ . We can construct a linking  $R: X \rightrightarrows Y$  from  $\sigma$  in the following way: for  $v \in S \cap T$ , we can let the trivial path  $v \in R$ . For  $v \in T \setminus S$ , there is some  $k \in \mathbb{N}$  such that  $\sigma^k(v) \notin V \setminus S$ : assume that for every  $k \in \mathbb{N}$ ,  $\sigma^k(v) \in V \setminus S$ , then  $\{\sigma^k(v) \mid k \in \mathbb{N}\} \subseteq V \setminus S$ , yet  $V \setminus S$  is finite. Thus there must be some  $k_0, k_1 \in \mathbb{N}$  with  $k_0 < k_1$  and  $\sigma^{k_0}(v) = \sigma^{k_1}(v)$ . Now let  $k_0, k_1 \in \mathbb{N}$  be integers with  $k_0 < k_1$  and  $\sigma^{k_0}(v) = \sigma^{k_1}(v)$  such that  $k_0$  is smallest possible. Clearly  $v \notin V \setminus T$ , so  $k_0 > 0$ . But then  $\sigma$  is a bijection, therefore the pre-images of  $\sigma^{k_0}(v)$  and  $\sigma^{k_1}(v)$  coincide. Now we have  $\sigma^{k_0-1}(v) = \sigma^{k_1-1}(v)$  which contradicts the minimality of  $k_0$ . Thus the trajectory of  $v$  under repetitions of  $\sigma$  has no cycle and therefore must be finite. Let  $k \in \mathbb{N}$  such that  $\sigma^k(v) \notin V \setminus S$ . The range of  $\sigma$  yields that  $\sigma^k(v) \in V \setminus T$  and therefore  $\sigma^k(v) \in S \setminus T$ . The construction of  $\mathcal{A}_{D,T}$  guarantees that for every  $i \in \{0, 1, \dots, k-1\}$  there is an arc  $(\sigma^i(v), \sigma^{i+1}(v)) \in A$ . Since  $\sigma$ -trajectories have no cycles, we can add the path  $\sigma^k(v)\sigma^{k-1}(v)\dots\sigma(v)v \in R$ . All paths obtained from the above constructions are pair-wise vertex disjoint, because  $\sigma$  is a bijection of finite sets, and so  $R$  is indeed a linking from  $S$  to  $T$  in  $D$ .  $\square$

We can extend Lemma 1.5.22 to routings in the natural way.

**Lemma 1.5.23.** *Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ . Then there is a routing from  $S$  to  $T$  in  $D$ , if and only if there is some  $T' \subseteq T$  such that  $V \setminus (S \cup T')$  is a transversal of the linkage system  $\mathcal{A}_{D,T}$ .*

*Proof.* Every routing  $R: S \rightrightarrows T$  in  $D$  consists of a linking from  $S$  to  $T_R = \{p_{-1} \mid p \in R\}$  and a set of unused targets  $T' = T \setminus T_R$ , and thus for every  $t' \in T'$ , we may add the trivial path  $t'$  to  $R$  and obtain the linking  $R': S \cup T' \rightrightarrows T$  where  $R' = R \cup \{t' \in \mathbf{P}(D) \mid t' \in T'\}$ . Therefore,  $R$  induces the transversal  $V \setminus (S \cup T')$  of  $\mathcal{A}_{D,T}$  by Lemma 1.5.22. Conversely, let  $T' \subseteq T$  such that  $V \setminus (S \cup T')$  is a transversal of  $\mathcal{A}_{D,T}$ . By Lemma 1.5.22 there is a linking  $R: S \cup T' \rightrightarrows T$  in  $D$ . Then  $R' = \{p \in R \mid p_1 \in S\}$  is a routing from  $S$  to  $T$  in  $D$ .  $\square$

### 1.5.2 Menger's Theorem

F. Göring published an intriguingly short and beautiful proof of Menger's Theorem [Gör00]. In this section, we present a slightly more verbose variant of this proof, which is transformed into the context of this work, along with two required yet straightforward definitions.

**Definition 1.5.24.** Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ . A set  $X \subseteq V$  is called  **$S$ - $T$ -separator** in  $D$ , if for every  $p \in \mathbf{P}(D)$  with  $p_1 \in S$  and  $p_{-1} \in T$ ,  $|p| \cap X \neq \emptyset$ . ■

It is easy to see that straightening out cycle paths from walks (Remark 1.5.16) yields paths using a subset of the original vertices, thus if  $X$  is an  $S$ - $T$ -separator, then for all  $w \in \mathbf{W}$  with  $w_1 \in S$  and  $w_{-1} \in T$  we also have  $|w| \cap X \neq \emptyset$ .

**Example 1.5.25.** Let  $D = (V, A)$  be a digraph,  $S \subseteq V$ . Then  $\partial S$  is a minimal  $S$ - $(V \setminus S)$ -separator in  $D$ : Since  $S \cap (V \setminus S) = \emptyset$ , any walk from  $s \in S$  to  $t \in V \setminus S$  must use an arc that starts in  $S$  but ends outside of  $S$ , and therefore it must visit an element of the outer margin  $\partial S$ . Now let  $v \in \partial S$ , then there is some  $u \in S$  such that  $(u, v) \in A$ . So  $uv \in \mathbf{P}(D)$  is a path from  $S$  to  $V \setminus S$ , yet  $\partial S \cap |uv| = \{v\}$ , therefore  $\partial S \setminus \{v\}$  is not an  $S$ - $(V \setminus S)$ -separator; thus  $\partial S$  is a minimal  $S$ - $(V \setminus S)$ -separator in  $D$ . ▲

Clearly, both  $S$  and  $T$  are  $S$ - $T$ -separators in every digraph  $D$ . Furthermore, every  $S$ - $T$ -separator in  $D$  is an  $S'$ - $T'$ -separator for every  $S' \subseteq S$  and  $T' \subseteq T$ .

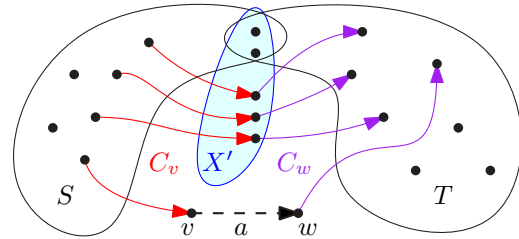
**Definition 1.5.26.** Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ . A routing  $Y \rightrightarrows T$  in  $D$  is called  **$S$ - $T$ -connector** in  $D$ , whenever  $Y \subseteq S$ . ■

**Theorem 1.5.27** (Menger's Theorem [Men27, Gör00]). Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$  subsets of vertices of  $D$ , and  $k \in \mathbb{N}$  the minimal cardinality of an  $S$ - $T$ -separator in  $D$ . There is an  $S$ - $T$ -connector  $R: Y \rightrightarrows T$  that consists of  $k$  paths.

*Proof.* By induction on  $|A|$ . If  $A = \emptyset$ , then there are only trivial paths in  $\mathbf{P}(D)$ . Thus  $S \cap T$  is a minimal  $S$ - $T$ -separator. Clearly,  $\{v \in \mathbf{P}(D) \mid v \in S \cap T\}$  is a routing from  $S \cap T$  to  $T$  in  $D$ .

For the induction step, let  $(v, w) = a \in A$ . The theorem holds for  $D' = (V, A \setminus \{a\})$  by

induction hypothesis, if  $D'$  has no  $S$ - $T$ -separator  $X'$  with  $|X'| < k$ , the claim follows directly from the induction hypothesis. Now assume that there is an  $S$ - $T$ -separator



$X'$  with  $|X'| < k$  in  $D'$ . Then  $X' \cup \{v\}$  as well as  $X' \cup \{w\}$  are  $S$ - $T$ -separators in  $D$ , therefore  $k \leq |X'| + 1$ . Furthermore, every  $(X' \cup \{w\})$ - $T$ -separator in  $D'$  and every  $S$ - $(X' \cup \{v\})$ -separator in  $D'$  is an  $S$ - $T$ -separator in  $D$ . By induction hypothesis there is an  $(X' \cup \{w\})$ - $T$ -connector  $C_w \subseteq \mathbf{P}(D') \subseteq \mathbf{P}(D)$  with  $|C_w| = k$  and  $p \in C_w$  where  $p_1 = w$  and there is an  $S$ - $(X' \cup \{v\})$ -connector  $C_v \subseteq \mathbf{P}(D') \subseteq \mathbf{P}(D)$  with  $|C_v| = k$  and  $q \in C_v$  where  $q_{-1} = v$ . Then

$$R = \{a.b \mid a \in C_v, b \in C_w, a_{-1} = b_1\} \cup \{qp\}$$

is an  $S$ - $T$ -connector in  $D$  with  $|R| = k$ : For any two  $r \in C_v$  and  $s \in C_w$ , we have  $|r| \cap |s| \subseteq X'$ , because otherwise there would be a walk from  $S$  to  $T$  in  $D'$  that does not hit the  $S$ - $T$ -separator  $X'$  – a contradiction. Therefore, for any two walks  $x, y \in R$  with  $x \neq y$ , we obtain  $|x| \cap |y| = \emptyset$ . The walks in  $R$  are paths because the concatenation  $p.q$  of two compatible paths  $p, q$  is a non-path walk if and only if  $|p| \cap |q| \supsetneq \{q_1\}$  and thus  $R$  is indeed a routing.  $\square$

It is immediate from the respective definitions that every  $S$ - $T$ -separator in  $D$  must hit every path of every  $S$ - $T$ -connector in  $D$  at least once, therefore Menger's Theorem is the non-trivial part of the following strong duality<sup>4</sup> theorem.

**Corollary 1.5.28.** *Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ . The maximal cardinality of an  $S$ - $T$ -connector in  $D$  equals the minimal cardinality of an  $S$ - $T$ -separator in  $D$ .*

Another immediate consequence is that every vertex of a minimal separator is hit by every maximal connector.

**Corollary 1.5.29.** *Let  $D = (V, A)$  be a digraph,  $S, T, X \subseteq V$ , such that  $X$  is an  $S$ - $T$ -separator of minimal cardinality. Every  $S$ - $T$ -connector  $R$  with maximal cardinality in  $D$  has the property*

$$\forall x \in X: \exists p \in R: x \in |p|.$$

### 1.5.3 Augmentation of $S$ - $T$ -Connectors

Menger's Theorem states that if  $R: X \rightrightarrows Y$  is an  $S$ - $T$ -connector in  $D$  with  $|R| < |C|$  for every  $S$ - $T$ -separator  $C$ , then there must be some bigger  $S$ - $T$ -connector in  $D$ . In this section, we prove that Menger's Theorem is still true if we consider only those  $S$ - $T$ -connectors  $R': X' \rightrightarrows Y'$  with  $X \subsetneq X'$ .

<sup>4</sup>Strong duality is a notion from linear programming, stating that if the primal and the dual linear optimization problems are both feasible, then their optimal values are attained and equal.

**Theorem 1.5.30.** *Let  $D = (V, A)$  be a digraph,  $S, T \subseteq V$ , and let  $A, B$  be two  $S$ - $T$ -connectors in  $D$  with  $|A| < |B|$ . Then there is an  $S$ - $T$ -connector  $C$  in  $D$ , such that  $|C| = |A| + 1$  and  $\{p_1 \mid p \in A\} \subseteq \{p_1 \mid p \in C\} \subseteq \{p_1 \mid p \in A \cup B\}$ .*

This proof is based on the argumentation found on p. 220 in [Wel76].

*Proof.* Let  $A_1 = \{p_1 \mid p \in A\}$  and  $B_1 = \{p_1 \mid p \in B\}$  be the initial vertices of paths in  $A$  and  $B$ , and let  $A_{-1} = \{p_{-1} \mid p \in A\}$  and  $B_{-1} = \{p_{-1} \mid p \in B\}$  be the terminal vertices of paths in  $A$  and  $B$ . By Lemma 1.5.22 we see that  $V \setminus A_1$  is a transversal of the linkage system  $\mathcal{A}_{D, A_{-1}}$  and  $V \setminus B_1$  is a transversal of the linkage system  $\mathcal{A}_{D, B_{-1}}$ . Consider the linkage system  $\mathcal{A}' = \mathcal{A}_{D, A_{-1} \cup B_{-1}}$ , clearly  $\mathcal{A}'$  is both a subfamily of  $\mathcal{A}_{D, A_{-1}}$  and  $\mathcal{A}_{D, B_{-1}}$ . Therefore,  $V \setminus A_1$  and  $V \setminus B_1$  both contain a maximal partial transversal of  $\mathcal{A}'$ , thus  $V \setminus A_1$  and  $V \setminus B_1$  each contain a base of  $M(\mathcal{A}')$ . By Lemma 1.2.30,  $A_1$  and  $B_1$  are independent sets of the dual matroid  $M(\mathcal{A}')^*$ , and by Lemma 1.2.7 there is a base  $X$  of  $A_1 \cup B_1$  in  $M(\mathcal{A}')^*$ , such that  $A_1 \subseteq X \subseteq A_1 \cup B_1$ . But then  $V \setminus X$  is a spanning set of  $M(\mathcal{A}')$ , therefore it contains a maximal partial transversal  $V \setminus P$  of  $\mathcal{A}'$  where  $X \subseteq P$ . By Lemma 1.5.21 we obtain that  $V \setminus P$  is also a transversal of  $\mathcal{A}'$ . Again it follows from Lemma 1.5.22 that there is a linking  $L: P \rightrightarrows (A_{-1} \cup B_{-1})$  in  $D$ . Now  $A_1 \subseteq X$ , furthermore  $X \cap (B_1 \setminus A_1) \neq \emptyset$  and  $X \subseteq P$ , thus  $A_1 \subseteq P$  and  $P \cap (B_1 \setminus A_1) \neq \emptyset$ . Therefore there is an element  $b \in P \cap (B_1 \setminus A_1)$  which can be used to filter the augmented  $S$ - $T$ -connector from  $P$ : The linking  $C = \{p \in L \mid p_1 \in A \cup \{b\}\}$  is the desired augmented  $S$ - $T$ -connector.  $\square$





# Chapter 2

## Gammoids

J.H. Mason first introduced the notions of a gammoid and of a strict gammoid. Both are matroids that arise from free matroids through matroid induction [Mas72]: Given a digraph  $D = (V, A)$  and a matroid  $N = (E, \mathcal{I})$ , the set of vertices, from which there is a routing onto some  $T \subseteq V$  in  $D$  with  $T \in \mathcal{I}$ , forms a family of independent sets of a matroid on the ground set  $V$ . The resulting matroid is called the matroid induced by  $D$  from  $N$ . The general case of matroids induced by  $D$  from  $N$  is connected to the special case of gammoids, where  $N$  is a free matroid, through the following generalization: The augmentation theorem for  $S$ - $T$ -connectors in  $D$  still holds if we restrict the class of all  $S$ - $T$ -connectors in  $D$  to the class of  $S$ - $T$ -connectors in  $D$  that link onto an independent set of a given matroid on  $T$  — a proof may be obtained by replacing the transversal matroid  $M(\mathcal{A}')$  presented by the linkage system  $\mathcal{A}' = \mathcal{A}_{D, A_{-1} \cup B_{-1}}$  in the proof of Theorem 1.5.30 with a suitable matroid  $M(\Delta', N)$  obtained through bipartite matroid induction with respect to the directed bipartite graph  $\Delta'$  associated with the linkage system  $\mathcal{A}'$  through Definition 1.4.10. But since we are most interested in a certain special case of matroid induction by directed graphs, we omit this concept for now and give a direct definition of gammoids instead.

## 2.1 Definition and Representations

**Definition 2.1.1.** Let  $D = (V, A)$  be a digraph,  $E \subseteq V$ , and  $T \subseteq V$ . The **gammoid represented by  $(D, T, E)$**  is defined to be the matroid  $\Gamma(D, T, E) = (E, \mathcal{I})$  where

$$\mathcal{I} = \{X \subseteq E \mid \text{there is a routing } X \rightrightarrows T \text{ in } D\}.$$

The elements of  $T$  are usually called **sinks** in this context, although they are not required to be actual sinks of the digraph  $D$ . To avoid confusion, we shall call the elements of  $T$  **targets** in this work. A matroid  $M' = (E', \mathcal{I}')$  is called **gammoid**, if there is a digraph  $D' = (V', A')$  and a set  $T' \subseteq V'$  such that  $M' = \Gamma(D', T', E')$ . ■

**Lemma 2.1.2.** Let  $D = (V, A)$  be a digraph,  $E \subseteq V$ , and  $T \subseteq V$ . Then  $\Gamma(D, T, E)$  is a matroid.

*Proof.* Let  $\Gamma(D, T, E) = (E, \mathcal{I})$ . Clearly, the empty routing  $\emptyset \subseteq \mathbf{P}(D)$  routes  $\emptyset$  to  $T$ , therefore  $\emptyset \in \mathcal{I}$ , so (I1) holds. Also, if  $R: X \rightrightarrows T$  is a routing from  $X$  to  $T$  in  $D$ , and if  $Y \subseteq X$ , then  $\{p \in R \mid p_1 \in Y\}$  is a routing from  $Y$  to  $T$  in  $D$ , therefore (I2) holds, too. Now let  $X, Y \in \mathcal{I}$  with  $|X| < |Y|$ . Then there are routings  $R: X \rightrightarrows T$  and  $S: Y \rightrightarrows T$  in  $D$ . We may regard  $R$  and  $S$  as  $(X \cup Y)$ - $T$ -connectors in  $D$ . Thus by Theorem 1.5.30 there is a routing  $C$  from  $X'$  to  $T$  such that  $X \subsetneq X' \subseteq X \cup Y$ , so there is an element  $y \in X' \setminus X \subseteq Y$  such that  $X \cup \{y\} \in \mathcal{I}$ . Therefore (I3) holds and, consequently,  $\Gamma(D, T, E)$  is a matroid. □

**Lemma 2.1.3.** Let  $D = (V, A)$ ,  $E \subseteq V$ ,  $T \subseteq V$ ,  $M = \Gamma(D, T, E)$ , and  $X \subseteq E$ . Then  $\text{rk}_M(X)$  equals the size of a maximal  $X$ - $T$ -connector in  $D$ .

*Proof.* Let  $(E, \mathcal{I}) = M$  and let  $C: X_0 \rightrightarrows T$  be a maximal-cardinality  $X$ - $T$ -connector, then clearly for all  $x \in X \setminus X_0$ , there is no routing  $X_0 \cup \{x\} \rightrightarrows T$  in  $D$ , therefore  $X_0 \cup \{x\} \notin \mathcal{I}$  for all  $x \in X \setminus X_0$ . Thus  $\text{rk}_M(X) < |C| + 1$ . We have  $X_0 \in \mathcal{I}$  since  $\Gamma(D, T, E)$  is defined that way. Thus  $|C| = |X_0| \leq \text{rk}_M(X)$ , which yields  $\text{rk}_M(X) = |C|$ . □

### 2.1.1 Switching Between Representations

From the definition of a gammoid  $M$ , it is clear that any given representation  $(D, T, E)$  of a gammoid cannot be unique for  $M$ , because the number of vertices of  $D$  is not constrained. Therefore every gammoid has a myriad of representations, and some of these representations are nicer than others, also depending on the purpose. In this

section, we deal with operations on representations  $(D, T, E)$  that leave the represented gammoid fixed.

Without loss of generality we may always assume that a gammoid is presented by some  $(D, T, E)$  where  $T$  consists only of sinks of  $D$ .

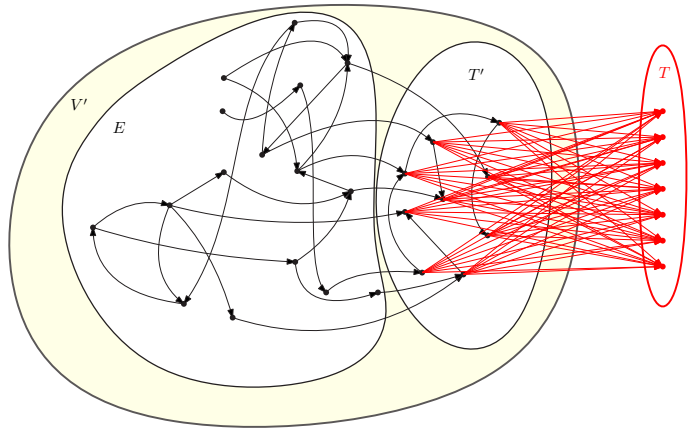
**Lemma 2.1.4.** *Let  $D = (V, A)$  be a digraph,  $E \subseteq V$ , and  $T \subseteq V$ . Furthermore, let  $D' = (V, A')$  where  $A' = A \setminus (T \times V)$ . Then  $\Gamma(D, T, E) = \Gamma(D', T, E)$ .*

*Proof.* Let  $M = \Gamma(D, T, E) = (E, \mathcal{I})$  and  $M' = \Gamma(D', T, E) = (E, \mathcal{I}')$ . Clearly every routing  $R$  in  $D'$  is also a routing in  $D$ , thus  $\mathcal{I}' \subseteq \mathcal{I}$ . Now let  $X \in \mathcal{I}$  and let  $R: X \rightrightarrows T$  be a routing in  $D$ . Then for every  $p \in R$  there is a minimal integer  $i(p)$  such that  $p_{i(p)} \in T$ . Let  $R' = \{p_1 p_2 \dots p_{i(p)} \mid p \in R\}$ .  $R'$  is a routing from  $X$  to  $T$  in  $D'$ . Thus  $X \in \mathcal{I}'$  and therefore  $\mathcal{I} \subseteq \mathcal{I}'$ , so  $M = M'$ .  $\square$

Without loss of generality, we may always assume that the cardinality of the target set equals the rank of the gammoid.

**Lemma 2.1.5.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then there is a digraph  $D = (V, A)$  and a subset  $T \subseteq V$ , such that  $|T| = \text{rk}_M(E)$  and  $M = \Gamma(D, T, E)$ .*

*Proof.* Let  $(D', T', E)$  be a representation of  $M$  where  $D' = (V', A')$ . There is an easy construction that achieves the claim: Remember that  $X \subseteq E$  is independent in  $\Gamma(D', T', E)$  if and only if there is a routing  $X \rightrightarrows T'$  in  $D'$ . Since  $|X| \leq \text{rk}(E)$ , at most  $\text{rk}(E)$  vertices of  $T'$  are visited by paths that belong to



$X \rightrightarrows T'$ . Thus we may extend the digraph  $D'$  to a digraph  $D = (V, A)$  by adding  $\text{rk}(E)$  new vertices  $T = \{t_1, \dots, t_{\text{rk}(E)}\}_{\neq}$  in such a way, that there is an arc  $(v, t) \in A$  for  $v \in V'$  and  $t \in T$  if and only if  $v \in T'$ . Formally, we let  $V = V' \dot{\cup} T$  and  $A = A' \cup (T' \times T)$ . By construction, every routing  $X \rightrightarrows T'$  in  $D'$  can be extended to a routing  $X \rightrightarrows T$  in  $D$ , as there are sufficient elements in  $T$  and arcs between  $T'$  and  $T$  in  $D$ . On the other hand, a routing  $X \rightrightarrows T$  in  $D$  implies that there is a routing  $X \rightrightarrows T'$  because every non-trivial path ending in  $T$  must visit some  $t' \in T'$ . Therefore,  $X \subseteq E$  is independent in  $\Gamma(D', T', E)$  if and only if  $X$  is independent in  $\Gamma(D, T, E)$ . Thus  $M = \Gamma(D, T, E)$ , so  $(D, T, E)$  represents  $M$  with  $|T| = \text{rk}(E)$ .  $\square$

We obtain the following from the previous proof:

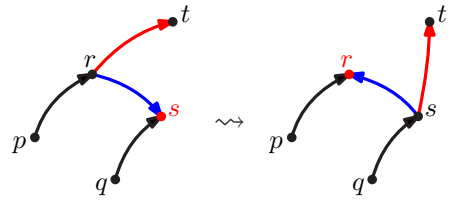
**Corollary 2.1.6.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then there is a digraph  $D = (V, A)$  and a subset  $T \subseteq V$  with  $T \cap E = \emptyset$ , such that  $|T| = \text{rk}_M(E)$  and  $M = \Gamma(D, T, E)$ . Furthermore, every  $t \in T$  is a sink in  $D$ .*

**Definition 2.1.7.** *Let  $D = (V, A)$  be a digraph,  $s \in V$  be a vertex of  $D$ , and  $r \in V$  be a vertex such that  $(r, s) \in A$  is an arc of  $D$ . The  **$r$ - $s$ -pivot of  $D$**  shall be the digraph  $D_{r \leftarrow s} = (V, A_{r \leftarrow s})$  where the arc set*

$$A_{r \leftarrow s} = (A \setminus (\{r\} \times V)) \cup \left( \left( \{s\} \times \overrightarrow{\{r\}}^D \right) \setminus \{(s, s)\} \right)$$

consists of arcs leaving  $s$  and entering  $x$  for every  $x \in \overrightarrow{\{r\}}^D \setminus \{s\}$ , i.e. for every  $x \neq s$  with either  $x = r$  or such that there is an arc from  $r$  to  $x$  in  $D$ , and all arcs  $(u, v) \in A$  of  $D$  which have a tail  $u \neq r$ . ■

**Example 2.1.8.** Consider the digraph  $D = (V, A)$  where  $V = \{p, q, r, s, t\}_{\neq}$  and  $A = \{(p, r), (q, s), (r, s), (r, t)\}$ . Clearly,  $s$  is a sink in  $D$  and  $(r, s) \in A$ , and thus the  $r$ - $s$ -pivot of  $D$  is  $D_{r \leftarrow s} = (V, A_{r \leftarrow s})$  with  $A_{r \leftarrow s} = \{(p, r), (q, s), (s, r), (s, t)\}$ . Let us examine the paths in  $D$  and  $D_{r \leftarrow s}$ :  $\mathbf{P}(D) = \{p, pr, prs, prt, q, qs, r, rs, rt, s, t\}$ , whereas  $\mathbf{P}(D_{r \leftarrow s}) = \{p, pr, q, qs, qsr, qst, r, s, sr, st, t\}$ .



The maximal routings in  $D$  with respect to set-inclusion, which are also maximal routings in  $D_{r \leftarrow s}$  are  $\{p, q, r, s, t\}$ ,  $\{p, qs, r, t\}$ , and  $\{pr, q, s, t\}$ ; the maximal routings in  $D$  which are not in  $D_{r \leftarrow s}$  are  $\{p, q, rs, t\}$ ,  $\{p, q, rt, s\}$ ,  $\{prs, q, t\}$ , and  $\{prt, q, s\}$ ; and those only in  $D_{r \leftarrow s}$  are  $\{p, q, r, st\}$ ,  $\{p, q, sr, t\}$ ,  $\{p, qsr, t\}$ , and  $\{p, qst, r\}$ . ▲

The next lemma is called *the fundamental theorem* by J.H. Mason in [Mas72].

**Lemma 2.1.9.** *Let  $D = (V, A)$  be a digraph,  $T \subseteq V$ ,  $s \in T$  a sink of  $D$ ,  $r \in V \setminus T$  with  $(r, s) \in A$ , and  $X \subseteq V$ . Then there is a routing  $X \rightrightarrows T$  in  $D$  if and only if there is a routing  $X \rightrightarrows (T \setminus \{s\}) \cup \{r\}$  in  $D_{r \leftarrow s}$ .*

*Proof.* First, we prove that a routing  $X \rightrightarrows T$  in  $D$  implies a routing  $X \rightrightarrows (T \setminus \{s\}) \cup \{r\}$  in  $D_{r \leftarrow s}$ . Let  $R: X \rightrightarrows T$  be a routing in  $D$ . If  $(r, s) \in \bigcup_{p \in R} |p|_A$ , i.e. the routing  $R$  has a path  $p = (p_i)_{i=1}^n$  that traverses the arc  $(r, s)$ ; then  $n > 1$  and  $\{r, s\} \cap |p'| = \emptyset$  for all  $p' \in R \setminus \{p\}$ . Let  $q = p_1 p_2 \dots p_{n-1}$  be the path that arises when the vertex  $s$  is

chopped off of  $p$ . Then  $R' = R \setminus \{p\} \cup \{q\}$  is a routing from  $X$  to  $(T \setminus \{s\}) \cup \{r\}$  in  $D_{r \leftarrow s}$ . Otherwise, we have that  $(r, s) \notin \bigcup_{p \in R} |p|_A$ . Let  $Q = \{r, s\} \cap \left( \bigcup_{p \in R} |p| \right)$  be the criterion for a case analysis. If  $Q = \emptyset$ , then  $R$  is obviously a routing in  $D_{r \leftarrow s}$ , because  $D$  and  $D_{r \leftarrow s}$  coincide on  $V \setminus \{r, s\}$ . Then no path  $p \in R$  has  $p_{-1} = s$ , thus  $R$  even is a routing from  $X$  to  $T \setminus \{s\}$  in  $D_{r \leftarrow s}$ . If  $Q = \{s\}$ , then there is a path  $p \in R$  with  $p_{-1} = s$ , yet no path of  $R$  visits  $r$ , therefore  $R \setminus \{p\} \cup \{pr\}$  is the desired routing in  $D_{r \leftarrow s}$ . If  $Q = \{r\}$ , then no path in  $R$  visits  $s$ , and there is a path  $p = (p_i)_{i=1}^n$  that visits  $r = p_j$  with  $j \in \{1, 2, \dots, n\}$ . Then  $R \setminus \{p\} \cup \{p_1 p_2 \dots p_j\}$  is the desired routing in  $D_{r \leftarrow s}$ . If  $Q = \{r, s\}$ , then there are two paths  $p, q \in R$  with  $p \neq q$  such that  $s \in |p|$  and  $r \in |q|$ . Let  $q = (q_i)_{i=1}^m$ , and let  $1 \leq j \leq m$  such that  $q_j = r$ . Since  $s$  is a sink in  $D$ , we have  $p_{-1} = s$ . Let  $p' = pq_{j+1}q_{j+2} \dots q_m$  be the path in  $D_{r \leftarrow s}$  that first follows  $p$  and then follows the end of  $q$ . We have  $|p'|_A \subseteq A_{r \leftarrow s}$  since  $(r, q_{j+1}) \in |q|_A \subseteq A$  thus  $(s, q_{j+1}) \in A_{r \leftarrow s}$ , and the digraphs  $D$  and  $D_{r \leftarrow s}$  have the same arcs on  $V \setminus \{r, s\}$ . Furthermore, let  $q' = q_1 q_2 \dots q_j$ , clearly  $q' \in \mathbf{P}(D_{r \leftarrow s})$ , thus  $(R \setminus \{p, q\}) \cup \{p', q'\}$  is the desired routing in  $D_{r \leftarrow s}$ .

The second implication of the lemma follows from the first implication together with the fact, that in the situation of the lemma where the operand  $s$  is a sink of  $D$ ,  $(D_{r \leftarrow s})_{s \leftarrow r} = D$  holds.  $\square$

**Theorem 2.1.10.** *Let  $M = (E, \mathcal{I})$  be a gammoid, and  $B$  a base of  $M$ . Then there is a digraph  $D = (V, A)$ , such that*

$$M = \Gamma(D, B, E)$$

and every  $b \in B$  is a sink in  $D$ .

*Proof.* Let  $D' = (V, A')$  be a digraph and  $T' \subseteq A'$  such that  $M = \Gamma(D', T', E)$ . We may assume that  $|T'| = |B| = \text{rk}_M(E)$  and that all  $t' \in T'$  are sinks in  $D'$  (Corollary 2.1.6). Let  $R: B \rightrightarrows T'$  a linking of  $B$  onto  $T'$  in  $D'$ . We prove the statement by induction on  $\left| \bigcup_{p \in R} |p| \right|$ . In the base case we have  $\left| \bigcup_{p \in R} |p| \right| = |B|$ , and therefore every path  $p \in R$  is trivial. Thus  $B = T$  and  $D = D'$  is the desired digraph. Now let  $\left| \bigcup_{p \in R} |p| \right| > |B|$ , thus there is a non-trivial path  $p = (p_i)_{i=1}^n \in R$  where  $n > 1$ . Let  $s = p_n$  and let  $r = p_{n-1}$ . The vertex  $s$  is a sink in  $D'$  since  $s \in T'$ , and clearly  $(r, s) \in |p|_A \subseteq A'$ . Since  $|B| = |T'|$ ,  $r \notin T'$ . The proof of Lemma 2.1.9 yields that  $R' = R \setminus \{p\} \cup \{p_1 p_2 \dots p_{n-1}\}$  is a linking of  $B$  onto  $(T' \setminus \{s\}) \cup \{r\}$  in  $D'_{r \leftarrow s}$  with  $\left| \bigcup_{p \in R'} |p| \right| < \left| \bigcup_{p \in R} |p| \right|$ . Furthermore Lemma 2.1.9 implies that  $\Gamma(D', T', E) = \Gamma(D'_{r \leftarrow s}, (T' \setminus \{s\}) \cup \{r\}, E)$  and the existence of the digraph  $D$  follows from the induction hypothesis for the linking  $R'$  with respect to the representation  $(D'_{r \leftarrow s}, (T' \setminus \{s\}) \cup \{r\}, E)$ .  $\square$

## 2.1.2 Number of Vertices Needed to Represent a Gammoid

In the paper *Representative Sets and Irrelevant Vertices: New Tools for Kernelization* [KW12], S. Kratsch and M. Wahlström proved the following upper bound result regarding the number of vertices in a given digraph, that suffice to be considered in order to find certain  $S$ - $T$ -separators of minimal cardinality. This bound may be used to derive a bound on the number of vertices needed in order to represent a gammoid on a ground set of given cardinality.

**Theorem 2.1.11** ([KW12], Theorem 3). *Let  $D = (V, A)$  be a digraph,  $E, T \subseteq V$ , and  $r > 0$  be the cardinality of a minimal  $E$ - $T$ -separator in  $D$ . There is a set  $Z \subseteq V$  with  $E \cup T \subseteq Z$  and  $|Z| = O(|E| \cdot |T| \cdot r)$  such that for all  $X \subseteq E$  and  $Y \subseteq T$  there is a minimal  $X$ - $Y$ -separator  $S$  in  $D$  with  $S \subseteq Z$ . The set  $Z$  can be found in randomized polynomial time with failure probability  $O(2^{-n})$ .*

For the proof, see [KW12].<sup>1</sup> The statement that  $E \cup T \subseteq Z$  is not part of the original theorem in [KW12], as well as the condition  $r > 0$ , but it is easy to see that these modifications are valid, since  $|E \cup T| = O(|E| \cdot |T| \cdot r)$  for  $r \geq 1$ .

**Remark 2.1.12.** In [KW12], the authors only give the  $O$ -behavior of the size of  $Z$  in Theorem 2.1.11, but it is possible to derive the factor hidden in the  $O$ -notation by inspecting their proof and the proof of Lemma 4.1 [Mar09] by D. Marx. We obtain

$$|Z| \leq \binom{r}{1} \cdot \binom{|E|}{1} \cdot \binom{|T|}{1} + |E| + |T| = r \cdot |E| \cdot |T| + |E| + |T|. \quad \bullet$$

**Corollary 2.1.13.** *Let  $M = (E, \mathcal{I})$  be a gammoid. There is a representation  $(D, T, E)$  of  $M$  with  $D = (V, A)$ , such that*

$$|V| = O(|E| \cdot \text{rk}_M(E)^2) \leq O(|E|^3).$$

*Proof.* Let  $(D, T, E)$  be a representation of  $M$  where  $|T| = \text{rk}_M(E)$  and  $D = (V, A)$  (Lemma 2.1.5). Let  $Z' \subseteq V$  be a subset of  $V$  as in the consequent of Theorem 2.1.11. Let  $D' = (Z', A')$  be the digraph, where for all  $x, y \in Z'$ , there is an arc

$$(x, y) \in A' \iff \exists p \in \mathbf{P}(D; x, y): |p| \cap Z' = \{x, y\}.$$

Thus there is an arc leaving  $y \in Z'$  and entering  $z \in Z'$  in  $D'$  if there is a path from  $y$  to  $z$  in  $D$  that never visits another vertex of  $Z'$ . Let  $p = (p_i)_{i=1}^n \in \mathbf{P}(D)$  be a path of length  $n$

<sup>1</sup>The actual proof starts on page 24.

from  $p_1 \in E$  to  $p_{-1} \in T$ . Let  $I' = \{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } p_i \in Z'\} = \{i_1, i_2, \dots, i_k\}_{\neq}$  with  $i_1 < i_2 < \dots < i_k$ . Then let  $p' = p_{i_1} p_{i_2} \dots p_{i_k}$ , i.e.  $p'$  is the path consisting of the vertices visited by  $p$  that are in  $Z'$ . Observe that  $p_1 = p'_1$ ,  $p_{-1} = p'_{-1}$ , and  $p' \in \mathbf{P}(D')$  holds. Let  $R: X \rightrightarrows T$  be a routing from  $X \subseteq E$  to  $T$  in  $D$ , and let  $R' = \{p' \in \mathbf{P}(D') \mid p \in R\}$  be the set of paths in  $D'$  that consists of all  $p'$  derived from  $p \in R$  as described above. By construction of  $D'$ , we see that  $R'$  is a routing from  $X$  to  $T$  in  $D'$ . Thus every independent set  $X \subseteq E$  of  $M = \Gamma(D, T, E)$  is also an independent set of  $N = \Gamma(D', T, E)$ . Now assume that there is some  $X \subseteq E$  that is independent in  $N$ , but not in  $M$ . Then  $D$  would have an  $X$ - $T$ -separator  $S$  with  $|S| < |X|$ , and by Theorem 2.1.11 we may assume that  $S \subseteq Z'$  holds. Thus  $S$  would be an  $X$ - $T$ -separator of  $D'$ , too, contradicting the assumption that  $X$  is independent with respect to  $N$ . Therefore every independent set of  $N$  is also independent with respect to  $M$ . Consequently,  $M = N$ . Thus  $(D', T, E)$  is a representation of  $M$  using only  $O(|E| \cdot \text{rk}_M(E)^2)$  vertices.  $\square$

**Remark 2.1.14.** In the light of Remark 2.1.12 we obtain that if  $M = (E, \mathcal{I})$  is a gammoid, there is a representation  $(D, T, E)$  where  $D = (V, A)$  such that  $|T| = \text{rk}_M(E)$  and such that

$$|V| \leq \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E| \leq 2|E|^3. \quad \bullet$$

### 2.1.3 Duality Respecting Representations

**Definition 2.1.15.** Let  $(D, T, E)$  be a representation of a gammoid. We say that  $(D, T, E)$  is a **duality respecting representation**, if

$$\Gamma(D^{\text{opp}}, E \setminus T, E) = (\Gamma(D, T, E))^*. \quad \blacksquare$$

**Example 2.1.16.** Consider the uniform matroid  $U = (\{a, b, c\}, \{\emptyset, \{a\}, \{b\}, \{c\}\})$  and the digraphs  $D_1 = (\{a, b, c\}, \{(a, b), (b, c)\})$  and  $D_2 = (\{a, b, c\}, \{(a, c), (b, c)\})$  (Fig. 2.1). Clearly  $\Gamma(D_1, \{c\}, \{a, b, c\}) = U = \Gamma(D_2, \{c\}, \{a, b, c\})$ , but  $\Gamma(D_1^{\text{opp}}, \{a, b\}, \{a, b, c\}) \neq U^*$ , since there is no routing from  $\{b, c\}$  to  $\{a, b\}$  in  $D_1^{\text{opp}}$ . On the other hand, such a routing exists in  $D_2^{\text{opp}}$ , and indeed we have  $U^* = \Gamma(D_2^{\text{opp}}, \{a, b\}, \{a, b, c\})$ . Therefore duality respecting representations exist, but not all representations have this property.  $\blacktriangle$

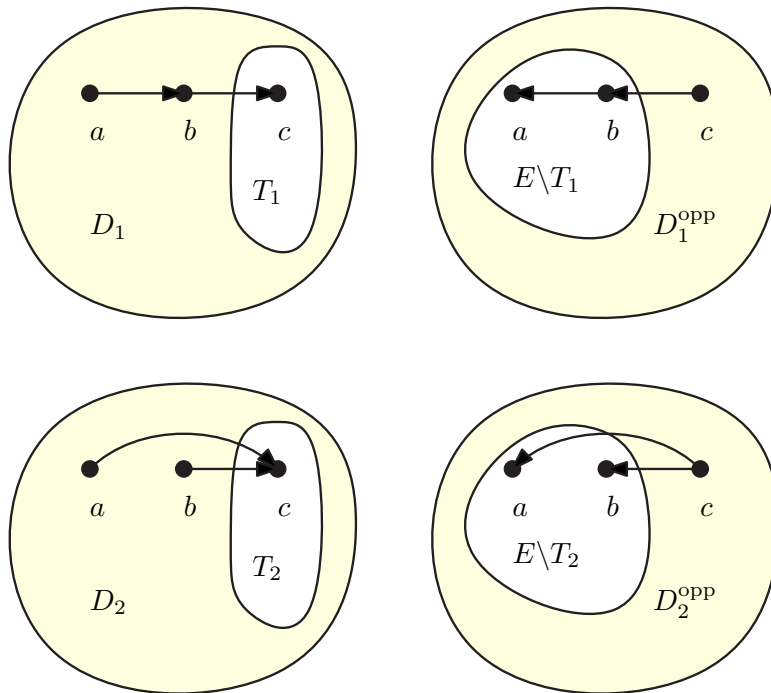


Fig. 2.1 Non-duality respecting and duality respecting representations of  $U$ .

**Lemma 2.1.17.** *Let  $M = (E, \mathcal{I})$  be a gammoid, and  $B \subseteq E$  a base of  $M$ . There is a digraph  $D = (V, A)$  such that the sinks of  $D$  are precisely the elements of  $B$ , the sources of  $D$  are precisely the elements of  $E \setminus B$ , and such that  $M = \Gamma(D, B, E)$ .*

*Proof.* There is a digraph  $D' = (V', A')$  such that  $M = \Gamma(D', B, E)$  (Theorem 2.1.10), and without loss of generality we may assume that the only sinks in  $D'$  are the elements of  $B$ , and that all sources in  $D'$  are elements of  $E$  — since sources not in  $E$  and sinks not in  $B$  cannot be part of a path that belongs to any routing from  $X \subseteq E$  to  $B$  in  $D'$ , and therefore may be dropped from  $D'$  without changing the represented gammoid. Clearly, we can give each  $e \in V' \cap (E \setminus B)$  a new name — say  $e''$  — in  $D'$ , yielding a digraph  $D'' = (V'', A'')$  where  $V'' \cap E = B$ . Then we can add the elements  $E \setminus B$  back to  $D''$  as isolated vertices, and after that, we add an arc leaving  $e$  and entering its renamed copy  $e''$  for every  $e \in E \setminus B$ . Let us denote the digraph that we just constructed by  $D = (V, A)$  where  $V = V'' \dot{\cup} (E \setminus B)$  and  $A = A'' \cup \{(e, e'') \mid e \in E \setminus B\}$ . Clearly, each routing  $R: X \rightrightarrows B$  with  $X \subseteq E$  in  $D'$  induces the routing  $R'' = \{p_1 p_1'' p_2 \dots p_n \mid p_1 p_2 \dots p_n \in R\}$  from  $X$  to  $B$  in  $D$ ; and conversely, each routing  $Q'': X \rightrightarrows B$  with  $X \subseteq E$  in  $D$  induces the routing  $Q = \{p_1 p_3 p_4 \dots p_n \mid p_1 p_2 \dots p_n \in Q''\}$  from  $X$  to  $B$  in  $D'$ . Therefore,  $\Gamma(D, B, E) = \Gamma(D', B, E) = M$ .  $\square$



**Lemma 2.1.18.** *Let  $(D, T, E)$  be a representation of a gammoid with  $T \subseteq E$ , and such that every  $e \in E \setminus T$  is a source of  $D$ , and every  $t \in T$  is a sink of  $D$ . Then  $(D, T, E)$  is a duality respecting representation.*

*Proof.* We have to show that the bases of  $N = \Gamma(D^{\text{opp}}, E \setminus T, E)$  are precisely the complements of the bases of  $M = \Gamma(D, T, E)$  (Corollary 1.2.31). Let  $B \subseteq E$  be a base of  $M$ , then there is a linking  $L: B \rightrightarrows T$  in  $D$ , and since  $T$  consists of sinks, we have  $\{x \in \mathbf{P}(D) \mid x \in T \cap B\} \subseteq L$ . Further, let  $L^{\text{opp}} = \{p_n p_{n-1} \dots p_1 \mid p_1 p_2 \dots p_n \in L\}$ . Then  $L^{\text{opp}}$  is a linking from  $T$  to  $B$  in  $D^{\text{opp}}$  which routes  $T \setminus B$  to  $B \setminus T$ . The special property of  $D$ , that  $E \setminus T$  consists of sources and that  $T$  consists of sinks, implies, that for all  $p \in L$ , we have  $|p| \cap E = \{p_1, p_{-1}\}$ . Observe that thus

$$R = \{p \in L^{\text{opp}} \mid p_1 \in T \setminus B\} \cup \{x \in \mathbf{P}(D^{\text{opp}}) \mid x \in E \setminus (T \cup B)\}$$

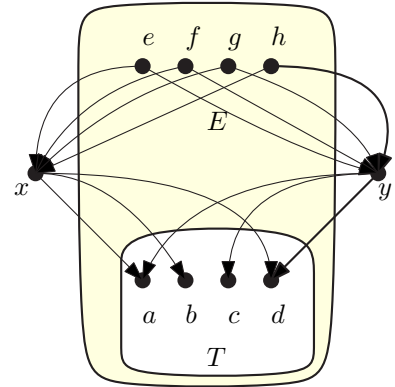
is a linking from  $E \setminus B = (T \dot{\cup} (E \setminus T)) \setminus B$  onto  $E \setminus T$  in  $D^{\text{opp}}$ , thus  $E \setminus B$  is a base of  $N$ . An analog argument yields that for every base  $B'$  of  $N$ ,  $E \setminus B'$  is a base of  $M$ . Therefore  $\Gamma(D^{\text{opp}}, E \setminus T, E) = (\Gamma(D, T, E))^*$ .  $\square$

**Corollary 2.1.19.** *Let  $M = (E, \mathcal{I})$  a gammoid. Then there is a duality respecting representation  $(D, T, E)$  with  $\Gamma(D, T, E) = M$ . Consequently,  $M^*$  is a gammoid if and only if  $M$  is a gammoid.*

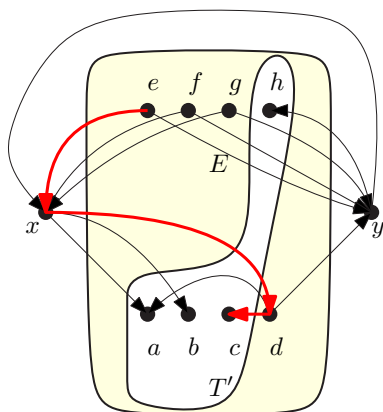
*Proof.* Immediate consequence of Lemmas 2.1.17 and 2.1.18.  $\square$

Unfortunately, the property of a representation to be duality respecting is not preserved by the digraph pivot operation. Thus we cannot take a duality respecting representation, pivot in a base as in the proof of Theorem 2.1.10 and then expect that the resulting representation is still duality respecting.

**Example 2.1.20.** Consider the gammoid  $M$  on the ground set  $E = \{a, b, c, d, e, f, g, h\}_{\neq}$  represented by the digraph  $D = (V, A)$  with the vertex set  $V = E \dot{\cup} \{x, y\}_{\neq}$  and the arcs  $A = (\{e, f, g, h\} \times \{x, y\}) \cup (\{x\} \times \{a, b, d\}) \cup (\{y\} \times \{a, c, d\})$  together with the target set  $T = \{a, b, c, d\}$ , i.e. we have  $M = \Gamma(D, T, E)$ . The bases of  $M$  are the set  $T = \{a, b, c, d\}$ , the sets of the form  $X \cup \{y\}$  where  $X \subseteq T$  with  $|X| = 3$  and  $y \in \{e, f, g, h\}$ , and the sets of the form  $X \cup Y$  where  $X \subseteq T$  with  $|X| = 2$  and  $Y \subseteq \{e, f, g, h\}$  with  $|Y| = 2$ . Clearly,  $(D, T, E)$  is duality respecting (Lemma 2.1.18).



Observe that there is only one routing that links  $\{a, b, c, h\}$  to  $T$  – up to symmetries of  $D$  that stabilize  $E$  – namely  $R = \{a, b, c, hxd\} \subseteq \mathbf{P}(D)$ .



If we use the routing  $R$  together with the procedure described in the proof of Theorem 2.1.10, we obtain the digraph  $D'$  depicted to the left. The vertex  $h$  is now a sink in  $D'$ , but  $d$  is not a source in  $D'$ , therefore Lemma 2.1.18 is not applicable to  $D'$ . There are two routings that link the base  $B' = \{a, b, e, h\}$  to the target set  $T' = \{a, b, c, h\}$  in  $D'$ ,  $R'_1 = \{a, b, exdc, h\}$  and  $R'_2 = \{a, b, eyxdc, h\}$ . Therefore every routing from  $B$  to  $T'$  uses the vertex  $d$  as an inner vertex of some path, so the construction from the proof of Lemma 2.1.18 breaks

at this point. Let  $B^* = E \setminus B' = \{c, d, f, g\}$  and  $T^* = E \setminus T' = \{d, e, f, g\}$ . There is no routing from  $B^*$  to  $T^*$  in  $(D')^{\text{opp}}$  because  $c$  can only be linked to  $d$ , and therefore  $\text{rk}_{\Gamma((D')^{\text{opp}}, T^*, E)}(\{c, d\}) = 1$ , thus  $B^*$  is not independent in  $\Gamma((D')^{\text{opp}}, T^*, E)$ . Consequently,  $(D', T', E)$  is not a duality respecting representation of  $M$ . There are two obvious ways to modify  $D'$  such that the resulting digraph is again duality respecting, but both methods introduce another arc. If we would like to use Lemma 2.1.18 as it is stated, we could rename  $d$  with  $x$ , add a new  $d$ -vertex and the arc  $(d, x)$  to  $D'$ , effectively forcing  $d$  to be a source again. Or we could add the arc  $(x, c)$  to  $D'$  — which corresponds to adding the arc  $(x, c)$  to  $D$  — then  $d$  is no longer on any essential path from  $x$  to any  $t \in T'$ . This would imply that for every  $X \subseteq E$  and every routing from  $X$  to  $T'$  that uses  $d$  as an inner vertex there is a routing  $R_{-d}$  from  $X$  to  $T'$  that omits  $d$  entirely. This routing  $R_{-d}$  could be used in the construction from the proof of Lemma 2.1.18, which yields a routing  $R^*$  linking  $E \setminus X$  to  $T^*$  in the opposite digraph. ▲

## 2.1.4 Complexity-Bounded Classes of Gammoids

In this section, we introduce three measures of complexity for gammoids that are related to a class of certain representations, and examine the corresponding classes of matroids with a bounded complexity measure.

**Definition 2.1.21.** *Let  $M$  be a gammoid and  $(D, T, E)$  with  $D = (V, A)$  be a representation of  $M$ . Then  $(D, T, E)$  is a **standard representation of  $M$** , if  $(D, T, E)$  is a duality respecting representation,  $T \subseteq E$ , every  $t \in T$  is a sink in  $D$ , and every  $e \in E \setminus T$  is a source in  $D$ . ■*

**Remark 2.1.22.** Lemmas 2.1.17 and 2.1.18 guarantee that every gammoid  $M$  has a standard representation. •

**Definition 2.1.23.** Let  $M$  be a gammoid. The **arc-complexity of  $M$**  is defined to be

$$C_A(M) = \min \left\{ |A| \mid ((V, A), T, E) \text{ is a standard representation of } M \right\}.$$

The **vertex-complexity of  $M$**  is defined to be

$$C_V(M) = \min \left\{ |V| \mid ((V, A), T, E) \text{ is a standard representation of } M \right\}. \quad \blacksquare$$

**Lemma 2.1.24.** Let  $M = (E, \mathcal{I})$  be a gammoid,  $X \subseteq E$ . Then

$$C_A(M|X) \leq C_A(M), \quad C_A(M.X) \leq C_A(M), \quad C_A(M) = C_A(M^*)$$

as well as

$$C_V(M|X) \leq C_V(M), \quad C_V(M.X) \leq C_V(M), \quad C_V(M) = C_V(M^*).$$

*Proof.* Let  $(D, T, E)$  be a standard representation of  $M$ . Then  $(D^{\text{opp}}, E \setminus T, E)$  is a standard representation of  $M^*$ : By Definition 2.1.15 we have  $M^* = \Gamma(D^{\text{opp}}, E \setminus T, E)$ , and since every sink of  $D$  is a source of  $D^{\text{opp}}$  and every source of  $D$  is a sink of  $D^{\text{opp}}$ , the set  $E \setminus T$  consists of sinks of  $D^{\text{opp}}$ , and the set  $T = E \setminus (E \setminus T)$  consists of sources of  $D^{\text{opp}}$ . Consequently,  $C_A(M^*) \leq C_A(M)$  and  $C_V(M^*) \leq C_V(M)$ . It follows that  $C_A(M) = C_A(M^*)$ , as well as  $C_V(M) = C_V(M^*)$ , since  $M = (M^*)^*$  (Corollary 1.2.32). Now let  $(D, T, E)$  be a standard representation of  $M$  where  $D = (V, A)$  such that  $|A| = C_A(M)$ . If  $T \subseteq X$ , then  $(D, T, X)$  is a standard representation of  $M|X$  and therefore  $C_A(M|X) \leq C_A(M)$ . Otherwise let  $Y = T \setminus X$ , and let  $B_0 \subseteq X$  be a set of maximal cardinality such that there is a routing  $R_0: B_0 \rightrightarrows Y$  in  $D$ . Let  $D' = (V, A')$  be the digraph that arises from  $D$  by a sequence of pivot operations as they are described in the proof of Theorem 2.1.10 with respect to the routing  $R_0$ . Observe that every  $b \in B_0$  is a sink in  $D'$  and that  $|A'| = |A|$ . We argue that  $(D', (T \cap X) \cup B_0, X)$  is a standard representation of  $M|X$ : Let  $Y_0 = \{p_{-1} \mid p \in R_0\}$  be the set of targets that are entered by the routing  $R_0$ . It follows from the proof of Theorem 2.1.10 that the triple  $(D', (T \cap X) \cup B_0 \cup (Y \setminus Y_0), E)$  is a representation of  $M$ . The chain of pivot operations we carried out on  $D$  preserves all those sources and sinks of  $D$ , which are not visited by a path  $p \in R_0$ . So we obtain that every  $e \in E \setminus (T \cup B_0)$  is a source in  $D'$ , and that every  $t \in T \cap X$  is a sink in  $D'$ . Thus the set  $T' = (T \cap X) \cup B_0$  consists of sinks in  $D'$ , and the set  $X \setminus T' \subseteq E \setminus (T \cup B_0)$  consists of sources in  $D'$ . Therefore  $(D', (T \cap X) \cup B_0, X)$  is a

standard representation, and we give an indirect argument that  $(D', (T \cap X) \cup B_0, X)$  represents  $M|X$ . Clearly,  $(D', (T \cap X) \cup B_0 \cup (Y \setminus Y_0), X)$  is a representation of  $M|X$ . Since we assume that  $(D', (T \cap X) \cup B_0, X)$  does not represent  $M|X$ , there must be a set  $X_0 \subseteq X$  such that there is a routing  $Q_0: X_0 \rightrightarrows (T \cap X) \cup B_0 \cup (Y \setminus Y_0)$  and such that there is no routing  $X_0 \rightrightarrows (T \cap X) \cup B_0$ . Thus there is a path  $q \in Q_0$  with  $q_{-1} \in Y \setminus Y_0$  and  $q_1 \in X$ . Consequently we have a routing  $Q'_1 = \{q\} \cup \{b \in \mathbf{P}(D') \mid b \in B_0\}$  in  $D'$ . This implies that there is a routing  $B_0 \cup \{q_1\} \rightrightarrows Y$  in  $D$ , a contradiction to the maximal cardinality of the choice of  $B_0$  above. Thus our assumption is wrong and  $(D', (T \cap X) \cup B_0, X)$  is a standard representation of  $M|X$ , so  $C_A(M|X) \leq C_A(M)$  holds. Finally, let  $(D, T, E)$  be a standard representation of  $M$  with  $D = (V, A)$  such that  $|V| = C_V(M)$ . By an analogue argument we obtain that  $C_V(M|X) \leq C_V(M)$  holds. The previous results combined with Lemma 1.2.46 yield that the dual inequalities  $C_A(M.X) \leq C_A(M)$  and  $C_V(M.X) \leq C_V(M)$  hold, too.  $\square$

**Lemma 2.1.25.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then  $C_V(M) \geq |E|$ .*

*Proof.* Clear, since  $E \subseteq V$  for every representation  $(D, T, E)$  with  $D = (V, A)$ .  $\square$

**Remark 2.1.26.** Let  $k \in \mathbb{N}$ . Clearly, the class of gammoids  $M$  with  $C_V(M) \leq k$  is closed under duality and arbitrary minors, but Lemma 2.1.25 shows that this class has only a finite number of pair-wise non-isomorphic matroids. Thus such a class of gammoids is trivially characterized by a finite number of excluded minors, because there are only finitely many non-isomorphic matroids with  $k+1$  elements.  $\bullet$

**Lemma 2.1.27.** *Let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  be gammoids with  $E \cap E' = \emptyset$ . Then  $M \oplus N$  is a gammoid,*

$$C_A(M \oplus N) \leq C_A(M) + C_A(N), \text{ and } C_V(M \oplus N) \leq C_V(M) + C_V(N).$$

*Proof.* Let  $(D, T, E)$  and  $(D', T', E')$  be standard representations of  $M$  and  $N$ , respectively, such that  $D = (V, A)$  and  $D' = (V', A')$  with  $|A| = C_A(M)$  and  $|A'| = C_A(N)$ , and such that  $V \cap V' = \emptyset$ . Let  $D_\oplus = (V \dot{\cup} V', A \dot{\cup} A')$ . Then  $M \oplus N = \Gamma(D_\oplus, T \dot{\cup} T', E \dot{\cup} E')$  because there are no arcs in  $D_\oplus$  connecting vertices from  $V$  with  $V'$  or vice versa. Thus every routing  $R_\oplus: X_\oplus \rightrightarrows T \dot{\cup} T'$  in  $D_\oplus$  is the disjoint union of the routings  $R = \{p \in R_\oplus \mid |p| \subseteq V\}$  and  $R' = \{p' \in R_\oplus \mid |p'| \subseteq V'\}$ , and conversely every pair of routings  $R: X \rightrightarrows T$  and  $R': X' \rightrightarrows T'$  yields a routing  $R_\oplus = R \dot{\cup} R'$  since  $V \cap V' = \emptyset$ . Thus a set  $X \subseteq E \dot{\cup} E'$  is independent in  $\Gamma(D_\oplus, T \dot{\cup} T', E \dot{\cup} E')$  if and only if  $X \cap E$  is independent in  $M$  and  $X \cap E'$  is independent in  $N$ .  $(D_\oplus, T \dot{\cup} T', E \dot{\cup} E')$  is a standard

representation of  $M \oplus N$  with  $C_A(M) + C_A(N)$  arcs, therefore

$$C_A(M \oplus N) \leq C_A(M) + C_A(N)$$

holds. The same construction applied to representations  $(D, T, E)$  and  $(D', T', E')$  with  $D = (V, A)$ ,  $D' = (V', A')$ ,  $V \cap V' = \emptyset$ ,  $|V| = C_V(M)$ , and  $|V'| = C_V(N)$  yields that  $C_V(M \oplus N) \leq C_V(M) + C_V(N)$  holds, too.  $\square$

**Corollary 2.1.28.** *Let  $M = (E, \mathcal{I})$  be a gammoid,  $F$  and  $L$  finite sets such that  $F \cap L = E \cap F = E \cap L = \emptyset$ . Then  $M \oplus (F, 2^F) \oplus (L, \{\emptyset\})$  is a gammoid and*

$$C_A(M \oplus (F, 2^F) \oplus (L, \{\emptyset\})) = C_A(M)$$

*Proof.* This is a direct consequence of Lemma 2.1.27 and the fact that

$$\Gamma((F, \emptyset), F, F) = (F, 2^F) \quad \text{and} \quad \Gamma((L, \emptyset), \emptyset, L) = (L, \{\emptyset\}). \quad \square$$

**Lemma 2.1.29.** *Let  $M = (E, \mathcal{I})$  be a gammoid with  $C_A(M) = 0$ . Then there is a subset  $X \subseteq E$  such that*

$$M = (X, 2^X) \oplus (E \setminus X, \{\emptyset\}).$$

*Proof.* Since there is a representation  $(D, T, E)$  of  $M$  with  $D = (V, \emptyset)$ , we obtain that the sets  $X \subseteq E$  that are linked to  $T$  are precisely the subsets of  $T$ . An element of  $E \setminus T$  can never be linked to  $T$  since  $\mathbf{P}(D)$  only consists of trivial paths. Thus  $\mathcal{I} = 2^T$  and obviously  $M = (T, 2^T) \oplus (E \setminus T, \{\emptyset\})$ .  $\square$

**Theorem 2.1.30.** *Let  $\mathcal{G}_0$  be the class of gammoids  $M$  with  $C_A(M) = 0$ . Then  $\mathcal{G}_0$  is closed under duality, minors, and direct sums; and  $\mathcal{G}_0$  is characterized by the excluded minor  $U = (E, 2^E \setminus \{E\})$  with  $E = \{a, b\} \neq \emptyset$ .*

*Proof.* Lemma 2.1.24 yields that  $\mathcal{G}_0$  is closed under duality and minors. Let  $M_1, M_2 \in \mathcal{G}_0$  with disjoint ground sets. By Lemma 2.1.27 we have

$$C_A(M_1 \oplus M_2) \leq C_A(M_1) + C_A(M_2) = 0,$$

so  $C_A(M_1 \oplus M_2) = 0$ , thus  $\mathcal{G}_0$  is closed under direct sums.

Now let  $X \subsetneq E$ , then  $U.X = (X, \{\emptyset\})$  and  $U|X = (X, \{X, \emptyset\})$ . Clearly,  $C_A(U.X) = 0$  and  $C_A(U|X) = 0$ . Thus every proper minor of  $U$  is in  $\mathcal{G}_0$ . Now let  $M \in \mathcal{G}_0$ , then  $M = (F, 2^F) \oplus (L, \{\emptyset\})$  for some finite sets  $F$  and  $L$ . Therefore  $\mathcal{C}(M) = \{\{l\} \mid l \in L\}$ , so every circuit of a matroid  $M \in \mathcal{G}_0$  has cardinality 1. But  $\mathcal{C}(U) = \{\{a, b\}\}$ , thus

$U \notin \mathcal{G}_0$ . Now let  $M = (Q, \mathcal{I})$  be any matroid. If there is some  $C \in \mathcal{C}(M)$  with  $|C| > 1$ , then  $M \notin \mathcal{G}_0$  and  $M|C = (C, 2^C \setminus \{C\})$  is a uniform matroid. Now, let  $c_1, c_2 \in C$  with  $c_1 \neq c_2$ , then  $(M|C) \cdot \{c_1, c_2\} = (\{c_1, c_2\}, \{\emptyset, \{c_1\}, \{c_2\}\})$  is a rank-1 uniform matroid on a 2-elementary ground set. Therefore  $(M|C) \cdot \{c_1, c_2\}$  is a minor of  $M$  that is isomorphic to  $U$ . If there is no  $C \in \mathcal{C}(M)$  with  $|C| > 1$ , then let  $L_Q = \{q \in Q \mid \{q\} \in \mathcal{C}(M)\}$  and  $F_Q = Q \setminus L_Q$ . Clearly,  $M = (F_Q, 2^{F_Q}) \oplus (L_Q, \{\emptyset\})$  and it is easy to see that  $C_A(M) = 0$ , thus  $M \in \mathcal{G}_0$ . Therefore the class  $\mathcal{G}_0$  is characterized by the single excluded minor  $U$ .  $\square$

**Lemma 2.1.31.** *Let  $k \in \mathbb{N}$  and  $M = (E, \mathcal{I})$  be a gammoid with  $C_A(M) = k$ . Then there is a partition  $E_1 \dot{\cup} E_2 \dot{\cup} E_3$  of  $E$  such that*

$$M = (M|E_1) \oplus (E_2, 2^{E_2}) \oplus (E_3, \{\emptyset\}),$$

and such that  $C_A(M|E_1) = k$ . Furthermore,  $|E_1| \leq 2k$ ,  $\text{rk}_M(E_1) \leq k$ , and there is a set  $X_0 \subseteq E_1$  with cardinality at most  $\text{rk}_M(E_1)$  such that for every  $X \subsetneq E_1$  with  $X_0 \subseteq X$

$$C_A(M|X) < k.$$

*Proof.* Let  $(D, T, E)$  be a standard representation of  $M$  with  $D = (V, A)$  and  $|A| = k$ . We may partition  $V$  into  $V_1 = \{v \in V \mid \exists u \in V : (u, v) \in A \text{ or } (v, u) \in A\}$ , the set of vertices incident with an arc, and  $V_2 = V \setminus V_1$ , the set of isolated vertices. There are no arcs in the induced digraph  $D' = (V_2, A \cap (V_2 \times V_2)) = (V_2, \emptyset)$ , thus we obtain that

$$M' = M|(E \cap V_2) = \Gamma(D, T, E \cap V_2) = \Gamma(D', T \cap V_2, E \cap V_2)$$

and consequently we have  $C_A(M|V_2) = 0$ . Therefore there are disjoint  $F, L \subseteq V_2$  such that  $M|V_2 = (F, 2^F) \oplus (L, \{\emptyset\})$  (Lemma 2.1.29). Now let  $X \subseteq V_1 \cap E$  with  $X \in \mathcal{I}$ . Then there is a routing  $R: X \rightrightarrows T$  in  $D$ , and since no arc of  $D$  is incident with  $v \in V_2$ , we obtain that  $|p| \subseteq V_1$  for all  $p \in R$ . Therefore we may conclude that  $X$  is independent in  $M'' = \Gamma(D'', T \cap V_1, E \cap V_1)$  for  $D'' = (V_1, A)$ , thus  $M'' = M|(E \cap V_1)$ . Therefore, for all  $X \subseteq E$  with  $X \in \mathcal{I}$ , we have that  $X \cap V_1$  is independent in  $M''$ , and that  $X \cap V_2$  is independent in  $M'$ . Thus

$$M = M'' \oplus M' = (M|(E \cap V_1)) \oplus (T \cap V_2, 2^{T \cap V_2}) \oplus (V_2 \setminus T, \{\emptyset\}).$$

Assume that  $C_A(M|(E \cap V_1)) < C_A(M)$ , then we could take a standard representation of  $M|(E \cap V_1)$  and augment it with isolated vertices in order to obtain a standard

representation of  $M$  with fewer than  $C_A(M)$  arcs — yielding a contradiction. Therefore  $C_A(M|(E \cap V_1)) = C_A(M)$ . Since every element of  $E_1 = E \cap V_1$  must be incident with at least one arc in  $D$ , and every arc is incident with two vertices, and since  $|A| = k$ , we obtain that  $|E_1| \leq |V_1| \leq 2k$ . Furthermore, every arc in  $D$  is incident with at most one source and at most one sink, thus  $|T \cap V_1| \leq k$ , and therefore  $\text{rk}_M(E_1) \leq k$ .

Now we show that  $C_A(M|X) < C_A(M)$  holds for every  $X \subsetneq E_1$  with  $T \cap V_1 \subseteq X$  by constructing a smaller representation. Let  $X \subsetneq E_1$  such that  $C_A(M|X) = C_A(M)$  and  $T \cap V_1 \subseteq X$ , and let  $x \in E_1 \setminus X$ . Since  $x \notin T \cap V_1$  we have  $x \notin T$ . Let

$$D_x = (V_1 \setminus \{x\}, A \cap ((V_1 \setminus \{x\}) \times (V_1 \setminus \{x\})))$$

be the digraph induced from  $D$  by removing the source  $x$ . Clearly,  $D_x$  has fewer arcs than  $D$  because at least one arc in  $D$  is incident with  $x$ . But then the contraction of  $M$  to  $X$  satisfies the equation  $M|X = \Gamma(D_x, T, X)$ , which implies  $C_A(M|X) < C_A(M)$ .  $\square$

**Theorem 2.1.32.** *Let  $k \in \mathbb{N}$ ,  $k \geq 1$ , and let  $\mathcal{G}_k$  be the class of gammoids  $M$  with  $C_A(M) \leq k$ . Then  $\mathcal{G}_k$  is closed under duality and minors, but not under direct sums; and  $\mathcal{G}_k$  is characterized by finitely many excluded minors.*

*Proof.* Let  $k \in \mathbb{N}$  be arbitrarily fixed from now on. Lemma 2.1.24 yields that  $\mathcal{G}_k$  is closed under duality and minors. Now let  $M_i = (\{a_i, b_i\}_{\neq}, \{\emptyset, \{a_i\}, \{b_i\}\})$  for  $i \in \{1, 2, \dots, k+1\}$ , such that  $\{a_i, b_i\}_{\neq} \cap \{a_j, b_j\}_{\neq} = \emptyset$  for all  $i, j \in \{1, 2, \dots, k+1\}$  with  $i \neq j$ . Then  $C_A(M_i) = 1$ , because  $M_i$  is neither free nor does  $M_i$  consist of loops, and it can be represented by  $(\{a_i, b_i\}, \{(a_i, b_i)\}, \{b_i\}, \{a_i, b_i\})$ . Now let  $N = \bigoplus_{i=1}^{k+1} M_i$ , and let  $(D, T, E)$  be a standard representation of  $N$  with  $D = (V, A)$ . Then  $|T| = \text{rk}_N(E) = k+1$  and  $|E| = 2k+2$ . Now assume that  $|A| \leq k$ , i.e. that  $N \in \mathcal{G}_k$ . There is some  $e \in E \setminus T$  such that  $e$  is not incident with an arc from  $A$ , thus  $\{e\}$  cannot be linked to  $T$  in  $D$ . But then  $\text{rk}_N(\{e\}) = 0$  follows, which is a contradiction to the fact that  $\text{rk}_{M_i}(\{e\}) = 1$  for the appropriate index  $i$ . Thus  $N \notin \mathcal{G}_k$ , and consequently,  $\mathcal{G}_k$  is not closed under direct sums.

Now let  $M = (E, \mathcal{I})$  be a matroid. If  $M \in \mathcal{G}_k$ , then Lemma 2.1.31 yields that there is a partition  $E_1 \dot{\cup} E_2 \dot{\cup} E_3 = E$  with

$$M = (M|E_1) \oplus (E_2, 2^{E_2}) \oplus (E_3, \{\emptyset\})$$

and  $|E_1| \leq 2k$  such that  $C_A(M|E_1) \leq k$ . Now let  $M = (E, \mathcal{I})$  be an excluded minor for  $\mathcal{G}_k$ . Then for all  $e \in E$  the restriction  $M|(E \setminus \{e\}) \in \mathcal{G}_k$ . Thus Lemma 2.1.27 yields



that for all  $e \in E$

$$M|(E \setminus \{e\}) \oplus (\{e\}, \{\emptyset, \{e\}\}) \neq M \neq M|(E \setminus \{e\}) \oplus (\{e\}, \{\emptyset\}),$$

i.e.  $M$  has neither a loop nor a coloop. In this case, Lemma 2.1.31 implies that  $|E \setminus \{e\}| \leq 2k$ , so  $|E| \leq 2k + 1$ , thus every excluded minor for  $\mathcal{G}_k$  has at most  $2k + 1$  elements. But up to isomorphism, there are only finitely many matroids on ground sets with at most  $2k + 1$  elements, so  $\mathcal{G}_k$  is characterized by finitely many excluded minors.  $\square$

We have seen that subclasses of gammoids, that are defined by limiting the number of arcs or the number of vertices available in a standard representation, merely consist of a finite number of matroids which may be extended with an arbitrary amount of loops and coloops. Moreover, except for  $\mathcal{G}_0$ , those classes are not closed under direct sums.

**Definition 2.1.33.** Let  $f: \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$  be a non-decreasing function, and let  $M = (E, \mathcal{I})$  be a gammoid. The  **$f$ -width of  $M$**  shall be

$$W_f(M) = \max \left\{ \frac{C_A((M.Y)|X)}{f(|X|)} \mid X \subseteq Y \subseteq E \right\}.$$

Let  $k \in \mathbb{N}$ , then the  **$k$ -width of  $M$**  shall be

$$W^k(M) = W_{f_k}(M)$$

where

$$f_k: \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}, n \mapsto \max \{1, k \cdot n\}. \quad \blacksquare$$

Clearly  $W^0(M) = C_A(M)$  for all gammoids  $M$ .

**Corollary 2.1.34.** Let  $M = (E, \mathcal{I})$  be a gammoid,  $X \subseteq Y \subseteq E$ . Then

$$W_f(M) = W_f(M^*) \quad \text{and} \quad W_f((M.Y)|X) \leq W_f(M).$$

*Proof.* The second inequality is a direct consequence of the Definition 2.1.33. Let  $M = (E, \mathcal{I})$  be a gammoid and  $X \subseteq Y \subseteq E$ , then

$$(M^*.Y)|X = ((M|Y).X)^* = ((M.E \setminus (Y \setminus X))|X)^*$$

holds due to Lemmas 1.2.46 and 1.2.47, and Remark 1.2.49. Since  $N$  and  $N^*$  share the same ground set and  $C_A(N) = C_A(N^*)$  for all gammoids  $N$  (Lemma 2.1.24), we



obtain that

$$W_f(M) = W_f(M^*). \quad \square$$

**Definition 2.1.35.** Let  $f: \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$  be a non-decreasing function. We say that  $f$  is **super-additive**, if for all  $n, m \in \mathbb{N} \setminus \{0\}$

$$f(n+m) \geq f(n) + f(m)$$

holds. ■

**Lemma 2.1.36.** Let  $f: \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$  be a non-decreasing and super-additive function, let  $k \in \mathbb{N}$ , and let  $\mathcal{W}_{f,k}$  denote the class of gammoids  $M$  with  $W_f(M) \leq k$ . Then  $\mathcal{W}_{f,k}$  is closed under duality, minors, and direct sums.

*Proof.* It is clear from Corollary 2.1.34 that  $\mathcal{W}_{f,k}$  is closed under minors and duality. Now, let  $M = (E, \mathcal{I})$  and  $N = (E', \mathcal{I}')$  with  $E \cap E' = \emptyset$  and  $M, N \in \mathcal{W}_{f,k}$ . Furthermore, let  $X \subseteq Y \subseteq E \cup E'$ . Then, by Lemmas 1.2.40 and 1.2.45, we have that

$$\begin{aligned} ((M \oplus N).Y)|X &= ((M.Y \cap E) \oplus (N.Y' \cap E))|X \\ &= ((M.Y \cap E)|X \cap E) \oplus ((N.Y \cap E')|X \cap E'). \end{aligned}$$

holds. With Lemma 2.1.27 we obtain

$$C_A(((M \oplus N).Y)|X) \leq C_A((M.Y \cap E)|X \cap E) + C_A((N.Y \cap E')|X \cap E').$$

We use the super-additivity of  $f$  in order to derive

$$\begin{aligned} \frac{C_A(((M \oplus N).Y)|X)}{f(|X|)} &\leq \frac{C_A((M.Y \cap E)|X \cap E) + C_A((N.Y \cap E')|X \cap E')}{f(|X|)} \\ &\leq \frac{k \cdot f(|X \cap E|) + k \cdot f(|X \cap E'|)}{f(|X|)} \\ &= \frac{k \cdot (f(|X \cap E|) + f(|X \cap E'|))}{f(|X|)} \\ &\leq k \cdot \frac{f(|X|)}{f(|X|)} = k, \end{aligned}$$

where the second inequality follows from the fact that

$$\frac{C_A(G)}{f(|F|)} \leq W_f(G) \leq k$$

holds for every  $G = (F, \mathcal{J}) \in \mathcal{W}_{f,k}$ , thus it holds for all minors of  $M$  and  $N$  (Corollary 2.1.34). As a consequence,  $W_f(M \oplus N) \leq k$ , and therefore  $M \oplus N \in \mathcal{W}_{f,k}$  holds.  $\square$

We may consider a class of matroids, that is closed under direct sums, and that contains a matroid, that is neither trivial nor free, to be truly infinite, as opposed to a class that consists of matroids, that are direct sums of free matroids, trivial matroids, and one matroid that is isomorphic to a member of a finite family of matroids.

**Theorem 2.1.37.** *Let  $(M_k)_{k \in \mathbb{N}}$  with  $M_k = (E_k, \mathcal{I}_k)$  be a sequence of gammoids with*

$$C_A(M_k) \geq k \cdot |E_k|.$$

*Then there is an infinite chain of strictly bigger classes of gammoids that are closed under duality, minors, and direct sums in the family of classes*

$$\mathcal{W}^{\mathbb{N}} = \left\{ \mathcal{W}^k \mid \mathcal{W}^k \text{ is the class of all gammoids } M \text{ with } W^k(M) \leq 1, k \in \mathbb{N} \right\}.$$

*Proof.* Clearly, we have that  $W^k(M) > W^{k'}(M)$  and  $W^k(M_{k'}) \geq \frac{k'}{k} > 1$  for all  $k, k' \in \mathbb{N}$  with  $k' > k$ , so every class  $\mathcal{W}^k$  contains at most  $k$  elements of the matroid sequence  $(M_k)_{k \in \mathbb{N}}$ , and every class  $\mathcal{W}^{k'}$  contains the class  $\mathcal{W}^k$  if  $k' > k$ . Furthermore,  $M_{k'}$  is contained in  $\mathcal{W}^{C_A(M_{k'})}$ , therefore every matroid of the sequence is eventually contained in some  $\mathcal{W}^k$ . Consequently,  $\mathcal{W}^{\mathbb{N}}$  must contain a countable chain of strictly bigger subclasses of gammoids.  $\square$

Conjecture 4.2.1 would imply that there is a strict chain of truly infinite subclasses of gammoids that are closed under minors and duality, and that  $\mathcal{W}^i$  is a proper subclass of  $\mathcal{W}^{i+1}$  for all  $i \in \mathbb{N}$ .

**Lemma 2.1.38.** *Let  $E$  be a finite set,  $r \in \mathbb{N}$  with  $r \leq |E|$ , and let*

$$U = \left( E, \left\{ X \subseteq E \mid |X| \leq r \right\} \right)$$

*be the uniform matroid of rank  $r$  on  $E$ . Then*

$$C_V(U) = |E| \quad \text{and} \quad C_A(U) \leq r \cdot (|E| - r).$$

*Proof.* Let  $T \subseteq E$  with  $|T| = r$  and let  $D = (E, A)$  be the digraph on the vertex set  $E$  where  $A = \{(e, t) \mid e \in E \setminus T, t \in T\}$ . Clearly,  $(D, T, E)$  is a standard representation with  $U = \Gamma(D, T, E)$ . Therefore  $C_V(U) \leq |E|$  and  $C_A(U) \leq r \cdot (|E| - r)$ . Obviously,

the vertex complexity is bounded from below by the size of the ground set, thus  $C_V(U) = |E|$ .  $\square$

The following kind of matroids is usually defined as matroids, whose ground sets consist of edges of undirected graphs, such that subsets of these edges are independent, if they contain no subgraph that consists of (i) two cycles with a single common vertex ( $\infty$ -graph), (ii) two cycles which share a common line segment ( $\Theta$ -graph), or (iii) two cycles each of which has a special vertex and those special vertices are connected by a line (hand-cuffs graph). We use L.R. Matthews's characterization in order to define bicircular matroids.

**Definition 2.1.39** ([Mat77], Corollary 3.3 and Theorem 3.5). *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  is a **bicircular matroid**, if there is a family  $\mathcal{A} = (A_i)_{i=1}^{\text{rk}_M(E)}$  of subsets of  $E$  with the property that  $|\{i \in I \mid e \in A_i\}| \in \{1, 2\}$  holds for all  $e \in E$ , and such that  $M = M(\mathcal{A})$ .  $\blacksquare$*

It is clear that bicircular matroids are special gammoids.

**Lemma 2.1.40.** *Let  $M = (E, \mathcal{I})$  be a bicircular matroid. Then*

$$C_A(M) \leq 2 \cdot |E| \quad \text{and} \quad C_V(M) \leq |E| + \text{rk}_M(E).$$

*Proof.* Let  $I$  be a set with  $|I| = \text{rk}_M(E)$  and  $\mathcal{A} = (A_i)_{i \in I}$  be a family of subsets of  $E$  such that  $M = M(\mathcal{A})$  and such that  $|\{i \in I \mid e \in A_i\}| \in \{1, 2\}$  for all  $e \in E$ . For technical reasons, let us further assume that  $I \cap E = \emptyset$ . Let  $D_0 = (V, A)$  with  $V = E \dot{\cup} I$  and  $A = \{(e, i) \mid e \in E, i \in I, e \in A_i\}$ . Then  $M = \Gamma(D_0, I, E)$  and  $|A| \leq 2 \cdot |E|$ . We obtain a standard representation of  $M$  by pivoting in an arbitrary base  $T \in \mathcal{B}(M)$  as it is done in the proof of Theorem 2.1.10. This operation does not introduce any new arcs or vertices, therefore  $C_A(M) \leq |A| \leq 2 \cdot |E|$  and  $C_V(M) \leq |E| + |I| = |E| + \text{rk}_M(E)$  holds.  $\square$

### 2.1.5 Essential Arcs and Vertices

Let  $(D, T, E)$  be a representation of a gammoid, and let  $D = (V, A)$ . In this section, we are concerned with the question when an arc  $a \in A$  or a vertex  $v \in V$  is essential for the representation of  $\Gamma(D, T, E)$ . It turns out that this kind of question may be answered by inspection of the family of independent sets of a derived gammoid.

**Definition 2.1.41.** Let  $(D, T, E)$  with  $D = (V, A)$  be a representation of the gammoid  $\Gamma(D, T, E) = (E, \mathcal{I})$ , and let  $a \in A$  be an arc of  $D$ . The arc  $a$  shall be called **essential arc of  $(D, T, E)$** , if there is some  $X \in \mathcal{I}$  such that  $X$  is not independent with respect to  $\Gamma(D_a, T, E)$  where  $D_a = (V, A \setminus \{a\})$ . ■

**Remark 2.1.42.** If  $(D, T, E)$  with  $D = (V, A)$  is a representation of  $M = \Gamma(D, T, E)$  such that  $|A| = C_A(M)$ , then every arc  $a \in A$  is essential. Also, the converse is not true: Let  $(D, T, E)$  be a representation of a gammoid such that every arc of  $D = (V, A)$  is essential. If we subdivide an arc of  $D$  with a newly introduced auxiliary vertex, then the resulting digraph  $D'$  still consists only of essential arcs with respect to  $(D', T, E)$  — but  $(D', T, E)$  can no longer have an arc set of minimal cardinality. •

**Lemma 2.1.43.** Let  $M = (E, \mathcal{I})$  be a gammoid, and let  $(D, T, E)$  be a representation of  $M$  with  $D = (V, A)$ . Let  $(u, v) \in A$  be an essential arc of  $(D, T, E)$ , and let  $N = \Gamma(D, T, V)$  and  $N' = \Gamma(D', T, V)$  where  $D' = (V, A \setminus \{(u, v)\})$ . There is a circuit  $C \in \mathcal{C}(N')$  with  $u \in C$  such that  $C$  is independent in  $N$ .

*Proof.* Clearly, if  $N = N'$ , then  $(u, v)$  is not an essential arc of  $(D, T, E)$ . Therefore there is a subset  $X \subseteq E \subseteq V$  that is independent in  $N$  yet dependent in  $N'$ . Since every routing in  $D$  is a routing in  $D'$  unless it traverses the arc  $(u, v)$ , we observe that every routing  $R: X \rightrightarrows T$  in  $D$  must traverse the arc  $(u, v)$ . Since  $X$  is dependent in  $N'$ , there is an minimum-cardinality  $X$ - $T$ -separator  $S'$  in  $D'$  with  $|S'| < |X|$ . With the previous observation we obtain that  $S = S' \cup \{u\}$  is an  $X$ - $T$ -separator in  $D$  with  $|S| = |X|$ . Furthermore, we see that  $u \notin S'$ , because otherwise  $S'$  would be an  $X$ - $T$ -separator in  $D$ , which would lead us to the contradiction  $\text{rk}_N(X) \leq |S'| < |X| = \text{rk}_N(X)$  — as  $X$  is an independent set of  $N$ . Corollary 1.5.29 yields that we may cut off the initial parts of the paths of a maximal  $X$ - $T$ -connector in  $D$  and thereby obtain a routing from  $S$  to  $T$  in  $D$ , so  $S$  is independent in  $N$ . We give an indirect argument that  $S$  is dependent in  $N'$ . Assume that  $S$  is independent in  $N'$ .  $S'$  is a minimal cardinality  $X$ - $T$ -separator in  $D'$ , thus  $S' \subseteq \text{cl}_{N'}(X)$  (Corollary 1.5.29). If there is a path  $p \in \mathbf{P}(D)$  with  $p_1 \in X \setminus S'$  and  $p_{-1} = u$  that does not visit a vertex  $s \in S'$ , then  $S' \cup \{p_1\}$  is independent, and so we obtain

$$\text{rk}_{N'}(X) = \text{rk}_{N'}(\text{cl}_{N'}(X)) \geq \text{rk}_{N'}(S' \cup \{p_1\}) = |S'| + 1 = |X|.$$

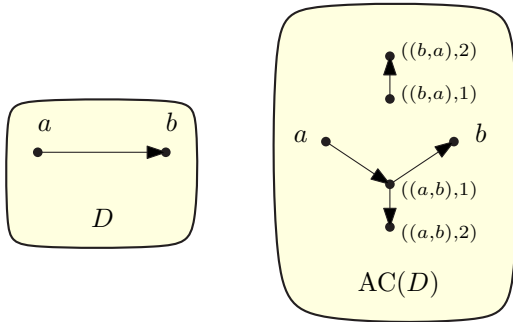
Thus  $X$  would be independent in  $N'$  — a contradiction. To avoid this contradiction, every path  $p \in \mathbf{P}(D)$  with  $p_1 \in X$  and  $p_{-1} = u$  must visit a vertex  $s \in S'$ . But then  $S'$  is an  $X$ - $S$ -separator in  $D$ , and since  $S$  is an  $X$ - $T$ -separator in  $D$ , we have that  $S'$  is an

$X$ - $T$ -separator in  $D$ . Again, this yields the contradiction  $\text{rk}_N(X) \leq |S'| < |X| = \text{rk}_N(X)$ . Therefore we may dismiss our assumption and we conclude that  $S$  is dependent in  $N'$ . Remember that  $S'$  is independent in  $N'$  because it is a minimal-cardinality  $X$ - $T$ -separator in  $D'$ , thus there is a circuit  $C \in \mathcal{C}(N')$  with  $C \subseteq S$  and  $C \not\subseteq S'$ , so  $u \in C$ ; and since  $S$  is independent in  $N$ , we obtain that  $C$  is independent in  $N$ , too.  $\square$

**Definition 2.1.44.** Let  $D = (V, A)$  be a digraph with  $V \cap ((V \times V) \times \{1, 2\}) = \emptyset$ . The **arc-cut digraph for  $D$**  shall be the digraph  $\text{AC}(D) = (V_D, A_D)$  where

$$\begin{aligned} V_D &= V \dot{\cup} (\{(u, v) \in V \times V \mid u \neq v\} \times \{1, 2\}) \quad \text{and} \\ A_D &= \{(u, ((u, v), 1)), ((u, v), 1), v) \mid (u, v) \in A, u \neq v\} \\ &\cup \{(((u, v), 1), ((u, v), 2)) \mid u, v \in V, u \neq v\}. \end{aligned}$$

In other words, for all  $u, v \in V$  with  $u \neq v$  we do the following in order to obtain  $\text{AC}(D)$  from  $D$ : If there is an arc  $(u, v)$  in  $D$ , we add two new vertices and turn it into a top-left-to-bottom-right-oriented  $\top$ -shaped-junction. If there is no arc  $(u, v)$  in  $D$ , we add two new vertices and connect one with the other.  $\blacksquare$



**Example 2.1.45.** Consider the digraph  $D = (\{a, b\}, \{(a, b)\})$ . Then  $\text{AC}(D) = (V_D, A_D)$  is the digraph where  $V_D = \{a, b, ((a, b), 1), ((a, b), 2), ((b, a), 1), ((b, a), 2)\}$  and where  $A_D = \{(a, ((a, b), 1)), (((a, b), 1), b), (((a, b), 1), ((a, b), 2)), (((b, a), 1), ((b, a), 2)), (((b, a), 2), ((b, a), 1))\}$ .  $\blacktriangle$

**Definition 2.1.46.** Let  $(D, T, E)$  be a representation of a gammoid where  $D = (V, A)$ , and such that  $V \cap ((V \times V) \times \{1, 2\}) = \emptyset$ . The **arc-cut matroid for  $(D, T, E)$**  shall be the matroid  $\text{AC}(D, T, E) = \Gamma(\text{AC}(D), T', E')$  where

$$E' = E \cup \{((u, v), i) \mid u, v \in V, u \neq v, i \in \{1, 2\}\}$$

and where

$$T' = T \cup \{((u, v), 2) \mid u, v \in V, u \neq v\}. \quad \blacksquare$$

**Lemma 2.1.47.** *Let  $(D, T, E)$  be a representation of a gammoid where  $D = (V, A)$ , and such that  $V \cap ((V \times V) \times \{1, 2\}) = \emptyset$ . Then  $X \subseteq E$  is independent with respect to  $\Gamma(D, T, E)$ , if and only if  $X' = X \cup \{((u, v), 2) \mid u, v \in V, u \neq v\}$  is independent with respect to  $\text{AC}(D, T, E)$ .*

*Proof.* Let  $X$  be independent with respect to  $M = \Gamma(D, T, E)$ . There is a routing  $R: X \rightrightarrows T$  in  $D$ . Thus we have a routing  $R' = \{p' \mid p \in R\} \cup \{((u, v), 2) \mid u, v \in V, u \neq v\}$  in  $D'$  where

$$p' = p_1((p_1, p_2), 1)p_2((p_2, p_3), 1) \dots p_{n-1}((p_{n-1}, p_n), 1)p_n$$

denotes the path in  $\text{AC}(D)$  that is obtained from  $p = (p_i)_{i=1}^n$  by subdividing every arc  $(u, v)$  traversed by  $p$  with  $((u, v), 1)$ . Consequently, the derived set  $X'$  is independent in  $N = \text{AC}(D, T, E)$ . Now let  $X$  be dependent in  $M$ , therefore there is no routing from  $X$  to  $T$  in  $D$ . Now assume that there is a routing  $R'$  from the derived set  $X'$  to  $T' = T \cup \{((u, v), 2) \mid u, v \in V, u \neq v\}$  in  $\text{AC}(D)$ , i.e. that  $X'$  is independent with respect to  $N$ . Then  $R'$  routes every  $x \in X$  to some element  $t_x \in T' \setminus (X' \setminus X) = T$  in  $\text{AC}(D)$ . By omitting the subdivision vertices in the corresponding paths  $p' \in R$ , we obtain a routing from  $X$  to  $T$  in  $D$  — a contradiction. Therefore  $X'$  is dependent in  $N$  if  $X$  is dependent in  $M$ .  $\square$

**Lemma 2.1.48.** *Let  $(D, T, E)$  be a representation of a gammoid where  $D = (V, A)$ , and such that  $V \cap ((V \times V) \times \{1, 2\}) = \emptyset$ . Furthermore, let  $a \in A$ . The arc  $a$  is an essential arc of  $(D, T, E)$  if and only if there is a circuit  $C \in \mathcal{C}(\text{AC}(D, T, E))$  with*

$$(a, 1) \in C \subseteq E \cup \{(a, 1)\} \cup \{((u, v), 2) \mid u, v \in V, u \neq v, a \neq (u, v)\}.$$

*Proof.* First, let us assume that  $a$  is an essential arc of  $(D, T, E)$ . Let  $X \subseteq E$  be independent with respect to  $\Gamma(D, T, E)$ , such that every routing  $R: X \rightrightarrows T$  in  $D$  traverses the arc  $a$ . Then every routing from  $X$  to  $T$  in  $\text{AC}(D)$  visits the vertex  $(a, 1)$ . Therefore every routing from  $X' = X \cup \{((u, v), 2) \mid u, v \in V, u \neq v, a \neq (u, v)\}$  to  $T' = T \cup \{((u, v), 2) \mid u, v \in V, u \neq v\}$  in  $\text{AC}(D)$  also has to visit the vertex  $(a, 1)$ . This implies that  $X' \cup \{(a, 1)\}$  must be dependent. From Lemma 2.1.47 we obtain that  $X'$  is independent in  $\text{AC}(D, T, E)$ , and consequently there is a circuit  $C \subseteq X' \cup \{(a, 1)\}$  such that  $(a, 1) \in C$ . Now assume that  $a$  is not an essential arc of  $(D, T, E)$ . Let  $X \subseteq E$  be independent with respect to  $\Gamma(D, T, E)$ , then there is a routing  $R: X \rightrightarrows T$  in  $D$  such that the arc  $a$  is not traversed by  $R$ . Thus there is a routing  $R'$  from  $X' = X \cup \{((u, v), 2) \mid u, v \in V, u \neq v, a \neq (u, v)\}$  to  $T'$  in  $\text{AC}(D)$

that does not visit the vertex  $(a,1)$ . It is clear from Definition 2.1.41 that such a routing  $R'$  cannot visit  $(a,2)$  either. Therefore  $R' \cup \{(a,1)(a,2)\}$  is a routing in  $AC(D)$  and  $X' \cup \{(a,1)\}$  is independent with respect to  $AC(D,T,E)$ . Consequently, if  $C \subseteq E \cup \{(a,1)\} \cup \{(u,v),2 \mid u,v \in V, u \neq v, a \neq (u,v)\}$  is a circuit of  $AC(D,T,E)$ , then  $C \cap E$  is dependent, therefore  $(a,1) \notin C$ .  $\square$

A.W. Ingleton and M.J. Piff showed the following nice theorem about representations of strict gammoids where every arc is essential, which they call *minimal presentation* of  $\Gamma(D,T,V)$ .

**Theorem 2.1.49** ([IP73], Theorem 3.12). *Let  $(D,T,V)$  be a representation of a gammoid where  $D = (V,A)$  and where all  $a \in A$  are essential arcs of  $(D,T,V)$ , and let  $u \in V \setminus T$ . Then*

$$S_u = \{v \in V \mid (u,v) \in A\} \cup \{u\} \in \mathcal{C}(\Gamma(D,T,V)).$$

For a proof, see [IP73] p.60.

**Corollary 2.1.50.** *Let  $D = (V,A)$  be a digraph,  $T \subseteq V$ , and  $E \subseteq V$ . Furthermore, let  $M = \Gamma(D,T,E)$  and  $N = \Gamma(D,T,V)$ . Then*

$$\begin{aligned} C_A(M) &\leq C_A(N) \leq |V \setminus T| + \sum_{u \in V \setminus T} \text{rk}_N(\{v \in V \mid (u,v) \in V\}) \\ &\leq (|V| - \text{rk}_N(V)) \cdot (\text{rk}_N(V) + 1). \end{aligned}$$

*Proof.* Since  $M$  is a minor of  $N$ , we have  $C_A(M) \leq C_A(N)$  (Lemma 2.1.24). The last inequality follows from Lemma 1.2.15. Let  $D' = (V,A')$  be a digraph obtained from  $D$  by successively removing one non-essential arc of  $(D',T,V)$  after another from  $A'$  until every remaining arc  $a \in A'$  is an essential arc of  $(D',T,V)$ . Let  $u \in V \setminus T$ , then Theorem 2.1.49 yields that  $S_u = \{v \in V \mid (u,v) \in A\} \cup \{u\} \in \mathcal{C}(N)$ , thus the process of removing non-essential arcs stops no sooner than when  $O_u = \{v \in V \mid (u,v) \in A'\}$  is independent in  $N$  for all  $u \in V \setminus T$ . Clearly, no arc leaving a vertex  $t \in T$  is essential for  $(D',T,V)$ . Thus

$$|A'| = \sum_{u \in V \setminus T} \text{rk}_N(\{v \in V \mid (u,v) \in V\})$$

holds. We may obtain a standard representation of  $N$  from  $(D',T,V)$  by first renaming all  $v \in V \setminus T$  to  $v'$  and then adding a new source  $v \in V \setminus T$  and a new arc  $(v,v')$  to  $D'$ .

Consequently,

$$C_A(N) \leq |V \setminus T| + \sum_{u \in V \setminus T} \text{rk}_N(\{v \in V \mid (u, v) \in V\}) \leq |V \setminus T| + \sum_{u \in V \setminus T} \text{rk}_N(V). \quad \square$$

Corollary 2.1.50 together with Remark 2.1.14 implies that every gammoid  $M = (E, \mathcal{I})$  may be represented on a digraph with at most  $k = \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E|$  vertices and with at most  $(k - \text{rk}_M(E)) \cdot (1 + \text{rk}_M(E))$  arcs.

**Lemma 2.1.51.** *Let  $r \in \mathbb{N}$ ,  $U = (E, \mathcal{I})$  be a uniform matroid with  $r \leq |E|$ , i.e.  $\mathcal{I} = \{X \subseteq E \mid |X| \leq r\}$ , and let  $(D, T, E)$  with  $D = (E, A)$  be a strict representation of  $U$ . Then*

$$|A| \geq r \cdot (|E| - r).$$

*Proof.* Without loss of generality we may assume that no digraph occurring in this proof contains a loop arc  $(v, v)$ . Let  $(D, T, E)$  with  $D = (E, A)$  be a strict representation of  $U$  with a minimal number of arcs among all such representations. Due to that minimality, every  $t \in T$  is a sink of  $D$ , and every arc  $a \in A$  is an essential arc. Observe that  $|C| = r + 1$  for all  $C \in \mathcal{C}(U)$ . Thus we obtain from Theorem 2.1.49 that

$$\begin{aligned} |A| &= \left| \bigcup_{u \in E \setminus T} \{(x, v) \in A \mid x = u\} \right| \\ &= \sum_{u \in E \setminus T} |\{(x, v) \in A \mid x = u\}| \\ &= \sum_{u \in E \setminus T} |\{v \in V \mid (u, v) \in A\}| \\ &\geq |E \setminus T| \cdot \min \{ |C| - 1 \mid C \in \mathcal{C}(U) \} = (|E| - r) \cdot r. \end{aligned}$$

Thus, every strict representation of  $U$  has at least  $r \cdot (|E| - r)$  arcs. The strict standard representation constructed in Lemma 2.1.38 yields that this bound is attained.  $\square$

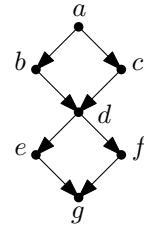
In general, strict gammoids that are not transversal matroids exist and such matroids cannot have a standard representation that is also a strict representation, because their duals do not have a strict representation.



**Definition 2.1.52.** Let  $(D, T, E)$  with  $D = (V, A)$  be a representation of the gammoid  $\Gamma(D, T, E) = (E, \mathcal{I})$ , and let  $q \in V$  be a vertex of  $D$ . Then  $q$  shall be called **essential vertex of  $(D, T, E)$** , if either  $q \in E$  or if  $q \in V \setminus E$  and there is some  $X \in \mathcal{I}$  such that  $X$  is not independent with respect to  $\Gamma(D_q, T, E)$  where

$$D_q = (V \setminus \{q\}, \{(u, v) \in A \mid u \neq q \text{ and } v \neq q\}). \quad \blacksquare$$

**Remark 2.1.53.** Clearly, if  $(u, v)$  is an essential arc of  $(D, T, E)$ , then  $u$  and  $v$  are essential vertices of  $(D, T, E)$ . On the other hand, not every essential vertex  $v$  of  $(D, T, E)$  is incident with an essential arc of  $(D, T, E)$ . For instance, let  $D = (V, A)$  be the digraph where  $V = \{a, b, c, d, e, f, g\}$  and  $A = \{(a, b), (a, c), (b, d), (c, d), (d, e), (d, f), (e, g), (f, g)\}$ . Then  $d$  is an essential vertex of  $(D, \{g\}, \{a\})$ , but  $(D, \{g\}, \{a\})$  has no essential arcs. •



**Lemma 2.1.54.** Let  $M = (E, \mathcal{I})$  be a gammoid, and let  $(D, T, E)$  be a representation of  $M$ , and let  $D = (V, A)$ . Let  $q \in V \setminus E$  be an essential vertex of  $(D, T, E)$ , and let  $N = \Gamma(D, T, V \setminus \{q\})$  and  $N' = \Gamma(D_q, T, V \setminus \{q\})$  where

$$D_q = (V \setminus \{q\}, \{(u, v) \in A \mid u \neq q \text{ and } v \neq q\}).$$

Then there is a circuit  $C \in \mathcal{C}(N')$  with  $C \cap \{u \in V \mid (u, q) \in A\} \neq \emptyset$  such that  $C$  is independent in  $N$ .

*Proof.* Without loss of generality we may assume that  $(q, q) \notin A$ , and we do induction on the number of arcs entering  $q$  in  $D$ . There has to be at least one arc  $(u, q) \in A$  with  $u \in V$ , because there is a subset  $X \subseteq E$ , such that every routing  $R: X \rightrightarrows T$  visits the vertex  $q \notin E$ . So  $q$  cannot be a source in  $D$ . Therefore, the base case of the induction is the case where precisely one arc  $(u, q)$  enters  $q$  in  $D$ . This arc is essential with respect to  $(D, T, E)$ , since it is traversed by every routing from  $X$  to  $T$  in  $D$ . Lemma 2.1.43 yields a desired circuit  $C$  with  $u \in C$ . If there is a non-essential arc entering  $q$ , then  $\Gamma(D, T, E) = \Gamma(D', T, E)$  where  $D' = (V, A \setminus \{(u, q)\})$  for an arbitrarily chosen non-essential arc  $(u, q) \in A$ , and then the existence of  $C$  follows by induction hypothesis on  $(D', T, E)$ . If all arcs entering  $q$  are essential for  $(D, T, E)$ , then we pick an arbitrary choice  $(u, q) \in A$ , and throw away all other arcs entering  $q$ . Let  $D'' = (V, A'')$  where  $A'' = \{(x, y) \in A \mid y \neq q\} \cup \{(u, q)\}$ , and let  $M' = \Gamma(D'', T, E)$ . Clearly, every independent set of  $M'$  is also independent in  $M$ , and if we delete  $q$  and all incident arcs from  $D''$  we obtain  $D_q$ . Furthermore, there is exactly one arc entering

$q$  in  $D''$ , and therefore we obtain a circuit  $C \in \mathcal{C}(N')$  with  $u \in C$  that is independent in  $\Gamma(D'', T, V \setminus \{q\})$  – and therefore independent in  $N$  – from the induction hypothesis applied to  $(D'', T, E)$ .  $\square$

## 2.1.6 Digraphs as Black Boxes

Lemma 2.1.17 states that each gammoid  $M$  may be represented by a triple  $(D, T, E)$ , where  $T \subseteq E$  is a base of  $M$  and where  $D = (V, A)$  is a digraph, such that all  $t \in T$  are sinks and all  $e \in E \setminus T$  are sources of  $D$ . Given such a representation, we may disregard the structure of  $D$ . Instead, we may regard  $D$  merely as a function, which assigns to each pair  $(X, S)$  — where  $X \subseteq E$  and where  $S \subseteq T$  such that  $X \cap T \subseteq S$  holds — the minimal cardinality of an  $X$ - $S$ -separator in  $D$ . Clearly, the value of this function with respect to  $D$  equals the rank of  $X$  with respect to the contraction  $M.(V \setminus T) \cup S$ , and therefore the function derived from  $D$  does not depend on the choice of the representation  $(D, T, E)$  of  $M$ , it is already determined by  $M$  alone. In this section, we will elaborate this idea.

**Definition 2.1.55.** Let  $M = (E, \mathcal{I})$  be a matroid,  $B \subseteq E$ ,  $\rho: 2^E \times 2^B \rightarrow \mathbb{N}$  a map. The pair  $(B, \rho)$  shall be called  **$M$ -black box**, if  $B$  is a base of  $M$  and if for all  $X \subseteq E$  and all  $S \subseteq B$  the equation

$$\rho(X, S) = \text{rk}_{M.(E \setminus B) \cup S}(X \setminus (B \setminus S))$$

is satisfied. If  $B$  is clear from the context, we also denote the  $M$ -black box  $(B, \rho)$  by  $\rho$  alone.  $\blacksquare$

Clearly, for every  $B \in \mathcal{B}(M)$ , there is a unique  $M$ -black box  $(B, \rho)$ .

**Definition 2.1.56.** Let  $D = (V, A)$  be a digraph, and  $X, Y \subseteq V$ . The **black box for  $(X, D, Y)$**  shall be the map

$$\lambda_{(X, D, Y)}: 2^X \times 2^Y \rightarrow \mathbb{N}$$

where for all  $S \subseteq X$  and all  $T \subseteq Y$

$$\lambda_{(X, D, Y)}(S, T) = \min \left\{ |C| \mid C \subseteq V \text{ s.t. } C \text{ is an } S\text{-}T\text{-separator in } D \right\}.$$

If  $(X, D, Y)$  is clear from the context, we may denote  $\lambda_{(X, D, Y)}$  by  $\lambda$ , too.  $\blacksquare$

**Definition 2.1.57.** Let  $X, Y$  be finite sets with  $X \cap Y = \emptyset$ , and let  $\lambda: 2^X \times 2^Y \rightarrow \mathbb{N}$  be a map. Then  $\lambda$  shall be called a  **$D$ -black box**, if there is a digraph  $D = (V, A)$  with  $X \cup Y \subseteq V$  such that for all  $X' \subseteq X$  and  $Y' \subseteq Y$  with  $X' \cap Y' \subseteq Y'$

$$\lambda(X', Y') = \lambda_{(X, D, Y)}(X', Y').$$

In this case, we say that  $\lambda$  is a  $D$ -black box represented by  $(X, D, Y)$ . ■

**Corollary 2.1.58.** Let  $M = (E, \mathcal{I})$  be a matroid,  $B \in \mathcal{B}(M)$  a base of  $M$ , and let  $(B, \rho)$  be the corresponding  $M$ -black box. Then  $M$  is a gammoid if and only if  $\rho$  is a  $D$ -black box.

*Proof.* There is a standard representation  $(D, T, E)$  of the gammoid  $M$  such that  $D = (V, A)$  and  $T = B$  (Remark 2.1.22). Then  $(E, D, T)$  represents  $\rho$  due to Menger's Theorem 1.5.27, Definition 2.1.1, and the fact that for all  $T' \subseteq T$ , we have the equality  $\Gamma(D, T, E).(E \setminus T') = \Gamma(D', T \setminus T', E \setminus T')$  where  $D' = (V \setminus T', A \cap ((V \setminus T') \times (V \setminus T')))$  — this is a special case of the construction used in the proof of Lemma 2.2.8. Therefore every  $M$ -black box is a  $D$ -black box. Conversely, if the  $M$ -black box  $\rho$  is represented by  $(X, D, Y)$ , then  $M = \Gamma(D, Y, X)$  and so  $M$  is a gammoid. □

**Definition 2.1.59.** Let  $D = (V, A)$  be a digraph. Then  $D$  shall be called ***cascade digraph***, if there is a partition  $V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_k = V$  such that  $A \subseteq \bigcup_{i=1}^{k-1} (V_i \times V_{i+1})$ . ■

**Definition 2.1.60** ([Mas72]). Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  is a ***cascade***, if there is a digraph  $D = (V, A)$ , such that there is a partition  $V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_k = V$  with  $A \subseteq \bigcup_{i=1}^{k-1} (V_i \times V_{i+1})$ , and such that

$$M = \Gamma(D, V_k, V_1). \quad \blacksquare$$

**Proposition 2.1.61** ([Mas72], [Mas70]<sup>2</sup>). Let  $\mathcal{CM}$  be the class of all cascades. Then

$$\{M^* \mid M \in \mathcal{CM}\} \not\subseteq \mathcal{CM} \quad \text{and} \quad \{M.E' \mid M = (E, \mathcal{I}) \in \mathcal{CM}, E' \subseteq E\} \not\subseteq \mathcal{CM}.$$

In other words, the class of cascades is neither closed under taking duals nor under contraction.

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<sup>2</sup>Unfortunately, we were not able to acquire a copy of J.H. Mason's thesis from within Europe before the printing deadline. The thesis contains the proof that the dual of a certain cascade is not a cascade itself, which is only cited in [Mas72]. It appears to be available at the Memorial Library, UW Madison Theses Basement North AWB M411 J655.

Clearly, every cascade digraph is acyclic, and it is also clear that the **transitive triple digraph**  $(\{x, y, z\}, \{(x, y), (y, z), (x, z)\})$  is an acyclic digraph but not a cascade digraph. But regarding  $D$ -black boxes, the class of cascade digraphs and the class of acyclic digraphs have the same expressiveness.

**Lemma 2.1.62.** *Let  $D = (V, A)$  be a digraph and let  $X, Y \subseteq V$ , and  $a = (u, w) \in A$ . Furthermore, let  $v \notin V$  be a new element. Then*

$$\lambda_{(X, D, Y)} = \lambda_{(X, D', Y)}$$

where

$$D' = (V \dot{\cup} \{v\}, A \setminus \{a\} \cup \{(u, v), (v, w)\})$$

denotes the digraph obtained from  $D$  by subdividing the arc  $a$  with the new vertex  $v$ .

*Proof.* Clearly,  $v \notin X \cup Y \subseteq V$ . The statement of the lemma follows from the fact that for all  $x \in X$  and  $y \in Y$  there is an obvious bijection

$$\varphi: \mathbf{P}(D; x, y) \longrightarrow \mathbf{P}(D'; x, y), p \mapsto \begin{cases} p & \text{if } a \notin |p|_A, \\ qvr & \text{otherwise,} \end{cases}$$

where  $q = (p_1, p_2, \dots, p_j)$  and  $r = (p_j, p_{j+1}, \dots, p_n)$  for  $p = (p_i)_{i=1}^n$  and  $j \in \{1, 2, \dots, n\}$  such that  $p_j = u$ , and consequently,  $p_{j+1} = w$ . Let  $X' \subseteq X$  and  $Y' \subseteq Y$ . The map  $\varphi$  yields that every  $X'$ - $Y'$ -separator in  $D$  is also an  $X'$ - $Y'$ -separator in  $D'$ , as well as every  $X'$ - $Y'$ -separator  $S$  in  $D'$  with  $v \notin S$  is an  $X'$ - $Y'$ -separator in  $D$ . Furthermore, if  $S$  is an  $X'$ - $Y'$ -separator in  $D'$  with  $v \in S$ , then  $S \setminus \{v\} \cup \{u\}$  is an  $X'$ - $Y'$ -separator in  $D$  of the same or less cardinality. Therefore  $\lambda_{(X, D, Y)} = \lambda_{(X, D', Y)}$ , since the values of those maps only depend on the cardinality of their respective minimal separators.  $\square$

**Lemma 2.1.63.** *Let  $D = (V, A)$  be an acyclic digraph, and let  $X, Y \subseteq V$ . Then there is a cascade digraph  $D'$  such that*

$$\lambda_{(X, D, Y)} = \lambda_{(X, D', Y)}.$$

*Proof.* Without loss of generality we may assume that  $X \cap Y = \emptyset$ , since otherwise we could introduce a copy  $v'$  for every vertex  $v \in X \cap Y$  and add a single arc leaving  $v$  and entering  $v'$  to  $D$ , and then continue with  $Y' = Y \setminus X \cup \{v' \mid v \in X \cap Y\}$ . Using similar constructions, we may also assume without loss of generality, that  $X$  consists of sources of  $D$  and  $Y$  consists of sinks of  $D$ , as well as that  $D$  has no sources and no

sinks in  $V \setminus (X \cup Y)$ . Possibly renaming elements from  $V$ , we may further assume that  $V \cap (A \times \mathbb{N}) = \emptyset$ . Since  $D$  is acyclic, there is a strict linear order<sup>3</sup>  $\prec$  on  $V$  such that  $u \prec v$  holds for all  $(u, v) \in A$ . Let  $D' = (V', A')$  be the digraph where

$$\begin{aligned} V' &= V \dot{\cup} \left\{ ((u, v), i) \in A \times \mathbb{N} \mid 1 \leq i \leq |\{x \in V \mid u \prec x \prec v\}| \right\} \text{ and} \\ A' &= \left\{ (u, v) \in A \mid \{x \in V \mid u \prec x \prec v\} = \emptyset \right\} \\ &\cup \left\{ (u, ((u, v), 1)) \mid (u, v) \in A, \{x \in V \mid u \prec x \prec v\} \neq \emptyset \right\} \\ &\cup \left\{ (((u, v), k), v) \mid (u, v) \in A, k = |\{x \in V \mid u \prec x \prec v\}| \neq 0 \right\} \\ &\cup \left\{ (((u, v), k), ((u, v), k+1)) \mid (u, v) \in A, k \in \mathbb{N}, 1 \leq k < |\{x \in V \mid u \prec x \prec v\}| \right\}. \end{aligned}$$

In words, every arc  $(u, v) \in A$  is subdivided by  $k$  new vertices, where  $k$  equals the number of vertices that the arc  $(u, v)$  skips with respect to the strict linear order  $\prec$  on  $V$ . For instance, if  $(u, v) \in A$  and  $u \prec x \prec y \prec v$  is a maximal chain connecting  $u$  with  $v$  in  $\prec$ , then the arc  $(u, v)$  is subdivided by the new vertices  $((u, v), 1)$  and  $((u, v), 2)$ . Repeated application of Lemma 2.1.62 yields that

$$\lambda_{(X, D, Y)} = \lambda_{(X, D', Y)}.$$

We define the map

$$\begin{aligned} \varphi: V' &\longrightarrow \{1, 2, \dots, |V \setminus (X \cup Y)| + 1\}, \\ v' &\mapsto \begin{cases} 1 & \text{if } v' \in X, \\ 1 + |\{u \in V \mid u \prec v'\}| & \text{if } v' \in V \setminus X, \\ 1 + i + |\{x \in V \mid x \prec u\}| & \text{if } v' = ((u, v), i) \in A \times \mathbb{N}. \end{cases} \end{aligned}$$

Let  $k = |V \setminus (X \cup Y)| + 1$ , then there is a partition  $V'_1, V'_2, \dots, V'_k$  of  $V'$  with

$$V'_i = \{v' \in V' \mid \varphi(v') = i\}$$

for all  $i \in \{1, 2, \dots, k\}$  that has the property that

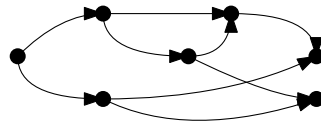
$$A' \subseteq \bigcup_{i=1}^{k-1} (V'_i \times V'_{i+1}).$$

Therefore,  $D'$  is a cascade digraph where the set  $X = V_1$  and the set  $Y = V_k$ . □

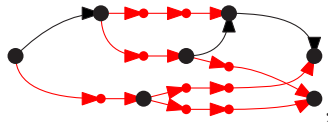
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<sup>3</sup>That is a binary relation, which is irreflexive, antisymmetric, and transitive.

**Example 2.1.64.** The construction from Lemma 2.1.63 applied to



yields



where the vertices and arcs that do not belong to the original digraph are depicted red. ▲

**Corollary 2.1.65.** *Let  $D = (V, A)$  be an acyclic digraph,  $E, T \subseteq V$ . Then there is a cascade digraph  $D' = (V', A')$  with a partition  $V'_1 \dot{\cup} V'_2 \dot{\cup} \dots \dot{\cup} V'_k = V'$  such that  $A' \subseteq \bigcup_{i=1}^{k-1} (V'_i \times V'_{i+1})$  and such that*

$$\Gamma(D, T, E) = \Gamma(D', V'_k, V'_1).$$

*Every gammoid that can be represented using an acyclic digraph is a cascade.*

*Proof.* Direct consequence of the proof of Lemma 2.1.63. □

**Remark 2.1.66.** Corollary 2.1.65, together with Proposition 2.1.61 stating that cascades are not closed under duality, implies that cycle walks are inevitable in representations of some gammoids, since the class of gammoids is closed under duality (Lemma 2.1.18). •

## 2.2 Strict Gammoids

**Definition 2.2.1.** Let  $M = (E, \mathcal{I})$  be a matroid.  $M$  is a **strict gammoid** if there is a digraph  $D = (V, A)$  and a set  $T \subseteq V$  such that  $M = \Gamma(D, T, V)$ .<sup>4</sup> ■

It is clear from this definition, that every gammoid is a deletion-minor of a strict gammoid, as  $\Gamma(D, T, E)$  for  $D = (V, A)$  is a deletion-minor of  $\Gamma(D, T, V)$ . Given a strict gammoid representation  $(D, T, V)$ , then  $T \subseteq V$  is a base of  $M$  and  $\text{rk}_M(V) = |T|$ . The following characterization of the rank function of a strict gammoid was given by C. McDiarmid in [McD72], where it is used in order to proof that gammoids are indeed matroids.

**Theorem 2.2.2.** Let  $D = (V, A)$  and  $T \subseteq V$ . Let  $M = \Gamma(D, T, V)$  be the strict gammoid represented by  $(D, T, V)$ . Then for  $X \subseteq V$ ,

$$\text{rk}(X) = \min_{U \subseteq V \setminus T} (|X \setminus \vec{U}| + |\partial U|).$$

*Proof.* Let  $X \subseteq V$  and  $U \subseteq V \setminus T$ . Trivially,  $\partial U$  separates  $X \cap \vec{U}$  from  $T$  in  $D$ . Let  $S = \partial U \cup (X \setminus \vec{U})$ , then  $S$  is an  $X$ - $T$ -separator in  $D$ : let  $x \in X \setminus S$ , then  $x \in \vec{U}$ . Since  $U \cap T = \emptyset$ , every path  $p \in \mathbf{P}(D)$  with  $p_1 = x$  and  $p_{-1} \in T$  must leave the set  $U$  at some point. Consequently, it must visit a vertex from  $\partial U$ , so  $|p| \cap S \neq \emptyset$ . It follows that

$$\text{rk}(X) \leq |S| = |\partial U \cup (X \setminus \vec{U})| \leq |X \setminus \vec{U}| + |\partial U|,$$

i.e. that the right-hand side of the equation in the lemma is an upper bound for the left-hand side of that equation.

Now let  $S \subseteq V$  be an  $X$ - $T$ -separator in  $D$  with  $|S| = \text{rk}(X)$ , its existence is guaranteed by Menger's Theorem 1.5.27. Let

$$U = \{p_{-1} \in V \mid p \in \mathbf{P}(D): p_1 \in X \text{ and } |p| \cap S = \emptyset\}$$

denote the set of vertices, that can be reached from any vertex  $x \in X$  by a path not visiting  $S$ . Clearly,  $U \subseteq V \setminus T$  holds because  $S$  is an  $X$ - $T$ -separator in  $D$ , and the outer margin  $\partial U$  is a subset of  $S$  whereas  $S \cap U = \emptyset$  by construction. Let  $S' = S \cap \partial U$  and let  $X' = S \setminus \vec{U}$ .  $X'$  is indeed a subset of  $X$ : Let  $s \in S \setminus X$ , since  $S$  is a minimal  $X$ - $T$ -separator, every maximal  $X$ - $T$ -connector  $R$  has a path  $p \in R$  with  $S \cap |p| = \{s\}$ . We

<sup>4</sup>Note that this implies that  $V = E$ , so  $D = (E, A)$  is a digraph where the ground set of  $M$  is the vertex set of  $D$ .

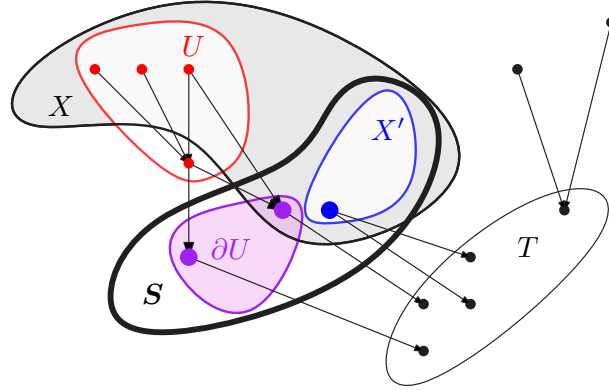


Fig. 2.2 Construction of  $U$  from an  $X$ - $T$ -separator  $S$ .

obtain that  $s = p_k$  for some  $k \in \mathbb{N}$ , and since  $s \notin X$  but  $p_1 \in X$ , we have  $k > 1$ . Therefore  $p_{k-1} \in U$  and then  $s \in \partial U \subseteq \vec{U}$ , so  $s \notin X'$  — this establishes  $X' \subseteq X$ . This yields  $X' = X \cap X' = X \cap (S \setminus \vec{U}) = (X \cap S) \setminus \vec{U}$ , and since  $X \setminus S \subseteq U \subseteq \vec{U}$ , we have  $X' = X \setminus \vec{U}$ . Since  $S \cap \vec{U} = S \cap (U \cup \partial U) = (S \cap U) \cup (S \cap \partial U) = S'$ , we have  $|S| = |X'| + |S'|$ . Now assume that  $S' \subsetneq \partial U$ , then there is a vertex  $u \in \partial U$  with  $u \notin S$ , such that there is a path  $p \in \mathbf{P}(D)$  with  $p_1 \in X$ ,  $|p| \cap S = \emptyset$ , and there is an arc  $(p_{-1}, u) \in A$ . But then we obtain that  $u \in U$ , and therefore  $u \notin \partial U$ : Since  $pu \in \mathbf{P}(D)$  is a path with  $|pu| = |p| \cup \{u\}$ , and since  $u \notin S$ , we have  $|pu| \cap S = \emptyset$ , and so  $u$  qualifies as a member of  $U$ . Therefore  $u \notin S$  cannot be the case and so  $S' = \partial U$  holds. Thus we obtain

$$\text{rk}(X) = |S| = |X'| + |S'| = |X \setminus \vec{U}| + |\partial U|$$

and therefore, on the right-hand side of the equation in the lemma, the minimum expression ranges over an upper bound that is equal to  $\text{rk}(X)$ , therefore both sides of the equation must be equal.  $\square$

J.H. Mason gives a necessary and sufficient condition for when a matroid  $M$  is a strict gammoid in [Mas72]. In order to present the proof, we need the Lemma 2.1 from [Mas72]. Here, we give a slightly more detailed version of J.H. Mason's proof. But first, we want to introduce the following notion of a special  $X$ - $T$ -separator.

**Definition 2.2.3.** Let  $D = (V, A)$  be a digraph, and let  $X \subseteq V$  and  $T \subseteq V$  be sets of vertices. The **barrier between  $X$  and  $T$  in  $D$**  is defined to be the set

$$\delta_D(X, T) = \{x \in X \mid (\partial_D\{x\}) \cap (V \setminus X) \neq \emptyset\} \cup (X \cap T). \quad \blacksquare$$



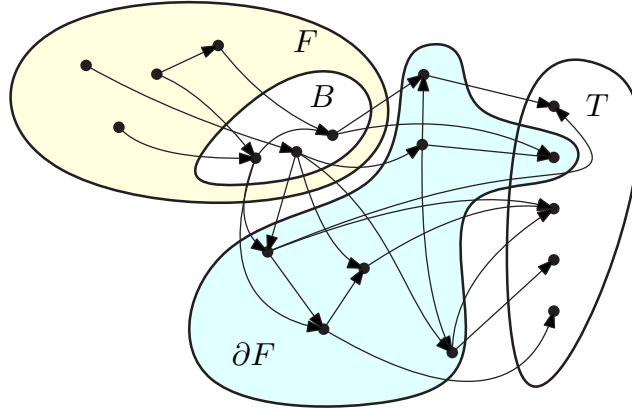


Fig. 2.3 Situation of the “right-most”  $F$ - $T$ -separator  $B$  in  $D$ .

**Lemma 2.2.4.** *Let  $D = (V, A)$  be a digraph, and let  $X \subseteq V$  and  $T \subseteq V$  be sets of vertices. Then the barrier  $\delta_D(X, T)$  is an  $X$ - $T$ -separator in  $D$ .*

*Proof.* Let  $R$  be an  $X$ - $T$ -connector, and let  $p = (p_i)_{i=1}^k \in R$  be a path that does not end in a vertex from  $X \cap T$ . Then there is a maximal integer  $1 \leq j < k$  such that  $p_j \in X$ . Then  $p_{j+1} \notin X$ , yet  $(p_j, p_{j+1}) \in A$ , thus  $p_{j+1} \in \partial\{p_j\}$  and so  $p_j \in \delta_D(X, T)$ . If otherwise  $p \in R$  is a path that ends in  $X \cap T$  we clearly have  $p_{-1} \in \delta_D(X, T)$  — thus  $\delta_D(X, T)$  is an  $X$ - $T$ -separator.  $\square$

**Lemma 2.2.5.** *Let  $D = (V, A)$  be a digraph,  $T \subseteq V$ ,  $F \in \mathcal{F}(\Gamma(D, T, V))$ , and let  $S \subseteq V$  be an  $F$ - $T$ -separator in  $D$  with minimal cardinality. Then  $S \subseteq F$ .*

*Proof.* Let  $M = \Gamma(D, T, V)$ , and let  $R: B_F \rightrightarrows T$  be a maximal  $F$ - $T$ -connector in  $D$  where  $B_F$  is a base of  $F$  in  $M$ . Then every  $s \in S$  is visited by a path  $p \in R$  (Corollary 1.5.29), thus any path  $q \in \mathbf{P}(D) \setminus R$  with  $q_1 \in S$  has the property that  $\{s\} \subseteq |q| \cap |p|$ , so  $R \cup \{q\}$  can never be a routing in  $D$ . Therefore  $R$  cannot be extended by a path starting in  $s$ , so  $R$  is also a maximal  $F \cup S$ - $T$ -connector in  $D$ . Therefore  $\text{rk}(F) = \text{rk}(F \cup S)$ , so  $S \subseteq \text{cl}(F) = F$ .  $\square$

**Lemma 2.2.6.** *Let  $M = (E, \mathcal{I})$  be a strict gammoid, and  $F \in \mathcal{F}(M)$  be a flat of  $M$ . Then the restriction  $M|_F$  is a strict gammoid. Furthermore, if  $D = (E, A)$  and  $T \subseteq E$  with  $M = \Gamma(D, T, E)$ , then the barrier  $\delta_D(F, T)$  is an  $F$ - $T$ -separator of minimal cardinality in  $D$ .*

*Proof.* Let  $M = \Gamma(D, T, E)$  for suitable  $D = (E, A)$  and  $T \subseteq E$ . Now let

$$B = \delta_D(F, T) = \left\{ f \in F \mid (\partial\{f\}) \cap (E \setminus F) \neq \emptyset \right\} \cup (F \cap T)$$

be the barrier between  $F$  and  $T$  in  $D$  (Fig. 2.3), i.e. the set which consists of those  $f \in F$ , that are either targets of the representation  $(D, T, E)$ , or that have an out-arc which leaves the flat  $F$ . Clearly,  $B$  is an  $F$ - $T$ -separator in  $D$  (Lemma 2.2.4). We give an indirect argument that  $B$  is a minimal  $F$ - $T$ -separator in  $D$ : Assume that  $B$  is not a minimal  $F$ - $T$ -separator, then there is a set  $S \subseteq F$ , which is a minimal  $F$ - $T$ -separator (Lemma 2.2.5), and there is an element  $b \in B \setminus S$ , since  $|B| > |S|$ . Clearly,  $b \notin T$  since  $F \cap T$  is a subset of every  $F$ - $T$ -separator. Further, there is an element  $e \in E \setminus F$  such that  $(b, e) \in A$  according to the definition of  $B$ , and since  $e \notin F = \text{cl}_M(F)$ , there is a maximal  $F$ - $T$ -connector  $R$  and a path  $p \in \mathbf{P}(D)$  with  $p_1 = e$  and  $p_{-1} \in T$ , such that  $R \cup \{p\}$  is a routing in  $D$ . Thus the path  $p$  does not visit any vertex that belongs to a minimal  $F$ - $T$ -separator in  $D$ , and therefore the path  $bp$  does not visit any vertex of  $S$ , too — which contradicts the assumption that  $S$  is an  $F$ - $T$ -separator, thus  $B$  must be a minimal  $F$ - $T$ -separator.

— Let  $D' = (F, A \cap (F \times F))$  be the restriction of  $D$  to  $F$ , and let  $M' = \Gamma(D', B, F)$  be the strict gammoid presented by the restriction of  $D$  and the target set  $B$ . Let  $R$  be a routing from  $X_0 \subseteq F$  to  $T$  in  $D$ , then every path  $p = (p_i)_{i=1}^k \in R$  has a smallest integer  $1 \leq j(p) \leq k$ , such that  $p_{j(p)} \in B$ . By construction of  $B$ , we have that  $\{p_1, p_2, \dots, p_{j(p)}\} \subseteq F$ . Thus  $R$  induces a routing  $R' = \{p_1 p_2 \dots p_{j(p)} \mid p \in R\}$  in  $D'$  which routes  $X_0$  to  $B$ . So  $\text{rk}_{M'}(X_0) \geq \text{rk}_M(X_0)$ . We give an indirect argument that the inequality  $\text{rk}_{M'}(X_0) \leq \text{rk}_M(X_0)$  holds, too. Let  $S \subseteq E$  be a minimal  $X_0$ - $T$ -separator in  $D$ , then we have  $S \subseteq \text{cl}_M(X_0) \subseteq F$  (Lemma 2.2.5). Assume that there is a routing  $R'$  from  $X_0$  to  $B$  in  $D'$ , such that  $|R'| > |S|$ . Then there must be some  $p \in R'$  such that  $|p| \cap S = \emptyset$  and  $p_{-1} \in B$ . If  $p_{-1} \in T$ , then  $p$  is a contradiction to  $S$  being an  $X_0$ - $T$ -separator in  $D$ . If otherwise  $p_{-1} \in B \setminus T$ , we have again the situation where there is some  $e \in E \setminus F$  with  $(p_{-1}, e) \in A$ , such that there is a path  $q \in \mathbf{P}(D)$  with  $q_1 = e$  and  $q_{-1} \in T$ , that avoids every  $F$ - $T$ -separator. So, consequently,  $|q| \cap F = \emptyset$ . The path  $pq \in \mathbf{P}(D)$  contradicts  $S$  being an  $X_0$ - $T$ -separator in  $D$ . Therefore  $|R'| \leq |S|$ , and we just proved, that for any  $X \subseteq F$  the equation  $\text{rk}_M(X) = \text{rk}_{M'}(X)$  holds. Thus  $M|F = \Gamma(D', B, F)$  is a strict gammoid.  $\square$

**Corollary 2.2.7.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then there is a strict gammoid  $M' = (V, \mathcal{I}')$  such that  $\text{rk}_M(E) = \text{rk}_{M'}(V)$  and  $M = M'|E$ .*

*Proof.* Let  $(D, T, E)$  be a representation of  $M$ , where  $D = (V, A)$ . Let  $M_0 = \Gamma(D, T, V)$  be the strict gammoid arising naturally from the representation of  $M$ , and let  $F = \text{cl}_{M_0}(E)$  be the smallest flat in  $M_0$  that contains  $E$ . Then

$\text{rk}_{M_0}(F) = \text{rk}_{M_0}(E) = \text{rk}_M(E)$  since  $M = M_0|E$ . Now, let  $M' = M_0|F$ .  $M'$  is a strict gammoid (Lemma 2.2.6), and since  $E \subseteq F$ , we have  $M = M_0|E = M'|E$ , thus  $M$  is the restriction of a strict gammoid of the same rank.  $\square$

**Lemma 2.2.8.** *Let  $M = (E, \mathcal{I})$  be a strict gammoid,  $C \subseteq E$ . Then  $M.C$  is a strict gammoid.*

*Proof.* Let  $B_0$  be a base of  $E \setminus C$  in  $M$ , and let  $B$  be a base of  $M$  with  $B_0 \subseteq B$  (Lemma 1.2.7). Let further  $D = (E, A)$  be a digraph, such that  $M = \Gamma(D, B, E)$  and such that  $B$  consists only of sinks in  $D$  (Theorem 2.1.10). We denote the family of independent sets of  $M.C$  by  $\mathcal{I}'$ . Then for every  $X \subseteq C$ , we have  $X \in \mathcal{I}'$  if and only if  $X \dot{\cup} B_0 \in \mathcal{I}$  (Lemma 1.2.42). But  $X \dot{\cup} B_0 \in \mathcal{I}$  if and only if there is a routing  $R: X \dot{\cup} B_0 \rightrightarrows B$  in  $D$ . Since  $B_0 \subseteq B$  consists of sinks in  $D$ , for every  $b_0 \in B_0$ , the trivial path  $b_0 \in \mathbf{P}(D)$  is a member of  $R$ . We give an indirect argument, that for every  $e \in (E \setminus C) \setminus B_0$  and every  $p \in R$ ,  $e \notin |p|$  holds: If there would be such a path  $p = (p_i)_{i=1}^n \in R$ , then for some  $j \in \mathbb{N}$ ,  $p_j = e$ . But then the path  $q = p_j p_{j+1} \dots p_n \in \mathbf{P}(D)$  yields a routing  $\{q\} \cup \{b_0 \in \mathbf{P}(D) \mid b_0 \in B_0\}$  which implies that  $\{e\} \cup B_0 \in \mathcal{I}$  – a contradiction to the maximality of the base  $B_0$  of  $E \setminus C$  in  $M$ . Thus the routing  $R' = \{p \in R \mid p_1 \notin B_0\}$  routes  $X$  to  $B \setminus B_0$  in  $D' = (C, A \cap (C \times C))$ , the sub-digraph of  $D$  induced by  $C$ . Conversely, every routing  $S: Y \rightrightarrows B \setminus B_0$  in  $D'$  induces the routing  $S \cup \{b_0 \in \mathbf{P}(D) \mid b_0 \in B_0\}$  from  $Y \dot{\cup} B_0$  to  $B$  in  $D$ , so  $M.C = \Gamma(D', B \setminus B_0, C)$ : the contraction is again a strict gammoid.  $\square$

### 2.2.1 Mason's $\alpha$ -Criterion

In the proof of Lemma 2.2.6 we have seen that the elements of a flat  $F$  of a strict gammoid  $M = \Gamma(D, T, V)$  fall into two disjoint categories: for some  $f \in F$ , we have  $\partial\{f\} \subseteq F$ , and for an independent subset  $I \subseteq F$ , we have  $\partial\{i\} \not\subseteq F$  for all  $i \in I$  – more precisely, there is a base  $B$  of  $F$  such that  $I = B \setminus T$ . Before we present Mason's criterion, we need one last definition.

**Notation 2.2.9.** *Let  $M = (E, \mathcal{I})$  be a matroid and  $X \subseteq E$ . The family of those flats of  $M$ , which are proper subsets of  $X$ , shall be denoted by*

$$\mathcal{F}(M, X) = \{F \in \mathcal{F}(M) \mid F \subsetneq X\}. \quad \blacksquare$$

**Definition 2.2.10.** Let  $M = (E, \mathcal{I})$  be a matroid. The  **$\alpha$ -invariant of  $M$**  shall be the map

$$\alpha_M: 2^E \longrightarrow \mathbb{Z}$$

that is uniquely characterized by the recurrence relation

$$\alpha_M(X) = |X| - \text{rk}_M(X) - \sum_{F \in \mathcal{F}(M, X)} \alpha_M(F).$$

If the matroid  $M$  is clear from the context, we also write  $\alpha(X)$  for  $\alpha_M(X)$ . ■

**Remark 2.2.11.** Clearly,  $\alpha(\emptyset) = 0$  for any matroid  $M$ . Furthermore, the value  $\alpha(X)$  for  $X \subseteq E$  may be calculated from the values  $\alpha(X')$  corresponding to proper subsets  $X' \subsetneq X$  and the rank of  $X$ , so  $\alpha$  is well-defined. •

Just like it is the case for the rank function, the family of bases, and the family of circuits of a matroid, we can use  $\alpha_M$  to reconstruct the matroid  $M$ ; thus  $M$  is already uniquely determined by  $\alpha_M$ .

**Definition 2.2.12.** Let  $E$  be a finite set and let  $\alpha: 2^E \longrightarrow \mathbb{Z}$  be a map. The **zero-family of  $\alpha$**  shall be

$$\mathcal{I}_\alpha = \{X \subseteq E \mid \forall Y \subseteq X: \alpha(Y) = 0\}.$$

The family of  **$\alpha$ -flats** shall be defined as

$$\mathcal{F}(\alpha) = \{F \subseteq E \mid \forall e \in E \setminus F, X \subseteq F: X \in \mathcal{I}_\alpha \text{ and } \{e\} \in \mathcal{I}_\alpha \Rightarrow X \cup \{e\} \in \mathcal{I}_\alpha\}.$$

Furthermore, we define the pair

$$M(\alpha) = (E, \mathcal{I}_\alpha). ■$$

**Lemma 2.2.13.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $\alpha = \alpha_M$  be its  $\alpha$ -invariant. Then  $\mathcal{I} = \mathcal{I}_\alpha$ ,  $\alpha(X) = 0$  for all  $X \in \mathcal{I}$ , and  $\alpha(C) = 1$  for all  $C \in \mathcal{C}(M)$ .

*Proof.* Let  $X \in \mathcal{I}$ , we show  $\alpha(X) = 0$  by induction on  $|X|$ . In the base case, we have

$$\alpha(\emptyset) = |\emptyset| - \text{rk}(\emptyset) = 0 - 0 = 0.$$

For the induction step, we may assume by induction hypothesis that for all  $Y \subsetneq X$  the equality  $\alpha(Y) = 0$  holds. Thus

$$\alpha(X) = |X| - \text{rk}(X) - \sum_{F \in \mathcal{F}(M, X)} \alpha(F) = |X| - |X| - \sum_{F \in \mathcal{F}(M, X)} 0 = 0.$$

Therefore we obtain  $\mathcal{I} \subseteq \mathcal{I}_\alpha$ . Now let  $X \subseteq E$  with  $X \notin \mathcal{I}$ . Then there is a circuit  $C \in \mathcal{C}(M)$  such that  $C \subseteq X$ . For all  $D \subsetneq C$ , we have  $D \in \mathcal{I}$ , therefore  $\alpha(D) = 0$ . So clearly

$$\alpha(C) = |C| - \text{rk}(C) - \sum_{F \in \mathcal{F}(M, C)} \alpha(F) = |C| - (|C| - 1) - \sum_{F \in \mathcal{F}(M, C)} 0 = 1,$$

which implies  $X \notin \mathcal{I}_\alpha$ , and we obtain that  $\mathcal{I} = \mathcal{I}_\alpha$ .  $\square$

**Corollary 2.2.14.** *Let  $M = (E, \mathcal{I})$  and  $N = (E, \mathcal{I}')$  be two matroids defined on the same ground set  $E$ . Then  $M = N$  if and only if  $\alpha_M = \alpha_N$ .*

**Remark 2.2.15.** We may express the rank of a matroid  $M = (E, \mathcal{I})$  in terms of the  $\alpha$ -invariant of  $M$  in different ways. Let  $X \subseteq E$ , then

$$\begin{aligned} \text{rk}(X) &= |X| - \sum_{F \in \mathcal{F}(M), F \subseteq X} \alpha(F) \\ &= \max \left\{ |I| \mid I \subseteq X, \forall J \subseteq I: \alpha(J) = 0 \right\} \\ &= \max \left\{ |I| \mid I \subseteq X, I \in \mathcal{I}_\alpha \right\}. \end{aligned}$$

We may use this equation in order to give an axiomatization of matroids in terms of its  $\alpha$ -invariant, which admittedly appears to be not very helpful.

(A1)  $\alpha(\emptyset) = 0$ .

(A2) For all  $X, Y \subseteq E$  with  $|X| < |Y|$  and for which the restrictions  $\alpha|_{2^X}$  and  $\alpha|_{2^Y}$  are constantly zero, there is an element  $y \in Y \setminus X$ , such that  $\alpha|_{2^{X'}}$  is constantly zero, where  $X' = X \cup \{y\}$ .

(A3) For all  $X \subseteq E$

$$\alpha(X) = |X| - \max \left\{ |I| \mid I \subseteq X, I \in \mathcal{I}_\alpha \right\} - \sum_{F \in \mathcal{F}(\alpha), F \subsetneq X} \alpha(F).$$

Clearly, (A2) resembles the augmentation axiom (I3) for  $\mathcal{I}_\alpha$  and (A1) guarantees that  $\emptyset \in \mathcal{I}_\alpha$ . (I2) trivially holds for  $\mathcal{I}_\alpha$  by construction, and so  $M(\alpha) = (E, \mathcal{I}_\alpha)$  is a matroid.

Then (A3) guarantees that  $\alpha = \alpha_{M(\alpha)}$ , i.e. that  $\alpha$  behaves like the  $\alpha$ -invariant for  $M(\alpha)$  on the dependent sets. •

**Theorem 2.2.16** ([Mas72], Theorem 2.2). *Let  $D = (V, A)$ ,  $T \subseteq V$ , and  $M = \Gamma(D, T, V)$  be a strict gammoid. Then for all  $X \subseteq V$ , we have  $\alpha(X) \geq 0$ . Furthermore, if  $F \in \mathcal{F}(M)$  then  $\alpha(F)$  is the number of elements of  $F \setminus T$  with the property, that  $\partial\{f\}$  is a subset of  $F$  but not of any proper sub-flat  $F' \subsetneq F$ .*

We present a slightly polished version of the proof in [Mas72].

*Proof.* For every  $X \subseteq E$  we define the subsets

$$\begin{aligned} X_1 &= \{x \in X \setminus T \mid \partial\{x\} \not\subseteq X\} \cup (X \cap T) = \delta_D(X, T), \\ X_2 &= \{x \in X \setminus T \mid \partial\{x\} \subseteq X \text{ and } \forall F \in \mathcal{F}(M): F \subsetneq X \Rightarrow \partial\{x\} \not\subseteq F\}, \text{ and} \\ X_3 &= \{x \in X \setminus T \mid \exists F \in \mathcal{F}(M): F \subsetneq X \text{ and } \partial\{x\} \subseteq F\}. \end{aligned}$$

Then  $X = X_1 \dot{\cup} X_2 \dot{\cup} X_3$  is the disjoint union of  $X_1$ ,  $X_2$ , and  $X_3$ . Furthermore  $X_3$  is the disjoint union of the sets  $F_2$ , where  $F$  ranges over all flats in  $M$  that are proper subsets of  $X$ , because if  $\partial\{x\} \subseteq F \in \mathcal{F}(M)$  and  $x \notin T$ , then every path from  $x$  to some  $t \in T$  must visit a vertex of  $F$ . Therefore every  $F$ - $T$ -separator in  $D$  is also an  $(F \cup \{x\})$ - $T$ -separator in  $D$ , so  $\text{rk}(F) = \text{rk}(F \cup \{x\})$ , thus  $x \in F$ , so  $x \in F_2$  for some flat  $F \subsetneq X$ . Now assume that  $x \in F_2 \cap G_2$  for some  $F, G \in \mathcal{F}(M)$ . Then  $F \cap G \in \mathcal{F}(M)$  (Lemma 1.2.19) and  $x \in F \cap G$ . But  $x \notin F_3$ , so  $F \cap G$  is not a proper subset of  $F$ , thus  $F = F \cap G$ . Analogously  $G = F \cap G$ , thus  $F = G$  whenever  $x \in F_2 \cap G_2$ . Therefore  $F_2 \cap G_2 = \emptyset$  for every  $F, G \in \mathcal{F}(M)$  with  $F \neq G$ .

— First, we prove the second claim by induction on the rank of the flat. Let  $O = \text{cl}(\emptyset)$  be the unique rank 0 flat of  $M$ . Then  $\alpha(O) = |O| - \text{rk}(O) = |O|$ . We have

$$O = \{v \in E \mid \nexists p \in \mathbf{P}(D): p_1 = v \text{ and } p_{-1} \in T\},$$

because  $O$  must consist precisely of those vertices of  $D$ , which cannot reach any target  $t \in T$ . Therefore  $\partial O = \emptyset$ , which implies that for every  $o \in O$ ,  $\partial\{o\} \subseteq O$ . Consequently,  $O = O_2$  as defined above, so  $\alpha(O) = |O_2|$  follows and the induction base is established.

— Now let  $F \in \mathcal{F}(M)$  be a flat, and by induction hypothesis we may assume that  $\alpha(F') = |F'_2|$  for all  $F' \in \mathcal{F}(M)$  with  $F' \subsetneq F$ . Thus we may assume the equation

$$|F_3| = \sum_{F' \in \mathcal{F}(M, F)} \alpha(F').$$

Furthermore,  $F_1 = \delta_D(F, T)$  is a minimal  $F$ - $T$ -separator in  $D$  (Lemma 2.2.6), therefore  $F_1$  is a base of  $F$ , and so  $|F_1| = \text{rk}(F)$ . We obtain

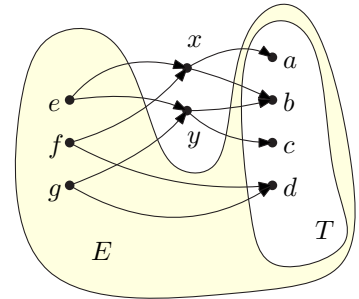
$$\begin{aligned} |F| &= |F_1| + |F_2| + |F_3| \\ &= \text{rk}(F) + |F_2| + \sum_{F' \in \mathcal{F}(M, F)} \alpha(F'), \text{ and so} \\ |F_2| &= |F| - \text{rk}(F) - \sum_{F' \in \mathcal{F}(M, F)} \alpha(F') = \alpha(F). \end{aligned}$$

— Now let  $X$  be a subset of  $E$  that is not necessarily a flat of  $M$ . Then  $X_1 = \delta_D(X, T)$  is still an  $X$ - $T$ -separator in  $D$  (Lemma 2.2.4), albeit not necessarily minimal. Therefore  $|X_1| \geq \text{rk}(X)$ . Thus we obtain

$$\begin{aligned} \alpha(X) &= |X| - \text{rk}(X) - \sum_{F \in \mathcal{F}(M, X)} \alpha(F) \\ &\geq |X| - |X_1| - |X_3| \\ &= |X_2| \geq 0. \end{aligned}$$

□

**Example 2.2.17.** Consider the digraph  $D = (V, A)$  with  $V = \{a, b, c, d, e, f, g, x, y\} \neq$  and  $A$  as depicted on the right. Let  $E = \{a, b, c, d, e, f, g\}$ ,  $T = \{a, b, c, d\}$  and  $M = \Gamma(D, T, E) = (E, \mathcal{I})$ . We argue that  $M$  – which is obviously a gammoid – is not a strict gammoid. In order to show this, we calculate some values of  $\alpha$ . Since  $\alpha(X) = 0$  for every  $X \in \mathcal{I}$ , we only have to consider summands in the recurrence relation of  $\alpha$ , that correspond to dependent flats of  $M$ . Let



$$\mathcal{F}' = \mathcal{F}(M) \setminus \mathcal{I} = \{E, \{a, b, c, e\}, \{a, b, d, f\}, \{b, c, d, g\}, \{d, e, f, g\}\}.$$

For any  $F \in \mathcal{F}' \setminus \{E\}$ , we have  $\alpha(F) = |F| - \text{rk}(F) = 4 - 3 = 1$ . Therefore

$$\alpha(E) = |E| - \text{rk}(E) - \sum_{F \in \mathcal{F}' \setminus \{E\}} \alpha(F) = 7 - 4 - 4 \cdot 1 = -1,$$

so  $M$  cannot be a strict gammoid (Theorem 2.2.16). ▲

**Definition 2.2.18.** Let  $M = (E, \mathcal{I})$  be a matroid. The  $\alpha$ -system of  $M$  is defined to be the family  $\mathcal{A}_M = (A_i)_{i \in I} \subseteq E$ , where

$$I = \{(F, n) \in \mathcal{F}(M) \times \mathbb{N} \mid 1 \leq n \leq \alpha_M(F)\}$$

and  $A_{(F,n)} = F$  for all  $(F, n) \in I$ . ■

J.H. Mason also proved that the condition  $\alpha_M \geq 0$  is sufficient for  $M$  to be a strict gammoid. First, we need a sufficient condition that allows us to recognize that a triple  $(D, T, V)$  satisfies the equality  $M = \Gamma(D, T, V)$ .

**Lemma 2.2.19** ([Mas72], Lemma 2.3). Let  $M = (V, \mathcal{I})$  be a matroid,  $D = (V, A)$  be a digraph and  $T \subseteq V$ . If for all  $X \subseteq V$ , the barriers  $\delta_D(X, T)$  have the property

$$\text{rk}_M(\delta_D(X, T)) = \text{rk}_M(X)$$

and if the barriers of flats are independent, i.e. for all  $F \in \mathcal{F}(M)$

$$|\delta_D(F, T)| = \text{rk}_M(F),$$

then  $M = \Gamma(D, T, V)$ .

*Proof.* Let  $N = \Gamma(D, T, V)$  throughout this proof. Let  $B \subseteq V$  be a base of  $M$ , and assume that  $B$  is not independent in  $N$ , i.e. that there is a  $B$ - $T$ -separator  $S$  in  $D$ , such that  $|S| < |B|$ . Let

$$X = \{p_{-1} \in V \mid p \in \mathbf{P}(D): p_1 \in B \text{ and } |p| \cap S = \emptyset\} \cup S.$$

By construction,  $X$  consists of  $S$  together with all vertices, that can be reached from some  $b \in B \setminus S$  without traversing an element from  $S$ . Since  $S$  separates  $B$  from  $T$  in  $D$ ,  $S$  is an  $X$ - $T$ -separator in  $D$ , too. By construction of  $S$ , we have  $\partial(X \setminus S) \subseteq S$  and  $X \cap T = S \cap T$ . Therefore, for all  $x \in X$  we have that  $\partial\{x\} \not\subseteq X$  implies that  $x \in S$ . Together with the fact that  $S$  is a minimal  $B$ - $T$ -separator and  $B \subseteq X$ , we arrive at  $\delta_D(X, T) = S$ . Using the premise of the lemma we arrive at the contradiction to (R1),

$$\text{rk}_M(\delta_D(X, T)) = \text{rk}_M(X) \geq \text{rk}_M(B) = |B| > |S| = |\delta_D(X, T)|.$$

In other words, if we assume that the base  $B$  of  $M$  is dependent in  $N$ , we can construct a set  $X$  that spans  $M$ , but that has a barrier in  $D$  which is smaller than  $|B|$ , and



therefore the barrier property  $\text{rk}_M(\delta_D(X, T)) = \text{rk}_M(X)$  cannot hold. Consequently,  $B$  must be independent in  $N$  and we have  $\text{rk}_N(X) \geq \text{rk}_M(X)$ .

Now let  $X \subseteq V$ , then let  $F = \text{cl}_M(X)$  be the smallest flat containing  $X$  in  $M$ . By Lemma 2.2.4,  $\delta_D(F, T)$  is an  $F$ - $T$ -separator in  $D$ , therefore

$$\text{rk}_N(X) \leq \text{rk}_N(F) \leq |\delta_D(F, T)| = \text{rk}_M(F) = \text{rk}_M(X).$$

Consequently,  $\text{rk}_M = \text{rk}_N$ , and so  $M = N = \Gamma(D, T, V)$ .  $\square$

**Theorem 2.2.20** ([Mas72], Theorem 2.4). *Let  $M = (E, \mathcal{I})$  be a matroid. If  $\alpha(X) \geq 0$  holds for all  $X \subseteq E$ , then  $M$  is a strict gammoid.*

J.H. Mason's proof [Mas72] uses the following line of arguments: First, observe that  $\alpha \geq 0$  is a sufficient condition for the  $\alpha$ -system of  $M$  to have a transversal  $T_0$ . Let  $T_0$  be such a transversal, and the map  $\sigma': T_0 \rightarrow \mathcal{F}(M)$  shall be the projection on the first coordinate of the bijection  $\sigma: T_0 \rightarrow I$  witnessing the transversal property of  $T_0$  with respect to  $\mathcal{A}_M$ . Then let  $T = E \setminus T_0$  be the target set, and let  $D = (E, A)$  be the digraph where  $(u, v) \in A$  if and only if  $u \in T_0$  and  $v \in \sigma'(u)$ . We have  $M = \Gamma(D, T, E)$ . Now, let us see this proof in detail.

*Proof.* Let  $I$  and  $\mathcal{A}_M$  be as in Definition 2.2.18. It follows from Hall's Theorem (Corollary 1.4.4) that  $\mathcal{A}_M$  has a transversal  $T_0$  if and only if for all  $J \subseteq I$  the inequality  $|\bigcup_{i \in J} A_i| \geq |J|$  holds. For  $\mathcal{A}_M$ , this is the case if and only if for all  $\mathcal{G} \subseteq \mathcal{F}(M)$ , the inequality

$$\left| \bigcup_{F \in \mathcal{G}} F \right| \geq \sum_{F \in \mathcal{G}} \alpha(F) \quad (2.1)$$

holds. For every  $X \subseteq E$ , the recurrence relation of  $\alpha$  (Definition 2.2.10) can be written as the equation

$$\alpha(X) + \sum_{F' \in \mathcal{F}(M, F)} \alpha(F') = |X| - \text{rk}(X).$$

Consequently, we obtain the inequality

$$|X| - \text{rk}(X) = \alpha(X) + \sum_{F' \in \mathcal{F}(M, X)} \alpha(F') \geq \sum_{F' \in \mathcal{F}(M), F' \subseteq X} \alpha(F'),$$

where equality holds whenever  $X \in \mathcal{F}(M)$  or  $\alpha(X) = 0$ . Now let  $\mathcal{G} \subseteq \mathcal{F}(M)$ , then we can use the last inequality together with the property that  $\alpha \geq 0$ , and we obtain

$$\left| \bigcup_{G \in \mathcal{G}} G \right| \geq \text{rk} \left( \bigcup_{G \in \mathcal{G}} G \right) + \sum_{F' \in \mathcal{F}(M), F' \subseteq \bigcup \mathcal{G}} \alpha(F') \geq \sum_{G \in \mathcal{G}} \alpha(G),$$

therefore the inequality 2.1 holds for  $\mathcal{G}$ . Consequently,  $\mathcal{A}_M$  has a transversal.

— Let  $T_0$  be a transversal of  $\mathcal{A}_M$ , and let  $\sigma: T_0 \rightarrow I$  be a bijective map with the property that for all  $t \in T_0$ ,  $t \in A_{\sigma(t)}$ . We set  $T = E \setminus T_0$  and define the map

$$\sigma': T_0 \rightarrow \mathcal{F}(M), t \mapsto F_t$$

where  $F_t \in \mathcal{F}(M)$  such that there is some  $i_t \in \mathbb{N}$  with  $\sigma(t) = (F_t, i_t)$ . Now, define  $D = (E, A)$  to be the digraph on  $E$  where  $(u, v) \in A$  if and only if  $u \in T_0$  and  $v \in \sigma'(u)$ . Let  $N = \Gamma(D, T, E)$  be the strict gammoid represented by  $(D, T, E)$ . We want to use Lemma 2.2.19 in order to show that  $M = N$ . Let  $X \subseteq E$  be a subset of  $E$ . We have to show that the set

$$B_X = \delta_D(X, T) = \left\{ x \in X \setminus T \mid \sigma'(x) \not\subseteq X \right\} \cup (X \cap T)$$

contains a base of  $X$  with respect to  $M$ , i.e. that  $\text{rk}_M(B_X) = \text{rk}_M(X)$ ; and further, if  $X \in \mathcal{F}(M)$ , we have to show that  $\text{rk}_M(B_X) = |B_X|$  holds, too. Assume for now, that  $B_X$  contains a base of  $X$ , and that  $X \in \mathcal{F}(M)$ . Then

$$|X| = \text{rk}_M(X) + \sum_{F \in \mathcal{F}(M), F \subseteq X} \alpha_M(F)$$

and the set  $X \setminus B_X$  consists of all elements  $t \in X \cap T_0$ , such that the flat  $\sigma'(t)$  is a subflat of  $X$ , therefore

$$|X \setminus B_X| = \sum_{F \in \mathcal{F}(M), F \subseteq X} \alpha_M(F),$$

and consequently  $|B_X| = \text{rk}_M(B_X)$ .

— We give an indirect argument that indeed  $\text{rk}_M(B_X) = \text{rk}_M(X)$ , so let us assume that  $\text{rk}_M(B_X) < \text{rk}_M(X)$ . Let  $Y \subseteq \text{cl}_M(X)$  be a subset that is maximal with respect to set-inclusion among all subsets of  $\text{cl}_M(X)$  with the property, that  $\text{rk}_M(B_Y) < \text{rk}_M(Y)$ , where  $B_Y = \delta_D(Y, T)$ . We show that  $\text{cl}_M(B_Y) \subseteq Y$  holds for the maximal choice  $Y$ .

Let  $Y' = Y \cup \text{cl}_M(B_Y)$ , then

$$B_{Y'} = \delta_D(Y', T) = \{y' \in Y' \setminus T \mid \sigma'(y') \not\subseteq Y'\} \cup (Y' \cap T) \subseteq \text{cl}_M(B_Y),$$

because  $Y' \cap T = (Y \cap T) \cup (\text{cl}_M(B_Y) \cap T) \subseteq \text{cl}_M(B_Y)$  for the reason that

$$Y \cap T \subseteq \delta_D(Y, T) = B_Y;$$

and because

$$\{y' \in Y' \setminus T \mid y' \notin \text{cl}_M(B_Y) \text{ and } \sigma'(y') \not\subseteq Y'\} \subseteq \{y \in Y \setminus T \mid \sigma'(y) \not\subseteq Y\} \subseteq B_Y.$$

This holds since for every  $y \in Y' \setminus \text{cl}_M(B_Y) \subseteq Y$ , the inequality  $\partial\{y\} \cap (T_0 \setminus Y') \neq \emptyset$  implies the inequality  $\partial\{y\} \cap (T_0 \setminus Y) \neq \emptyset$  due to  $Y \subseteq Y'$  — so the left-most set above is actually empty, because  $B_Y \subseteq \text{cl}_M(B_Y)$ . Since  $\text{cl}_M$  does not change the rank, we obtain

$$\text{rk}_M(B_{Y'}) \leq \text{rk}_M(B_Y) < \text{rk}_M(Y) = \text{rk}_M(Y'),$$

that means  $Y' = Y \cup \text{cl}_M(B_Y)$  also satisfies  $\text{rk}_M(B_{Y'}) < \text{rk}_M(Y')$ , and consequently,  $\text{cl}_M(B_Y) \subseteq Y$  for the  $\subseteq$ -maximal subset  $Y$ .

— Now, we want to show that for the  $\subseteq$ -maximal choice  $Y$ , the barrier  $B_Y$  is independent in  $M$ . We give an indirect argument. Assume that  $B_Y$  is not independent, therefore there is a circuit  $C \subseteq B_Y$ . Clearly,  $\text{cl}_M(C) \subseteq \text{cl}_M(B_Y) \subseteq Y$ . Let  $F \in \mathcal{F}(M)$  such that  $F \subseteq \text{cl}_M(C)$ . From the definition of  $B_Y$ , it is clear, that any  $e \in E$  with  $\sigma'(e) = F$  has the property, that  $e \in F \setminus B_Y \subseteq Y \setminus B_Y$ . So  $\text{cl}_M(C)$  has at least as many elements as the sum of the  $\alpha$ -values of all (not necessarily proper) subflats of  $\text{cl}_M(C)$  plus the number of elements of  $B_Y \cap \text{cl}_M(C)$ . Thus we obtain

$$\begin{aligned} |\text{cl}_M(C)| &\geq |\text{cl}_M(C) \cap B_Y| + \sum_{F \in \mathcal{F}(M), F \subseteq \text{cl}_M(C)} \alpha_M(F) \\ &\geq \text{rk}_M(C) + 1 + \sum_{F \in \mathcal{F}(M), F \subseteq \text{cl}_M(C)} \alpha_M(F) \\ &= |\text{cl}_M(C)| + 1, \end{aligned}$$

and arrive at a contradiction, where the second inequality is due to the fact that  $C \subseteq B_Y$  and  $\text{rk}_M(C) = |C| - 1$ . Therefore  $B_Y \in \mathcal{I}$  is independent in  $M$ .

— Now, observe that with  $\alpha_M \geq 0$ , we obtain

$$\begin{aligned} \text{rk}_M(Y) &> \text{rk}_M(B_Y) = |B_Y| \\ &= |Y| - \sum_{F \in \mathcal{F}(M), F \subseteq Y} \alpha_M(Y) \\ &\geq \text{rk}_M(Y) + \alpha_M(Y) \\ &\geq \text{rk}_M(Y). \end{aligned}$$

This contradiction yields, that the assumption, that there is a maximal subset  $Y$  of  $\text{cl}_M(X)$  with  $\text{rk}_M(B_Y) < \text{rk}_M(Y)$ , is wrong. Consequently,  $\text{rk}_M(B_X) < \text{rk}_M(X)$  cannot be the case. Thus  $\text{rk}_M(B_X) = \text{rk}_M(X)$  and all premises of Lemma 2.2.19 are met. We just established  $M = N = \Gamma(D, T, E)$ , so  $M$  is a strict gammoid.  $\square$

**Corollary 2.2.21.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  is a strict gammoid if and only if for all  $X \subseteq E$  the inequality  $\alpha(X) \geq 0$  holds.*

*Proof.* Combine the Theorems 2.2.16 and 2.2.20.  $\square$

We just saw that we obtain a strict representation of a strict gammoid from every transversal of its  $\alpha$ -system. The converse holds, too, in the sense that every representation of a strict gammoid yields a transversal of its  $\alpha$ -system.

**Lemma 2.2.22.** *Let  $D = (V, A)$  be a digraph,  $T \subseteq V$ ,  $M = \Gamma(D, T, V)$  be a strict gammoid, and  $\mathcal{A}_M = (A_i)_{i \in I}$  be the  $\alpha$ -system of  $M$ . Let further  $X = V \setminus T$  and*

$$\varphi: X \longrightarrow \mathcal{F}(M), u \mapsto \text{cl}\left(\{v \in V \mid (u, v) \in A\}\right).$$

*Then  $X$  is a transversal of  $\mathcal{A}_M$ , and there is a bijection  $\psi: X \longrightarrow I$  such that*

$$x \in A_{\psi(x)} = \varphi(x)$$

*for all  $x \in X$ .*

*Proof.* Without loss of generality we may assume that there is no loop arc  $(v, v) \in A$  in  $D$ . Let  $M = (V, \mathcal{I})$ ,  $x \in V \setminus T$ , and let  $S_x = \{y \in V \mid (x, y) \in A\}$ . Clearly  $S_x$  is an  $\{x\} \cup S_x$ - $T$ -separator in  $D$ , because every path from  $x$  to  $t \in T$  must use an arc that

leaves  $x$ , and thus this arc visits a vertex from  $S_x$ . Consequently,  $x \in \text{cl}(S_x)$ . Since  $x \notin S_x$ , we obtain that

$$\text{rk}(S_x) \leq |S_x| < |S_x \cup \{x\}| \leq |\text{cl}(S_x)|.$$

Therefore  $\varphi(x) = \text{cl}(S_x)$  is a dependent flat of  $M$  with  $x \in \varphi(x)$ . Let  $F \in \mathcal{F}(M)$ , let  $I_F = \{(F', k) \in I \mid F' \subseteq F\}$ , and let  $X_F = \{x \in X \mid \varphi(x) \subseteq F\}$ . We show that  $X_F$  is a partial transversal of the subfamily  $\mathcal{A}_F = (A_i)_{i \in I_F}$  of  $\mathcal{A}_M$ , by induction on the nullity  $|F| - \text{rk}(F)$  of  $F$ . If  $F \in \mathcal{I}$  then  $X_F = \emptyset$ , since  $\text{cl}(S_x)$  is dependent for all  $x \in V \setminus T$ . We give an indirect argument for the induction step and assume that  $|X_F| > |F| - \text{rk}(F)$ . There is an  $F$ - $T$ -separator  $S_F$  in  $D$  with minimal cardinality  $|S_F| = \text{rk}(F)$ . Clearly,  $S_F \in \mathcal{I}$  and  $S_F \subseteq \text{cl}(F) = F$  (Lemma 2.2.5). Since  $|X_F| > |F| - \text{rk}(F)$  and  $X_F \subseteq F$ , we obtain that  $X_F \cap S_F \neq \emptyset$ . Let  $f \in X_F \cap S_F$ , then  $S_f = \{g \in V \mid (f, g) \in A\} = \varphi(f) \subseteq F$ . Since  $f \in X_F \subseteq V \setminus T$  we have  $f \notin T$ . Now let  $f' \in F$  and  $t \in T$ , then every path  $p \in \mathbf{P}(D; f', t)$  with  $f \in |p|$  must also visit another element  $f'' \in S_f \subseteq F$  as it continues to  $t$ . Thus every such  $p$  must visit an element from  $S_F \setminus \{f\}$  after visiting  $f$  — a contradiction to the fact that  $S_F$  is an  $F$ - $T$ -separator with minimal cardinality in  $D$ . Thus  $|X_F| \leq |F| - \text{rk}(F)$ . Consequently,  $X_F$  is a partial transversal of  $\mathcal{A}_F$ . Observe that

$$|X| = |V \setminus T| = |V| - \text{rk}(V) = \sum_{F \in \mathcal{F}(M)} \alpha_M(F) = |I|.$$

So  $X$  is a transversal of  $\mathcal{A}_M$  with the property, that  $|\{x \in X \mid \varphi(x) = F\}| = \alpha_M(F)$  holds for all  $F \in \mathcal{F}(M)$ , thus  $X$  has the claimed property.  $\square$

## 2.3 Transversal Matroids

The notion of transversal matroids has been introduced in section 1.4.2. In this section, we develop the theory of transversal matroids a little further.

**Lemma 2.3.1.** *Let  $E$  be a finite set,  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a finite family of subsets of  $E$ , and  $M = M(\mathcal{A})$  be the transversal matroid presented by  $\mathcal{A}$ . Then  $M$  is a gammoid.*

*Proof.* Without loss of generality, we may assume that  $E \cap I = \emptyset$ . Let  $D = (V, A)$  be the digraph where  $V = E \dot{\cup} I$  and  $(e, i) \in A$  if and only if  $e \in E$ ,  $i \in I$  and  $e \in A_i$ . Then  $M(\mathcal{A}) = \Gamma(D, I, E)$ : The routings  $R: X_0 \rightrightarrows I$  in  $D$  with  $X_0 \subseteq E$  are in correspondence to the injections  $\iota: X_0 \rightarrow I$  that have the property  $x \in A_{\iota(x)}$  for all  $x \in X_0$ , where  $R(\iota) = \{x\iota(x) \in \mathbf{P}(D) \mid x \in X_0\}$  is the routing induced by a partial transversal  $X_0$  of  $\mathcal{A}$  with injective map  $\iota$ ; and  $P(R) = \{p_1 \in E \mid p \in R\}$  is the partial transversal of  $\mathcal{A}$  induced from a routing  $R: X_0 \rightrightarrows I$  with  $X_0 \subseteq E$  in  $D$ .  $\square$

**Lemma 2.3.2.** *Let  $M = (E, \mathcal{I})$  be a matroid. If  $M$  is a strict gammoid, then  $M^*$  is a transversal matroid.*

*Proof.* It is an immediate consequence from the fact that there is a linking from  $X$  onto  $T$  in  $D$  if and only if  $V \setminus X$  is a transversal of  $\mathcal{A}_{D,T}$  (Lemma 1.5.22), that the bases of  $\Gamma(D, T, V)$  are precisely those subsets of  $V$ , for which their complement in  $V$  is a base of the transversal matroid  $M(\mathcal{A}_{D,T})$  defined by the linkage system of  $D$  to  $T$ . Thus  $M^* = M(\mathcal{A}_{D,T})$ .  $\square$

The converse statement holds, too.

**Lemma 2.3.3.** *Let  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of sets, and  $M = M(\mathcal{A})$  the transversal matroid presented by  $\mathcal{A}$ . Then  $M^*$  is a strict gammoid.*

*Proof.* Without loss of generality, we may assume that  $E \cap I = \emptyset$ . We define the family  $\hat{\mathcal{A}} = (\hat{A}_i)_{i \in I} \subseteq E \dot{\cup} I$  by setting  $\hat{A}_i = A_i \dot{\cup} \{i\}$  for all  $i \in I$ . Further, let  $D = (E \dot{\cup} I, A)$  where

$$A = \{(e, i) \in E \times I \mid e \in A_i\}.$$

It is easy to see that the linkage system  $\mathcal{A}_{D,E}$  of  $D$  to  $E$  is precisely the family  $\hat{\mathcal{A}}$ . Therefore  $M(\hat{\mathcal{A}})^* = \Gamma(D, E, E \dot{\cup} I)$  is a strict gammoid. On the other hand,  $M(\hat{\mathcal{A}})|E = M(\mathcal{A})$  is evident from the construction. With Lemma 1.2.46 we obtain

$$M(\mathcal{A})^* = \left(M(\hat{\mathcal{A}})|E\right)^* = \left(M(\hat{\mathcal{A}})^*\right).E,$$

where the last term is the contraction of a strict gammoid, therefore  $M^*$  is a strict gammoid (Lemma 2.2.8).  $\square$

**Corollary 2.3.4.** *Let  $M$  be a matroid. Then  $M$  is a transversal matroid if and only if  $M^*$  is a strict gammoid.*

**Corollary 2.3.5.** *Let  $E, I$  be finite sets and  $\mathcal{A} = (A_i)_{i \in I} \subseteq E$  be a family of subsets. If  $M = M(\mathcal{A})$  is the transversal matroid presented by  $\mathcal{A}$  and  $r = \text{rk}_M(E)$ , then there is a family of subsets  $\mathcal{A}' = (A'_i)_{i=1}^r \subseteq E$ , such that  $M = M(\mathcal{A}')$ .*

*Proof.* By Lemma 2.3.3 the dual  $M^*$  is a strict gammoid of rank  $|E| - r$ . Thus there is a digraph  $D = (E, A)$  and a base  $T \subseteq E$  of  $M^*$  such that  $M^* = \Gamma(D, T, E)$ . Let  $\mathcal{A}_{D,T}$  be the linkage system of  $D$  to  $T$ , then  $M(\mathcal{A}_{D,T})^* = M^*$ , and  $\mathcal{A}_{D,T} = (A_i^{(D,T)})_{i \in E \setminus T}$  consists of  $|E| - |T| = r$  sets  $A_i$ , which may be renumbered by the integers from 1 through  $r$  yielding the desired  $\mathcal{A}'$ .  $\square$

**Example 2.3.6.** It is particularly easy to obtain a duality respecting representation of a transversal matroid from representations that are in the form of Corollary 2.3.5. Let  $M = (E, \mathcal{I})$  be a transversal matroid,  $r = \text{rk}(E)$ , and  $\mathcal{A} = (A_i)_{i=1}^r \subseteq E$  be a representation of  $M$ , i.e.  $M = M(\mathcal{A})$ . Then there is a base  $B \in \mathcal{B}(M)$  and a bijective map  $\varphi: B \rightarrow \{1, 2, \dots, r\}$  such that  $b \in A_{\varphi(b)}$  holds for all  $b \in B$ . Furthermore, if  $I = \{i_1, i_2, \dots, i_r\} \neq \emptyset$  is a set with  $I \cap E = \emptyset$ , then the digraph  $D = (E \dot{\cup} I, A)$  with

$$A = \left\{ (i_{\varphi(b)}, b) \mid b \in B \right\} \cup \left\{ (i_{\varphi(b)}, i_k) \mid b \in B, k \in \{1, 2, \dots, r\} \setminus \{\varphi(b)\} : b \in A_k \right\} \\ \cup \left\{ (e, i_k) \mid k \in \{1, 2, \dots, r\}, e \in A_k \setminus B \right\}$$

has the property, that  $M = \Gamma(D, B, E)$ , because it is the digraph that arises from the digraph described in Lemma 2.3.1 and the construction from the proof of Theorem 2.1.10 with respect to the basis  $B$ . Since the premises of Lemma 2.1.18 are satisfied,  $(D, B, E)$  is a duality respecting representation of  $M$ .  $\blacktriangle$

**Corollary 2.3.7.** *Let  $M$  be a transversal matroid. There is a representation  $(D, T, E)$  where  $D = (V, A)$  with  $M = \Gamma(D, T, E)$  and  $|V| \leq |E| + \text{rk}(E)$ . There even is a representation that uses a digraph with  $|V| < 2 \cdot |E|$ .*

*Proof.* A representation with  $|V| \leq |E| + \text{rk}(E)$  has been constructed in Example 2.3.6. If  $\text{rk}(E) = |E|$ , then  $M = (E, 2^E)$ , i.e.  $M$  is the free matroid on  $E$ , and therefore the digraph  $D' = (E, \emptyset)$  yields a representation  $M = \Gamma(D', E, E)$  with strictly fewer than  $2 \cdot |E|$  elements.  $\square$

## 2.4 Constructions within the Class of Gammoids

In this section, we explore methods of obtaining new gammoids from old ones. The main application of this section is the following: If we know that a matroid  $M$  may be constructed from a matroid  $N$  using a construction that does not leave the class of gammoids, then we may conclude that  $M$  is a gammoid whenever  $N$  is a gammoid.

Let us start with a well-known result of J.H. Mason [Mas72].

**Theorem 2.4.1.** *The class consisting of all gammoids is closed under minors, duality, and direct sums.*

*Proof.* Let  $M = (E, \mathcal{I})$  be a matroid. It is clear from Definition 2.1.1 that the representation  $(D, T, E)$  of  $M$  yields the representation  $(D, T, X)$  of  $M|X$  for all  $X \subseteq E$ . Thus the class of all gammoids is closed under restriction. Corollary 2.1.19 yields that if  $M$  is a gammoid, then so is its dual  $M^*$ . Consequently, the class of all gammoids is closed under duality. It follows with Lemma 1.2.46 that the class of gammoids is also closed under contraction. We showed in Lemma 2.1.27 that the class of gammoids is closed under direct sums.  $\square$

Remember that Corollary 2.1.28 established, that every extension of a gammoid  $M$  by a loop or a coloop is again a gammoid. Therefore  $M$  is a gammoid if and only if  $M|X$  is a gammoid, where  $X$  consists of all elements of the ground set of  $M$ , that are neither loops nor coloops.

**Lemma 2.4.2.** *Let  $M = (E, \mathcal{I})$  be a gammoid,  $e \notin E$ , and let  $N \in \mathcal{X}(M, e)$  such that*

$$C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$$

*has at most one  $\subseteq$ -minimal element. Then  $N$  is a gammoid. If  $M$  is a strict gammoid, then  $N$  is a strict gammoid.*

*Proof.* Let  $(D, T, E)$  with  $D = (V, A)$  be a strict representation of  $M$  if  $M$  is a strict gammoid, otherwise let  $(D, T, E)$  be a representation of  $M$ . If  $C = \emptyset$ , then  $e$  is a coloop in  $N$ . So  $(D', T \cup \{e\}, E \cup \{e\})$  with  $D' = (V \cup \{e\}, A)$  is a representation of  $N$ , which is a strict representation if  $M$  is a strict gammoid. Otherwise let  $F_0 = \bigcap C$  be the unique  $\subseteq$ -minimal element of  $C$ . Then  $D'' = (V \cup \{e\}, A'')$  with

$$A'' = A \cup (\{e\} \times F_0)$$

yields the representation  $(D'', T, E \cup \{e\})$  of  $N$  – which is strict if  $M$  is a strict gammoid: Let  $N' = \Gamma(D'', T, E \cup \{e\})$ , and let  $X \subseteq E \cup \{e\}$  be independent in  $N'$ . If  $X \subseteq E$ ,



then  $X$  is independent in  $N$ , because by construction, no path  $p \in \mathbf{P}(D''; x, t)$  for any  $x \in E$  and any  $t \in T$  visits  $e$ . Thus every routing  $X \rightrightarrows T$  in  $D''$  is also a routing with respect to  $D$ . If  $e \in X$  for  $X$  independent in  $N'$ , then the fact that  $F_0 \not\subseteq \text{cl}_M(X \setminus \{e\})$  follows from the way  $D''$  is constructed from  $D$ : Every path from  $e$  to any  $t \in T$  visits some element from  $f \in F_0$ . Thus every routing  $X \rightrightarrows T$  in  $D''$  induces a routing  $(X \setminus \{e\}) \cup \{f\} \rightrightarrows T$  in  $D$  for some  $f \in F_0 \setminus X$ . Therefore there is some  $f \in F_0 \setminus X$  such that  $(X \setminus \{e\}) \cup \{f\}$  is independent in  $M$ , consequently  $f \notin \text{cl}_M(X \setminus \{e\})$ . We obtain  $\text{rk}_N(X) = \text{rk}_M(X \setminus \{e\}) + 1$  and so  $X$  is independent in  $N$ . Now let  $X \subseteq E \cup \{e\}$  be independent in  $N$ . If  $e \notin X$  holds, then  $X$  is independent in  $M$ . So  $X$  is independent in  $N'$ , too, because  $A \subseteq A''$ . If  $e \in X$  and  $X$  is independent in  $N$ , then  $X' = X \setminus \{e\}$  is independent in  $M$  and  $F_0 \not\subseteq \text{cl}_M(X')$ . But then there is some  $f \in F_0 \setminus X'$  such that  $\text{rk}_M(X' \cup \{f\}) > \text{rk}_M(X')$ , thus  $X' \cup \{f\}$  is independent in  $M$ , too. Now let  $R: X' \cup \{f\} \rightrightarrows T$  be a corresponding routing in  $D$ , and let  $p^{(f)} \in R$  be the path of that routing where  $p_1^{(f)} = f$ . Then  $R' = (R \setminus \{p^{(f)}\}) \cup \{ep^{(f)}\}$  is a routing from  $X' \cup \{e\}$  to  $T$  in  $D''$ . It follows that  $X$  is independent in  $N'$ , and consequently  $N = N'$ .  $\square$

**Definition 2.4.3.** Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$ . The restriction  $N = M|X$  shall be a **deflate of  $M$** , if  $E \setminus X = \{e_1, e_2, \dots, e_m\} \neq \emptyset$  can be ordered naturally, such that for all  $i \in \{1, 2, \dots, m\}$  the modular cut

$$C_i = \left\{ F \in \mathcal{F} \left( M|(X \cup \{e_1, e_2, \dots, e_{i-1}\}) \right) \mid e_i \in \text{cl}_M(F) \right\}$$

has precisely one  $\subseteq$ -minimal element.  $\blacksquare$

**Definition 2.4.4.** Let  $M = (E, \mathcal{I})$  be a matroid.  $M$  shall be called **deflated**, if the only deflate of  $M$  is  $M$  itself.  $\blacksquare$

**Lemma 2.4.5.** Let  $M$  be an excluded minor for the class of gammoids. Then  $M$  is deflated.

*Proof.* We give an indirect proof and assume that  $M = (E, \mathcal{I})$  is an excluded minor for the class of gammoids, and  $M$  is not deflated. Then there is an element  $e \in E$  such that

$$C = \left\{ F \in \mathcal{F} (M|(E \setminus \{e\})) \mid e \in \text{cl}_M(F) \right\}$$

has the property that

$$C = \left\{ F \in \mathcal{F} (M|(E \setminus \{e\})) \mid F_0 \subseteq F \right\}$$

where  $F_0 = \bigcap C$ . Since  $M$  is an excluded minor, the restriction  $N = M|(E \setminus \{e\})$  is a gammoid. In this situation, Lemma 2.4.2 yields that  $M$  is a gammoid — a contradiction. Thus every excluded minor for the class of gammoids is deflated.  $\square$

**Lemma 2.4.6.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$  and let  $N = M|X$  be a deflate of  $M$ . Then  $M$  is a gammoid if and only if  $N$  is a gammoid.*

*Proof.* If  $M$  is a gammoid, then  $N$  is a gammoid, too (Theorem 2.4.1). Now let  $N$  be a gammoid, and let  $E \setminus X = \{e_1, e_2, \dots, e_m\}_{\neq}$  be implicitly ordered with the properties required in Definition 2.4.3. Lemma 2.4.2 yields that  $M|(X \cup \{e_1, e_2, \dots, e_i\})$  is a gammoid whenever  $M|(X \cup \{e_1, e_2, \dots, e_{i-1}\})$  is a gammoid. Thus, by induction on  $m$ , we obtain that  $M$  is a gammoid whenever  $N$  is a gammoid — a fact that we assumed.  $\square$

The former situation is a special case of the following situation:

**Definition 2.4.7.** *Let  $T = (T_0, \mathcal{T})$  be a matroid,  $D = (V, A)$  be a digraph with  $T_0 \subseteq V$ , and let  $E \subseteq V$  be any set. The **matroid on  $E$  induced by  $D$  from  $T$**  shall be the pair  $I(D, T, E) = (E, \mathcal{I})$ , where  $X \in \mathcal{I}$  if and only if there is a routing  $R: X \rightrightarrows T_0$  in  $D$ , such that  $\{p_{-1} \mid p \in R\} \in \mathcal{T}$ . In other words  $X$  is independent in  $I(D, T, E)$  if and only if there is a linking from  $X$  onto an independent set of  $T$  in  $D$ .  $\blacksquare$*

It is a result of J.H. Mason that this generalization of Definition 2.1.1 always produces a matroid.

**Theorem 2.4.8** ([Mas72], Theorem 1.1). *Let  $T = (T_0, \mathcal{T})$  be a matroid,  $D = (V, A)$  be a digraph with  $T_0 \subseteq V$ , and let  $E \subseteq V$  be any set. Then  $I(D, T, E)$  is indeed a matroid.*

For a proof<sup>5</sup>, see [Mas72], p.58; J.H. Mason constructs the linkage system with respect to two routings from  $X$ , and  $Y$ , respectively, onto independent subsets of  $T_0$  and then uses Theorem 1.4.11 in order to show that if  $|X| < |Y|$  the augmentation axiom (I3) holds for  $I(D, T, V)$ . The axioms (I1) and (I2) follow easily from the definition, thus  $I(D, T, V)$  is a matroid, and consequently  $I(D, T, E) = I(D, T, V)|E$  is a matroid, too. Now let us present the general form of the non-trivial implication of Lemma 2.4.6.

**Lemma 2.4.9.** *Let  $T = (T_0, \mathcal{T})$  be a matroid,  $D = (V, A)$  be a digraph with  $T_0 \subseteq V$ , and let  $E \subseteq V$ . If  $T$  is a gammoid, then  $I(D, T, E)$  is a gammoid.*

<sup>5</sup>We chose to omit the full proof, because for pure logical reasons, we do not need this theorem for our purposes in this work: the construction in Lemma 2.4.9 works if we define a triple  $(M, D, T)$  to be a digraph induction whenever  $M$  and  $T$  are matroids, such that  $X$  is independent in  $M$  if and only if it can be linked to an independent set of  $T$  in  $D$ . Theorem 2.4.8 states that for every matroid  $T$  and every digraph  $D$ , there is a matroid  $M$  on every subset of the vertex set of  $D$  such that  $(M, D, T)$  is such a digraph induction.

*Proof.* If  $T$  is a gammoid, then there is a representation  $(D', S', T_0)$  with  $D' = (V', A')$  such that  $T = \Gamma(D', S, T_0)$ . Let  $v \mapsto \tilde{v}$  denote a renaming scheme such that the renamed vertices are disjoint from  $V$ , i.e.  $\tilde{V}' \cap V = \emptyset$  where  $\tilde{V}' = \{\tilde{v} \mid v \in V'\}$ . Let  $\tilde{X} = \{\tilde{x} \mid x \in X\}$  for all  $X \subseteq V'$ . We define the digraph  $D_I = (V \dot{\cup} \tilde{V}', A_I)$  where

$$A_I = A \dot{\cup} \{(\tilde{u}, \tilde{v}) \mid (u, v) \in A'\} \dot{\cup} \{(t, \tilde{t}) \mid t \in T_0\}.$$

We show that  $I(D, T, E) = \Gamma(D_I, \tilde{S}, E)$ . First, observe that  $T_0$  is a  $V$ - $\tilde{S}$ -separator in  $D_I$ , because every arc between  $V$  and  $\tilde{V}'$  leaves  $t$  and enters  $\tilde{t}$  for some  $t \in T_0$ . Therefore there is a routing  $R_I: X \rightrightarrows \tilde{S}$  in  $D_I$  with  $X \subseteq E$  if and only if there are a linking  $R: X \rightrightarrows T_X$  with  $T_X \subseteq T_0$  in  $D$  and a routing  $R': T_X \rightrightarrows S$  in  $D'$  — we may construct  $R$  and  $R'$  from  $R_I$  by splitting every  $p_I \in R_I$  into its  $V$ - and  $\tilde{V}'$ -components. Conversely, if we have a pair of routings  $R$  and  $R'$  such that  $\{p_{-1} \mid p \in R\} = \{p'_1 \mid p' \in R'\}$ , then we may obtain  $R_I$  by joining the corresponding paths in  $D_I$ . The latter routing  $R'$  exists if and only if  $T_X \in \mathcal{T}$ , therefore  $X \subseteq E$  is independent in  $\Gamma(D_I, \tilde{S}, E)$  if and only if  $X$  may be linked onto an independent subset of  $T_0$  with respect to  $T$ , i.e. if and only if  $X$  is independent in  $I(D, T, E)$ .  $\square$

The following corollary is the corresponding generalization of Lemma 2.4.6.

**Corollary 2.4.10.** *Let  $T = (T_0, \mathcal{T})$  be a matroid,  $D = (V, A)$  be a digraph with  $T_0 \subseteq V$ , such that every vertex  $t \in T_0$  is a sink in  $D$ . Let further  $E \subseteq V$  such that  $T_0 \subseteq E$ . Then  $T$  is a gammoid if and only if  $I(D, T, E)$  is a gammoid.*

*Proof.* If  $T$  is a gammoid, then  $I(D, T, E)$  is a gammoid (Lemma 2.4.9). Since  $\mathbf{P}(D; t, v) = \emptyset$  for all  $t \in T_0$  and all  $v \in V \setminus \{t\}$ , we obtain that  $X$  is independent in  $I(D, T, E)$  if and only if  $X$  is independent in  $T$  for all  $X \subseteq T_0$ . Thus  $I(D, T, E)|_{T_0} = T$ , and, consequently, if  $I(D, T, E)$  is a gammoid, then so is  $T$  (Theorem 2.4.1).  $\square$

## 2.5 The Recognition Problem

First, we give a formal definition of what we mean when we talk about the problem of recognizing a gammoid.

**Definition 2.5.1.** *Let  $\mathcal{M}$  be a class of matroids. The **gammoid recognition problem for  $\mathcal{M}$**  – or shorter  $\text{Rec}\Gamma_{\mathcal{M}}$  – is the problem of computing the image of  $M \in \mathcal{M}$  under the class-map*

$$\Gamma_{\mathcal{M}}: \mathcal{M} \longrightarrow \{0, 1\}, \quad M \mapsto \begin{cases} 1 & \text{if } M \text{ is a gammoid,} \\ 0 & \text{otherwise.} \end{cases}$$

The elements  $M \in \mathcal{M}$  are called the **instances of  $\text{Rec}\Gamma_{\mathcal{M}}$** . ■

In less formal words, the gammoid recognition problem is the problem that given an instance of a matroid  $M$ , determine whether  $\Gamma_{\mathcal{M}}(M) = 1$  or  $\Gamma_{\mathcal{M}}(M) = 0$  by application of some algorithm. Thus we are interested in algorithms that compute  $\Gamma_{\mathcal{M}}$  and naturally we are also interested in the run-time complexity of those algorithms as well as lower bounds for the complexity of these algorithms. Obviously, there is no constant-time algorithm for the computation of  $\Gamma_{\mathcal{M}}(M)$ , therefore we would like to fix a certain way to encode a matroid  $M$  — up to renaming elements of its ground set, yet preserving the implicit linear order of its ground set  $E = \{e_1, e_2, \dots, e_n\}_{\neq}$ .

**Definition 2.5.2.** *Let  $n, r \in \mathbb{N}$  with  $n \geq r$ . We fix the bijection*

$$\text{kth}(n, r): \left\{ 1, 2, \dots, \binom{n}{r} \right\} \longrightarrow \binom{\{1, 2, \dots, n\}}{r}$$

with the defining property that for all  $i, j \in \mathbb{N}$  with  $1 \leq i, j \leq \binom{n}{r}$  we have

$$\min(\text{kth}(n, r)(i) \Delta \text{kth}(n, r)(j)) \in \text{kth}(n, r)(i) \iff i < j,$$

where  $\Delta$  denotes the symmetric difference of sets. In words, we enumerate all  $r$ -elementary subsets of  $\{1, 2, \dots, n\}$  in ascending order with respect to the linear order that relates a subset  $A$  with a subset  $B$  whenever the smallest element in  $(A \cup B) \setminus (A \cap B)$  belongs to  $A$ . ■

For  $n \geq r + 1$  we have  $\text{kth}(n, r)(1) = \{1, 2, \dots, r\}$ ,  $\text{kth}(n, r)(2) = \{1, 2, \dots, r - 1, r + 1\}$ , and  $\text{kth}(n, r)\left(\binom{n}{r}\right) = \{n - r, n - r + 1, \dots, n\}$ .

**Definition 2.5.3.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $E = \{e_1, e_2, \dots, e_n\}_{\neq}$  bear an implicit linear order. We define the **binary encoding of  $M$**  to be the vector

$$\mathbf{b}(M) = (\mathbf{b}(M, i))_{i=1}^N \in \{0, 1\}^N$$

where  $N = |E| + 2 + \binom{|E|}{\text{rk}_M(E)}$  is the encoding length of  $M$  and where

$$\mathbf{b}(M, i) = \begin{cases} 1 & \text{if } i \leq \text{rk}_M(E), \\ 0 & \text{if } i = \text{rk}_M(E) + 1, \\ 1 & \text{if } \text{rk}_M(E) + 1 < i \leq |E| + 1, \\ 0 & \text{if } i = |E| + 2, \\ 1 & \text{if } i > |E| + 2 \text{ and } \kappa(i - |E| - 2) \in \mathcal{I}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\kappa(k) = \{e_i \in E \mid i \in \text{kth}(|E|, \text{rk}_M(E))(k)\}$ . In other words,  $\mathbf{b}(M)$  consists of a unary encoding of  $\text{rk}_M(E)$ , followed by a unary encoding of  $|E| - \text{rk}_M(E)$ , followed by  $\binom{|E|}{\text{rk}_M(E)}$  bits encoding which of the  $\text{rk}_M(E)$ -elementary subsets of  $E$  are bases of  $M$ , in the ascending order with respect to the implicit linear order on  $E$ . Furthermore, the **encoding length of  $M$**  shall be denoted by

$$\mathbf{N}(M) = |E| + 2 + \binom{|E|}{\text{rk}_M(E)}. \quad \blacksquare$$

**Remark 2.5.4.** Clearly, we can restore a matroid isomorphic to  $M = (E, \mathcal{I})$  from  $\mathbf{b}(M)$ . Furthermore, the laws of the binomial coefficients yield that

$$\mathbf{N}(M) = \mathbf{N}(M^*)$$

since  $\binom{n}{k} = \binom{n}{n-k}$ , and for all  $X \subsetneq E$

$$\mathbf{N}(M|X) = \mathbf{N}(M.X) < \mathbf{N}(M)$$

since  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ . When  $|E| \geq 4$ , a rather rough estimate is  $\mathbf{N}(M) \leq 2^{|E|}$ . •

**Remark 2.5.5.** R. Pendavingh and J. van der Pol give the following lower bound for the number of matroids of rank  $r$  on an  $n$ -elementary ground set in [PvdP17], let  $s_{r,n}$  denote this lower bound. Then

$$\log(s_{n,r}) \geq \frac{1}{n-r+1} \cdot \binom{n}{r} \cdot \log(c^{1-r}(n-r+1)(1+o(1)))$$

for some constant  $c$  independent of  $r$  and  $n$  (Lemma 9 (3), [PvdP17] p.4). Thus, if we write a big list of all matroids with  $n$  elements and rank  $r$ , and then use the corresponding list index, encoded as a binary number, in order to represent the base vector of the matroid, we would still have the binomial  $\binom{|E|}{\text{rk}_M(E)}$  as a factor in the encoding length. Therefore our encoding  $b(M)$  from Definition 2.5.3 may be considered not excessively-bloated. •

First, we shall examine how easy it is to extract matroid information from an encoded matroid. Throughout this work, we assume that checking, whether a set of the correct cardinality is a base of  $M$ , can be done in  $O(1)$  time by reading the corresponding bit from  $b(M)$ .

### Algorithm 2.5.6. Check For Independence

**Input** (1) A matroid  $M = (E, \mathcal{I})$  given by  $b(M)$ .  
 (2) A subset  $X \subseteq E$ , given by a vector of  $2^{\{1,2,\dots,|E|\}}$ .  
**Output** 1 if  $X \in \mathcal{I}$ , 0 otherwise.

```

for  $i = 1 \dots \binom{|E|}{\text{rk}_M(E)}$  do
  if  $b(M, |E| + 1 + i) = 1$  and  $X \subseteq \text{kth}(|E|, \text{rk}_M(E))(i)$  then do
    return 1 and stop
  end if  $b(M, |E| + 1 + i) = 1$  and  $X \subseteq \text{kth}(|E|, \text{rk}_M(E))(i)$ 
next  $i$ 
end for  $i$ 
return 0.

```

In order to check for independence we have to test whether  $X$  is the subset of a base of  $M$ . Therefore we iterate over at most  $O(\mathbf{N}(M))$  base candidates  $B$  and for each candidate we check whether  $B \in \mathcal{B}(M)$  and  $B \subseteq X$ , which can be done in  $O(|E|)$  bit-comparisons. Thus the overall run-time is  $O(|E| \cdot \mathbf{N}(M)) = O(n^2)$  where  $n = \mathbf{N}(M) + |E|$  is the total bit-length of the input. ◀

*Proof of correctness.* Lemma 1.2.7 states that every independent set  $X \in \mathcal{I}$  can be extended to a base  $B \in \mathcal{B}(M)$ . The execution invariant at the “next  $i$ ” instruction is that none of the bases in  $\mathcal{B}(M) \cap \{\{e_k \mid k \in \text{kth}(|E|, \text{rk}_M(E))(j)\} \mid j \leq i\}$  contains  $X$ . Thus the invariant at the “end for  $i$ ”-instruction is that there is no  $B \in \mathcal{B}(M)$  with  $X \subseteq B$ . And then the return value is 0. Otherwise, if the algorithm returns 1, then the set  $\{e_k \mid k \in \text{kth}(|E|, \text{rk}_M(E))(i)\}$  is a base of  $M$  that proves  $X \in \mathcal{I}$ .  $\square$

**In order to reduce the technicalities of notation**, we identify the ground set  $E = \{e_1, e_2, \dots, e_n\}_{\neq}$  with the set  $\{1, 2, \dots, n\}_{\neq}$  through the bijection  $i \mapsto e_i$ ; therefore we identify  $\{e_i \mid i \in \text{kth}(n, r)(j)\}$  with  $\text{kth}(n, r)(j)$  for the treatise of algorithms.

### Algorithm 2.5.7. Compute the Rank

**Input** (1) A matroid  $M = (E, \mathcal{I})$  given by  $\text{b}(M)$ .  
 (2) A subset  $X \subseteq E$ , given by a vector  $2^{\{1, 2, \dots, |E|\}}$ .

**Output**  $\text{rk}_M(X)$ .

```

r := 0
for i = 1 ...  $\binom{|E|}{\text{rk}_M(E)}$  do
  r := max { r,  $\text{b}(M, |E| + 1 + i) \cdot |\text{kth}(|E|, \text{rk}_M(E))(i) \cap X|$  }
  next i
end for i
return r.

```

In order to compute the rank, we have to do  $O(\mathbf{N}(M))$  iterations of the main loop which consists of one multiplication, one comparison of values bounded above by  $|E|$ , possibly a copy operation of these values, and possibly a calculation of the intersection between  $\text{kth}(|E|, \text{rk}_M(E))(i)$  and  $X$ , which can be done in  $O(|E|)$ . Thus the overall run-time is  $O(|E| \cdot \mathbf{N}(M)) = O(n^2)$  where  $n = \mathbf{N}(M) + |E|$  is the total bit-length of the input.  $\blacktriangleleft$

*Proof of correctness.* Lemma 1.2.7 and Definition 1.2.14 yield that the rank of  $X \subseteq E$  equals the maximum cardinality of the intersection of  $X$  with a base  $B \in \mathcal{B}(M)$ . Clearly, the invariant for the value of  $r$  at the “next  $i$ ” instruction is

$$r = \max \left\{ |X \cap B| \mid B \in \mathcal{B}(M) \cap \{\text{kth}(|E|, \text{rk}_M(E))(j) \mid j \leq i\} \right\}.$$

Therefore the invariant at the “end for  $i$ ” instruction is

$$r = \max \{ |X \cap B| \mid B \in \mathcal{B}(M) \},$$

thus the returned value  $r = \text{rk}_M(X)$  is correct.  $\square$

### Algorithm 2.5.8. Compute the Closure

**Input** (1) A matroid  $M = (E, \mathcal{I})$  given by  $b(M)$ .  
 (2) A subset  $X \subseteq E$ , given by a vector of  $2^{\{1,2,\dots,|E|\}}$ .  
**Output**  $\text{cl}_M(X)$ .

```

C := X
r := rk_M(X)
for e ∈ E \ X do
  if rk_M(X ∪ {e}) = r then C := C ∪ {e}
next e
end for e
return C.

```

In order to compute the closure, we have to compute the ranks of subsets of  $E$  precisely  $|E \setminus X| + 1$  times, therefore we accumulate a running time of  $O(|E| \cdot n^2) = O(n^3)$  where  $n = \mathbf{N}(M) + |E|$  is the total bit-length of the input.  $\blacktriangleleft$

*Proof of correctness.* First, we show that

$$\text{cl}(X) = X \cup \{e \in E \setminus X \mid \text{rk}(X \cup \{e\}) = \text{rk}(X)\}.$$

The closure of  $X$  is the intersection of all flats  $F \in \mathcal{F}(M)$  with  $X \subseteq F$  (Definition 1.2.16), and flats are those sets  $F \subseteq E$ , such that  $\text{rk}(F \cup \{e\}) > \text{rk}(F)$  holds for all  $e \in E \setminus F$  (Definition 1.2.16). Since  $\text{rk}(X) = \text{rk}(\text{cl}(X))$  (Lemma 1.2.18), we obtain that for all  $x \in E \setminus X$  with  $\text{rk}(X \cup \{x\}) = \text{rk}(X)$ , we have the implication  $X \subseteq F \Rightarrow x \in F$  for all flats  $F \in \mathcal{F}(M)$ . Consequently,  $x \in \text{cl}(X)$  whenever  $\text{rk}(X \cup \{x\}) = \text{rk}(X)$ . On the other hand, if  $\text{rk}(X \cup \{x\}) > \text{rk}(X)$ , then  $\text{cl}(X \cup \{x\}) \in \mathcal{F}(M)$  is the smallest flat that contains  $X \cup \{x\}$ , but  $\text{rk}(\text{cl}(X \cup \{x\})) > \text{rk}(\text{cl}(X))$ , thus  $x \notin \text{cl}(X)$  whenever  $\text{rk}(X \cup \{x\}) \neq \text{rk}(X)$ .



Let  $E \setminus X = \{e_1, e_2, \dots, e_k\}_{\neq}$  with the implicit order of occurrence in the “for  $e$ ”-loop, and let  $i \in \{1, 2, \dots, k\}$  be the index of the iteration of the loop corresponding to  $e$ , i.e.  $e_i = e$ . The invariant at the “next  $e$ ”-instruction with regard to  $C$  is  $C = \text{cl}_M(X) \cap (X \cup \{e_1, e_2, \dots, e_i\})$ . Thus the invariant at the “end for  $e$ ”-instruction is  $C = \text{cl}_M(X) \cap (X \cup (E \setminus X)) = \text{cl}_M(X)$ .  $\square$

### Algorithm 2.5.9. Naive Test for Strict Gammoids

**Input** (1) A matroid  $M = (E, \mathcal{I})$  given by  $b(M)$ .  
**Output** 1, if  $\alpha_M \geq 0$ , or 0 otherwise.

```

initialize  $\alpha_M \in \mathbb{Z}^{2^E}$  with  $\alpha_M \equiv 0$ 
initialize  $\mathcal{F} \in \{0, 1\}^{2^E}$  with  $\mathcal{F} \equiv 0$ 
for  $k = 0 \dots |E|$  do
  for  $X \in \binom{E}{k}$  do
     $a := k - \text{rk}_M(X)$ 
    for  $Y \subsetneq X$  do
      if  $\mathcal{F}(Y) = 1$  then do
         $a := a - \alpha_M(Y)$ 
        if  $a < 0$  then do
          return 0 and stop
        end if  $a < 0$ 
      end if  $\mathcal{F}(Y) = 1$ 
    next  $Y$ 
  end for  $Y$ 
   $\alpha_M(X) := a$ 
  if  $X = \text{cl}_M(X)$  then  $\mathcal{F}(X) := 1$ 
next  $X$ 
end for  $X$ 
next  $k$ 
end for  $k$ 
return 1.

```

The algorithm calculates  $\alpha_M$  bottom-up using the recurrence relation and simultaneously keeping track of the family of flats of  $M$ . If  $\alpha_M < 0$  at some point, then the algorithm stops early with a negative answer. Otherwise we have to calculate all values

of  $\alpha_M$  in order to be sure that  $M$  is a strict gammoid, therefore iterating  $2^{|E|}$  different values of  $X$ . All values that are assigned to  $\alpha_M(X)$  are non-negative integers that are bounded by  $|X| - \text{rk}_M(X)$ , because the algorithm stops if  $a < 0$  before assigning the negative value to  $\alpha_M(X)$ . For the same reason we have  $|a| \leq |X|$ . Thus the calculation of the correct value of  $\alpha_M(X)$  needs at most  $2^{|X|}$  subtractions, each with a run-time in  $O(\log(|X|))$ , and  $2^{|X|}$  tests of the flat property in  $O(1)$ . In order to determine the value of  $\alpha_M(X)$  for a single instance  $X$ , we need  $O((\log(|X|) + 1) \cdot 2^{|X|}) = O(\log(|E|) \cdot 2^{|E|})$ -time. The flat book-keeping needs one closure operation that takes  $O(|E|^2 \cdot \mathbf{N}(M))$ , and  $|E|$  bit-comparisons, thus the book-keeping is in  $O(|E|^2 \cdot \mathbf{N}(M))$ . If  $M$  is a strict gammoid, this has to be done for all  $2^{|E|}$  subsets  $X \subseteq E$ , thus the total run-time is

$$O(\log(|E|) \cdot 2^{2^{|E|}} + |E|^2 \cdot \mathbf{N}(M) \cdot 2^{|E|}) = O(\log(n) \cdot 2^{2^n})$$

where  $n = \mathbf{N}(M)$  is the bit-length of the input. ◀

*Proof of correctness.* It suffices to show that the algorithm correctly computes the values  $\alpha_M(X)$ , and that the algorithm returns 1, if  $\alpha_M(X) \geq 0$  holds for all  $X \subseteq E$ , and 0 otherwise (Corollary 2.2.21). Let  $\mathcal{X} = \{X_1, X_2, \dots, X_{2^{|E|}}\}_{\neq}$  be the family of all subsets of  $E$  in the order of occurrence with respect to the “for  $X$ ”-instruction, and let  $i \in \{1, 2, \dots, 2^{|E|}\}$  such that  $X = X_i$ . The invariant at the “ $a := k - \text{rk}_M(X)$ ”-instruction is, that for all  $j \in \{1, 2, \dots, i-1\}$  the value of  $\alpha_M(X_j)$  is correctly assigned and non-negative, and that we have that  $\mathcal{F}(X_j) = 1$  if and only if  $X_j$  is a flat. Furthermore,  $a = |X_i| - \text{rk}_M(X_i)$ . Now let  $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_K\}_{\neq}$  be the proper subsets of  $X$  in their order of occurrence with respect to the “for  $Y$ ”-instruction, and let  $k \in \{1, 2, \dots, K\}$  be the index such that  $Y = Y_k$ . The invariant at the “next  $Y$ ”-instruction is

$$a = |X_i| - \text{rk}_M(X_i) - \sum_{F \in \mathcal{F}(M) \cap \{Y_1, Y_2, \dots, Y_k\}, F \subsetneq X} \alpha_M(F).$$

Thus the invariant at the “end for  $Y$ ”-instruction is  $a = \alpha_M(X_i) \geq 0$ , and the invariant at the “return 1”-instruction is that  $M$  is a strict gammoid. The invariant at the “return 0 and stop”-instruction is that

$$a = |X_i| - \text{rk}_M(X_i) - \sum_{F \in \mathcal{F}(M) \cap \{Y_1, Y_2, \dots, Y_k\}, F \subsetneq X} \alpha_M(F) < 0,$$

and since  $\alpha_M(Y) \geq 0$  for all  $Y \subsetneq X$ , we may conclude that  $\alpha_M(X_i) \leq a < 0$ , which implies that  $M$  is not a strict gammoid. Therefore the output of the algorithm is 1 if  $M$  is a strict gammoid, and 0 if  $M$  is not a strict gammoid. ◻

**Corollary 2.5.10.** *Given a matroid  $M$  via its encoding  $b(M)$ , we can decide whether  $M$  is a transversal matroid and whether  $M$  is a strict gammoid in  $O\left(\log(\mathbf{N}(M))2^{2\mathbf{N}(M)}\right)$  time.*

*Proof.* We may use Algorithm 2.5.9 on  $M$  to test whether  $M$  is a strict gammoid, and on  $M^*$  in order to test whether  $M$  is a transversal matroid. The encoding  $b(M^*)$  can be obtained in  $O(\mathbf{N}(M))$  time: the location of the first zero has to be moved from  $\text{rk}_M(E) + 1$  to  $|E| - \text{rk}_M(E) + 1$ , and the encoding of the bases in  $b(M)$  has to be brought in reverse order to obtain an encoding of  $M^*$ . Then we can test whether  $M^*$  is a strict gammoid to obtain the result (Lemma 2.3.2).  $\square$

### 2.5.1 Special Cases

In this section, we give a quick overview over some special classes of matroids and gammoids, where there is an easy way to answer the question whether a matroid, that exhibits the additional properties, is a gammoid with other special properties — or whether it is not.

**Proposition 2.5.11** ([IP73], Proposition 4.8 and Corollary 4.9). *Let  $M = (E, \mathcal{I})$  be a matroid. Then*

- (i) *If  $\text{rk}_M(E) \leq 2$ , then  $M$  is a strict gammoid.*
- (ii) *If  $\text{rk}_M(E) = 3$ , then  $M$  is a gammoid if and only if  $M$  is a strict gammoid.*
- (iii) *If  $\text{rk}_M(E) = |E| - 3$ , then  $M$  is a gammoid if and only if  $M$  is a transversal matroid.*
- (iv) *If  $\text{rk}_M(E) \geq |E| - 2$ , then  $M$  is a transversal matroid.*

We omit the proof here as it does not provide any further guidance for the unconstrained problem of deciding whether a given matroid is a gammoid or not. The reader interested in a proof of this proposition should read A.W. Ingleton and M.J. Piff's paper [IP73]. Certainly, we should keep in mind the consequence of this proposition: We can expect the most general flavor of gammoids to unfold only with matroids  $M = (E, \mathcal{I})$  where  $4 \leq \text{rk}_M(E) \leq |E| - 4$ . For matroids with  $\text{rk}_M(E) \in \{0, 1, 2, |E| - 2, |E| - 1, |E|\}$ , the answer is always that  $M$  is a gammoid. For  $\text{rk}_M(E) \in \{3, |E| - 3\}$ , we may use Mason's  $\alpha$ -criterion with respect to  $M$ , or  $M^*$ , respectively, in order to decide whether  $M$  is a gammoid (Corollary 2.2.21) in  $O\left(\log(n)2^{2n}\right)$  time (Corollary 2.5.10).

Instead of limiting the class of allowed input instances, we also might consider the related problem of determining whether a given matroid  $M$  belongs to some subclass  $\mathcal{G}_P$  of the class of gammoids, where  $\mathcal{G}_P$  consists of all gammoids that have the additional property  $P$ . It is folklore, that the problem of deciding whether a given matroid  $M$  belongs to a minor-closed class of matroids  $\mathcal{M}_P$ , which is characterized by finitely many excluded minors, has a solution algorithm that runs in polynomial time. First, we present the following theorem, that gives us a hint where such an excluded minor must appear in any  $M \notin \mathcal{M}_P$ .

**Theorem 2.5.12** (Scum Theorem, [Oxl11] p.113). *Let  $M = (E, \mathcal{I})$  be a matroid and let  $N = (E', \mathcal{I}')$  be a minor of  $M$ . There is a subset  $Z \subseteq E \setminus E'$  such that  $M.(E \setminus Z)$  has the same rank as  $N$ , and such that*

$$N = (M.(E \setminus Z))|E'.$$

*If  $N$  has no loop, then we may choose  $Z \in \mathcal{F}(M)$ .*

For the proof, please refer to J.G. Oxley's book [Oxl11]. It is easy to see that given such a  $Z \subseteq E \setminus E'$ , every base  $Z' \in \mathcal{B}_M(Z)$  also has the property that  $N = (M.(E \setminus Z'))|E'$  (Lemma 1.2.42).

**Algorithm 2.5.13. Test for Minor**

**Input** (1) A matroid  $M = (E, \mathcal{I})$  given by  $b(M)$ .  
(2) A matroid  $N = (E', \mathcal{I}')$  given by  $b(N)$ .

**Output** 1, if  $N$  is isomorphic to a minor of  $M$ ,  
0, otherwise.

```

if  $\text{rk}_N(E', \mathcal{I}') > \text{rk}_M(E, \mathcal{I})$  then return 0 and stop
for  $Z \in \binom{E}{\text{rk}_M(E) - \text{rk}_N(E')}$  do
  if  $Z \notin \mathcal{I}$  then next  $Z$ 
  for every  $\varphi: E' \rightarrow E \setminus Z$  injective map do
    for  $X \in \binom{E'}{\text{rk}_N(E')}$  do
      if not  $X \in \mathcal{B}(N) \Leftrightarrow \varphi[X] \cup Z \in \mathcal{B}(M)$  then next  $\varphi$ 
    next  $X$ 
  end for  $X$ 
  return 1 and stop
end for  $\varphi$ 

```

```

    next Z
  end for Z
return 0.

```

Let  $n = \mathbf{N}(M) + \mathbf{N}(N)$ ,  $n_M = \mathbf{N}(M)$ , and  $n_N = \mathbf{N}(N)$  be the respective encoding lengths. The “for  $Z$ ”-instruction loops through at most  $\binom{|E|}{\text{rk}_M(E) - \text{rk}_N(E')}$  iterations. For  $k, m \in \mathbb{N}$  with  $k \leq m$ , we may estimate  $\binom{m}{k-1} \leq k \cdot \binom{m}{k}$ , because we obtain a  $(k-1)$ -elementary subset of an  $m$ -elementary set  $X$  by first choosing an  $k$ -elementary subset of  $X$  and then choosing one element to drop. This way we obtain all  $(k-1)$ -elementary subsets provided that there is some  $k$ -elementary subset of  $X$ . Consequently, we may estimate the number of  $Z$ -iterations by

$$\binom{|E|}{\text{rk}_M(E) - \text{rk}_N(E')} \leq |E|^{\text{rk}_N(E')} \cdot \binom{|E|}{\text{rk}_M(E)} = O\left((n_M)^{n_N+1}\right).$$

The test whether  $Z \in \mathcal{I}$  can be done in  $O\left((n_M)^2\right)$  (Algorithm 2.5.6). The generation of the injective maps  $\varphi$  for a single instance of  $Z$  as lookup tables has a combined run-time in

$$O\left(|E|^{|E'|} \cdot \log(|E|)\right) = O\left((n_M)^{n_N} \cdot \log(n_M)\right).$$

The “for  $\varphi$ ”-instruction loops through at most  $|E|^{|E'|} = O\left((n_M)^{n_N}\right)$  iterations, and the “for  $X$ ”-instruction loops through at most  $O(n_N)$  iterations. Calculating  $\varphi[X]$  can be done with  $|X| = \text{rk}_N(E') = O(n_N)$  table lookups and corresponding bit-set operations, calculating  $\varphi[X] \cup Z$  can be achieved with an  $|E|$ -bit bitwise-or operation in  $O(n_M)$ -time. Checking whether  $X \in \mathcal{B}(N) \Leftrightarrow \varphi[X] \cup Z \in \mathcal{B}(M)$  can be done in  $O(1)$ -time by bit comparison. This yields a combined run-time of the algorithm in

$$O\left((n_M)^{2n_N+2} \cdot (n_N)^2\right) = O\left(n^{2n+4}\right). \quad \blacktriangleleft$$

Thus deciding whether  $M$  has a minor isomorphic to  $N$  can be done in polynomial time with respect to  $\mathbf{N}(M)$  for a fixed matroid  $N$ .

*Proof of correctness.* Assume that  $M = (E, \mathcal{I})$  has a minor  $L = (D, \mathcal{J})$  that is isomorphic to  $N = (E', \mathcal{I}')$ , then there is a set  $Z_L \subseteq E \setminus D$  such that  $L = (M \cdot (E \setminus Z_L)) | D$  as guaranteed by the Scum Theorem 2.5.12. Thus for every base  $B_L$  of  $Z_L$  in  $M$ , we have the property, that a set  $X \subseteq D$  is a base of  $L$  if and only if  $B_L \cup X$  is a base of  $M$  (Lemma 1.2.42). Furthermore, the minor  $L$  is isomorphic to  $N$ , if and only if there is a matroid isomorphism  $\varphi': E' \rightarrow D$  between  $L$  and  $N$ , i.e. a bijective map  $\varphi'$  with

the property that  $\varphi'[X] \in \mathcal{B}(L) \Leftrightarrow X \in \mathcal{B}(N)$  holds. Assume that  $M$  has a minor  $L$  isomorphic to  $N$ . Let further  $\varphi'$  be the corresponding matroid isomorphism, and let  $Z_L \subseteq E \setminus D$  and  $B_L \subseteq Z_L$  be derived from the Scum Theorem 2.5.12 as above. Since  $B_L \subseteq Z_L \subseteq E \setminus D$ , we have  $B_L \cap D = \emptyset$ . By extension of the codomain of  $\varphi'$  we obtain an injective map  $\hat{\varphi}: E' \rightarrow E \setminus B_L$ , where  $\hat{\varphi}(e') = \varphi'(e')$  for all  $e' \in E'$ . Either the algorithm returns 1 early, or at some point, the “for  $Z$ ”-instruction starts an iteration where  $Z = B_L$  since  $|B_L| = \text{rk}_M(Z_L) = \text{rk}_M(E) - \text{rk}_L(D) = \text{rk}_M(E) - \text{rk}_N(E')$ . In this iteration, we have  $Z = B_L \in \mathcal{I}$ , therefore we enter the “for  $\varphi$ ”-loop. Again, either the algorithm returns 1 early, or we reach the iteration where  $\varphi = \hat{\varphi}$ . In this iteration, we have the equivalence  $X \in \mathcal{B}(N) \Leftrightarrow \varphi[X] \cup Z = \varphi'[X] \cup B_L \in \mathcal{B}(M)$ , therefore we reach the “end for  $X$ ”-instruction. In the next instruction, we return the correct value 1. Now assume that  $M$  has no minor isomorphic to  $N$ . If the algorithm reaches the “return 0”-instruction the result is correct. We give an indirect argument for this to happen, and assume that the algorithm reaches the “return 1”-instruction. But then the “for  $X$ ”-loop must have finished without reaching the “next  $\varphi$ ”-instruction. This, together with the property (B2) that all bases of a matroid have the same cardinality, implies that  $X \in \mathcal{B}(N) \Leftrightarrow \varphi[X] \cup Z \in \mathcal{B}(M)$  holds for all  $X \subseteq E'$ . Thus  $(M.(E \setminus Z))|_{\varphi[E']}$  is a minor of  $M$  isomorphic to  $N$ , contradicting our assumption that  $M$  has no minor isomorphic to  $N$ . Therefore we may conclude that the algorithm returns 0 if  $M$  has no minor isomorphic to  $N$ .  $\square$

**Theorem 2.5.14.** *Let  $\mathcal{G}$  be a minor-closed class of matroids that is characterized by the excluded minors  $N_1, N_2, \dots, N_k$ , and let  $K = \max\{\mathbf{N}(N_1), \mathbf{N}(N_2), \dots, \mathbf{N}(N_k)\}$  be the maximal encoding length of the excluded minors. If  $M = (E, \mathcal{I})$  is an arbitrary matroid and  $n = \mathbf{N}(M)$  is its encoding length, then we may decide whether  $M \in \mathcal{G}$  in  $O(n^{K+2})$ -time.*

*Proof.* For each  $i \in \{1, 2, \dots, k\}$  we may use Algorithm 2.5.13 in order to test whether  $M$  has a minor isomorphic to  $N_i$  in  $O(n^{2\mathbf{N}(N_i)+2})$ -time. On the other hand,  $M \in \mathcal{G}$  if and only if  $M$  has no minor isomorphic to one of the matroids  $N_1, N_2, \dots, N_k$ . Thus if Algorithm 2.5.13 returns 1 for any  $N_i$ , then  $M \notin \mathcal{G}$ , and if Algorithm 2.5.13 returns 0 for all  $N_i$ , then  $M \in \mathcal{G}$ . Therefore we have to run at most  $k$  tests in  $O(n^{2K+2})$ -time in order to decide whether  $M \in \mathcal{G}$ .  $\square$

A consequence of this theorem is the following: Let  $k \in \mathbb{N}$ , then we may decide in polynomial time whether a matroid  $M$  is (a) a gammoid with  $C_V(M) \leq k$  and (b) a gammoid with  $C_A(M) \leq k$  (Remark 2.1.26 and Theorem 2.1.32).

## 2.5.2 The General Recognition Problem

For the rest of this chapter, we let  $\mathcal{M}$  be the class of all matroids. Now, let us investigate the problem  $\text{Rec}\Gamma_{\mathcal{M}}$ . In order to present the most obvious algorithm that computes  $\Gamma_{\mathcal{M}}(M)$ , we need a way to verify whether a given vector  $b \in \{0,1\}^{\binom{n}{r}}$  codes the bases of a rank- $r$  matroid on an  $n$ -elementary ground set.

### Algorithm 2.5.15. Test Base Axioms

**Input** (1)  $r \in \mathbb{N}$ , given as unary encoded bit-stream.  
 (2)  $(e-r) \in \mathbb{N}$ , given as unary encoded bit-stream.  
 (3)  $B \in \{0,1\}^{\binom{\{1,2,\dots,e\}}{r}}$  family of  $r$ -elementary sets, as a vector of  $2^{\binom{e}{r}}$ .

**Output** 1, if  $B$  is the characteristic vector of a family of bases of a matroid with rank  $r$ ,  
 0, otherwise.

```

g := 0
for X ∈  $\binom{\{1,2,\dots,e\}}{r}$  do
  if B(X) = 0 then next X
  g := 1
  for Y ∈  $\binom{\{1,2,\dots,e\}}{r}$  do
    if X = Y or B(Y) = 0 then next Y
    for x ∈ X \ Y do
      for y ∈ Y \ X do
        if B((X \ {x}) ∪ {y}) = 1 then next x
      next y
    end for y
  return 0 and stop
end for x
next Y
end for Y
next X
end for X
return g.

```

Observe that the input resembles the format of an encoding of a rank- $r$  matroid defined on an  $e$ -elementary ground set, the total bit-length of the input is  $n = 2 + e + \binom{e}{r}$ . A

rough estimate of the run-time is the following: the “for  $X$ ”-instruction iterates over  $\binom{e}{r}$  sets, the “for  $Y$ ”-instruction iterates over  $\binom{e}{r}$  sets, too, the “for  $x$ ”-instruction iterates over  $\leq r$  elements of  $X$ , the “for  $y$ ”-instruction iterates over  $\leq r$  elements of  $Y$ . In total, we have to do less than

$$\binom{e}{r} \cdot \binom{e}{r} \cdot r^2 + \binom{e}{r} \cdot \binom{e}{r} + \binom{e}{r}$$

bit-comparisons involving the vector  $B$ . Thus the run-time is in  $O\left(r^2 \cdot \binom{e}{r}^2\right) = O(n^3)$ , since  $r^2 = O\left(\binom{e}{r}\right) = O(n)$ . ◀

*Proof of correctness.* Clearly, the input format guarantees that (B2) holds for all inputs. For every matroid  $M$  of rank  $r$  on the ground set  $\{1, 2, \dots, e\}$ , at least one set  $X \in \binom{\{1, 2, \dots, e\}}{r}$  must be a base of  $M$ , and the variable  $g$  obviously keeps track of the existence of this set, and consequently, whether (B1) holds. In other words, upon reaching the “return  $g$ ”-instruction,  $g = 0$  if and only if  $B \equiv 0$ , i.e.  $B$  is the zero vector. Let  $X, Y \in \binom{\{1, 2, \dots, e\}}{r}$ . The “for  $x$ ”-instruction is reached for  $X$  and  $Y$  if and only if  $B(X) = B(Y) = 1$ , i.e.  $X$  and  $Y$  are supposed to be bases of the input matroid candidate. The loops “for  $x$ ” and “for  $y$ ” test whether  $(X \setminus \{x\}) \cup y$  is a base. If for a given  $x \in X \setminus Y$  an exchange partner  $y \in Y \setminus X$  is found, the “next  $x$ ”-instruction is reached. Otherwise, if no  $y \in Y \setminus X$  has this property for a given  $x \in X \setminus Y$ , the “end for  $y$ ”-instruction is reached. In this case,  $B$  violates the base exchange axiom (B3) and therefore the input candidate does not correspond to a matroid. In this case, the output of the algorithm is 0. When the algorithm reaches the “end for  $X$ ”-instruction, then it is established that the axiom (B3) holds for the input candidate. In this case, the input candidate is a matroid if and only if  $B \neq 0$ , which is correctly reflected by the value of  $g$ . Thus the output of the algorithm is 1 if the input base vector candidate is a base vector of a matroid of rank  $r$  with  $e$  elements, and 0 otherwise. ◻

Given any matroid  $M = (E, \mathcal{I})$ , we can combine Remark 2.1.14 and the Algorithms 2.5.15 and 2.5.9 with the brute-force exhaustive search algorithm in order to compute  $\Gamma_{\mathcal{M}}(M)$ : We generate all candidate families of subsets of  $\binom{E'}{\text{rk}_M(E)}$  with respect to a set  $E'$  of cardinality  $\text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E|$  with  $E \subseteq E'$ , that coincide with  $\mathcal{B}(M)$  when intersected with  $2^E$ . Then we use Algorithm 2.5.15 in order to determine whether the generated family corresponds to an actual matroid  $M'$  on  $E'$ . If this is the case, we test whether the generated matroid  $M'$  is a strict gammoid. If so, then  $M$  is a gammoid, and  $M'$  certifies this. Otherwise we continue until we exhausted all possibilities to



generate candidate families. If we have not found any strict gammoid among the candidates, then  $M$  is not a gammoid.

**Algorithm 2.5.16. Compute  $\Gamma_{\mathcal{M}}(M)$  (Brute-Force Search)**

**Input** (1)  $M = (E, \mathcal{I})$  matroid, given by its encoding  $b(M)$ .  
**Output** 1 if  $M$  is a gammoid,  
 0 otherwise.

```

let  $E' := \{1, 2, \dots, \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E|\}$ 
let  $\mathcal{E} := \binom{E'}{\text{rk}_M(E)}$ 
let  $\mathcal{Y} := \{Y \in 2^{\mathcal{E}} \mid Y \cap 2^{\{1, 2, \dots, |E|\}} = \mathcal{B}(M)\}$ 
for  $B \in \mathcal{Y}$  do
  if  $B$  satisfies the base axioms then do
    let  $N := (E', \{I \subseteq E' \mid \exists X \in \mathcal{E}: I \subseteq X \text{ and } B(X) = 1\})$ 
    if  $N$  is a strict gammoid then return 1 and stop
  end if  $B$  satisfies the base axioms
end for  $B$ 
return 0.

```

In the worst case  $M$  is not a gammoid and we have to iterate over

$$2^{|\mathcal{E}| - \binom{|E|}{\text{rk}_M(E)}} = O\left(2^{|E'|^{\text{rk}_M(E)}}\right)$$

possible values for  $B$  in the “for  $B$ ”-loop. The test whether  $B$  corresponds to a matroid can be carried out by Algorithm 2.5.15 and is possible within

$$O\left(\text{rk}_M(E)^2 \cdot \binom{|E'|}{\text{rk}_M(E)}\right)\text{-time.}$$

The test whether  $N$  is a strict gammoid may be done with Algorithm 2.5.9 and therefore can be done in

$$O\left(\log(|E'|) \cdot 2^{2|E'|} + |E'|^2 \cdot \mathbf{N}(N) \cdot 2^{|E'|}\right)\text{-time.}$$

Let  $n = \mathbf{N}(M)$  be the bit-length of  $b(M)$ , clearly  $|E| \leq n$  and  $\text{rk}_M(E) \leq n$ . We have  $|E'| \leq n^3 + 2n$ . Thus one test of the base axioms can be done in

$$O\left(n^2 \cdot \binom{n^3 + 2n}{n}\right) = O(n^{2.001n})\text{-time.}$$

Since the bit-length  $\mathbf{N}(N) \leq 2^{|E'|} \leq 2^{n^3 + 2n}$ , we obtain that each strict gammoid test can be carried out in

$$O\left(n^6 \cdot 2^{2n^3 + 4n}\right)\text{-time.}$$

R. Pendavingh and J. van der Pol give the following upper bound for the number of matroids  $m(k, r)$  on  $k$ -elementary ground sets with rank  $r$  in [PvdP17]

$$\log(m(k, r)) \leq \frac{1}{k - r + 1} \binom{k}{r} \cdot \log(c \cdot (k - r + 1))$$

under the mild condition that  $r \geq 3$  and  $k \geq r + 12$ , and where  $c$  denotes a constant factor that does not depend on  $k$  or  $r$ . We can use this bound in order to determine how often we have to decide whether  $N$  is a strict gammoid or not, let  $i$  denote the number of strict gammoid tests, then

$$\begin{aligned} \log(i) &\leq \log(m(n^3 + 2n, \text{rk}_M(E))) \\ i &= O\left(2^{n^{(3\text{rk}_M(E) - 3) \cdot \log(c \cdot (n^3 + 2n))}}\right) \\ &= O\left(2^{\binom{n^{3\text{rk}_M(E)}}{r}}\right) = O\left(2^{n^{3n}}\right). \end{aligned}$$

A naive upper bound for the strict gammoid tests can be derived from the number of  $B$ -iterations, it is

$$O\left(2^{(n^3 + 2n)^{3n}}\right) = O\left(\prod_{i=0}^{3n} 2^{\binom{3n}{i} \cdot (n^{2i + 3n})}\right),$$

and it obviously is a looser upper bound than the one derived from [PvdP17], since it has a factor  $2^{(n^{9n})}$ . Thus we have to account

$$O\left(2^{(n^3 + 2n)^{3n}} \cdot n^{2.001n}\right)$$

for the base exchange axiom tests and

$$O\left(n^6 \cdot 2^{n^{3n} + 2n^3 + 4n}\right)$$

for the strict gammoid tests. Clearly,  $O\left(n^6 \cdot 2^{n^{3n}+2n^3+4n}\right) = O\left(2^{(n^3+2n)^{3n}}\right)$ , so we may estimate the total-run time of the algorithm to be in

$$O\left(2^{(n^3+2n)^{3n}} \cdot n^{2.001n}\right) = O\left(2^{(n^{9n+1})}\right). \quad \blacktriangleleft$$

*Proof of correctness.* If  $M$  is a gammoid, then there is a representation  $(D, T, E)$  with  $D = (V, A)$  such that  $|V| \leq \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E| = |E'|$  (Remark 2.1.14). But then  $N = \Gamma(D, T, V)$  is a strict gammoid with the property  $N|E = \Gamma(D, T, E) = M$ , thus  $\{X \subseteq E \mid X \in \mathcal{B}(N)\} = \mathcal{B}(M)$ . The “for  $B$ ”-instruction iterates through all possible and impossible  $B = \mathcal{B}(N)$  with that restriction-property, then tests whether  $B$  is indeed a matroid base family, and then tests whether the corresponding matroid is a strict gammoid. If so, the algorithm gives the truthful output 1. If no such  $B$  is found, the algorithm returns 0, and by the above consideration we may conclude that in this case  $M$  is not a gammoid.  $\square$

No one would expect that the brute-force method would be of any practical use for determining whether a given matroid is a gammoid, and it apparently is not. One obvious problem with Algorithm 2.5.16 is that it does not make use of any of the structural results for matroid extensions, instead it guesses matroid extensions, and this takes so much time that the actual testing for the strict gammoid property in  $O\left(2^{n^{3n+1}}\right)$  does not have a significant impact on the estimation. The other obvious problem is that we are not using any information from the  $\alpha_M$ -vector in order to guide our search for a strict gammoid extensions of  $M$ . Furthermore, it seems to be excessive to compute all  $\alpha$ -vectors of strict gammoid extension candidates from scratch.<sup>6</sup> We give a straight-forward back-tracking algorithm that also computes  $\Gamma_{\mathcal{M}}(M)$ .

<sup>6</sup>There is a second brute-force search method for finding the strict gammoid extension of a gammoid  $M = (E, \mathcal{I})$ , which guesses the arcs of a digraph, then calculates the ranks for all  $\text{rk}_M(E)$ -elementary subsets of  $E$  of the gammoid represented by the candidate digraph: if all of these values are correct, then  $M$  is a gammoid represented by the candidate digraph, if otherwise we run out of candidates, then  $M$  is not a gammoid. This method is clearly better than Algorithm 2.5.16 as it does eliminate the check whether a candidate is indeed a matroid. With the currently known bounds for arcs and vertices there are still  $\left(\sum_{k=0}^r \binom{n^3+2n}{k}\right)^{n^3+n}$  candidate digraphs on  $n^3+2n$  vertices (Remark 2.1.14) with at most  $r = O(n)$  leaving arcs per non-target vertex (Theorem 2.1.49), so this brute-force method is still not a practical solution — and the computation of the bases of a given strict gammoid involves a more complicated algorithm, therefore we will not give more details for this method. Of course, the refined brute-search method still wastes a lot of time because usually a considerable amount of digraphs represent the same gammoid.

**Algorithm 2.5.17. Compute  $\Gamma_{\mathcal{M}}(M)$  (Digraph Backtracking)**

**Input** (1)  $M = (E, \mathcal{I})$  matroid, given by its encoding  $b(M)$ .  
**Output** 1 if  $M$  is a gammoid,  
 0 otherwise.

```

let  $V$  be a set with  $E \subseteq V$  and  $|V| = \text{rk}_M(E)^2 \cdot |E| + \text{rk}_M(E) + |E|$ 
let  $\{(u_1, v_1), (u_2, v_2), \dots, (u_{|V|^2 - |V|}, v_{|V|^2 - |V|})\}_{\neq} = V \times V \setminus \{(v, v) \mid v \in V\}$ 
let  $K := (V, V \times V)$ 
let  $B$  be an arbitrary base of  $M$ 
declare state variable  $A \subseteq V \times V$ 
declare state variable  $i \in \{1, 2, \dots, |V|^2 - |V|\}$ 
declare state variable  $\mathcal{P} \subseteq \mathbf{P}(K)$ 
declare state variable  $\mathcal{R} \subseteq 2^{\mathbf{P}(K)}$ 
declare state variable  $\mathcal{B} \subseteq \binom{V}{\text{rk}_M(E)}$ 
 $A := \emptyset$ 
 $i := 1$ 
 $\mathcal{P} := \{v \in \mathbf{P}(K) \mid v \in V\}$ 
 $\mathcal{R} := \{\{b \in \mathbf{P}(K) \mid b \in B\}\}$ 
 $\mathcal{B} := \{B\}$ 
push state to stack
 $d := 1$ 
while  $d > 0$  do
  if  $\mathcal{B} = \mathcal{B}(M)$  return 1 and stop
  if  $i > |V|^2 - |V|$  or  $\mathcal{B} \not\subseteq \mathcal{B}(M)$  then do
    pop state from stack
     $d := d - 1$ 
     $i := i + 1$ 
  else do
    push state to stack
     $d := d + 1$ 
     $\mathcal{P}' := \{lr \mid l, r \in \mathcal{P}, l_{-1} = u_i, r_1 = v_i, |l| \cap |r| = \emptyset\}$ 
     $\mathcal{R}' := \left\{ (R \setminus \{r\}) \cup \{l.r\} \mid \begin{array}{l} R \in \mathcal{R}, r \in R, l \in \mathcal{P}', l_{-1} = r_1, \\ |l| \cap (\cup_{p \in R} |p|) = \{r_1\}, l_1 \in E \end{array} \right\}$ 
     $\mathcal{B}' := \{\{p_1 \mid p \in R\} \mid R \in \mathcal{R}'\}$ 

```

```

     $A := A \cup \{(u_i, v_i)\}$ 
     $\mathcal{P} := \mathcal{P} \cup \mathcal{P}'$ 
     $\mathcal{R} := \mathcal{R} \cup \mathcal{R}'$ 
     $\mathcal{B} := \mathcal{B} \cup \mathcal{B}'$ 
  end if
end while  $d > 0$ 
return 0.

```

We give a rough estimate of the worst-case run-time behavior of this algorithm relative to the run-time of two major blocks of instructions. First, let  $\varphi(d)$  denote a worst-case run-time estimate for the instructions inclusively between the “push state stack”-instruction and the “ $\mathcal{B} := \mathcal{B} \cup \mathcal{B}'$ ”-instruction. This is the time it takes to update the paths, maximal routings, and bases of the digraph when adding the arc  $(u_i, v_i)$ , and this operation clearly depends on the number of arcs in  $A \setminus \{(u_i, v_i)\}$ , which is a function of the value of  $d$  at the start of the instruction block. We would expect  $\varphi(d)$  to grow with  $|\mathcal{P}|$  and  $|\mathcal{R}|$ . Clearly, we have the very loose upper bound  $|\mathcal{P}| \leq |V| + d!$ , since every non-trivial path consists of a non-repeating sequence of arcs with further constraints.<sup>7</sup> We further have  $|\mathcal{R}| \leq d!$  because we may associate a routing  $R \in \mathcal{R}$  with a non-repeating sequence of arcs obtained from its paths: if  $R = \{p_1, \dots, p_r\}_{\neq}$ , we first list all arcs of  $p_1$  in the order of appearance, then the arcs of  $p_2$ , and so on until we reach  $p_r$ , and then we list all arcs from  $A$  that are not traversed by any path  $p \in R$ . Since all  $R \in \mathcal{R}$  route onto  $B$ , we can reconstruct  $R$  from the arc sequence we just constructed. Again, this bound is very loose. Furthermore, let  $\psi(d)$  denote a worst-case run-time estimate for the instructions inclusively between the “pop state stack”-instruction and the “ $i := i + 1$ ”-instruction. For the worst-case analysis, we assume that the backtracking method does traverse every digraph candidate  $(V, A)$  with  $A \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$  — which clearly is impossible for any input matroid  $M$ . With this assumption we obtain a run-time in

$$O\left(\sum_{i=0}^{2n^3-1} \binom{2n^3}{i} (\varphi(i) + \psi(i+1))\right).$$

It is clear that this estimation is overly pessimistic and does not convey a realistic picture of the run-time behavior of the digraph backtracking algorithm. Therefore we implemented a version of this algorithm in SageMath and measured its performance

<sup>7</sup>P. Seymour and B.D. Sullivan give an upper bound for the number of 4-vertex paths in digraphs without a cycle walk of length  $\leq 4$ , which is  $\frac{4}{75}|V|^4$  [SS10].

on a few sample inputs (see Listing 5.1). It is an open research task to find conditions and estimates for how often the above algorithm does prune a large chunk of candidate solutions, as well as good implementations of the update procedure, that exceeds the scope of this work. ◀

*Proof of correctness.* We have the following invariants at the “**while**  $d > 0$ ”-instruction:  $d = |A|$ , let  $D = (V, A)$ , then  $\mathcal{P} = \mathbf{P}(D)$ ,  $\mathcal{R}$  is the family of all linkings from a subset of  $E$  onto  $B$  in  $D$ , and  $\mathcal{B}$  is the family of subsets of  $E$  that can be linked onto  $B$  by a routing  $R \in \mathcal{R}$ . Furthermore, the stack consists of  $d$  sets of previously pushed assignments of the variables  $A, i, \mathcal{P}, \mathcal{R}, \mathcal{B}$ . The instructions in the “**while**  $d > 0$ ”-loop recursively test or dismiss all digraphs  $D' = (V, A')$  for the property  $\Gamma(D', B, E) = M$ . First, the algorithm tests whether there is a digraph  $D'$  representing  $M$  with  $(u_i, v_i) \in A$ ; if it can be ruled out that there is such a digraph  $D'$ , the algorithm tests whether there is a digraph  $D'$  representing  $M$  with  $(u_i, v_i) \notin A$ . On the other hand, if the loop finds a representation, it returns 1 and the algorithm ends. Therefore, if we reach the “**end while**  $d > 0$ ”-instruction, we may conclude that there is no digraph on  $V$  that represents  $M$ . With Remark 2.1.14 and Theorem 2.1.10 we then may conclude that  $M$  is not a gammoid, and the next instruction correctly returns 0.

Now let us show in detail that the “**while**  $d > 0$ ”-loop indeed has the property stated above. Clearly, if  $\mathcal{B} = \mathcal{B}(M)$ , then  $\Gamma(D, B, E) = M$  and therefore we may safely return 1. First, if there is some  $X \in \mathcal{B}$  which is not a base of  $M$ , then there is a routing  $X \rightrightarrows B$  in every digraph  $D' = (V, A')$  with  $A \subseteq A'$ , consequently  $X$  is independent in  $\Gamma(D', B, E)$  for all such  $D'$ . Thus we may dismiss all such candidate digraphs. The same holds if  $i > |V|^2 - |V|$ , in this case we are out of arcs that we may add to  $A$ , but  $\mathcal{B} \subsetneq \mathcal{B}(M)$ , i.e. there is still a base  $Y$  of  $M$  which is not a base of  $\Gamma(D, B, E)$ . In other words, if  $M$  is a gammoid, then there is an arc  $a \in A$  that obstructs the addition of some other arcs, one of which is needed to represent  $M$ . In both cases, we have to undo our last addition of an arc. We achieve this by popping the assignments of  $A, i, \mathcal{P}, \mathcal{R}, \mathcal{B}$  from the stack, which were pushed before the last arc had been added. Afterwards, we have to decrease  $d$  in order to reflect the new stack size, and increase  $i$  in order to prevent adding the same arc again. Now assume that neither  $i > |V|^2 - |V|$  nor  $\mathcal{B} \subsetneq \mathcal{B}(M)$ , thus we enter the “**else do**”-branch of the second **if**-instruction in the “**while**  $d > 0$ ”-loop. In this case there is a base  $Y$  of  $M$  that is not a base of  $\Gamma(D, B, E)$ . We try to fix this by adding the arc  $(u_i, v_i)$  to  $A$ , after we pushed the current state to the stack and adjusted  $d$  accordingly. Let  $D = (V, A)$  denote the digraph before adding the arc, i.e.  $(u_i, v_i) \notin A$ , and let  $D' = (V, A \cup \{(u_i, v_i)\})$ . Clearly  $\mathbf{P}(D') \setminus \mathbf{P}(D)$  consists of all paths  $p \in \mathbf{P}(D')$  with  $(u_i, v_i) \in |p|_A$ . But every such  $p$  can be written

as  $lr$  where  $l, r \in \mathbf{P}(D)$  with  $|l| \cap |r| = \emptyset$  and such that  $l$  ends in  $u_i$  and  $r$  starts in  $v_i$ . Remember that  $\mathcal{P} = \mathbf{P}(D)$ , thus  $\mathcal{P}' = \mathbf{P}(D') \setminus \mathbf{P}(D)$ . Now let  $R: X \rightrightarrows B$  with  $X \subseteq E$  be a routing in  $D'$  which is not a routing in  $D$ , then  $R \cap \mathcal{P}' \neq \emptyset$ . If we cut off the path of  $R$  that uses the new arc  $(u_i, v_i)$  at  $u_i$ , we obtain a routing that is also a routing in  $D$ . Therefore, all routings of  $D'$  that start in a subset  $X \subseteq E$  and that are not routings of  $D$  are members of the family  $\mathcal{R}'$ . Consequently, all bases of  $\Gamma(D', B, E)$  that are not bases of  $\Gamma(D, B, E)$  are members of  $\mathcal{B}'$ . Thus the above invariants at the “while  $d > 0$ ”-instruction hold after the update operations on  $A, \mathcal{P}, \mathcal{R}, \mathcal{B}$ . The correctness of the algorithm is therefore established.  $\square$

**Remark 2.5.18.** Algorithm 2.5.17 is obviously faster than the brute-force search method, as it does not generate non-matroid solution candidates. Yet, it still has to dismiss all possibilities of arranging arcs in a big digraph in order to determine that  $M$  is not a gammoid. The dismissal of a chunk of solution candidates may only take place as soon as it can be proven, that every gammoid corresponding to a digraph that contains a certain set of arcs has some independent set  $X \subseteq E$  which is dependent in  $M$ .<sup>8</sup> Alas, this happens quite late in the process: At least as long as none of the auxiliary vertices in  $V \setminus E$  has a leaving arc that enters any  $e \in E$ , there is no way to detect excess connectivity in partial solutions. Therefore Algorithm 2.5.17 traverses all candidate arc sets  $A$  that cover more than  $2^{(|V|-1) \cdot |V \setminus E|}$  digraphs without loop-arcs on  $V$  with  $A \cap ((V \setminus E) \times E) = \emptyset$ . This implies a lower bound of the run-time for all inputs  $M = (E, \mathcal{I})$  with  $M$  not a gammoid<sup>9</sup> in  $\Omega\left(2^{|E|^{5.999}}\right)$ .

We implemented a less naive version of Algorithm 2.5.17 (see Listing 5.1), where we use an implicit linear order on  $V = \{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_m\}_{\neq}$  such that  $E = \{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_{|E|}\}$  holds. We keep track of the smallest index  $i_0$  that belongs to a vertex  $\hat{v}_{i_0} \in V \setminus E$  that is not entered by any arc  $a \in A$ . Let  $i, j \in \{1, 2, \dots, |V|^2 - |V|\}$  and let  $(u_i, v_i) = (\hat{v}_{i_1}, \hat{v}_{i_2})$  and  $(u_j, v_j) = (\hat{v}_{j_1}, \hat{v}_{j_2})$ . We require that the implicit linear order on the arcs has the property that we have  $\max\{i_1, i_2\} < \max\{j_1, j_2\}$  or  $(\max\{i_1, i_2\} = \max\{j_1, j_2\}$  holds, and either  $i_2 = j_1 = \max\{i_1, i_2\}$  or  $\min\{i_1, i_2\} < \min\{j_1, j_2\}$  holds), if and only if  $i < j$  holds. In other words, we enumerate  $V \times V \setminus \{(v, v) \mid v \in V\}$  in the following order:  $(\hat{v}_1, \hat{v}_2), (\hat{v}_2, \hat{v}_1), (\hat{v}_1, \hat{v}_3), (\hat{v}_2, \hat{v}_3), (\hat{v}_3, \hat{v}_1), (\hat{v}_3, \hat{v}_2), (\hat{v}_1, \hat{v}_4), \dots$ . Now we may implement a shortcut and backtrack as soon as  $i_2 > i_0$  holds for  $(u_i, v_i) = (\hat{v}_{i_1}, \hat{v}_{i_2})$ . The rationale behind this is that if we have to add a new arc that enters a previously unentered vertex,

<sup>8</sup>A tempting modification of Algorithm 2.5.17 would be to check whether  $\{\{p_1 \mid p \in R\} \cap E \mid R \in \mathcal{R}'\} \not\subseteq \mathcal{I}$  holds instead of  $\mathcal{B} \not\subseteq \mathcal{B}(M)$ , but the Augmentation Lemma 1.2.7 implies that this does not occur any earlier than  $\mathcal{B} \not\subseteq \mathcal{B}(M)$ .

<sup>9</sup>For instance, we may use the arbitrary large non-gammoids  $M(K_4) \oplus (\{X\}, 2^X)$  for growing finite sets  $X$  to approach this run-time bound (see also Example 2.5.25).

we can always choose the vertex entered to be the one with the lowest index among all unentered vertices. Although the improvement corresponding to this adjustment is quite measurable in practice, the algorithm still has to try more than  $2^{|V \setminus E|^2 - |V \setminus E|}$  — e.g. the number of digraphs  $D' = (V', A')$  on  $V' = V \setminus E$  with  $A' \subseteq (V' \times V') \setminus \{(v, v) \mid v \in V'\}$  — different candidate arc sets that have the property  $A \cap ((V \setminus E) \times E) = \emptyset$  for every  $Q = A \cap (E \times (V \setminus E))$  with  $Q_A \subseteq Q$  where we let  $Q_A = \{(\hat{v}_1, w) \mid w \in V \setminus E\}$ . Since there are  $2^{(|E|-1) \cdot |V \setminus E|}$  different possibilities for such  $Q$ , the adjusted algorithm still exposes the  $\Omega\left(2^{|E|^{5.999}}\right)$  behavior.

In theory, there is a possibility to speed up the algorithm a little further, since Theorem 2.1.49 guarantees that no vertex has more than  $\text{rk}_M(E)$  leaving arcs. Clearly, the problem that the backtracking information is only available late in the process is not remedied by limiting the number of arcs leaving each vertex. The number of digraph candidates, that have to be processed before any target is connected, still is at least

$$\left( \sum_{k=0}^{\text{rk}_M(E)} \binom{\text{rk}_M(E)^2 \cdot |E| + |E|}{k} \right)^{\text{rk}_M(E)^2 \cdot |E|} = \Omega\left(2^{\text{rk}_M(E)^2 \cdot |E|}\right).$$

Thus such an adjusted algorithm still exposes  $\Omega\left(2^{|E|^{2.999}}\right)$  behavior. When we implemented forced bounds on the number of leaving arcs, the run-time actually increased. •

Therefore it is clearly indicated that we examine how Mason's criterion and matroid extensions play along with each other in order to gain better understanding of the problem. This understanding is an essential milestone for the research in better algorithms for determining  $\Gamma_{\mathcal{M}}(M)$ . Before we devote ourselves to that, we want to make a remark on potentially more easy subclasses of gammoids.

**Remark 2.5.19.** Let  $k \in \mathbb{N}$ . For the subclasses  $\mathcal{W}^k$  that consists of all gammoids  $G$  with  $W^k(G) \leq 1$ , the problem of deciding class membership appears to be dramatically more easy. Let  $M = (E, \mathcal{I})$  be a matroid. If  $M \in \mathcal{W}^k$  there is a representation using at most  $k \cdot |E|$  arcs. Therefore there is a representation with at most  $2k \cdot |E|$  auxiliary vertices. So if  $M \in \mathcal{W}^k$ , then  $M = \Gamma(D, T, E)$  where  $D = (V, A)$  with  $|V| \leq (2k+1) \cdot |E|$  and  $|A| \leq k \cdot |E|$ . Thus there are at most

$$\sum_{i=0}^{k \cdot |E|} \binom{9k^2 \cdot |E|^2}{i}$$

candidate digraphs for  $M$  that we may have to regard. Furthermore,  $M$  is not in  $\mathcal{W}^k$  as soon as  $M|X$  cannot be represented with a digraph on  $3k \cdot |X|$  vertices with



at most  $k \cdot |X|$  arcs, this may open up possibilities for effective divide and conquer approaches. •

### 2.5.3 Violations of Mason’s $\alpha$ -Criterion

Some of the ideas and results presented in this section have been published in I. Albrecht’s *On Finding New Excluded Minors for Gammoids* [Alb17], where an equivalent yet different version of Mason’s  $\alpha$ -criterion is used.

**Definition 2.5.20.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $V \subseteq E$ .  $V$  shall be an  $\alpha_M$ -violation, if  $\alpha_M(V) < 0$  and for all  $V' \subsetneq V$ ,  $\alpha_M(V') \geq 0$ . The family of all  $\alpha_M$ -violations is denoted by*

$$\mathcal{V}(M) = \left\{ V \subseteq E \mid \alpha_M(V) < 0 \text{ and } \forall V' \subsetneq V: \alpha_M(V') \geq 0 \right\}. \quad \blacksquare$$

Clearly, an  $\alpha_M$  violation is an inclusion minimal set  $X$ , for which the inequality  $\alpha_M(X) \geq 0$  does not hold.

**Corollary 2.5.21.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  is a strict gammoid, if and only if  $\mathcal{V}(M) = \emptyset$ .*

*Proof.* Immediate from Corollary 2.2.21 and Definition 2.5.20. □

**Example 2.5.22.** Let  $k \in \mathbb{N} \setminus \{0, 1\}$  be an arbitrary choice, let

$$X = \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid i, j < k, j \in \{i, (i + 1) \bmod k\} \right\},$$

and let  $E = X \cup \{a, b\} \neq \emptyset$  where  $a, b \notin X$ . Furthermore, let

$$\mathcal{H}_a = \left\{ \{a, (i, i), (i, (i + 1) \bmod k)\} \mid i \in \mathbb{N}, i < k \right\}$$

and

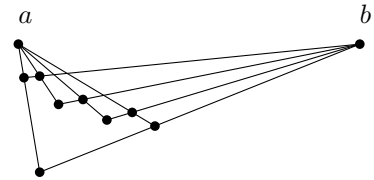
$$\mathcal{H}_b = \left\{ \{b, ((j - 1) \bmod k, j), (j, j)\} \mid j \in \mathbb{N}, j < k \right\}.$$

Let  $M = (E, \mathcal{I})$  be the matroid of rank 3 such that  $\mathcal{H} = \mathcal{H}_a \cup \mathcal{H}_b$  is the family of its dependent hyperplanes.<sup>10</sup> We show that the matroid  $M$  exists by postulating that

$$\mathcal{C}(M) = \left\{ C \in \binom{E}{4} \mid \nexists H \in \mathcal{H}: H \subseteq C \right\} \cup \mathcal{H}.$$

---

<sup>10</sup> $M$  is a paving matroid, see [Wel76], Section 2.3. The figure depicts the affine configuration of  $M$  for  $k = 4$ .



Clearly,  $\mathcal{C}(M) \neq \emptyset$ , and for  $C_1, C_2 \in \mathcal{C}(M)$  we have  $C_1 \subseteq C_2$  if and only if  $C_1 = C_2$ . Let  $C_1, C_2 \in \mathcal{C}(M)$  with  $C_1 \neq C_2$ . Then  $C_1 \cup C_2$  has at least 5 elements, thus for every  $e \in C_1 \cap C_2$ , the set  $(C_1 \cup C_2) \setminus \{e\}$  has four elements. So, by construction of  $\mathcal{C}(M)$ , the set  $(C_1 \cup C_2) \setminus \{e\}$  contains a circuit. Consequently,  $\mathcal{C}(M)$  satisfies the circuit elimination axiom. Therefore  $\mathcal{C}(M)$  satisfies all circuit axioms for matroids ([Oxl11], Theorem 1.1.4, p.9) and the matroid  $M$  with the above properties exists. Since every dependent hyperplane is a circuit, we obtain that  $\alpha_M(H) = 1$  for all  $H \in \mathcal{H}$ . A flat  $F \in \mathcal{F}(M)$  is either independent, a hyperplane, or the whole ground set of  $M$ . For independent  $F \in \mathcal{F}(M)$  we have  $\alpha_M(F) = 0$ . Therefore

$$\alpha_M(E) = |E| - \text{rk}_M(E) - \sum_{F \in \mathcal{F}(M), F \neq E} \alpha_M(F) = 2k - 1 - |\mathcal{H}| = -1.$$

Since there are at least  $1 + |W|$  dependent hyperplanes  $H \in \mathcal{H}$  with  $H \cap W \neq \emptyset$  for all  $W \subseteq E$  with  $W \neq \emptyset$ , we obtain that  $\alpha_M(E \setminus W) \geq 1 + \alpha_M(E) \geq 0$  for all  $\emptyset \neq W \subseteq E$ . Consequently  $E$  is an  $\alpha_M$ -violation and  $\mathcal{V}(M) = \{E\}$ . Let us fix an arbitrary element  $e \in E$  for now and let  $N = M|(E \setminus \{e\})$ . Then at least two dependent hyperplanes of  $M$  are no longer hyperplanes of  $N$ . All dependent flats of  $N$  are still hyperplanes of  $N$ , consequently  $\alpha_N \geq 0$  and  $N$  is a strict gammoid (Corollary 2.2.21).  $M|(E \setminus \{e\})$  has rank 2 and therefore is a strict gammoid (Proposition 2.5.11). Furthermore, Proposition 2.5.11 yields that  $M$  is not a gammoid, and therefore  $M$  is an excluded minor of the class of gammoids. By Theorem 2.4.1 we obtain that  $M^*$  is an excluded minor for the class of gammoids, too. The circuits of  $M^*$  are the complements of hyperplanes of  $M$ , and the hyperplanes of  $M^*$  are the complements of circuits of  $M$ . Thus every  $D \in \mathcal{C}(M^*)$  has at least  $|E| - 3$  elements. Furthermore  $\text{rk}_{M^*}(E) = |E| - \text{rk}_M(E) = |E| - 3$ , therefore all dependent hyperplanes of  $M^*$  are circuits, and these hyperplane circuits of  $M^*$  are the complements of the dependent hyperplanes of  $M$ .<sup>11</sup> Therefore  $M^*$  has  $|\mathcal{H}| = 2k$  dependent hyperplanes of the form  $H' = E \setminus H$  for  $H \in \mathcal{H}$ . In this situation, we have  $\alpha_{M^*}(H') = 1$ . Furthermore, if  $F \in \mathcal{F}(M^*)$  such that  $F \neq E \setminus H$  for all  $H \in \mathcal{H}$ , then

<sup>11</sup>We admit that this situation might be slightly confusing. In both matroids  $M$  and  $M^*$ , every circuit has one of two cardinalities, and every hyperplane has one of two cardinalities. The hyperplanes with the higher cardinality are circuits and therefore dependent, whereas the hyperplanes with smaller cardinality are independent. The circuits with smaller cardinality are hyperplanes, the circuits with higher cardinality have full rank. The hyperplanes with higher cardinality – the dependent hyperplanes – are therefore complements of smaller cocircuits, which happen to be cohyperplanes. Therefore the dependent hyperplanes are complements of codependent cohyperplanes, a situation that might appear very special, but it is rather not: Matroids with this property are called *sparse paving matroids*, and R. Pendavingh and J. van der Pol showed that they are quite abundant [PvdP15].

either  $F = E$  or  $F$  is independent. Thus we may calculate

$$\alpha_{M^*}(E) = |E| - \text{rk}_{M^*}(E) - \sum_{F \in \mathcal{F}(M^*), F \neq E} \alpha_{M^*}(F) = 3 - |\mathcal{H}| = -(2k - 3).$$

Now, let  $x \in X$ , then there are precisely 2 dependent hyperplanes  $H$  of  $M$  with  $x \in H$ . Consequently, there are  $|H| - 2 = 2k - 2$  dependent hyperplanes of  $M^*$  which contain  $x$ . Similarly, there are  $|H| - k = k$  dependent hyperplanes of  $M^*$  which contain  $y \in \{a, b\}$ . Also, for all  $W \subseteq X$  with  $|W| = 2$  there are at least  $2k - 1$  dependent hyperplanes of  $M^*$  which have non-empty intersection with  $W$ , and for all  $W \subseteq X$  with  $|W| \geq 3$  every dependent hyperplane of  $M^*$  has non-empty intersection with  $W$ . Thus  $\alpha_{M^*}(E \setminus W) \geq 0$  for all  $W \subseteq E$  with  $X \cap W \neq \emptyset$ . Furthermore

$$\alpha_{M^*}(E \setminus \{a\}) = \alpha_{M^*}(E \setminus \{b\}) = -(2k - 3) + (k - 1) = -(k - 2),$$

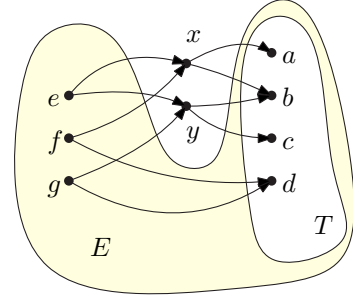
thus  $\mathcal{V}(M^*) = \{E\}$  if  $k = 2$ , and  $\mathcal{V}(M^*) = \{E \setminus \{a\}, E \setminus \{b\}\}$  whenever  $k > 2$ . Let us consider the case where  $k > 2$ , and let  $N = (M^*)|(E \setminus \{b\})$ . Then we also have  $\alpha_N(E \setminus \{b\}) = -(k - 2)$ ,  $\mathcal{V}(N) = \{E \setminus \{b\}\}$ , and  $N$  is the dual of a strict gammoid — so  $N$  is a transversal matroid. ▲

**Remark 2.5.23.** Let  $\mathcal{M}$  be the class of matroids where  $M \in \mathcal{M}$  if and only if  $M$  is the matroid constructed in Example 2.5.22 for some  $k \in \mathbb{N} \setminus \{0, 1\}$ . Then  $\mathcal{M}$  is an infinite family of excluded minors of rank 3 for the family of gammoids. The derived family  $\mathcal{M}^* = \{M^* \mid M \in \mathcal{M}\}$  is also an infinite family of excluded minors and yields excluded minors with rank  $2k - 1$ . We see that excluded minors for the class of gammoids may have multiple  $\alpha_M$ -violations, and that the value of  $\alpha_M(E)$  for  $E \in \mathcal{V}(M)$  may become arbitrarily low for excluded minors of the class of gammoids as well as for gammoids that are non-strict. The matroids  $N = (M^*)|(E \setminus \{b\})$  and  $M^*$  have very similar  $\alpha$ -violation structure:  $M^*$  contains two violations, and if we restrict  $M^*$  to any of these two violations, we obtain  $N$  — thus  $M^*$  essentially has two isomorphic copies of the unique  $\alpha_N$ -violation. Consequently, we cannot decide whether a matroid is a gammoid or not by just considering one violation of  $M$  at a time, instead we have to consider the interaction between violations in  $M$  as well. •<sup>12</sup>

<sup>12</sup>We may consider the property that a matroid is a gammoid to be global with respect to the proper violation-restrictions  $M|X$  for  $X \in \mathcal{V}(M)$  in the same sense as the chromatic number of a graph is a global property with respect to proper induced sub-graphs.

Before we start developing the theory of  $\alpha_M$ -violations, we should familiarize ourselves some more with the two different kinds of violations that arise in matroids — violations in non-gammoids and violations in gammoids that are not strict.

**Example 2.5.24.** We examine the situation with respect to the gammoid  $M = \Gamma(D, T, E)$  with the ground set  $E = \{a, b, c, d, e, f, g\} \neq$  as presented in Example 2.2.17. We have  $\mathcal{F}(M) \setminus \mathcal{I} = \{\{a, b, c, e\}, \{a, b, d, f\}, \{b, c, d, g\}, \{d, e, f, g\}, E\}$  and, clearly,  $\mathcal{V}(M) = \{E\}$  and we have  $\alpha_M(E) = -1$ . But since we know that  $M = \Gamma(D, T, V)|E$ , we know that the violation  $E$  can be resolved by adding the elements  $x$  and  $y$  to the matroid  $M$ . Let  $M_x = \Gamma(D, T, E \cup \{x\}) = (E \cup \{x\}, \mathcal{I}_x)$ . Then the new dependent flats of  $M_x$  with respect to  $M$  are



$$(\mathcal{F}(M_x) \setminus \mathcal{I}_x) \setminus (\mathcal{F}(M) \setminus \mathcal{I}) = \{\{a, b, x\}, \{d, f, x\}, \{a, b, g, x\}, \{c, d, f, x\}, \\ \{a, b, c, e, x\}, \{a, b, d, f, x\}, \{d, e, f, g, x\}, E \cup \{x\}\},$$

and the dependent flats of  $M$  that vanish in  $M_x$  are

$$(\mathcal{F}(M) \setminus \mathcal{I}) \setminus (\mathcal{F}(M_x) \setminus \mathcal{I}_x) = \{\{a, b, c, e\}, \{a, b, d, f\}, \{d, e, f, g\}, E\}.$$

We now have

$$\begin{aligned} \alpha_{M_x}(\{a, b, x\}) &= \alpha_{M_x}(\{d, f, x\}) = \alpha_{M_x}(\{b, c, d, g\}) = \alpha_{M_x}(\{a, b, g, x\}) = \\ &\alpha_{M_x}(\{a, b, c, e\}) = \alpha_{M_x}(\{a, b, d, f\}) = \alpha_{M_x}(\{d, e, f, g\}) = 1, \\ &\alpha_{M_x}(\{a, b, c, e, x\}) = 5 - 3 - \alpha_{M_x}(\{a, b, x\}) = 1, \\ &\alpha_{M_x}(\{d, e, f, g, x\}) = 5 - 3 - \alpha_{M_x}(\{d, f, x\}) = 1, \\ &\alpha_{M_x}(\{a, b, d, f, x\}) = 5 - 3 - \alpha_{M_x}(\{a, b, x\}) - \alpha_{M_x}(\{d, f, x\}) = 0, \\ &\alpha_{M_x}(E) = 7 - 4 - \alpha_{M_x}(\{b, c, d, g\}) = 2, \text{ and} \\ \alpha_{M_x}(E \cup \{x\}) &= 8 - 4 - \alpha_{M_x}(\{a, b, x\}) - \alpha_{M_x}(\{d, f, x\}) - \alpha_{M_x}(\{d, e, f, g, x\}) \\ &\quad - \alpha_{M_x}(\{a, b, c, e, x\}) - \alpha_{M_x}(\{b, d, f, g\}) = -1. \end{aligned}$$

Thus  $\mathcal{V}(M_x) = \{E \cup \{x\}\}$ . So we still have a violation if we just add  $x$  to the ground set of the gammoid, and it is easy to tell from the symmetric design of  $D$ , that the same holds when we would just add  $y$ . Although  $E$  is no longer a violation, we seem to just have shifted the problem to  $E \cup \{x\} = \text{cl}_{M_x}(E)$ . Yet, we made some

progress by adding  $x$  to  $M$ :  $M_x$  has a modular cut that is generated by three rank 2 flats, namely  $\mathcal{C}_y = \{F \in \mathcal{F}(M_x) \mid \{b, c\} \subseteq F \text{ or } \{d, g\} \subseteq F \text{ or } \{e, x\} \subseteq F\}$ , whereas  $M$  has no such modular cut. So the violation  $E \cup \{x\}$  of  $M_x$  is less rigid than the violation  $E$  of  $M$ . Now let  $N = \Gamma(D, T, V)$ . Then  $\mathcal{V}(N) = \emptyset$ , and  $\alpha_N(X) = 1$ , if  $X \in \{\{a, b, x\}, \{b, c, y\}, \{d, f, x\}, \{d, g, y\}, \{e, x, y\}\}$ , otherwise  $\alpha_N(X) = 0$ . So we see how the gammoid  $M$  violates Mason's  $\alpha$ -criterion: by deleting  $x$  and  $y$ , the nullity of the rank 2 flats disappears, and so a common reason for nullity in the hyperplanes goes below the radar, resulting in excess negative terms for  $\alpha_M(E)$ , which then create an  $\alpha_M$ -violation.  $\blacktriangle$

**Example 2.5.25.** Let us now consider the matroid  $M(K_4) = (E, \mathcal{I})$  which shall be defined on the ground set  $E = \{a, b, c, d, e, f\}_{\neq}$  and which has the following circuits

$$\mathcal{C}(M(K_4)) = \{\{a, b, d\}, \{a, c, e\}, \{b, c, f\}, \{d, e, f\}\}.$$

Every circuit of  $M(K_4)$  is also a hyperplane of  $M(K_4)$ , and therefore a flat. We calculate

$$\begin{aligned} \alpha_{M(K_4)}(\{a, b, d\}) &= \alpha_{M(K_4)}(\{a, c, e\}) = \\ \alpha_{M(K_4)}(\{b, c, f\}) &= \alpha_{M(K_4)}(\{d, e, f\}) = 1 \\ \alpha_{M(K_4)}(E) &= 6 - 3 - 4 \cdot 1 = -1. \end{aligned}$$

Thus  $\mathcal{V}(M(K_4)) = \{E\}$  and by Proposition 2.5.11 and Corollary 2.2.21, we obtain that  $M(K_4)$  is not a gammoid. Unlike Example 2.5.24, the  $\alpha_{M(K_4)}$ -violation  $E$  does not allow any particular progress by adding elements to  $M(K_4)$ . First, consider an extension  $N \in \mathcal{X}(M(K_4), g)$ , that corresponds to a modular cut  $\{F \in \mathcal{F}(M) \mid g \in \text{cl}_N(F)\}$  which is the principal filter of a flat  $F_g \in \mathcal{F}(M(K_4))$  in  $\mathcal{F}(M(K_4))$ . Such an extension always has the violation  $E \cup \{g\} \in \mathcal{V}(N)$ . Furthermore, we do not gain any headroom in the sense of allowing new qualities of modular cuts that are not available with respect to  $M(K_4)$ : If  $A, B \in \mathcal{F}(N)$  is not a modular pair in  $N$ , then  $A \setminus \{g\}, B \setminus \{g\}$  is not a modular pair in  $M(K_4)$ . The modular cuts of  $M(K_4)$  that are not principal filters in  $\mathcal{F}(M(K_4))$  are the cuts of  $\mathcal{F}(M(K_4))$  generated by the two- and three-elementary subsets of  $Q = \{\{a, f\}, \{b, e\}, \{c, d\}\}$ . Now none of the hyperplanes of  $M(K_4)$  belong to such cuts, because no hyperplane is a subset of any element of  $Q$ . Therefore, the hyperplanes of  $M(K_4)$  are still flats in the corresponding extension, and so  $E$  is still a violation.  $\blacktriangle$

### 2.5.4 The $\alpha$ -Invariant and Single Element Extensions

In this section we take a look at how single element extensions of a matroid interact with the  $\alpha$ -invariant. We start with an easy observation.

**Lemma 2.5.26.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  be a single-element extension of  $M$  such that  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$ . Let further  $X \subseteq E$  such that  $\alpha_N(X) \geq 0$ . Then there is a set  $C_0 \subseteq \{F \in C \mid F \subsetneq X\}$  such that*

$$\forall F \in C_0: \alpha_M(F) > 0 \quad \text{and} \quad \sum_{F \in C_0} \alpha_M(F) \geq -\alpha_M(X).$$

*Proof.* Clearly,  $\alpha_N(X) - \alpha_M(X) \geq -\alpha_M(X)$ . It follows from Lemma 1.3.8 and Definition 2.2.10 that  $\alpha_N(Y) = \alpha_M(Y)$  for all  $Y \subseteq E$  with  $\text{cl}_M(Y) \notin C$ . Furthermore,  $\{F \in \mathcal{F}(N) \mid F \subsetneq X, F \in C\} = \emptyset$ .

$$\begin{aligned} \alpha_N(X) - \alpha_M(X) &= |X| - \text{rk}_N(X) - \sum_{F \in \mathcal{F}(N), F \subsetneq X} \alpha_N(F) \\ &\quad - |X| + \text{rk}_M(X) + \sum_{F \in \mathcal{F}(M), F \subsetneq X} \alpha_M(F) \\ &= - \left( \sum_{F \in \mathcal{F}(N), F \subsetneq X} \alpha_N(F) \right) + \left( \sum_{F \in \mathcal{F}(M), F \subsetneq X} \alpha_M(F) \right) \\ &= \sum_{F \in C, F \subsetneq X} \alpha_M(F) \leq \sum_{F \in C_0} \alpha_M(F) \end{aligned}$$

where  $C_0 = \{F \in C \mid F \subsetneq X, \alpha_M(F) > 0\}$  is a subset of  $C$  with the desired property.  $\square$

Unfortunately, if  $M$  is a matroid and  $C$  is a modular cut that satisfies the consequent of Lemma 2.5.26 with respect to every  $X \subseteq E$ , and if  $N \in \mathcal{X}(M, e)$  is the extension corresponding to  $C$ , then  $N$  may still not be a strict gammoid. On the other hand, if there is a subset  $X \subseteq E$  which violates the consequent of Lemma 2.5.26, we know that  $N$  is definitely not a strict gammoid. If we tried to extend a given matroid in order to obtain a strict gammoid, then it would be quite natural to first try modular cuts which satisfy the consequent of Lemma 2.5.26 for as many  $X \subseteq E$  with  $\alpha_M(X) < 0$  as possible.

**Definition 2.5.27.** Let  $M = (E, \mathcal{I})$  be a matroid. We define the  $\alpha_M$ -poset as the pair  $(\mathbf{A}_M, \sqsubseteq_M)$  where  $\mathbf{A}_M = 2^E$  and where for all  $X, Y \in \mathbf{A}_M$

$$X \sqsubseteq_M Y \iff X = Y \text{ or } X \in \mathcal{F}(M, Y)$$

holds. If  $M$  is clear from the context, we also write  $\mathbf{A}$  for  $\mathbf{A}_M$  and  $\sqsubseteq$  for  $\sqsubseteq_M$ . ■

**Remark 2.5.28.**  $(\mathbf{A}_M, \sqsubseteq)$  is obviously a poset: for all  $X \in \mathbf{A}_M$  we have  $X \sqsubseteq X$ . Furthermore, if  $X \sqsubseteq Y$  and  $Y \sqsubseteq X$  holds for  $X, Y \in \mathbf{A}_M$ , then  $X = Y$  must hold because all elements of  $\mathcal{F}(M, Y)$  are proper subsets of  $Y$  and therefore  $X \in \mathcal{F}(M, Y)$  and  $Y \in \mathcal{F}(M, X)$  contradict each other. Now let  $X, Y, Z \in \mathbf{A}_M$  such that  $X \sqsubseteq Y$  and  $Y \sqsubseteq Z$ . If  $X = Y$  or  $Y = Z$ , there is nothing to show. Otherwise,  $X \sqsubseteq Y \sqsubseteq Z$  implies  $X, Y \in \mathcal{F}(M)$ . Since  $X \subsetneq Y \subsetneq Z$  we obtain  $X \in \mathcal{F}(M, Z)$ , thus  $X \sqsubseteq Z$ . •

**Lemma 2.5.29.** Let  $M = (E, \mathcal{I})$  be a matroid, and let

$$\nu: 2^E \longrightarrow \mathbb{Z}, X \mapsto |X| - \text{rk}(X).$$

Then

$$\alpha_M = \nu * \mu_{\mathbf{A}}$$

where  $\mu_{\mathbf{A}}$  is the Möbius-function of the  $\alpha_M$ -poset  $(\mathbf{A}, \sqsubseteq)$ .

*Proof.* From the recurrence relation of the  $\alpha$ -invariant (Definition 2.2.10) and the definition of the  $\alpha$ -poset (Definition 2.5.27) we obtain

$$\nu(X) = |X| - \text{rk}(X) = \alpha(X) + \sum_{F \in \mathcal{F}(M, X)} \alpha(F) = \sum_{Y \sqsubseteq X} \alpha(Y)$$

for all  $X \subseteq E$ . The zeta-matrix of  $(\mathbf{A}, \sqsubseteq)$  (Definition 1.1.14) allows us to write

$$\nu = \alpha * \zeta_{\mathbf{A}}.$$

We multiply with the Möbius-function of  $(\mathbf{A}, \sqsubseteq)$  and use Lemma 1.1.15 in order to obtain

$$\nu * \mu_{\mathbf{A}} = \alpha * \zeta_{\mathbf{A}} * \mu_{\mathbf{A}} = \alpha * \text{id}_{\mathbb{Z}}(2^E) = \alpha. \quad \square$$

**Corollary 2.5.30.** Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ , and  $N \in \mathcal{X}(M, e)$  a single element extension of  $M$ . Then

$$\alpha_N|_{2^E} = \alpha_M * \zeta_{\mathbf{A}_M} * (\mu_{\mathbf{A}_N}|_{2^E} \times 2^E).$$

*Proof.* Let  $\nu_N \in \mathbb{Z}^{2^E}$  and  $\nu_M \in \mathbb{Z}^{2^{E \cup \{e\}}}$  be the maps where  $\nu_M(X) = |X| - \text{rk}_M(X)$  and  $\nu_N(X) = |X| - \text{rk}_N(X)$  holds for all  $X \subseteq E$ , or  $X \subseteq E \cup \{e\}$ , respectively. Then  $\nu_N|_{2^E} = \nu_M = \alpha_M * \zeta_{A_M}$  because  $N$  is an extension of  $M$ . Furthermore, for  $X \subseteq E$  and  $Y \subseteq E \cup \{e\}$  with  $e \in Y$ , we have  $Y \not\sqsubseteq_N X$ , and therefore  $\mu_{A_N}(Y, X) = 0$  (\*), thus we may restrict the equation from Lemma 2.5.29 in the following way:

$$\begin{aligned} \alpha_N|_{2^E} &= (\nu_N * \mu_{A_N})|_{2^E} &&= \nu_N * (\mu_{A_N}|_{2^{E \cup \{e\}}} \times 2^E) \\ &\stackrel{(*)}{=} (\nu_N|_{2^E}) * (\mu_{A_N}|_{2^E} \times 2^E) &&= \alpha_M * \zeta_{A_M} * (\mu_{A_N}|_{2^E} \times 2^E). \quad \square \end{aligned}$$

Let us explain the above equations a little further. Here, we interpret  $\alpha_N|_{2^E}$  as a vector in the  $2^{|E|}$ -dimensional  $\mathbb{Z}$ -module  $\mathbb{Z}^{2^E}$ . The term  $\nu_N * \mu_{A_N}$  denotes a vector of the  $\mathbb{Z}$ -module  $\mathbb{Z}^{2^{E \cup \{e\}}}$ , and for all  $X \subseteq E \cup \{e\}$ ,

$$(\nu_N * \mu_{A_N})(X) = \sum_{W \subseteq E \cup \{e\}} \nu_N(W) \cdot \mu_{A_N}(W, X) = \alpha_N(X)$$

by Lemma 2.5.29, therefore the equation also holds for the vector restricted to  $\mathbb{Z}^{2^E}$ . The vector  $\nu_N * (\mu_{A_N}|_{2^{E \cup \{e\}}} \times 2^E)$  on the right arises by first restricting  $\mu_{A_N}$  to  $2^{E \cup \{e\}} \times 2^E$ , effectively dropping all rows  $(\mu_{A_N})_R$  from  $\mu_{A_N}$  where  $e \in R \subseteq E \cup \{e\}$ , and only afterwards calculating the product. For all  $X \subseteq E$ , we still have to compute

$$(\nu_N * (\mu_{A_N}|_{2^{E \cup \{e\}}} \times 2^E))(X) = \sum_{W \subseteq E \cup \{e\}} \nu_N(W) \cdot \mu_{A_N}(W, X).$$

For the next equation, we need the property (\*) that allows us to drop all the summands that belong to  $W \subseteq E \cup \{e\}$  with  $e \in W$  on the left-hand side:

$$\begin{aligned} \sum_{W \subseteq E \cup \{e\}} \nu_N(W) \cdot \mu_{A_N}(W, X) &\stackrel{(*)}{=} \sum_{W \subseteq E} \nu_N(W) \cdot \mu_{A_N}(W, X) \\ &= ((\nu_N|_{2^E}) * (\mu_{A_N}|_{2^E} \times 2^E))(X). \end{aligned}$$

**Lemma 2.5.31.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  a single element extension of  $M$ , and  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut. Further, let  $(A_M, \sqsubseteq_M)$  be the  $\alpha_M$ -poset, and  $(A_N, \sqsubseteq_N)$  be the  $\alpha_N$ -poset. Then for all  $X \subseteq E$  and all  $Y \subseteq E \cup \{e\}$  with  $X \neq Y$*

$$X \sqsubseteq_N Y \iff X \sqsubseteq_M Y \text{ and } X \notin C.$$



*Proof.* Lemma 1.3.8 yields  $\mathcal{F}(N) \cap 2^E = \mathcal{F}(M) \setminus C$  and the statement of this lemma follows from Definition 2.5.27.  $\square$

**Lemma 2.5.32.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  be an extension of  $M$ ,  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut, and let  $\mathbf{A}_M$  and  $\mathbf{A}_N$  denote the  $\alpha_M$ - and  $\alpha_N$ -posets, respectively. Then for all  $X, Y \subseteq E$ , we have*

(i)  $X \sqsubseteq_{\mathbf{A}_N} Y \cup \{e\}$  holds if and only if  $X \sqsubseteq_{\mathbf{A}_M} Y$  and  $X \notin C$ .

Furthermore, if  $Y \sqsubseteq_{\mathbf{A}_N} Y \cup \{e\}$  then  $Y \cup \{e\} \in \mathcal{F}(N)$ .

(ii)  $X \cup \{e\} \sqsubseteq_{\mathbf{A}_N} Y \cup \{e\}$  holds if and only if  $X \sqsubseteq_{\mathbf{A}_M} Y$  and  $X \notin \partial C$  where

$$\partial C = \{F \in \mathcal{F}(M) \setminus C \mid \exists x \in E \setminus F: \text{cl}_M(F \cup \{x\}) \in C\}.$$

*Proof.* This is clear from Lemma 1.3.8 and Definition 2.5.27, too.  $\square$

**Remark 2.5.33.** As we have just seen, the  $\mathbf{A}_N$ -down-sets of subsets of  $E$  are the corresponding down-sets of the  $\alpha_M$ -poset  $\mathbf{A}_M$  where the upper part, that corresponds to the modular cut  $C$  of the single element extension  $N$  of  $M$ , has been cut off. Since the values of the Möbius-function  $\mu_P(X, Y)$  for an arbitrary poset  $P$  only depend on the  $P$ -down-sets of elements of the  $P$ -down-set of  $Y$  (Definition 1.1.14), we see that for  $X \subseteq E$  and  $Y \subseteq E$  with  $C \cap 2^Y \subseteq \{Y\}$  we have  $\mu_{\mathbf{A}_M}(X, Y) = \mu_{\mathbf{A}_N}(X, Y)$  and consequently  $\alpha_M(Y) = \alpha_N(Y)$ .  $\bullet$

**Corollary 2.5.34.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  a single element extension of  $M$ , and  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut. Then for all  $X, Y \subseteq E$*

$$\mu_{\mathbf{A}_N}(X, Y) = \mu_{\mathbf{A}_M}(X, Y) + \sum_{Z \in C, X \subseteq Z \subsetneq Y} \mu_{\mathbf{A}_M}(X, Z).$$

*Proof.* The first equation is a direct consequence of Lemma 2.5.31 and Remark 2.5.33:

$$\begin{aligned} - \sum_{X \subseteq_M Z \subseteq_M Y} \mu_{\mathbf{A}_M}(X, Z) &= - \left( \sum_{X \subseteq_N Z \subseteq_N Y} \mu_{\mathbf{A}_N}(X, Z) \right) \\ &\quad - \left( \sum_{X \subseteq_M Z \subseteq_M Y, Z \in C} \mu_{\mathbf{A}_M}(X, Z) \right) \end{aligned}$$

holds for all  $X, Y \subseteq E$ . Thus we may expand the terms  $\mu_{\mathbf{A}_N}(X, Y)$  and  $\mu_{\mathbf{A}_M}(X, Y)$  with Definition 2.2.10, and then cancel in the above equation.  $\square$

**Definition 2.5.35.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $\mathcal{M}(M)$  be the class of all modular cuts of  $M$ . The  **$\Delta\alpha$ -invariant of  $M$**  shall be defined as

$$\Delta\alpha_M: \mathcal{M}(M) \times 2^E \longrightarrow \mathbb{Z},$$

$$(C, X) \mapsto \sum_{Y \subseteq X} \left( (|Y| - \text{rk}(Y)) \cdot \sum_{Z \in C, Y \subseteq Z \subsetneq X} \mu_{\mathbf{A}}(Y, Z) \right),$$

where  $\mu_{\mathbf{A}}$  denotes the Möbius-function of the  $\alpha_M$ -poset. If the matroid  $M$  is clear from the context, we will denote  $\Delta\alpha_M$  simply by  $\Delta\alpha$ .  $\blacksquare$

**Lemma 2.5.36.** Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  be an extension of  $M$ ,  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut, and  $X \subseteq E$ . Then

$$\alpha_N(X) = \alpha_M(X) + \Delta\alpha_M(C, X).$$

*Proof.* Let  $X \subseteq E$ , and let  $\mathbf{A}_M$  and  $\mathbf{A}_N$  denote the  $\alpha_M$ - and  $\alpha_N$ -posets, respectively. From Corollary 2.5.30 and Lemmas 2.5.29 and 1.1.15 we obtain the equation

$$\alpha_N(X) = \sum_{Y \subseteq X} \left( (|Y| - \text{rk}_M(Y)) \cdot \mu_{\mathbf{A}_N}(Y, X) \right).$$

Corollary 2.5.34 yields

$$\mu_{\mathbf{A}_N}(Y, X) = \mu_{\mathbf{A}_M}(Y, X) + \sum_{Z \in C, Y \subseteq Z \subsetneq X} \mu_{\mathbf{A}_M}(Y, Z)$$

and therefore applying the distributive law of  $\mathbb{Z}$  together with Definition 2.5.35 yields the desired equation

$$\begin{aligned} \alpha_N(X) &= \sum_{Y \subseteq X} \left( (|Y| - \text{rk}_M(Y)) \cdot \left( \mu_{\mathbf{A}_M}(Y, X) + \sum_{Z \in C, Y \subseteq Z \subsetneq X} \mu_{\mathbf{A}_M}(Y, Z) \right) \right) \\ &= \alpha_M(X) + \Delta\alpha_M(C, X). \end{aligned} \quad \square$$

**Lemma 2.5.37.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  be an extension of  $M$ ,  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut, and  $X \subseteq E$  such that  $\text{rk}_M(F \cap X') < \text{rk}_M(F)$  for all  $F \in C$  and all proper subsets  $X' \subsetneq X$ . Then*

$$\alpha_N(X \cup \{e\}) = \begin{cases} 0 & \text{if } X \in \mathcal{F}(M) \text{ and } \text{cl}_M(X) \notin C, \\ \alpha_M(X) & \text{if } X \notin \mathcal{F}(M) \text{ and } \text{cl}_M(X) \notin C, \\ 1 + \alpha_M(X) & \text{if } \text{cl}_M(X) \in C. \end{cases}$$

*Proof.* Let  $(\mathbf{A}_M, \sqsubseteq_M)$  and  $(\mathbf{A}_N, \sqsubseteq_N)$  denote the  $\alpha_M$ - and  $\alpha_N$ -poset, respectively. Let  $W \subseteq X$ , then  $W$  satisfies the premises of this lemma whenever  $X$  satisfies the premises. Furthermore, if for some  $F \in C$  the equality  $\text{rk}_M(F \cap X) = \text{rk}_M(F)$  holds, then  $F = \text{cl}_M(X)$  and conversely, if  $\text{cl}_M(X) \notin C$ , then  $\text{rk}_M(F \cap X) < \text{rk}_M(F)$  for all  $F \in C$ .

Now, we prove the statement for all  $X \in \mathcal{F}(M)$  with  $\text{cl}_M(X) \notin C$  by induction on  $\text{rk}_M(X)$ . Let  $O = \text{cl}_M(\emptyset)$  be the unique rank-0 flat of  $M$ . Then the down-set  $\downarrow_{\mathbf{A}_N}(O \cup \{e\}) = \{O, O \cup \{e\}\}$ . Thus, by Definitions 2.2.10 and 2.5.27, we have

$$\begin{aligned} \alpha_N(O \cup \{e\}) &= |O \cup \{e\}| - \text{rk}_N(O \cup \{e\}) - \alpha_N(O) \\ &= |O| + 1 - 1 - (|O| - \text{rk}_N(O)) = 0. \end{aligned}$$

Now let  $X \in \mathcal{F}(M)$  be a flat with  $\text{rk}_M(X) > 0$ . Lemma 2.5.32 yields that

$$\downarrow_{\mathbf{A}_N}(X \cup \{e\}) = \{F, F \cup \{e\} \mid F \in \downarrow_{\mathbf{A}_M} X\}.$$

Note that  $X \cup \{e\}$  may or may not be a flat in  $N$ , as we have  $X \cup \{e\} \notin \mathcal{F}(N)$  if  $X \in \mathcal{F}(M)$  and  $X$  is covered by a flat from  $C$  — but  $X \cup \{e\}$  is still an element of the above down-set. The assumption, that  $\text{rk}_M(F \cap X) < \text{rk}_M(F)$  for all  $F \in C$ , guarantees that all  $F \in \mathcal{F}(M, X)$  are flats of  $N$ , too. Furthermore, we have

$$\alpha_N(X \cup \{e\}) = |X \cup \{e\}| - \text{rk}_N(X \cup \{e\}) - \sum_{F \sqsubset_N X} \alpha_N(F).$$

Using the induction hypothesis, we obtain

$$\begin{aligned}\alpha_N(X \cup \{e\}) &= |X \cup \{e\}| - \text{rk}_N(X \cup \{e\}) - \left( \sum_{F \sqsubset_N X} \alpha_N(F) \right) - \alpha_N(X) \\ &= |X| - \text{rk}_N(X) - \left( \sum_{F \sqsubset_N X} \alpha_N(F) \right) - \left( |X| - \text{rk}_N(X) - \sum_{F \sqsubset_N X} \alpha_N(F) \right) \\ &= 0.\end{aligned}$$

Now let  $X \subseteq E$  with  $\text{cl}_M(X) \notin C$  and  $X \notin \mathcal{F}(M)$ . Then

$$\downarrow_{A_N}(X \cup \{e\}) = \{F, F \cup \{e\} \mid F \in \downarrow_{A_M} X\} \setminus \{X\}.$$

Analogously to the above calculation we obtain

$$\begin{aligned}\alpha_N(X \cup \{e\}) &= |X \cup \{e\}| - \text{rk}_N(X \cup \{e\}) - \sum_{F \sqsubset_N X} \alpha_N(F) \\ &= \alpha_N(X) = \alpha_M(X),\end{aligned}$$

where the last equation is due to the fact that  $F \not\subseteq X$  holds for all  $F \in C$ , which implies that  $N|X = M|X$  and therefore  $\alpha_N(X) = \alpha_{N|X}(X) = \alpha_{M|X}(X) = \alpha_M(X)$  (Definition 2.2.10).

Now assume that  $\text{cl}_M(X) \in C$ . If  $X \in \mathcal{F}(M)$ , then  $e \in \text{cl}_N(X)$ , thus  $X \notin \mathcal{F}(N)$ . Otherwise  $X \notin \mathcal{F}(M)$  and therefore  $X \notin \mathcal{F}(N)$ , too. In both cases we obtain that

$$\{F \subseteq E \cup \{e\} \mid F \sqsubset_N X \cup \{e\}\} = \{F, F \cup \{e\} \mid F \in \downarrow_{A_M} X\} \setminus \{X, X \cup \{e\}\}.$$

Furthermore, for all  $X' \subsetneq X$  we have  $\text{cl}_M(X') \notin C$ , because  $\text{rk}_M(F \cap X') < \text{rk}_M(F)$  for all  $F \in C$ . This implies that if  $F \cup \{e\} \sqsubset_{A_N} X$  for some  $F \in \mathcal{F}(M)$ , then  $\alpha_N(F \cup \{e\}) = 0$ . Consequently, with Lemma 1.3.8, we obtain

$$\sum_{F \sqsubset_N X \cup \{e\}} \alpha_N(F) = \sum_{F \sqsubset_N X \cup \{e\}, e \notin F} \alpha_N(F) = \sum_{F \sqsubset_N X} \alpha_N(F) = \sum_{F \sqsubset_M X} \alpha_M(F).$$

Since  $e \in \text{cl}_N(X)$ , we have  $\text{rk}_N(X \cup \{e\}) = \text{rk}_N(X)$ . This yields the desired equation

$$\begin{aligned} \alpha_N(X \cup \{e\}) &= |X \cup \{e\}| - \text{rk}_N(X \cup \{e\}) - \sum_{F \sqsubset_N X \cup \{e\}} \alpha_N(F) \\ &= 1 + |X| - \text{rk}_M(X) - \sum_{F \sqsubset_M X} \alpha_M(F) = 1 + \alpha_M(X). \quad \square \end{aligned}$$

In order to determine the values of  $\alpha_N(X)$  of the extension  $N$  of  $M$  by  $e$  when  $e \in X$  and  $e \in \text{cl}_N(X \setminus \{e\})$ , we have to keep track of the flats  $F$  of  $M$  that are proper subsets  $X$  with the additional property that  $e \in \text{cl}_N(F)$ .

**Definition 2.5.38.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $C \in \mathcal{M}(M)$  be a modular cut of  $M$ . We define the **extension poset of  $C$  with respect to  $M$**  as the pair  $(\mathbf{B}_M^C, \sqsubseteq_M^C)$  where  $\mathbf{B}_M^C = 2^E$  and where

$$X \sqsubseteq_M^C Y \iff X = Y \text{ or } (X \in C \text{ and } X \subseteq Y)$$

holds for all  $X, Y \subseteq E$ . If  $M$  is clear from the context, we will denote  $\mathbf{B}_M^C$  by  $\mathbf{B}^C$  and  $\sqsubseteq_M^C$  by  $\sqsubseteq^C$ , too.  $\blacksquare$

**Remark 2.5.39.** Clearly,  $\sqsubseteq_M^C$  is reflexive, the anti-symmetry of  $\mathbf{B}_M^C$  follows from the anti-symmetry of  $\subseteq$ . Let  $X \sqsubset_M^C Y \sqsubset_M^C Z$ . Then  $X, Y \in C$  and  $X \subsetneq Y \subsetneq Z$ . Therefore  $X \sqsubset_M^C Z$  holds, and  $\mathbf{B}_M^C$  is indeed a poset.  $\bullet$

**Definition 2.5.40.** Let  $M = (E, \mathcal{I})$  be a matroid and let  $\mathcal{M}(M)$  be the class of all modular cuts of  $M$ . The  **$\tilde{\Delta}\alpha$ -invariant of  $M$**  shall be defined as

$$\begin{aligned} \tilde{\Delta}\alpha_M: \mathcal{M}(M) \times 2^E &\longrightarrow \mathbb{Z}, \\ (C, X) &\mapsto \begin{cases} -\alpha_M(X) & \text{if } X \in \mathcal{F}(M) \text{ and } \text{cl}_M(X) \notin C, \\ 0 & \text{if } X \notin \mathcal{F}(M) \text{ and } \text{cl}_M(X) \notin C, \\ 1 - \sum_{F \sqsubset^C X} \tilde{\Delta}\alpha_M(C, F) & \text{otherwise,} \end{cases} \end{aligned}$$

where  $(\mathbf{B}^C, \sqsubseteq^C)$  denotes the extension poset of  $C$  with respect to  $M$ . If the matroid  $M$  is clear from the context, we will denote  $\tilde{\Delta}\alpha_M$  simply by  $\tilde{\Delta}\alpha$ .  $\blacksquare$

**Lemma 2.5.41.** Let  $M = (E, \mathcal{I})$  be a matroid,  $e \notin E$ ,  $N \in \mathcal{X}(M, e)$  be an extension of  $M$ ,  $C = \{F \in \mathcal{F}(M) \mid e \in \text{cl}_N(F)\}$  the corresponding modular cut. Then

$$\alpha_N(X \cup \{e\}) = \alpha_M(X) + \tilde{\Delta}\alpha_M(C, X).$$

*Proof.* Let  $X \subseteq E$ . The cases where  $\text{cl}_M(X) \notin C$  are covered by Lemma 2.5.37. Furthermore, if  $X$  is  $\subseteq$ -minimal with the property that  $\text{cl}_M(X) \in C$ , then  $\downarrow_{\mathcal{B}_M^C} X = \{X\}$  and therefore  $\tilde{\Delta}\alpha(C, X) = 1 = \alpha_N(X \cup \{e\}) - \alpha_M(X)$  holds, too, by Lemma 2.5.37. For the general case where  $\text{cl}_M(X) \in C$ , remember that we saw in the proof of Lemma 2.5.29 that the equations

$$|X| - \text{rk}_M(X) = \sum_{F \subseteq_M X} \alpha_M(F)$$

and

$$|X \cup \{e\}| - \text{rk}_N(X \cup \{e\}) = \sum_{F \subseteq_N X \cup \{e\}} \alpha_N(F)$$

hold. Thus we obtain

$$(*) \quad \left( \sum_{F \subseteq_M X} \alpha_M(F) \right) + 1 = \sum_{F \subseteq_N X \cup \{e\}} \alpha_N(F).$$

We prove the missing part of the statement by induction on the length  $k$  of a maximal chain  $C_1 \subsetneq C_2 \subsetneq \dots \subsetneq C_k \subsetneq X$  with  $C_1, \dots, C_k \in C$ . The base case with  $k = 0$  has been established above. Using Lemma 1.3.8 we obtain that  $\downarrow_{\mathcal{A}_N}(X \cup \{e\}) = Q \dot{\cup} R \dot{\cup} S \dot{\cup} T$  where

$$\begin{aligned} Q &= \{Y \mid Y \in \mathcal{F}(M) \setminus C, Y \subseteq X\}, \\ R &= \{Y \cup \{e\} \mid Y \in \mathcal{F}(M) \setminus C, Y \subsetneq X, \forall f \in E \setminus Y: \text{cl}_M(Y \cup \{f\}) \notin C\}, \\ S &= \{Y \cup \{e\} \mid Y \in C, Y \subsetneq X\}, \text{ and} \\ T &= \{X \cup \{e\}\}. \end{aligned}$$

Clearly,  $Q \subseteq \downarrow_{\mathcal{A}_M} X$ , and Lemma 2.5.36 and Definition 2.5.35 yield that

$$\sum_{F \in Q} \alpha_N(F) = \sum_{F \in Q} \alpha_M(F).$$

Lemma 2.5.37 yields that  $\sum_{F \in R} \alpha_N(F) = 0$ . All  $F \in S$  have  $F \setminus \{e\} \in C$  with  $F \setminus \{e\} \subsetneq X$  and therefore those sets  $F \setminus \{e\}$  have shorter maximal descending chains in  $C$  than  $X$ . The induction hypothesis applied to each summand yields that

$$\sum_{F \in S} \alpha_N(F) = \sum_{F \in S} \left( \alpha_M(F \setminus \{e\}) + \tilde{\Delta}\alpha_M(C, F \setminus \{e\}) \right).$$

Furthermore, observe that  $X \notin Q$  because  $\text{cl}_M(X) \in C$  holds, and so we have the equivalence

$$F \sqsubset_M X \iff F \in Q \text{ or } F \cup \{e\} \in S$$

for all  $F \subseteq E$ : Elements  $F$  of the  $\mathbf{A}_M$ -down-set of  $X$  have either  $F \in \mathcal{F}(M) \setminus C$  or  $F \in C$ , thus either  $F \in Q$  or  $F \cup \{e\} \in S$ . Therefore we may cancel the corresponding summands of  $\downarrow_{\mathbf{A}_M} X$  and drop the zero summands from  $R$  in the equation (\*) and obtain

$$\alpha_M(X) + 1 = \alpha_N(X) + \sum_{F \in S} \tilde{\Delta} \alpha_M(C, F \setminus \{e\}).$$

Since all  $F \in S$  have  $e \in F$ , and since

$$\{F \setminus \{e\} \mid F \in S\} = \{F \in C \mid F \subsetneq X\} = \{F \subseteq E \mid F \sqsubset^C X\}$$

we obtain the desired equation

$$\alpha_N(X \cup \{e\}) = \alpha_M(X) + 1 - \sum_{F \sqsubset^C X} \tilde{\Delta} \alpha_M(C, F) = \alpha_M(X) + \tilde{\Delta} \alpha_M(C, X). \quad \square$$

We implemented and tested the performance of determining the  $\alpha_N$ -invariant for single element extensions  $N \in \mathcal{X}(M, e)$  by means of the formulas given in Lemmas 2.5.37 and 2.5.41. For details, please refer to Listing 5.2.

## 2.6 Matroid Tableaux

In this section, we present a general framework for the decision of  $\text{Rec}\Gamma_{\mathcal{M}}$  instances<sup>13</sup> by searching the domain of matroids defined on ground sets with bounded cardinality by the means of tableaux and derivations.

**Definition 2.6.1.** A *matroid tableau* is a tuple  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  where

- (i)  $G$  is a matroid, called the **goal of  $\mathbf{T}$** ,
- (ii)  $\mathcal{G}$  is a family of matroids, called the **gammoids of  $\mathbf{T}$** ,
- (iii)  $\mathcal{M}$  is a family of matroids, called the **intermediates of  $\mathbf{T}$** ,
- (iv)  $\mathcal{X}$  is a family of matroids, called the **excluded matroids of  $\mathbf{T}$** , and where
- (v)  $\simeq$  is an equivalence relation on  $\{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$ , called the **equivalence of  $\mathbf{T}$** . ■

**Definition 2.6.2.** Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a matroid tableau.  $\mathbf{T}$  shall be **valid**,

- (i) if all matroids in  $\mathcal{G}$  are indeed gammoids,
- (ii) if no matroid in  $\mathcal{M}$  is a strict gammoid,
- (iii) if all matroids in  $\mathcal{X}$  are indeed matroids which are not gammoids, and
- (iv) if for every equivalency classes  $[M]_{\simeq}$  of  $\simeq$  we have that either  $[M]_{\simeq}$  is fully contained in the class of gammoids or  $[M]_{\simeq}$  does not contain a gammoid. ■

**Definition 2.6.3.** Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a matroid tableau.  $\mathbf{T}$  shall be **decisive**, if  $\mathbf{T}$  is valid and if either of the following holds:

- (i) There is a matroid  $M \in \mathcal{G}$  such that  $G \simeq M$ .
- (ii) There is a matroid  $X \in \mathcal{X}$  that is isomorphic to a minor of  $G$ .
- (iii) For every extension  $N = (E', \mathcal{I}')$  of  $G = (E, \mathcal{I})$  with

$$|E'| = \text{rk}_G(E)^2 \cdot |E| + \text{rk}_G(E) + |E|$$

there is a matroid  $M \in \mathcal{M}$  that is isomorphic to  $N$ . ■

<sup>13</sup>Remember that in this chapter starting from Section 2.5.2,  $\mathcal{M}$  denotes the class of all matroids.



**Lemma 2.6.4.** *Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a decisive matroid tableau. Then  $G$  is a gammoid if and only if there is a matroid  $M \in \mathcal{G}$  such that  $G \simeq M$ .*

*Proof.* Assume that such an  $M \in \mathcal{G}$  exists. From Definition 2.6.2 we obtain that  $M$  is a gammoid, and that in this case  $G \simeq M$  implies that  $G$  is a gammoid, too. Now assume that no  $M \in \mathcal{G}$  has the property  $G \simeq M$ . Since  $\mathbf{T}$  is decisive, either case (ii) or (iii) of Definition 2.6.3 holds. If case (ii) holds, then  $G$  cannot be a gammoid since it has a non-gammoid minor, but the class of gammoids is closed under minors (Theorem 2.4.1). If case (iii) holds but not case (ii), then no extension of  $G = (E, \mathcal{I})$  with  $k = \text{rk}_G(E)^2 \cdot |E| + \text{rk}_G(E) + |E|$  elements is a strict gammoid. Now assume that  $G$  is a gammoid, then there is a digraph  $D = (V, A)$  with  $|V| \leq k$  vertices, such that  $G = \Gamma(D, T, E)$  for some  $T \subseteq V$  (Remark 2.1.14). Let  $N' = \Gamma(D, T, V) \oplus (\{|V|, |V|+1, \dots, k\}, \{\emptyset\})$ . Clearly,  $N'$  is an extension of  $G$  on a ground set with  $k$  elements, which is also a strict gammoid, a contradiction to the assumption that  $N'$  is isomorphic to some  $N \in \mathcal{M}$ , since  $\mathcal{M}$  is a family which consists of matroids that are not strict gammoids. Therefore we may conclude that in case (iii) the matroid  $G$  is not a gammoid.  $\square$

### 2.6.1 Valid Derivations

A **derivation** is an operation on a finite number of input tableaux and possible additional parameters with constraints that produces an output tableau. Furthermore, a derivation is **valid**, if the output tableau is valid for all sets of valid input tableaux and possible additional parameters that satisfy the constraints. The valid derivations presented here are fairly straight-forward consequences of the concepts presented earlier in this work.

**Definition 2.6.5.** *Let  $\mathbf{T}_i = (G_i, \mathcal{G}_i, \mathcal{M}_i, \mathcal{X}_i, \simeq^{(i)})$  be matroid tableaux for  $i \in \{1, 2, \dots, n\}$ . The **joint tableau** shall be the matroid tableaux*

$$\bigcup_{i=1}^n \mathbf{T}_i = (G_1, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$$

where

$$\mathcal{G} = \bigcup_{i=1}^n \mathcal{G}_i, \quad \mathcal{M} = \bigcup_{i=1}^n \mathcal{M}_i, \quad \mathcal{X} = \bigcup_{i=1}^n \mathcal{X}_i,$$

and where  $\simeq$  is the smallest equivalence relation such that  $M \simeq^{(i)} N$  implies  $M \simeq N$  for all  $i \in \{1, 2, \dots, n\}$ . In other words,  $\simeq$  is the equivalence relation on the family of matroids  $\{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$  which is generated by the relations  $\simeq^{(1)}, \simeq^{(2)}, \dots, \simeq^{(n)}$ .  $\blacksquare$

**Lemma 2.6.6.** *The derivation of the joint tableau is valid.*

*Proof.* Clearly,  $\mathcal{G}$ ,  $\mathcal{M}$ , and  $\mathcal{X}$  inherit their desired properties of Definition 2.6.2 from the valid input tableaux  $\mathbf{T}_i$  where  $i \in \{1, 2, \dots, n\}$ . Now let  $M \simeq N$  with  $M \neq N$ . Then there are matroids  $M_1, M_2, \dots, M_k$  and indexes  $i_0, i_1, \dots, i_k \in \{1, 2, \dots, n\}$  such that there is a chain of  $\simeq^{(i)}$ -relations

$$M \simeq^{(i_0)} M_1 \simeq^{(i_1)} M_2 \simeq^{(i_2)} \dots \simeq^{(i_{k-1})} M_k \simeq^{(i_k)} N.$$

The assumption that the input tableaux are valid yields that  $M$  is a gammoid if and only if  $M_1$  is a gammoid, if and only if  $M_2$  is a gammoid, and so on. Therefore it follows that  $M$  is a gammoid if and only if  $N$  is a gammoid, thus  $\simeq$  has the desired property of Definition 2.6.2. Consequently,  $\bigcup_{i=1}^n \mathbf{T}_i$  is a valid tableau.  $\square$

**Definition 2.6.7.** *Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  and  $\mathbf{T}' = (G, \mathcal{G}', \mathcal{M}', \mathcal{X}', \simeq')$  be matroid tableaux. We say that  $\mathbf{T}$  is a **sub-tableau of  $\mathbf{T}'$**  if  $\mathcal{G} \subseteq \mathcal{G}'$ ,  $\mathcal{M} \subseteq \mathcal{M}'$ , and  $\mathcal{X} \subseteq \mathcal{X}'$  holds, and if  $M \simeq N$  implies  $M \simeq' N$ .  $\blacksquare$*

**Lemma 2.6.8.** *The derivation of a sub-tableau is valid.*

*Proof.* Clearly  $\mathbf{T}$  inherits the properties of Definition 2.6.2 from the validity of  $\mathbf{T}'$ .  $\square$

**Definition 2.6.9.** *Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a matroid tableau. We shall call the tableau  $[\mathbf{T}]_{\simeq} = (G, \mathcal{G}', \mathcal{M}, \mathcal{X}', \simeq)$  **expansion tableau of  $\mathbf{T}$**  whenever*

$$\mathcal{G}' = \bigcup_{M \in \mathcal{G}} [M]_{\simeq} \quad \text{and} \quad \mathcal{X}' = \bigcup_{M \in \mathcal{X}} [M]_{\simeq}. \quad \blacksquare$$

**Lemma 2.6.10.** *The derivation of the expansion tableau is valid.*

*Proof.* If  $M' \in \mathcal{G}'$ , then there is some  $M \in \mathcal{G}$  such that  $M \simeq M'$ . Since we assume  $\mathbf{T}$  to be valid, we may infer that  $M'$  is a gammoid if and only if  $M$  is a gammoid, and the latter is the case since  $M \in \mathcal{G}$ . Therefore  $M'$  is a gammoid. An analogous argument yields that if  $M' \in \mathcal{X}'$ , then  $M'$  is not a gammoid.  $\square$

**Definition 2.6.11.** Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a matroid tableau. We shall call the tableau  $[\mathbf{T}]_{\equiv} = (G, \mathcal{G}', \mathcal{M}', \mathcal{X}', \simeq')$  **extended tableau of  $\mathbf{T}$**  whenever

$$\mathcal{G}' = \mathcal{G} \cup \{M^* \mid M \in \mathcal{G}\}, \quad \mathcal{X}' = \mathcal{X} \cup \{M^* \mid M \in \mathcal{X}\}, \quad \mathcal{M}' = \mathcal{M} \cup \mathcal{X}',$$

and when  $\simeq'$  is the smallest equivalence relation that contains the relations  $\simeq$  and  $\sim$ ; where  $M \sim N$  if and only if  $N$  is isomorphic to  $M$  or  $M^*$ . ■

**Lemma 2.6.12.** The derivation of the extended tableau is valid.

*Proof.* By Theorem 2.4.1 the class of gammoids is closed under duality, therefore a matroid  $M$  is a gammoid if and only if  $M^*$  is a gammoid. So  $\mathcal{G}'$  and  $\mathcal{X}'$  inherit their desired properties of Definition 2.6.2 from the validity of  $\mathbf{T}$ . If  $M \in \mathcal{M}' \setminus \mathcal{M}$ , then  $M \in \mathcal{X}'$ , therefore  $M$  cannot be a strict gammoid. □

**Definition 2.6.13.** Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a decisive matroid tableau. The tableau  $\mathbf{T}! = (G, \mathcal{G}', \mathcal{M}, \mathcal{X}', \simeq)$  shall be the **conclusion tableau for  $\mathbf{T}$**  if either

- (i)  $\mathcal{G}' = \mathcal{G} \cup \{G' \mid G' \text{ is a minor of } G\}$ ,  $\mathcal{X}' = \mathcal{X}$ , and the tableau  $\mathbf{T}$  satisfies case (i) of Definition 2.6.3; or
- (ii)  $\mathcal{G}' = \mathcal{G}$ ,  $\mathcal{X}' = \mathcal{X} \cup \{G\}$ , and  $\mathbf{T}$  satisfies case (ii) or (iii) of Definition 2.6.3. ■

**Corollary 2.6.14.** The derivation of the conclusion tableau is valid.

*Proof.* Easy consequence of Lemma 2.6.4. □

**Definition 2.6.15.** Let  $\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  be a matroid tableau, let  $M_1 = (E_1, \mathcal{I}_1)$  and  $M_2 = (E_2, \mathcal{I}_2)$  be matroids of the tableau, i.e.

$$\{M_1, M_2\} \subseteq \{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}.$$

Furthermore, let  $E'_1$  and  $E'_2$  be finite sets,  $D_1 = (V_1, A_1)$  and  $D_2 = (V_2, A_2)$  be digraphs such that  $E_1 \cup E'_2 \subseteq V_1$  and  $E'_1 \cup E_2 \subseteq V_2$ , and such that the induced matroid  $I(D_1, M_1, E'_2)$  is isomorphic to  $M_2$  and the induced matroid  $I(D_2, M_2, E'_1)$  is isomorphic to  $M_1$ . The tableau

$$\mathbf{T}(M_1 \simeq M_2) = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq')$$

is called **identified tableau for  $\mathbf{T}$  with respect to  $M_1$  and  $M_2$**  if the relation  $\simeq'$  is the smallest equivalence relation, such that  $M_1 \simeq' M_2$  holds, and such that  $M' \simeq N'$  implies  $M' \simeq N'$ . ■

**Lemma 2.6.16.** *The derivation of an identified tableau is valid.*

*Proof.* From Lemma 2.4.9 we obtain that  $M'_2 = I(D_1, M_1, E'_2)$  is a gammoid if  $M_1$  is a gammoid, and that  $M'_1 = I(D_2, M_2, E'_1)$  is a gammoid if  $M_2$  is a gammoid. Therefore  $M_1$  is a gammoid if and only if  $M_2$  is a gammoid. Consequently,  $\simeq'$  satisfies the properties of Definition 2.6.2, and thus the identified tableau  $\mathbf{T}(M_1 \simeq M_2)$  is valid for every valid input tableau  $\mathbf{T}$ .  $\square$

## 2.6.2 Valid Tableaux

In this section we present a variety of valid tableaux which may be used as inputs for valid derivation operations. Trivially, if  $M$  is a gammoid, then  $(M, \{M\}, \emptyset, \emptyset, \langle \rangle)$  is a valid tableau, and if  $M$  is not a gammoid, then  $(M, \emptyset, \emptyset, \{M\}, \langle \rangle)$  is a valid tableau; where  $\langle \cdot \rangle$  denotes the generated equivalence relation defined on the set of matroids occurring in the respective tableau. Thus exactly one of these two tableaux is valid. Unfortunately, in order to know which one is valid, we have to decide whether  $M$  is a gammoid first — in general this is not easier than determining  $\Gamma_{\mathcal{M}}(M)$ , but there are special cases which we should not ignore.

**Corollary 2.6.17.** *Let  $M = (E, \mathcal{I})$  be a matroid with  $\alpha_M \geq 0$ . Then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \{M, M^*\}$ ,  $\mathcal{M} = \emptyset$ ,  $\mathcal{X} = \emptyset$ , and  $M \simeq N \Leftrightarrow M = N$ .*

*Proof.* See Corollary 2.2.21.  $\square$

**Corollary 2.6.18.** *Let  $M = (E, \mathcal{I})$  be a matroid with  $\text{rk}_M(X) = 3$ ,  $X \subseteq E$  with  $\alpha_M(X) < 0$ . Then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \emptyset$ ,  $\mathcal{M} = \emptyset$ ,  $\mathcal{X} = \{M, M^*\}$ , and  $M \simeq N \Leftrightarrow M = N$ .*

*Proof.* See Corollary 2.2.21 and Proposition 2.5.11.  $\square$

**Remark 2.6.19.** Let  $M = (E, \mathcal{I})$  be a matroid,  $X \subseteq E$  with  $\alpha_M(X) < 0$ . Then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \emptyset$ ,  $\mathcal{M} = \{M\}$ ,  $\mathcal{X} = \emptyset$ , and  $M \simeq N \Leftrightarrow M = N$ .  $\bullet$

**Theorem 2.6.20** ([Ing77], Theorem 13; [Bry71], [Bry75], [Ing71a]). *Let  $\mathbb{F}_2$  be the two-elementary field,  $E, C$  finite sets, and let  $\mu \in \mathbb{F}_2^{E \times C}$  be a matrix. Then  $M(\mu)$  is a gammoid if and only if there is no minor  $N$  of  $M(\mu)$  which is isomorphic to  $M(K_4)$ . The latter is the case if and only if  $M(\mu)$  is isomorphic to the polygon matroid of a series-parallel network.*

For proofs of a sufficient set of implications which establish the equivalency stated, refer to [Bry71], [Bry75], and [Ing71a].

**Theorem 2.6.21** ([Oxl11], Theorem 6.5.4). *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  is isomorphic to  $M(\mu)$  for some matrix  $\mu \in \mathbb{F}_2^{E \times C}$  if and only if  $M$  has no minor isomorphic to the uniform matroid  $U_{2,4} = (E', \mathcal{I}')$ , where  $E' = \{a, b, c, d\}_{\neq}$  and  $\mathcal{I}' = \{X \subseteq E' \mid |X| \leq 2\}$ .*

See [Oxl11], pp.193f, for a proof.

**Corollary 2.6.22.** *Let  $M = (E, \mathcal{I})$  be a matroid. If  $M$  has no minor isomorphic to  $M(K_4)$  and no minor isomorphic to  $U_{2,4}$ , then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \{M, M^*\}$ ,  $\mathcal{M} = \emptyset$ ,  $\mathcal{X} = \emptyset$ , and  $M \simeq N \Leftrightarrow M = N$ .*

*Proof.* Direct consequence of Theorems 2.6.20 and 2.6.21. □

**Definition 2.6.23.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  shall be **strongly base-orderable**, if for every pair of bases  $B_1, B_2 \in \mathcal{B}(M)$  there is a bijection  $\varphi: B_1 \rightarrow B_2$  such that*

$$(B_1 \setminus X) \cup \varphi[X] \in \mathcal{B}(M)$$

*holds for all  $X \subseteq B_1$ . This property is also referred to as **full exchange property**. ■*

**Lemma 2.6.24** ([Mas72], Corollary 4.1.4). *Let  $M = (E, \mathcal{I})$  be a gammoid. Then  $M$  is strongly base-orderable.*

*Proof.* Let  $B_1, B_2 \in \mathcal{B}(M)$  be any two bases of  $M$ , and let  $(D, B_1, E)$  be a representation of  $M$  (Theorem 2.1.10). Since  $B_2 \in \mathcal{I}$  and  $|B_1| = |B_2|$ , there is a linking  $R: B_2 \rightrightarrows B_1$  in  $D$ . Let  $\varphi: B_1 \rightarrow B_2$  be the unique bijection with the property that  $p_1 = \varphi(p_{-1})$  for all  $p \in R$ . Let  $X \subseteq B_1$ , then the derived linking

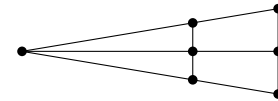
$$R_X = \{p \in R \mid p_1 \in \varphi[X]\} \cup \{b \in B_1 \mid b \notin X\}$$

proves that  $(B_1 \setminus X) \cup \varphi[X] \in \mathcal{B}(M)$ . Thus  $M$  is strongly base-orderable. □

**Corollary 2.6.25.** *Let  $M = (E, \mathcal{I})$  be a matroid,  $B_1, B_2 \in \mathcal{B}(M)$  be bases of  $M$  such that for every bijection  $\varphi: B_1 \setminus B_2 \rightarrow B_2 \setminus B_1$  there is a set  $X \subseteq B_1 \setminus B_2$  with the property  $(B_1 \setminus X) \cup \varphi[X] \notin \mathcal{B}(M)$ . Then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \emptyset$ ,  $\mathcal{M} = \emptyset$ ,  $\mathcal{X} = \{M, M^*\}$ , and  $M \simeq N \Leftrightarrow M = N$ .*

*Proof.* Direct consequence of the proof of Lemma 2.6.24. □

**Example 2.6.26.** Consider the matroid  $P_7$  ([Oxl11], p.644), its affine configuration is depicted on the right. It is strongly base-orderable but it is not a strict gammoid.



Since  $P_7$  has rank 3, it follows that  $P_7$  is a strongly base-orderable non-gammoid (Proposition 2.5.11). ▲

**Example 2.6.27.** The Vámos matroid ([Oxl11], p.649) is strongly base-orderable, but not representable over the reals  $\mathbb{R}$ . ▲

**Theorem 2.6.28** ([Ing71b], [MNW09]). *Let  $M = (E, \mathcal{I})$  be a matroid such that there is a field  $\mathbb{F}$  and a matrix  $\mu \in \mathbb{F}^{E \times C}$  with  $M = M(\mu)$ . Let further  $W, X, Y, Z \subseteq E$ . Then*

$$\begin{aligned} & \text{rk}(W) + \text{rk}(X) + \text{rk}(W \cup X \cup Y) + \text{rk}(W \cup X \cup Z) + \text{rk}(Y \cup Z) \\ & \leq \text{rk}(W \cup X) + \text{rk}(W \cup Y) + \text{rk}(W \cup Z) + \text{rk}(X \cup Y) + \text{rk}(X \cup Z). \end{aligned}$$

For a proof, see [Ing71b]. We mention A.W. Ingleton’s theorem here because it has been used by D. Mayhew in [May16] in order to prove that certain matroids are excluded minors of the class of gammoids. P. Nelson and J. van der Pol [NvdP17] showed that A.W. Ingleton’s necessary condition for representability over any field is rather weak: It is quite improbable for matroids on large ground sets that a matroid which satisfies this condition is indeed representable over any field, because there are double-exponentially many matroids satisfying the condition with respect to the cardinality of the ground set, yet there are only exponentially many representable matroids with respect to the cardinality of the ground set. Furthermore, L. Guillé, T. Chan, and A. Grant found a unique minimal subset of  $\frac{6^n}{4} - O(5^n)$  inequalities that imply A.W. Ingleton’s condition if satisfied [GCG11].

**Corollary 2.6.29.** *Let  $M = (E, \mathcal{I})$  be a matroid, and let  $W, X, Y, Z \subseteq E$  such that*

$$\begin{aligned} & \text{rk}(W) + \text{rk}(X) + \text{rk}(W \cup X \cup Y) + \text{rk}(W \cup X \cup Z) + \text{rk}(Y \cup Z) \\ & > \text{rk}(W \cup X) + \text{rk}(W \cup Y) + \text{rk}(W \cup Z) + \text{rk}(X \cup Y) + \text{rk}(X \cup Z). \end{aligned}$$

*Then the matroid tableau  $\mathbf{T}$  is valid, where  $\mathbf{T} = (M, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$  with  $\mathcal{G} = \emptyset$ ,  $\mathcal{M} = \emptyset$ ,  $\mathcal{X} = \{M, M^*\}$ , and  $M \simeq N \Leftrightarrow M = N$ .*

*Proof.* Consequence of Theorems 2.6.28 and 2.7.13. □

This strict inequality dualizes to

$$\begin{aligned} & \nu(W') + \nu(X') + \nu(W' \cap X' \cap Y') + \nu(W' \cap Y' \cap Z') + \nu(Y' \cap Z') \\ & < \nu(W' \cap X') + \nu(W' \cap Y') + \nu(W' \cap Z') + \nu(X' \cap Y') + \nu(X' \cap Z') \end{aligned}$$

where  $\nu(X) = |X| - \text{rk}(X)$ . A. Cameron showed that  $M$  satisfies A.W. Ingleton's condition if and only if  $M^*$  satisfies it ([Cam14], Lemma 4.5, p.26), therefore the dualized inequality does not provide any valid matroid tableaux that cannot be derived using Corollary 2.6.29.

### 2.6.3 Derivation of a Decisive Tableau

First of all, it is clear that we may derive a decisive matroid tableau for any given matroid  $G = (E, \mathcal{I})$  by simply determining all extensions of  $G$  with  $\text{rk}_G(E)^2 \cdot |E| + \text{rk}_G(E) + |E|$  elements (Remark 2.1.14). For each such extension  $N$ , there is a valid tableau, which depends on whether  $N$  is a strict gammoid (Corollary 2.6.17) or not (Remark 2.6.19). Thus we may derive the joint tableau of all valid tableaux of the extensions of  $G$ . It is clear from Definition 2.6.3 that this joint tableau is decisive: either case (i) or case (iii) holds. Thus we may always decide  $\Gamma_{\mathcal{M}}(G)$  using the matroid tableau method. Unfortunately, we cannot guarantee that there is no excluded minor  $X$  for the class of gammoids, where the only feasible way to refute, that  $X$  is a gammoid, requires to employ the tiresome case (iii). Now, let us provide a glimpse of the art of employing matroid tableaux.

**Example 2.6.30.** Consider the matroid  $G = G_{8,4,1} = (E, \mathcal{I})$  where  $E = \{1, 2, \dots, 8\}$  and where  $\mathcal{I} = \{X \subseteq E \mid |X| \leq 4, X \notin \mathcal{H}\}$  with

$$\mathcal{H} = \{\{1, 3, 7, 8\}, \{1, 5, 6, 8\}, \{2, 3, 6, 8\}, \{4, 5, 6, 7\}, \{2, 4, 7, 8\}\}.$$

Clearly,  $\alpha_G(H) = 1$  for all  $H \in \mathcal{H}$ , and consequently  $\alpha_G(E) = 4 - 5 = -1$ . The dual matroid  $G^* = (E, \mathcal{I}^*)$  has a similar structure:  $\mathcal{I}^* = \{X \subseteq E \mid |X| \leq 4, X \notin \mathcal{H}^*\}$  with

$$\mathcal{H}^* = \{\{1, 2, 3, 8\}, \{1, 3, 5, 6\}, \{1, 4, 5, 7\}, \{2, 3, 4, 7\}, \{2, 4, 5, 6\}\}.$$

Thus  $\alpha_{G^*}(H') = 1$  for all  $H' \in \mathcal{H}^*$ , and so  $\alpha_{G^*}(E) = 4 - 5 = -1$ , too. It turns out that neither  $G$  nor  $G^*$  have any minors of rank 3 which are not strict gammoids. Furthermore, both  $G$  and  $G^*$  are strongly base-orderable, and both  $G$  and  $G^*$  have a  $U_{2,4}$  minor. For the rest of this example, we will refer to ‘single-element extensions of the same rank’ simply by the word ‘extension’. There are 11962 different isomorphism classes of extensions of  $G$ , and 11495 different isomorphism classes of extensions of  $G^*$ . No extension of  $G$  or  $G^*$  is a strict gammoid or a transversal matroid. 8643 isomorphism classes of  $G$ -extensions either have non-gammoid rank-3 minors, or they are not strongly base-orderable, the same holds for 7892 isomorphism classes of  $G^*$ -extensions. This leaves 3319 classes of  $G$ -extensions and 3603 classes of  $G^*$ -extensions which may or may not be classes of gammoids — so extending and backtracking may not be our best approach here.

We have seen before that there is no easy way to decide whether  $G$  or  $G^*$  is a gammoid, therefore we start with the valid tableaux

$$\mathbf{T}_G = (G, \emptyset, \{G\}, \emptyset, \langle \cdot \rangle) \text{ and } \mathbf{T}_{G^*} = (G^*, \emptyset, \{G^*\}, \emptyset, \langle \cdot \rangle),$$

where  $\langle \cdot \rangle$  denotes the generated equivalence relation defined on the set of matroids occurring in the respective tableau. We may derive the extended joint tableau

$$\mathbf{T}_1 = [\mathbf{T}_G \cup \mathbf{T}_{G^*}]_{\equiv} = (G, \emptyset, \{G, G^*\}, \langle G \simeq G^* \rangle).$$

Now observe that although  $G$  is deflated,  $G^*$  is not deflated. We have

$$\begin{aligned} C_8^* &= \{F \in \mathcal{F}(G^* | \{1, 2, \dots, 7\}) \mid 8 \in \text{cl}_{G^*}(F)\} \\ &= \{F \in \mathcal{F}(G^* | \{1, 2, \dots, 7\}) \mid \{1, 2, 3\} \subseteq F\}. \end{aligned}$$

Let  $G_7^* = G^* | \{1, 2, \dots, 7\}$ . We have  $\alpha_{G_7^*}(\{1, 2, \dots, 7\}) = -1$ , thus  $G_7^*$  is not a strict gammoid, and thus

$$\mathbf{T}_{G_7^*} = (G_7^*, \emptyset, \{G_7^*\}, \emptyset, \langle \cdot \rangle)$$



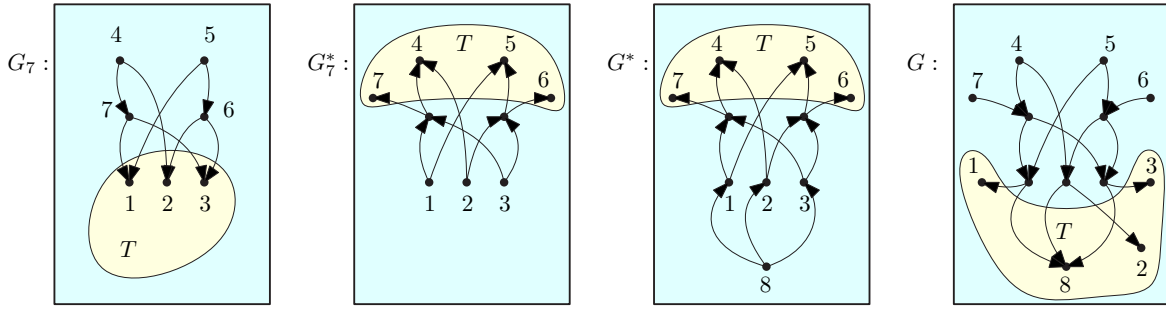


Fig. 2.4 Reconstruction of a representation of  $G_{8,4,1}$  from the matroid tableaux in Example 2.6.30.

is a valid tableau. Since  $G_7^*$  is a deflate of  $G^*$ , each of them is an induced matroid with respect to the other. Therefore we may identify  $G^*$  and  $G_7^*$  in the joint tableau

$$\mathbf{T}_2 = (\mathbf{T}_1 \cup \mathbf{T}_{G_7^*}) (G^* \simeq G_7^*) = (G, \emptyset, \{G, G^*, G_7^*\}, \emptyset, \langle G \simeq G^* \simeq G_7^* \rangle).$$

Now let  $G_7 = (G_7^*)^*$ , and we have  $\alpha_{G_7} \geq 0$ . Thus

$$\mathbf{T}_{G_7} = (G_7, \{G_7\}, \emptyset, \emptyset, \langle \rangle)$$

is a valid tableau. We now may derive the decisive tableau

$$\mathbf{T}_3 = [\mathbf{T}_2 \cup \mathbf{T}_{G_7}]_{\equiv} = (G, \{G_7\}, \{G, G^*, G_7^*, G_7\}, \emptyset, \langle G \simeq G^* \simeq G_7^* \simeq G_7 \rangle)$$

where case (i) of Definition 2.6.3 holds. Consequently,  $G$  is a gammoid. ▲

The representation of  $G_{8,4,1}$  given in Figure 2.4 can obviously be reduced by two vertices if we move both the vertices 6 and 7 one step along their only incident arcs and delete the now superfluous sources. So  $G_{8,4,1}$  may be represented with 11 vertices, and it is still possible that there is a representation of  $G_{8,4,1}$  with 10 vertices. Clearly, 9 vertices do not suffice since no single-element extension of  $G_{8,4,1}$  is a strict gammoid.

Based on our experience, let us provide our best procedure for determining whether a given matroid  $G = (E, \mathcal{I})$  is a gammoid. We start the procedure with the valid initial tableau  $\mathbf{T} := (G, \emptyset, \emptyset, \emptyset, \langle \rangle)$ .

**Step 1.** If  $\mathbf{T}$  is decisive, stop.

**Step 2.** Choose an intermediate goal  $M \in (\{G' \mid G' \text{ is a minor of } G\} \cup \mathcal{M}) \setminus (G \cup \mathcal{X})$ , preferably one with  $M \simeq G$  which is small both in rank and cardinality.

**Step 3.** If  $\mathbf{T}_M = (M, \emptyset, \emptyset, \emptyset, \langle \rangle) \cup \mathbf{T}$  is decisive, then set  $\mathbf{T} := \left[ [\mathbf{T} \cup (\mathbf{T}_M!) ]_{\equiv} \right]_{\simeq}$  and continue with Step 1.

**Step 4.** Determine whether  $M$  has a minor that is isomorphic to  $M(K_4)$ . If this is the case, then  $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_M]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

Since  $M(K_4) = (M(K_4))^*$ , we have that  $M(K_4)$  is neither a minor of  $M$  nor of  $M^*$  when reaching the next step.

**Step 5.** Determine whether  $M$  has a minor that is isomorphic to  $U_{2,4}$ . If this is not the case, then  $\mathbf{T}_M = (M, \{M, M^*\}, \emptyset, \emptyset, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_M]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

Since  $U_{2,4} = (U_{2,4})^*$ , we have that  $U_{2,4}$  is neither a minor of  $M$  nor of  $M^*$  when reaching the next step.

**Step 6.** If  $M \in \mathcal{M}$ , continue immediately with Step 7. Determine whether  $\alpha_M \geq 0$ . If this is the case, then  $\mathbf{T}_M = (M, \{M, M^*\}, \emptyset, \emptyset, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_M]_{\equiv} \right]_{\simeq}$  and continue with Step 1.

**Step 7.** If  $M^* \in \mathcal{M}$ , continue immediately with Step 8. Determine whether  $\alpha_{M^*} \geq 0$ . If this is the case, then  $\mathbf{T}_{M^*} = (M^*, \{M, M^*\}, \emptyset, \emptyset, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_{M^*}]_{\equiv} \right]_{\simeq}$  and continue with Step 1.

**Step 8.** Determine whether  $M$  is strongly base-orderable. If this is not the case, then  $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_M]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

The class of strong base-orderable matroids is closed under duality and minors [Ing71a], therefore  $M^*$  and all minors of  $M$  and  $M^*$  are strongly base-orderable upon reaching the next step.

**Step 9.** Let  $M = (E, \mathcal{I})$ . Determine whether there is some  $X \in \mathcal{I}$  with  $|X| = \text{rk}_M(E) - 3$  and some  $Y \subseteq E \setminus X$  such that  $\alpha_{M.(E \setminus X)}(Y) < 0$ . If this is the case, then the tableau  $\mathbf{T}_M = (M, \emptyset, \emptyset, \{M, M^*\}, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_M]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

**Step 10.** Let  $M^* = (E, \mathcal{I}^*)$ . Determine whether there is some  $X \in \mathcal{I}^*$  with  $|X| = \text{rk}_{M^*}(E) - 3$  and some  $Y \subseteq E \setminus X$  such that  $\alpha_{M^*.(E \setminus X)}(Y) < 0$ . If this is the case, then the tableau  $\mathbf{T}_{M^*} = (M^*, \emptyset, \emptyset, \{M, M^*\}, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_{M^*}]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

The next step may be omitted or carried out sloppily<sup>14</sup>, because it may take a considerable amount of time for larger matroids and it does not seem to be worth the computational effort in practice.

**Step 11.** Let  $M = (E, \mathcal{I})$ . Determine whether there are  $W, X, Y, Z \in \mathcal{I}$  such that

$$\begin{aligned} & \text{rk}(W) + \text{rk}(X) + \text{rk}(W \cup X \cup Y) + \text{rk}(W \cup X \cup Z) + \text{rk}(Y \cup Z) \\ & > \text{rk}(W \cup X) + \text{rk}(W \cup Y) + \text{rk}(W \cup Z) + \text{rk}(X \cup Y) + \text{rk}(X \cup Z). \end{aligned}$$

If this is the case, then the tableau  $\mathbf{T}_{M^*} = (M^*, \emptyset, \emptyset, \{M, M^*\}, \langle \rangle)$  is valid, we set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_{M^*}]_{\equiv} \right]_{\simeq}$  and then continue with Step 1.

**Step 12.** Determine whether  $M$  is deflated. If not, then find a deflate  $N$  of  $M$  with a ground set of minimal cardinality, set  $\mathbf{T} := \left[ [(\mathbf{T} \cup \mathbf{T}_N)(M \simeq N)]_{\equiv} \right]_{\simeq}$  where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle \rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle \rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1.

**Step 13.** Determine whether  $M^*$  is deflated. If not, then find a deflate  $N$  of  $M^*$  with a ground set of minimal cardinality, set  $\mathbf{T} := \left[ [(\mathbf{T} \cup \mathbf{T}_N)(M^* \simeq N)]_{\equiv} \right]_{\simeq}$  where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle \rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle \rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1.

This is the point where we may try creative ways of determining whether  $M$  is a gammoid or not. If we are successful, then we augment the tableau  $\mathbf{T}$  accordingly, and continue with Step 1. In the best case, we might guess a representation of  $M$ , or find a considerably smaller matroid  $M'$  such that  $M$  is induced from  $M'$  by some digraph  $D$ . In theory, it is also possible to find a known non-gammoid  $X'$  that is induced from  $M$  by some digraph  $D$ , which then implies that  $M$  must be a non-gammoid. Since the class of strongly base-orderable matroids is closed under matroid induction by digraphs,  $X' \notin \{M(K_4), P_7\}$ , because we know since Step 8 that  $M$  is strongly base-orderable. In practice, we never managed to successfully show that some known excluded minor

<sup>14</sup>It clearly would be sloppy to just consider  $W, X, Y$ , and  $Z$  with  $\max\{|W|, |X|, |Y|, |Z|\} \leq k$  for some  $k \in \mathbb{Z} \setminus \{0, 1\}$ , or even to just check whether  $M$  has a Vámos-matroid as a minor.

may be induced from a candidate matroid  $M$  under examination. Currently,  $P_8^-$  ([Oxl11], p.651) is the only excluded minor for the class of gammoids (J. Bonin, [Bon]) that we know of, which is strongly base-orderable yet neither has rank or co-rank 3. Furthermore,  $P_8^-$  is isomorphic to its dual  $(P_8^-)^*$  which makes it a rather special matroid. Therefore we think it is reasonable to assume that the odds are clearly in favor of  $M$  being a gammoid upon reaching the next step.<sup>15</sup>

**Step 14.** Try to find an extension  $N$  of  $M$  with at most  $\text{rk}_G(E)^2 \cdot |E| + \text{rk}_G(E) + |E|$  elements such that  $N$  is not isomorphic to any  $M' \in \mathcal{G} \cup \mathcal{M} \cup \mathcal{X}$ . Set  $\mathbf{T} := \left[ [\mathbf{T} \cup \mathbf{T}_N]_{\equiv} \right]_{\simeq}$  where

$$\mathbf{T}_N = \begin{cases} (N, \{N, N^*\}, \emptyset, \emptyset, \langle \rangle) & \text{if } \alpha_N \geq 0, \\ (N, \emptyset, \{N\}, \emptyset, \langle \rangle) & \text{otherwise,} \end{cases}$$

and continue with Step 1. If no such extension of  $M$  exists, then set  $M := G$  and continue with Step 5.

Clearly, if we continue this process long enough, then Step 14 ensures that the tableau  $\mathbf{T}$  will eventually become decisive for  $G$  by exhausting all isomorphism classes of extensions of  $G$  with at most  $\text{rk}_G(E)^2 \cdot |E| + \text{rk}_G(E) + |E|$  elements.

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<sup>15</sup> Or, more pessimistically, we might not know sufficiently general excluded minors for the class of gammoids to assess the situation here more realistically.

## 2.7 Representation over $\mathbb{R}$

There are many ways to arrive at the fact that every gammoid can be represented by a matrix over a field  $\mathbb{K}$  whenever  $\mathbb{K}$  has enough elements. Or, to be more precise, for every field  $\mathbb{F}$  and every gammoid  $M$  there is an extension field  $\mathbb{K}$  of  $\mathbb{F}$ , such that  $M$  can be represented by a matrix over  $\mathbb{K}$ . For the sake of simplicity, we only consider representations of gammoids over the field of the reals  $\mathbb{R}$ . In [Ard06], F. Ardila points out that the Lindström Lemma yields an easy method to construct a matrix  $\mu \in \mathbb{R}^{E \times B}$  from the digraph  $D = (V, A)$  such that  $\Gamma(D, T, E) = M(\mu)$ ; the construction is universal in the sense that it works with indeterminates and thus yields a representation over  $\mathbb{F}$  whenever these indeterminates can be replaced with elements from  $\mathbb{F}$  without zeroing out any nonzero subdeterminants of  $\mu$ .

**Definition 2.7.1.** *Let  $D = (V, A)$  be a digraph and  $w: A \rightarrow \mathbb{R}$ . Then  $w$  shall be called **indeterminate weighting of  $D$** , whenever the set  $\{w(a) \mid a \in A\}$  is  $\mathbb{Z}$ -independent.* ■

**Example 2.7.2.** Let  $D = (V, A)$  be any digraph, then  $|A| < \infty$ . Thus there is a set  $X \subseteq \mathbb{R}$  that is  $\mathbb{Z}$ -independent with  $|X| = |A|$  (Lemma 1.1.11). Then any bijection  $\sigma: A \rightarrow X$  induces an indeterminate weighting  $w: X \rightarrow \mathbb{R}$  with  $w(x) = \sigma(x)$ , thus indeterminate weightings exist for all digraphs. ▲

**Notation 2.7.3.** *Let  $D = (V, A)$  be a digraph and  $w: A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$ . Let  $q = (q_i)_{i=1}^n \in \mathbf{W}(D)$ , we shall write*

$$\prod q = \prod_{i=1}^{n-1} w((q_i, q_{i+1})).$$
 ■

**Lemma 2.7.4** (Lindström [Lin73]). *Let  $D = (V, A)$  be an acyclic digraph,  $n \in \mathbb{N}$  a natural number,  $S = \{s_1, s_2, \dots, s_n\}_{\neq} \subseteq V$  and  $T = \{t_1, t_2, \dots, t_n\}_{\neq} \subseteq V$  be equicardinal subsets of  $V$ , and let  $w: A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$ . Furthermore,  $\mu \in \mathbb{R}^{V \times V}$  shall be the matrix with*

$$\mu(u, v) = \sum_{p \in \mathbf{P}(D; u, v)} \prod p.$$

Then

$$\det(\mu|_{S \times T}) = \sum_{L: S \xrightarrow{\neq} T} \left( \operatorname{sgn}(L) \prod_{p \in L} (\prod p) \right)$$

where  $\text{sgn}(L) = \text{sgn}(\sigma)$  for the unique permutation  $\sigma \in \mathfrak{S}_n$  with the property that for every  $i \in \{1, 2, \dots, n\}$  there is a path  $p \in L$  with  $p_1 = s_i$  and  $p_{-1} = t_{\sigma(i)}$ . Furthermore,

$$\det(\mu|S \times T) = 0$$

if and only if there is no linking from  $S$  to  $T$  in  $D$ .

As suggested by F. Ardila, we present the following bijective proof given by I.M. Gessel and X.G. Viennot [GV89].

*Proof.* The Leibniz formula (Definition 1.1.7) yields

$$\begin{aligned} \det(\mu|S \times T) &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n \mu(s_i, t_{\sigma(i)}) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n \left( \sum_{p \in \mathbf{P}(D; s_i, t_{\sigma(i)})} \prod p \right) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \left( \sum_{K \in Q_\sigma} \prod_{p \in K} (\prod p) \right), \end{aligned}$$

where

$$Q_\sigma = \left\{ K \in \binom{\mathbf{P}(D)}{n} \mid \forall i \in \{1, 2, \dots, n\}: \exists p \in K: p_1 = s_i \text{ and } p_{-1} = t_{\sigma(i)} \right\}$$

consists of all families of paths connecting  $s_i$  with  $t_{\sigma(i)}$  for all  $i \in \{1, 2, \dots, n\}$ . Clearly, for  $\sigma, \tau \in \mathfrak{S}_n$  with  $\sigma \neq \tau$ , the sets  $Q_\sigma \cap Q_\tau = \emptyset$  are disjoint, therefore the following map with the domain  $Q = \bigcup_{\sigma \in \mathfrak{S}_n} Q_\sigma$  is well defined:

$$\text{sgn}: Q \longrightarrow \{-1, 1\}, \quad K \mapsto \text{sgn}(\sigma) \quad \text{where } \sigma \in \mathfrak{S}_n \text{ such that } K \in Q_\sigma.$$

Thus we may write

$$\det(\mu|S \times T) = \sum_{K \in Q} \text{sgn}(K) \prod_{p \in K} (\prod p).$$

Furthermore, if  $L: S \rightrightarrows T$  is a linking from  $S$  to  $T$  in  $D$ , then  $L \in Q_\sigma$  where  $\sigma \in \mathfrak{S}_n$  is the unique permutation mapping the indexes of the initial vertices of the paths in  $L$  to the indexes of the terminal vertices of the paths in  $L$ . Let us denote the routings in  $Q$  by

$$R = \{L \in Q \mid L \text{ is a routing}\}.$$

We prove the first statement of the lemma by showing that there is a bijection  $\varphi: Q \setminus R \rightarrow Q \setminus R$ , such that for all  $K \in Q \setminus R$ ,

$$\prod_{p \in K} (\prod p) = \prod_{p \in \varphi(K)} (\prod p)$$

and  $\text{sgn}(K) = -\text{sgn}(\varphi(K))$ . We construct such a map  $\varphi$  now. Let

$$K' = \left\{ p \in K \mid \exists q \in K \setminus \{p\}: |p| \cap |q| \neq \emptyset \right\}$$

be the set of paths in  $K$  that meet a vertex of another path, clearly  $|K'| \geq 2$  since  $K$  is not a routing. There is a total order on  $K'$ : let  $p, q \in K'$ , then  $p \leq q$  if and only if  $i \leq j$  where  $p_1 = s_i$  and  $q_1 = s_j$ . Now let  $p = (p_i)_{i=1}^{n(p)} \in K'$  be chosen such that  $p$  is the minimal element with respect to the above order. Let  $j(p) \in \{1, 2, \dots, n(p)\}$  be the smallest index, such that there is some  $q \in K' \setminus \{p\}$  with  $p_{j(p)} \in |q|$ . Now let  $q = (q_i)_{i=1}^{n(q)} \in \{k \in K' \setminus \{p\} \mid p_{j(p)} \in |q|\}$  be the minimal choice with respect to the above order on  $K'$ , and let  $j(q) \in \{1, 2, \dots, n(q)\}$  such that  $q_{j(q)} = p_{j(p)}$ . Now let  $p' = p_1 p_2 \dots p_{j(p)} q_{j(q)+1} q_{j(q)+2} \dots q_{n(q)}$  and  $q' = q_1 q_2 \dots q_{j(q)} p_{j(p)+1} p_{j(p)+2} \dots p_{n(p)}$ . Since  $D$  is acyclic, all walks are paths in  $D$ , so  $\mathbf{W}(D) = \mathbf{P}(D)$ . Therefore we may set  $\varphi(K) = (K \setminus \{p, q\}) \cup \{p', q'\} \in Q \setminus R$ . Clearly,  $\varphi(\varphi(K)) = K$ , therefore  $\varphi$  is bijective and self-inverse. Furthermore, if  $K \in Q_\sigma$ , then  $\varphi(K) \in Q_{\sigma \cdot (xy)}$  for a suitable cycle  $(xy) \in \mathfrak{S}_n$ . Thus  $\text{sgn}(\varphi(K)) = \text{sgn}(\sigma) \text{sgn}((xy)) = -\text{sgn}(\sigma) = -\text{sgn}(K)$ . Clearly,  $K$  and  $\varphi(K)$  traverse the same arcs, therefore  $\prod_{p \in K} (\prod p) = \prod_{p \in \varphi(K)} (\prod p)$ . The bijection  $\varphi$  implies that the summands  $K \in Q \setminus R$  add up to zero, thus we have

$$\det(\mu|S \times T) = \sum_{L \in R} \text{sgn}(L) \prod_{p \in L} (\prod p).$$

The second statement of the lemma follows from the fact that for two routings  $L_1, L_2 \in R$ , we have  $L_1 = L_2$  if and only if  $\bigcup_{p \in L_1} |p|_A = \bigcup_{p \in L_2} |p|_A$ . For the non-trivial direction: assume we have a set of arcs  $L_A$  that are traversed by the paths of a linking, and let  $V_A = \{u, v \mid (u, v) \in L_A\}$ . Then the initial vertices of that linking are the elements of the set  $S_A = \{u \in V_A \mid \forall (v, w) \in L_A: u \neq w\}$ . The terminal vertices are the elements of the set  $T_A = \{w \in V_A \mid \forall (u, v) \in L_A: u \neq w\}$ , and the paths can be reconstructed from the initial vertices  $v \in S_A$  by following the unique arcs  $(v, w), (w, x), \dots \in L_A$  until a vertex  $t \in T_A$  is reached. Clearly, for  $L \in R$ ,  $\prod_{p \in L} (\prod p) \neq 0$ , and since  $w$  is an indeterminate weighting, two summands  $L, L' \in R$  can only cancel each other when the corresponding monomials are equal, i.e.  $\prod_{p \in L} (\prod p) = \prod_{p \in L'} (\prod p)$ ; but then  $L_A = L'_A$

holds, and so  $L = L'$ . Thus no summand in the determinant formula which belongs to a routing from  $R$  can be cancelled out by another summand belonging to another routing from  $R$ . Therefore  $\det(\mu|S \times T) = 0$  if and only if  $R = \emptyset$ , i.e. there is no linking from  $S$  to  $T$  in  $D$ .  $\square$

**Corollary 2.7.5.** *Let  $D = (V, A)$  be an acyclic digraph,  $T, E \subseteq V$ , and  $w: A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$ . Furthermore, let  $\mu \in \mathbb{R}^{E \times T}$  be the matrix with*

$$\mu(e, t) = \sum_{p \in \mathbf{P}(D; e, t)} (\prod p).$$

Then  $\Gamma(D, T, E) = M(\mu)$ .

*Proof.* This is straightforward from the Definition 1.2.55 and the Lindström Lemma 2.7.4.  $\square$

Clearly, for an arbitrary gammoid  $M = \Gamma(D, T, E)$ , we cannot assume that  $D$  is acyclic (Remark 2.1.66). There are several ways to work around this. Either *a*)<sup>16</sup> we adjust our definition of a routing such that routings with non-path walks are allowed, making the class of routings in  $D$  infinite whenever there is a cycle in  $D$ . Then we could use power series to calculate the entries of  $\mu$  as well as its sub-determinants, where convergence is sufficiently guaranteed if  $\prod p \in (0, 1)$  holds for every cycle walk  $p \in \mathbf{W}(D)$ . A sufficient condition would be to use a weighting  $w$  where  $0 < w(a) < 1$  for all  $a \in A$ . The construction of  $\varphi$  in the proof of the Lindström Lemma would still go through, but for the second statement we would have to choose the indeterminate weights more carefully, since a cycle walk  $q \in \mathbf{W}(D)$  gives rise to the formal power series  $\sum_{i=0}^{\infty} (\prod q)^i$  which converges to  $\frac{1}{1 - \prod q}$ . Clearly, a similar cardinality-argument as in Lemma 1.1.11 guarantees that we can find a sufficient number of carefully chosen indeterminates in  $\mathbb{R}$ . Or *b*) we could try to find a construction that removes cycles from  $D$ , possibly changing the gammoid represented by the resulting digraph  $D'$ , then use the Lindström Lemma to obtain a matrix  $\nu$ , and then revert the constructions in order to obtain  $\mu$  from  $\nu$ ; which is what we will do now.

**Definition 2.7.6.** *Let  $D = (V, A)$  be a digraph,  $x, t \notin V$  be distinct new elements, and let  $c = (c_i)_{i=1}^n \in \mathbf{W}(D)$  be a cycle walk. The **lifting of  $c$  in  $D$  by  $(x, t)$**  is the digraph  $D_{(x, t)}^{(c)} = (V \dot{\cup} \{x, t\}, A')$  where*

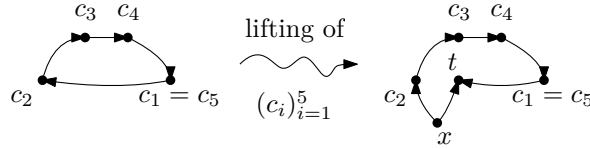
$$A' = A \setminus \{(c_1, c_2)\} \cup \{(c_1, t), (x, c_2), (x, t)\}. \quad \blacksquare$$

<sup>16</sup>This is what is implied by the rationale given in [Ard06].



Observe that the cycle walk  $c \in \mathbf{W}(D)$  is not a walk in the lifting of  $c$  in  $D$  anymore.

**Example 2.7.7.** Consider  $D = (\{c_1, c_2, c_3, c_4\}, \{(c_1, c_2), (c_2, c_3), (c_3, c_4), (c_4, c_1)\})$ . Then  $c_1c_2c_3c_4c_1 \in \mathbf{W}(D)$  is a cycle. The lifting of  $c$  in  $D$  by  $(x, t)$  is then defined to be the digraph  $D' = (\{c_1, c_2, c_3, c_4, x, t\}, \{(c_1, t), (c_2, c_3), (c_3, c_4), (c_4, c_1), (x, c_2), (x, t)\})$ .



▲

**Lemma 2.7.8.** Let  $D = (V, A)$  be a digraph,  $x, t \notin V$ , and  $c = (c_i)_{i=1}^n \in \mathbf{W}(D)$  a cycle walk, and let  $D' = D_{(x,t)}^{(c)}$  be the lifting of  $c$  in  $D$  by  $(x, t)$ . If  $c' \in \mathbf{W}(D')$  is a cycle walk, then  $c' \in \mathbf{W}(D)$ . In other words, the lifting of cycle walks does not introduce new cycle walks.

*Proof.* Let  $D' = (V', A')$ . Clearly,  $x$  is a source in  $D_{(x,t)}^{(c)}$  and  $t$  is a sink in  $D_{(x,t)}^{(c)}$ . Thus  $x, t \notin |c'|$ . But then  $|c'|_A \subseteq A' \cap (V \times V)$  and therefore  $c'$  is also a cycle walk in  $D$ .  $\square$

**Definition 2.7.9.** Let  $D = (V, A)$  be a digraph. A **complete lifting of  $D$**  is an acyclic digraph  $D' = (V', A')$  for which there is a suitable  $n \in \mathbb{N}$  such that there is a set  $X = \{x_1, t_1, x_2, t_2, \dots, x_n, t_n\} \neq \emptyset$  with  $X \cap V = \emptyset$ , a family of digraphs  $D^{(i)} = (V^{(i)}, A^{(i)})$  for  $i \in \{0, 1, \dots, n\}$  where  $D' = D^{(n)}$ ,  $D^{(0)} = D$ , and for all  $i \in \{1, 2, \dots, n\}$

$$D^{(i)} = \left( D^{(i-1)} \right)_{(x_i, t_i)}^{(c_i)}$$

with respect to a cycle walk  $c_i \in \mathbf{W}(D^{(i-1)})$ . In this case, we say that the set

$$R = \{(x_i, t_i) \mid i \in \{1, 2, \dots, n\}\}$$

**realizes** the complete lifting  $D'$  of  $D$ .  $\blacksquare$

**Lemma 2.7.10.** Let  $D = (V, A)$  be a digraph. Then  $D$  has a complete lifting.

*Proof.* By induction on the number of cycle walks in  $D$ . If  $D$  has no cycle walk,  $D$  is a complete lifting of  $D$ . Now let  $c \in \mathbf{W}(D)$  be a cycle walk, and let  $x, t \notin V$ . Let  $D' = D_{(x,t)}^{(c)}$ . By construction  $c \notin \mathbf{W}(D')$ , thus  $D'$  has strictly fewer cycle walks than  $D$  (Lemma 2.7.8), therefore there is a complete lifting  $D''$  of  $D'$  by induction hypothesis. Since  $D'$  is a lifting of  $D$ ,  $D''$  is also a complete lifting of  $D$ .  $\square$

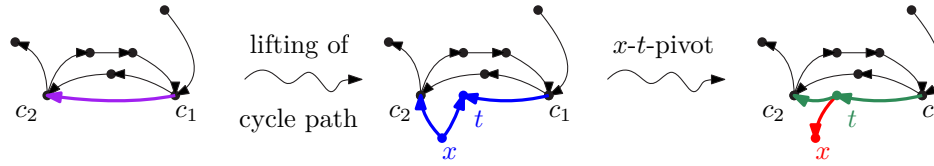


Fig. 2.5 Constructions involved in Lemma 2.7.11.

**Lemma 2.7.11.** *Let  $D = (V, A)$ ,  $E, T \subseteq V$ ,  $c \in \mathbf{W}(D)$  a cycle,  $x, t \notin V$ , and let  $D' = D_{(x,t)}^{(c)}$  be the lifting of  $c$  in  $D$ . Then  $\Gamma(D, T, E) = \Gamma(D', T \cup \{t\}, E \cup \{x\}).E$ .*

*Proof.* Let  $M = \Gamma(D, T, V)$  be the strict gammoid induced by the representation  $(D, T, E)$  of the not necessarily strict gammoid  $\Gamma(D, T, E)$ , and let  $M' = \Gamma(D', T \cup \{t\}, V')$  be the strict gammoid obtained from the lifting of  $c$ . Then  $M'' = (M').(V \cup \{t\})$  is a strict gammoid that is represented by  $(D'', T, V \cup \{t\})$  where the digraph  $D'' = (V_0 \setminus \{x\}, A_0 \setminus (V_0 \times \{x\}))$  is induced from the  $x$ - $t$ -pivot  $D_0$  of  $D'$ , i.e.  $D_0 = D'_{x \leftarrow t} = (V_0, A_0)$ . This follows from the proof of Lemma 2.2.8 along with the single-arc routing  $\{xt\}: \{x\} \rightrightarrows T \cup \{t\}$  in  $D'$ . Let  $A''$  denote the arc set of  $D''$ . It is easy to see from the involved constructions (Fig. 2.5), that  $A'' = (A \setminus \{(c_1, c_2)\}) \cup \{(c_1, t), (t, c_2)\}$ . Clearly, a routing  $R$  in  $D$  can have at most one path  $p \in R$  such that  $(c_1, c_2) \in |p|_A$ , and since  $t \notin V$ , we obtain a routing  $R' = (R \setminus \{p\}) \cup \{qtr\}$  for  $q, r \in \mathbf{P}(D)$  such that  $p = qr$  with  $q_{-1} = c_1$  and  $r_1 = c_2$ . Clearly,  $R'$  routes  $X$  to  $Y$  in  $D''$  whenever  $R$  routes  $X$  to  $Y$  in  $D$ . Conversely, let  $R': X' \rightrightarrows Y'$  be a routing in  $D''$  with  $t \notin X'$ . Then there is at most one  $p \in R'$  with  $t \in |p|$ . We can invert the construction and let  $R'' = (R' \setminus \{p\}) \cup \{qr\}$  for the appropriate paths  $q, r \in \mathbf{P}(D'')$  with  $p = qtr$ . Then  $R''$  is a routing from  $X'$  to  $Y'$  in  $D'$ . Thus we have shown that  $M''|V = M$ , and consequently, with  $E \subseteq V$  and Lemma 1.2.47, it follows that

$$\begin{aligned} \Gamma(D, T, E) = M|E &= (M'')|E = (\Gamma(D', T \cup \{t\}, V').(V \cup \{t\}))|E \\ &= \Gamma(D', T \cup \{t\}, E \cup \{x\}).E. \end{aligned} \quad \square$$

**Corollary 2.7.12.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then there is an acyclic digraph  $D = (V, A)$  and sets  $T, E' \subseteq V$  such that  $M = \Gamma(D, T, E').E$  and such that*

$$|T| = \text{rk}_M(E) + |E' \setminus E|.$$

*Proof.* Let  $M = \Gamma(D', T', E)$  with  $|T'| = \text{rk}_M(E)$ . Then let  $D$  be a complete lifting of  $D'$  (Lemma 2.7.10), and let  $D^{(0)}, D^{(1)}, \dots, D^{(n)}$  be the family of digraphs and  $c_1, c_2, \dots, c_n$

be the cycle walks that correspond to the complete lifting  $D$  of  $D'$  as required by Definition 2.7.9, and let  $\{x_1, t_1, \dots, x_n, t_n\}_{\neq}$  denote the new elements such that

$$D^{(i)} = \left( D^{(i-1)} \right)_{(x_i, t_i)}^{(c_i)}$$

holds for all  $i \in \{1, 2, \dots, n\}$ . Induction on the index  $i$  with Lemma 2.7.11 yields that

$$\Gamma(D', T, E) = \Gamma(D^{(i)}, T \cup \{t_1, t_2, \dots, t_i\}, E \cup \{x_1, x_2, \dots, x_i\}).E$$

holds for all  $i \in \{1, 2, \dots, n\}$ . Clearly,

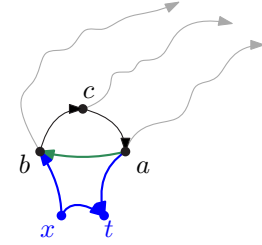
$$|T \cup \{t_1, t_2, \dots, t_n\}| = |T| + n = \text{rk}_M(E) + n = \text{rk}_M(E) + |\{x_1, x_2, \dots, x_n\}|. \quad \square$$

**Theorem 2.7.13.** *Let  $M = (E, \mathcal{I})$  be a gammoid,  $T = \{t_1, t_2, \dots, t_{\text{rk}_M(E)}\}_{\neq}$ . Then there is a matrix  $\mu \in \mathbb{R}^{E \times T}$  such that  $M = M(\mu)$ .*

*Proof.* By Corollary 2.7.12, there is an acyclic digraph  $D = (V, A)$  and there are sets  $E', T' \subseteq V$ , such that  $M = N.E$  where  $N = \Gamma(D, T', E')$  and  $|T'| = \text{rk}_M(E) + |E' \setminus E|$ . Remember that  $E' \setminus E$  is independent in  $N$ , and every base  $B$  of  $M$  induces a base  $B \cup (E' \setminus E)$  of  $N$ . The Lindström Lemma 2.7.4 yields a matrix  $\nu \in \mathbb{R}^{E' \times T'}$  such that  $N = M(\nu)$ . In Lemma 1.2.60 and Remark 1.2.61 we have seen that we can pivot in the independent set  $E' \setminus E$  in  $\nu$ , which yields a new matrix  $\nu' \in \mathbb{R}^{E' \times T'}$ . Let  $T_0 = \{t' \in T' \mid \forall e' \in E' \setminus E: \nu'(e', t') = 0\}$  denote the remaining columns of  $\nu'$  that have not been used to pivot in an element of  $E' \setminus E$ . We set  $\mu = \nu'|_{E \times T_0}$ . Thus  $M(\mu) = M(\nu).E = N.E = M$ .  $\square$

Let us compare the two methods *a)* and *b)* mentioned above. In our opinion, both methods are connected to aspects of the same underlying phenomenon that cycle paths do not interfere with the existence of linkings between given sets of vertices in a digraph.

**Example 2.7.14.** Let  $D = (V, A)$  be a digraph, such that the only cycle walk in  $D$  is  $abca \in \mathbf{W}(D)$ , and let  $x, t \notin V$ . Now chose an arbitrary target node  $t_0 \in V$ . Let  $D' = (V', A') = D_{(x,t)}^{(abca)}$  be the lifting of  $abca$  in  $D$ , clearly  $D'$  is acyclic. Let  $w: A' \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D'$ . We introduce the following sums



$$\alpha = \sum_{p \in P_{(a,t_0)}} \prod p, \quad \beta = \sum_{p \in P_{(b,t_0)}} \prod p, \quad \text{and} \quad \gamma = \sum_{p \in P_{(c,t_0)}} \prod p,$$

where for  $u \in \{a, b, c\}$ ,

$$P_{(u,t_0)} = \left\{ p \in \mathbf{P}(D') \mid p_1 = u, p_{-1} = t_0, \text{ and } \{(b,c), (c,a)\} \cap |p|_A = \emptyset \right\},$$

i.e.  $P_{(u,t_0)}$  consists of the paths from  $u$  to  $t_0$  not visiting another element from  $\{a, b, c\}$ . Now let  $\mu \in \mathbb{R}^{V' \times \{t_0, t\}}$  be the matrix obtained from the Lindström Lemma 2.7.4 for the strict gammoid  $M = \Gamma(D', \{t_0, t\}, V')$ . We set  $w_b = w(b, c)$ ,  $w_c = w(c, a)$ ,  $w_x = w(x, b)$ . Clearly, we have

$$\begin{aligned} \mu(a, t_0) &= \alpha, \\ \mu(c, t_0) &= \gamma + w_c \cdot \alpha, \\ \mu(b, t_0) &= \beta + w_b \cdot \gamma + w_b \cdot w_c \cdot \alpha, \text{ and} \\ \mu(x, t_0) &= w_x \cdot \beta + w_x \cdot w_b \cdot \gamma + w_x \cdot w_b \cdot w_c \cdot \alpha. \end{aligned}$$

Furthermore, we set  $w'_x = w(x, t)$  and  $w'_a = w(a, t)$ . With respect to the new target  $t$  introduced by the lifting of  $abca$  in  $D$ , we have  $\mu(x, t) = w'_x$ ,  $\mu(a, t) = w'_a$ ,  $\mu(c, t) = w_c \cdot w'_a$ , and  $\mu(b, t) = w_b \cdot w_c \cdot w'_a$ . Let  $N = M \cdot (V' \setminus \{t\})$  and  $\nu \in \mathbb{R}^{(V' \setminus \{t\}) \times \{t_0\}}$  be as in Lemma 1.2.60 with  $M(\nu) = N$ . Then

$$\begin{aligned} \nu(a, t_0) &= \mu(a, t_0) - \frac{\mu(a, t)}{\mu(x, t)} \mu(x, t_0) \\ &= \alpha - \frac{w'_a \cdot w_x}{w'_x} (\beta + w_b \cdot \gamma + w_b \cdot w_c \cdot \alpha) \end{aligned}$$

— Now let  $w': A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$  where  $w'(q) = w(q)$  for all  $q \in A \setminus \{(a, b)\}$ . We set  $w_a = w'(a, b)$ . We calculate

$$\alpha' = \sum_{p \in \mathbf{P}(D; a, t_0)} \prod p = \alpha + w_a \cdot \beta + w_a \cdot w_b \cdot \gamma.$$

If we further assume that  $0 < w_a \cdot w_b \cdot w_c < 1$ , then we have convergence in the following equation

$$\alpha'' = \sum_{w \in \mathbf{W}(D; a, t_0)} \left( \prod w \right) = \sum_{i=0}^{\infty} (w_a \cdot w_b \cdot w_c)^i \cdot \alpha' = \frac{\alpha + w_a \cdot \beta + w_a \cdot w_b \cdot \gamma}{1 - w_a \cdot w_b \cdot w_c}$$

The second equation holds because  $abca$  is the only cycle walk in  $D$ , therefore all non-path walks from  $a$  to  $t_0$  must be of the form  $(abc)^i p$  for  $i \in \mathbb{N} \setminus \{0\}$  and  $p \in \mathbf{P}(D; a, t_0)$ ; the summand where  $i = 0$  corresponds to the paths  $\mathbf{P}(D; a, t_0) \subseteq \mathbf{W}(D; a, t_0)$ . Therefore  $\alpha'' = \mu'(a, t_0)$ , where  $\mu'$  is the matrix that we would have obtained from  $D$  using the Lindström Lemma method, operating with formal power series and a convergent indeterminate weighting as in version  $a$ ) above.

— We argue that that for a given indeterminate weighting  $w$  of  $D$  — which has been chosen such that every formal power series involved in the construction of the Lindström Lemma matrix converges; and such that whenever the power series of a sub-determinant of the matrix converges to zero, then the power series of that determinant is the zero series — there is an indeterminate weighting  $w'$  of  $D'$ , with  $w(q) = w'(q)$  for all  $q \in A \cap A'$  such that  $\nu(a, t_0) = \mu'(a, t_0)$ . The formal equation  $\nu(a, t_0) = \mu'(a, t_0)$  may be solved for

$$w'_a = \frac{-w'_x \cdot w_a}{w_x \cdot (1 - w_a \cdot w_b \cdot w_c)} = \frac{P'}{Q'}$$

yielding the non-trivial polynomial equation  $Q' \cdot w'_a - P' = 0$  where  $P'$  and  $Q'$  are integer-coefficient polynomials. Therefore we may extend and restrict  $w$  to an indeterminate weighting  $\tilde{w}: (A \cup A') \setminus \{(a, t), (a, b)\} \rightarrow \mathbb{R}$ , and then set  $w'(x) = \tilde{w}(x)$  for  $x \in A' \setminus \{(a, t)\}$ , and  $w'((a, t)) = w'_a$  as in the equation above, calculated with respect to  $\tilde{w}$ . This yields the desired indeterminate weighting, because  $w'((a, t)) \mathbb{Z}$ -depends on  $\tilde{w}((a, b)) = w((a, b))$  which is  $\mathbb{Z}$ -independent of  $\tilde{w}[A \setminus \{(a, b)\}]$ .  $\blacktriangle$

In the paper *A parameterized view on matroid optimization problems* [Mar09], D. Marx shows that there is a randomized polynomial time algorithm with respect to the size of the ground set of a gammoid, that constructs a matrix  $\mu$  from  $(D, T, E)$  such

that  $M(\mu) = \Gamma(D, T, E)$ . The method of D. Marx starts with the construction of the dual  $N^*$  of the underlying strict gammoid  $N = \Gamma(D, T, V)$  for a given representation  $(D, T, E)$  with  $D = (V, A)$  through the linkage system of  $D$  to  $T$  (Definition 1.5.20 and Lemma 1.5.22). Then a matrix  $\nu$  with  $M(\nu) = N^*$  is constructed with a small probability of failure (see Proposition 2.7.17 below), which in turn is converted into a standard representation (Remark 1.2.63) of the form  $(I_r A^\top)^\top$  using Gaussian Elimination. Then  $(-A I_{n-r})^\top$  is the desired representation of  $M$ . Before we present the main proposition that leads to this result, we need the following lemma.

**Lemma 2.7.15** ([Sch80], Corollary 1). *Let  $\mathbb{F}$  be a field,  $X = \{x_1, x_2, \dots, x_n\}_{\neq}$ , let  $p \in \mathbb{F}[X]$  be a polynomial with  $p \neq 0$ . Furthermore, let  $F \subseteq \mathbb{F}$  be a finite subset of elements of the coefficient field with  $|F| \geq c \cdot \deg(p)$  for some  $c \in \mathbb{Q}$  with  $c > 0$ . Then*

$$\left| \left\{ \xi \in F^X \mid p[X = \xi] = 0 \right\} \right| \leq \frac{|F|^n}{c}.$$

For a formal proof, we refer the reader to J.T. Schwartz's *Fast Probabilistic Algorithms for Verification of Polynomial Identities* [Sch80]. The proof idea is to do induction on the number of variables involved. The base case is the fact that a polynomial in a single variable of degree  $d$  can have at most  $d$  different roots. In the induction step, we fix the values of all but one variable, if the resulting polynomial in a single variable is the zero polynomial, we may choose any value from  $F$  for that variable. Otherwise, there are at most the degree of the resulting polynomial many choices for the last variable such that the polynomial evaluates to zero.

**Lemma 2.7.16** ([Mar09], Lemma 1, [Sch80], [Zip79]). *Let  $\mathbb{F}$  be a field, let  $X = \{x_1, x_2, \dots, x_n\}_{\neq}$  be a finite set, let  $p \in \mathbb{F}[X]$  be a polynomial with  $p \neq 0$ , and let  $F \subseteq \mathbb{F}$  be a finite set. Let  $\xi$  be a random variable sampled from a uniform distribution on the set  $F^X$ . Then the probability that  $\xi$  is a zero of  $p$  may be estimated by*

$$\Pr(p[X = \xi] = 0) \leq \frac{\deg(p)}{|F|}.$$

*Proof.* <sup>17</sup> In Lemma 2.7.15 we set  $c = \frac{|F|}{\deg(p)}$  and get

$$\frac{\left| \left\{ \xi \in F^X \mid p[X = \xi] = 0 \right\} \right|}{|F^X|} \leq \frac{|F|^n}{c \cdot |F|^n} = \frac{1}{c} = \frac{\deg(p)}{|F|}. \quad \square$$

<sup>17</sup>D. Marx omits the proof and instead cites [Sch80] and [Zip79].

**Proposition 2.7.17** ([Mar09], Proposition 3.11). *Let  $E$  be a finite set,  $r \in \mathbb{N}$ , and  $\mathcal{A} = (A_i)_{i=1}^r \subseteq E$  be a family of subsets of  $E$ . Then a matrix  $\mu \in \mathbb{R}^{E \times \{1,2,\dots,r\}}$  with  $M(\mu) = M(\mathcal{A})$  can be constructed in randomized polynomial time.*

*Proof.* For all  $k \in \mathbb{N}$  with  $k > 1$ , we write  $\text{unif}(k)$  in order to denote an integer that has been randomly sampled from a uniform distribution on  $\{1, 2, \dots, k\}$ . Several instances of  $\text{unif}(k)$  shall denote independently sampled random variables. Let  $p \in \mathbb{N}$  be an arbitrary parameter. We define the random matrix  $\mu \in \mathbb{R}^{E \times \{1,2,\dots,r\}}$  by<sup>18</sup>

$$\mu(e, i) = \begin{cases} \text{unif}(2^p \cdot r \cdot Q) & \text{if } e \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$Q = \left( \begin{array}{c} |E| \\ \lceil \frac{|E|}{2} \rceil \end{array} \right).$$

Clearly,  $Q \leq 2^{|E|}$  with equality if  $|E| = 1$ . Observe that sampling  $\text{unif}(2^k)$  can be done by sampling  $k$  bits from a uniform distribution. Thus  $\mu$  can be obtained by sampling at most  $|E| \cdot r \cdot (p + \lceil \log_2(Q+r) \rceil)$  uniform random bits. We show that  $\Pr(M(\mu) \neq M(\mathcal{A})) \leq \frac{1}{2^p}$ . Let  $X \subseteq E$  be an independent set with respect to  $M(\mu)$ . Then  $\text{idet}(M|X \times \{1, 2, \dots, r\}) = 1$ , so there is an injective map  $\varphi: X \rightarrow \{1, 2, \dots, r\}$  such that  $\mu(x, \varphi(x)) \neq 0$  for all  $x \in X$ . By construction of  $\mu$  we obtain that in this case  $x \in A_{\varphi(x)}$ . Therefore  $X$  is a partial transversal of  $\mathcal{A}$ , and so  $X$  is independent in  $M(\mathcal{A})$ , too.

Now let  $X \subseteq E$  be a base of  $M(\mathcal{A})$ . Thus  $X$  is a maximal partial transversal of  $\mathcal{A}$  and there is an injective map  $\varphi: X \rightarrow \{1, 2, \dots, r\}$  such that  $x \in A_{\varphi(x)}$  for all  $x \in X$ . Let  $X = \{x_1, x_2, \dots, x_k\}_{\neq}$ , then we may define the matrix  $\nu \in \mathbb{R}[X]^{X \times \varphi[X]}$  where

$$\nu(x, i) = \begin{cases} x_i & \text{if } i = \varphi(x), \\ \mu(x, i) & \text{otherwise.} \end{cases}$$

Then  $\det(\nu)$  is a polynomial of degree  $|X| = k \leq r$  with leading monomial  $x_1 x_2 \cdots x_k$  in  $\mathbb{R}[X]$ , and if  $\xi \in \mathbb{R}^X$  is the vector where  $\xi(x) = \mu(x, \varphi(x))$  for all  $x \in X$ , we have the equality

$$\det(\mu|X \times \varphi[X]) = (\det(\nu)) [X = \xi].$$

<sup>18</sup>D. Marx uses samples from  $\text{unif}(2^p \cdot |E| \cdot 2^{|E|})$  and uses the argument that there are at most  $2^{|E|}$  independent sets. This line of arguments is valid, yet it does not use the fact that if  $X$  is independent in  $M(\mu)$ , then all subsets of  $X$  are independent in  $M(\mu)$ , too; consequently, the probability of failure is overestimated.

Remember that each value  $\xi(x)$  has been uniformly sampled from a set with cardinality  $2^p \cdot r \cdot Q$ , thus Lemma 2.7.16 yields

$$\Pr\left(\det(\mu|X \times \varphi[X]) = 0\right) \leq \frac{|X|}{2^p \cdot r \cdot Q} \leq \frac{1}{2^p \cdot Q}.$$

There are at most  $\binom{|E|}{\text{rk}_{M(\mathcal{A})}(E)}$  different bases in  $M(\mathcal{A})$ , and the family of all subsets of  $E$  with cardinality  $\lceil \frac{|E|}{2} \rceil$  is a maximal-cardinality anti-chain in the power set lattice of  $E$ . Therefore, there are at most  $Q$  different bases in  $M(\mathcal{A})$  needed to detect failure of  $M(\mathcal{A}) = M(\mu)$ . Thus we obtain

$$\Pr\left(M(\mu) \neq M(\mathcal{A})\right) \leq \sum_{B \in \mathcal{B}(M(\mathcal{A}))} \frac{1}{2^p \cdot Q} \leq \frac{1}{2^p}. \quad \square$$

Clearly, if the rank of  $M(\mathcal{A})$  is known, we may use the better factor  $Q = \binom{|E|}{\text{rk}_M(E)}$  in the probabilistic construction of  $\mu$  given in the proof of Proposition 2.7.17. If we also know the number of bases, we may even use  $Q = |\mathcal{B}(M(\mathcal{A}))|$ .



# Chapter 3

## Oriented Matroids

### 3.1 Quick Introduction to Oriented Matroids

Let us consider a matroid  $M(\mu)$  where  $\mu \in \mathbb{R}^{E \times \{1,2,\dots,r\}}$  is a finite matrix over the reals with full column rank, i.e. such that  $\text{rk}_{M(\mu)}(E) = r$ . Whenever  $C \in \mathcal{C}(M(\mu))$  is a circuit, there are coefficients  $\alpha: C \rightarrow \mathbb{R}$  such that  $\alpha(c) \neq 0$  for all  $c \in C$  and such that

$$\sum_{c \in C} \alpha(c) \cdot \mu_c = 0$$

holds in the vector space  $\mathbb{R}^r$ . Furthermore,  $\alpha$  is uniquely determined by  $\{\mu_c \mid c \in C\}$  up to a homogeneous factor  $\lambda \in \mathbb{R} \setminus \{0\}$ , i.e. whenever the equality  $\sum_{c \in C} \beta(c) \cdot \mu_c = 0$  holds for  $\beta: C \rightarrow \mathbb{R}$  with  $\beta$  not constantly zero on  $C$ , then there is some  $\lambda \in \mathbb{R} \setminus \{0\}$  with  $\alpha(c) = \lambda\beta(c)$  for all  $c \in C$ . Therefore, the signs of the coefficients are determined up to a possible negation of all signs by the circuit  $C$  and the matrix  $\mu$ .

**Definition 3.1.1.** *Let  $E$  be a set. A **signed subset of  $E$**  is a map*

$$X: E \rightarrow \{-1, 0, 1\}.$$

*We denote the **positive elements of  $X$**  by  $X_+ = \{x \in E \mid X(x) = 1\}$ , the **negative elements of  $X$**  shall be denoted by  $X_- = \{x \in E \mid X(x) = -1\}$ , the **support of  $X$**  is defined as  $X_\pm = \{x \in E \mid X(x) \neq 0\}$ , and the **zero-set of  $X$**  is denoted by  $X_0 = E \setminus X_\pm$ . The **negation of  $X$**  is the signed subset  $-X$  where  $-X: E \rightarrow \{-1, 0, 1\}$ ,  $e \mapsto -X(e)$ . Let  $C \subseteq E$  and  $\alpha: C \rightarrow \mathbb{R}$  be a vector of coefficients. The **signs of  $\alpha$  over  $E$**  shall*

be denoted by  $E_\alpha$ , which is defined to be the map

$$E_\alpha: E \longrightarrow \{-1, 0, 1\}, e \mapsto \begin{cases} 0 & \text{if } e \notin C \text{ or } \alpha(e) = 0, \\ -1 & \text{if } \alpha(e) < 0, \\ 1 & \text{if } \alpha(e) > 0. \end{cases}$$

The **class of all signed subsets of  $E$**  shall be denoted by  $\sigma E$ . Let  $X, Y \in \sigma E$  be signed subsets of  $E$ . We say that  **$X$  is a signed subset of  $Y$** , if  $X_+ \subseteq Y_+$  and  $X_- \subseteq Y_-$ . We denote this fact by writing  $X \subseteq_\sigma Y$ . Furthermore, we write  $X \subsetneq_\sigma Y$  whenever  $X \subseteq_\sigma Y$  and  $X_\pm \subsetneq Y_\pm$  holds. The **empty signed subset** of  $E$  is the map  $\emptyset_{\sigma E}: E \longrightarrow \{-1, 0, 1\}, e \mapsto 0$ . ■

**Notation 3.1.2.** Let  $E$  be a finite set, and let  $C \in \sigma E$  such that  $C_+ = \{p_1, p_2, \dots, p_m\} \neq \emptyset$  and  $C_- = \{n_1, n_2, \dots, n_k\} \neq \emptyset$ . We shall denote  $C$  by both

$$\{p_1, p_2, p_3, \dots, p_m, -n_1, -n_2, \dots, -n_k\}$$

and

$$\{+p_1, +p_2, +p_3, \dots, +p_m, -n_1, -n_2, \dots, -n_k\},$$

i.e. we write a list of  $C_\pm$  where every element from  $C_+$  has either no prefix or a  $+$ -sign, and where every element of  $C_-$  has a  $-$ -sign as prefix. The elements of  $C_0$  are not listed. As with normal sets, we disregard the order in which the elements of  $C_\pm$  are listed. ■

**Example 3.1.3.** With regard to a fixed representation  $\mu \in \mathbb{R}^{E \times \{1, 2, \dots, r\}}$ , every circuit  $C \in \mathcal{C}(M(\mu))$  gives rise to two different signed subsets of  $E$ : Let  $\alpha: C \longrightarrow \mathbb{R}$  be not constantly zero on  $C$  with  $\sum_{c \in C} \alpha(c) \cdot \mu_c = 0$ , then  $E_\alpha$  and  $-E_\alpha$  are the signed subsets of  $E$  that correspond to the signs of non-trivial coefficients  $\alpha: C \longrightarrow \mathbb{R}$  with  $\sum_{c \in C} \alpha(c) \cdot \mu_c = 0$ . ▲

**Definition 3.1.4.** Let  $E$  be a finite set,  $C, D \in \sigma E$  be signed subsets of  $E$ . We define the **separator of  $C$  and  $D$**  to be the set

$$\text{sep}(C, D) = (C_+ \cap D_-) \cup (C_- \cap D_+). \quad \blacksquare$$

There is a notion of orthogonality for signed subsets which generalizes the ordinary orthogonality in vector spaces (see [BLS<sup>+</sup>99], p.115; [Nic12], p.27).

**Definition 3.1.5.** Let  $E$  be a finite set,  $C, D \in \sigma E$  be signed subsets of  $E$ . Then  $C$  and  $D$  shall be called **orthogonal signed subsets**, if either

(i) there are  $e, f \in E$ , such that

$$C(e) \cdot D(e) = -C(f) \cdot D(f) \neq 0$$

holds; or

(ii) for all  $e \in E$ , the equation

$$C(e) \cdot D(e) = 0$$

holds.

We write  $X \perp Y$  in order to denote that  $X$  and  $Y$  are orthogonal, and  $X \not\perp Y$  to denote that  $X$  and  $Y$  are not orthogonal. In the latter case  $X_{\pm} \cap Y_{\pm} \neq \emptyset$  and the common elements of the supports of  $X$  and  $Y$  all have the same relative sign with respect to  $X$  and  $Y$ , i.e.  $X(e) = \alpha \cdot Y(e)$  for all  $e \in X_{\pm} \cap Y_{\pm}$  and some  $\alpha \in \{-1, 1\}$  that does not depend on the choice of  $e$ . ■

**Lemma 3.1.6.** Let  $E$  be a finite set,  $C, D \in \sigma E$ . Then  $C \perp D$  if and only if  $(-C) \perp D$  if and only if  $C \perp (-D)$  if and only if  $(-C) \perp (-D)$ .

*Proof.* Since  $\perp$  is obviously a symmetric relation, it suffices to show that  $C \perp D$  implies  $(-C) \perp D$ . But for every  $e \in E$ ,  $(-C)(e) = -C(e)$ , therefore both properties (i) and (ii) of Definition 3.1.5 carry over from  $C$  to  $-C$ . □

**Lemma 3.1.7.** Let  $E$  be a finite set,  $\alpha, \beta \in \mathbb{R}^E$  with  $\langle \alpha, \beta \rangle = 0$ . Then  $E_{\alpha} \perp E_{\beta}$ .

*Proof.* If  $(E_{\alpha})_{\pm} \cap (E_{\beta})_{\pm} = \emptyset$ , then (ii) of Definition 3.1.5 holds, thus  $E_{\alpha} \perp E_{\beta}$ . Otherwise, there is some  $e \in E$  with  $\alpha(e) \cdot \beta(e) \neq 0$ . Let

$$E_e = \{e' \in E \mid \operatorname{sgn}(\alpha(e) \cdot \beta(e)) = \operatorname{sgn}(\alpha(e') \cdot \beta(e'))\}.$$

Since  $\langle \alpha, \beta \rangle = 0$ , we have

$$-\sum_{e' \in E_e} \alpha(e') \cdot \beta(e') = \langle \alpha, \beta \rangle - \sum_{e' \in E_e} \alpha(e') \cdot \beta(e') = \sum_{f \in E \setminus E_e} \alpha(f) \cdot \beta(f).$$

We give an indirect argument and assume that (i) does not hold. Then for all  $f \in E \setminus E_e$ , we have  $\alpha(f) \cdot \beta(f) = 0$ . Thus  $-\sum_{e' \in E_e} \alpha(e') \cdot \beta(e') = 0$ , but the sign of  $\alpha(e') \cdot \beta(e')$  is the same for every  $e' \in E_e$ . Therefore  $\alpha(e') \cdot \beta(e') = 0$  for all  $e' \in E_e$ , contradicting the assumption that  $\alpha(e) \cdot \beta(e) \neq 0$ , so (i) must hold. Thus  $E_{\alpha} \perp E_{\beta}$ . □

**Definition 3.1.8.** Let  $E$  be a finite set,  $\mathcal{C} \subseteq \sigma E$  and  $\mathcal{C}^* \subseteq \sigma E$ . The triple  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  is called **oriented matroid**, if the following properties hold:

- (C1)  $\emptyset_{\sigma E} \notin \mathcal{C}$ ,
- (C2) for all  $C \in \sigma E$ ,  $C \in \mathcal{C}$  if and only if  $-C \in \mathcal{C}$ ,
- (C3) for all  $X, Y \in \mathcal{C}$ ,  $X_{\pm} \subseteq Y_{\pm}$  implies  $X = Y$  or  $X = -Y$ ,
- (C4) for all  $X, Y \in \mathcal{C}$  with  $X \neq -Y$  and all  $e \in X_+ \cap Y_-$  and  $f \in X_{\pm} \setminus \text{sep}(X, Y)$ , there is some  $Z \in \mathcal{C}$  such that  $e \notin Z_{\pm}$ ,  $Z(f) = X(f)$ ,  $Z_+ \subseteq X_+ \cup Y_+$  and  $Z_- \subseteq X_- \cup Y_-$ ;
- (C\*1)  $\emptyset_{\sigma E} \notin \mathcal{C}^*$ ,
- (C\*2) for all  $C' \in \sigma E$ ,  $C' \in \mathcal{C}^*$  if and only if  $-C' \in \mathcal{C}^*$ ,
- (C\*3) for all  $X', Y' \in \mathcal{C}^*$ ,  $X'_{\pm} \subseteq Y'_{\pm}$  implies  $X' = Y'$  or  $X' = -Y'$ ,
- (C\*4) for all  $X', Y' \in \mathcal{C}^*$  with  $X' \neq -Y'$  and all  $e \in X'_+ \cap Y'_-$  and  $f \in X'_{\pm} \setminus \text{sep}(X', Y')$ , there some is  $Z' \in \mathcal{C}^*$  such that  $e \notin Z'_{\pm}$ ,  $Z'(f) = X'(f)$ ,  $Z'_+ \subseteq X'_+ \cup Y'_+$  and  $Z'_- \subseteq X'_- \cup Y'_-$ ;
- (O1) for all  $C \in \mathcal{C}$  and  $C' \in \mathcal{C}^*$ , we have  $C \perp C'$ ,
- (O2) there is a matroid  $M = (E, \mathcal{I})$ , such that

$$\{C_{\pm} \mid C \in \mathcal{C}\} = \mathcal{C}(M) \quad \text{and} \quad \{C'_{\pm} \mid C' \in \mathcal{C}^*\} = \mathcal{C}(M^*).$$

In this case, the elements  $C \in \mathcal{C}$  shall be called **signed circuits of  $\mathcal{O}$** , and  $\mathcal{C}$  shall be the **family of signed circuits of  $\mathcal{O}$** . Likewise, the elements  $C' \in \mathcal{C}^*$  shall be called **signed cocircuits of  $\mathcal{O}$** , and  $\mathcal{C}^*$  shall be the **family of signed cocircuits of  $\mathcal{O}$** . Furthermore,  $M(\mathcal{O})$  shall denote the **underlying matroid of  $\mathcal{O}$** , whose existence and uniqueness is guaranteed by (O2). ■

**Remark 3.1.9.** The above definition of an oriented matroid is redundant in the sense that some of the properties follow from other properties easily. For instance, (O1) and (O2) imply all other properties from Definition 3.1.8 (see [Oxl11], p.401). We give a quick overview over the most common cryptomorphic ways to define oriented matroids via signed circuits. For full disclosure on these less redundant definitions of oriented matroids, we refer the reader to [BLS<sup>+</sup>99], [BV78], and [FL78].

In [BV78], R.G. Bland and M. Las Vergnas define oriented matroids to be pairs  $(E, \mathcal{C})$  such that  $\mathcal{C} \subseteq \sigma E$  has the properties (C1), (C2), (C3), and the property

(C4') for all  $X, Y \in \mathcal{C}$  with  $X \neq -Y$  and all  $e \in X_+ \cap Y_-$ , there is some  $Z \in \mathcal{C}$  such that  $e \notin Z_\pm$ ,  $Z_+ \subseteq X_+ \cup Y_+$  and  $Z_- \subseteq X_- \cup Y_-$ .

In Theorem 2.1 [BV78], R.G. Bland and M. Las Vergnas prove that if we assume (C1), (C2), (C3), then the properties (C4) and (C4') are equivalent. In Theorem 2.2 [BV78], they prove that if  $(E, \mathcal{C}, \mathcal{C}^*)$  is an oriented matroid as in our Definition 3.1.8, then  $\mathcal{C}$  uniquely determines  $\mathcal{C}^*$  and vice versa. Therefore, in order to define an oriented matroid on the ground set  $E$ , it suffices to determine  $\mathcal{C}$ , and show that (C1), (C2), (C3), and (C4') hold. Since the underlying matroid  $M(\mathcal{O})$  is already uniquely determined by the supports of the elements of  $\mathcal{C}$ , we can reconstruct the supports of  $\mathcal{C}^*$  by examining the cocircuits of  $M(\mathcal{O})$ . In order to find the correct signatures of  $D \in \mathcal{C}^*$ , we can set the sign  $D(d)$  for an arbitrarily chosen  $d \in D_\pm$  to +1, or to -1 in order to generate the corresponding negation  $-D$ . If  $D_\pm = \{d\}$ , we are done. If  $|D_\pm| > 1$ , then Lemma 1.2.36 with respect to the dual matroid  $M(\mathcal{O})^*$  indicates that for every  $c \in D_\pm \setminus \{d\}$ , there is a circuit  $C \in \mathcal{C}$  such that  $C_\pm \cap D_\pm = \{c, d\}$ . We let  $D(c) = D(d)$  if  $C(c) \neq C(d)$ , and  $D(c) = -D(d)$  if  $C(c) = C(d)$ . Clearly, this is the only possibility which yields  $D \perp \mathcal{C}$ .

In [FL78], J. Folkman and J. Lawrence independently defined their version of oriented matroids to be triples  $(E_\sigma, \mathcal{C}, -)$  subject to basically the same properties as R.G. Bland's and M. Las Vergnas's version of oriented matroids, but where the latter used explicit signs + and -, the former used a fix-point free involution on  $E_\sigma$  that maps  $+e$  to  $-e$  and vice versa. Thus the oriented matroids of J. Folkman and J. Lawrence correspond to *reorientation classes* of oriented matroids in this work. •

**Example 3.1.10.** Let  $E$  be a finite set and  $M = (E, 2^E)$  be the free matroid on  $E$ . Then

$$\mathcal{O} = \left( E, \emptyset, \left\{ \{e\}, \{-e\} \mid e \in E \right\} \right)$$

is the only possible oriented matroid with  $M(\mathcal{O}) = M$ . ▲

It is straightforward, that applying an  $M(\mathcal{O})$ -automorphism to the signed circuits  $\mathcal{C}$  and cocircuits  $\mathcal{C}^*$  of an oriented matroid  $\mathcal{O}$  yields another oriented matroid.

**Definition 3.1.11.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid, and let  $\varphi: E \rightarrow E$  be an  $M(\mathcal{O})$ -automorphism. The **relabeling of  $\mathcal{O}$  by  $\varphi$**  shall be the triple

$$\varphi[\mathcal{O}] = \left( E, \mathcal{C}_\varphi, \mathcal{C}_\varphi^* \right)$$

where  $\mathcal{C}_\varphi = \{C \circ \varphi \in \sigma E \mid C \in \mathcal{C}\}$  and  $\mathcal{C}_\varphi^* = \{C' \circ \varphi \in \sigma E \mid C' \in \mathcal{C}^*\}$ . ■

**Lemma 3.1.12.** Let  $\mathcal{O}$  be an oriented matroid and  $\varphi$  an  $M(\mathcal{O})$ -automorphism. Then  $\varphi[\mathcal{O}]$  is an oriented matroid.

*Proof.* For  $X \in \sigma E$ , we have  $(X \circ \varphi)_+ = \varphi[X_+]$ ,  $(X \circ \varphi)_- = \varphi[X_-]$ ,  $(X \circ \varphi)_\pm = \varphi[X_\pm]$ , and  $-(X \circ \varphi) = (-X) \circ \varphi$ . As a consequence, for  $C, D \in \sigma E$ , we have that  $C \perp D$  if and only if  $C \circ \varphi \perp D \circ \varphi$  as well as  $\text{sep}(C \circ \varphi, D \circ \varphi) = \varphi[\text{sep}(C, D)]$ . Furthermore, for  $Y \subseteq E$ , we have  $Y \in \mathcal{C}(M(\mathcal{O}))$  if and only if  $\varphi[Y] \in \mathcal{C}(M(\mathcal{O}))$ , as well as  $Y \in \mathcal{C}(M(\mathcal{O})^*)$  if and only if  $\varphi[Y] \in \mathcal{C}(M(\mathcal{O})^*)$ . With these properties, it is straightforward yet tiresome to verify using the definition of  $\varphi[\mathcal{O}]$ , that the axioms for  $\mathcal{O}$  carry over to  $\varphi[\mathcal{O}]$ .  $\square$

**Definition 3.1.13.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. The **dual oriented matroid of  $\mathcal{O}$**  is the triple  $\mathcal{O}^* = (E, \mathcal{C}^*, \mathcal{C})$ .  $\blacksquare$

**Lemma 3.1.14.** Let  $\mathcal{O}$  be an oriented matroid. Then  $\mathcal{O}^*$  is an oriented matroid, too.

*Proof.* Observe that the axioms  $(\mathcal{C}i)$  for  $\mathcal{O}$  are equivalent to the axioms  $(\mathcal{C}^*i)$  for  $\mathcal{O}^*$ , analogously  $(\mathcal{C}^*i)$  for  $\mathcal{O}$  are equivalent to  $(\mathcal{C}i)$  for  $\mathcal{O}^*$ ; where  $i \in \{1, 2, 3, 4\}$ . Furthermore,  $(\mathcal{O}1)$  is symmetric in itself, so it holds for  $\mathcal{O}$  if and only if it holds for  $\mathcal{O}^*$ . Moreover, every witness  $M(\mathcal{O})$ , that certifies  $(\mathcal{O}2)$  for  $\mathcal{O}$ , yields a witness  $(M(\mathcal{O}))^*$ , that certifies  $(\mathcal{O}2)$  for  $\mathcal{O}^*$ , and vice versa. Therefore, the triple  $\mathcal{O}$  is an oriented matroid if and only if the triple  $\mathcal{O}^*$  is an oriented matroid, so we obtain that  $\mathcal{O}^*$  is an oriented matroid from the premises of this lemma.  $\square$

**Definition 3.1.15.** Let  $E$  be a finite set and  $\mu \in \mathbb{R}^{E \times \{1, 2, \dots, r\}}$  be a matrix. The **oriented matroid represented by  $\mu$**  is the uniquely determined oriented matroid  $\mathcal{O}(\mu) = (E, \mathcal{C}_\mu, \mathcal{C}_\mu^*)$  where

$$\mathcal{C}_\mu = \left\{ C \in \mathcal{D}_\mu \mid \forall C' \in \mathcal{D}_\mu: C'_\pm \subseteq C_\pm \Rightarrow C'_\pm = C_\pm \right\}$$

and

$$\mathcal{D}_\mu = \left\{ E_\alpha \in \sigma E \setminus \{\emptyset_{\sigma E}\} \mid \alpha \in \mathbb{R}^E, \alpha \neq 0, \sum_{e \in E} \alpha(e) \cdot \mu_e = 0 \right\}.$$

**Lemma 3.1.16.** Let  $E$  be a finite set and  $\mu \in \mathbb{R}^{E \times \{1, 2, \dots, r\}}$  be a matrix. Then  $\mathcal{O}(\mu)$  is indeed an oriented matroid.

*Proof.* By Remark 3.1.9, it suffices to show that  $(\mathcal{C}1)$ ,  $(\mathcal{C}2)$ ,  $(\mathcal{C}3)$ , and  $(\mathcal{C}4')$  hold for  $\mathcal{C}_\mu$  in Definition 3.1.15.  $(\mathcal{C}1)$  is obvious from the construction. Let  $C \in \mathcal{C}_\mu$ , then there is a vector  $\alpha \in \mathbb{R}^E$  such that  $\sum_{e \in E} \alpha(e) \cdot \mu_e = 0$ , thus there is  $\alpha' \in \mathbb{R}^E$  with  $\alpha'(e) = -\alpha(e)$  such that

$$\sum_{e \in E} \alpha'(e) \cdot \mu_e = - \sum_{e \in E} \alpha(e) \cdot \mu_e = 0.$$

Clearly,  $E_{\alpha'} = -E_\alpha$  and therefore  $(E_\alpha)_\pm = (E_{\alpha'})_\pm$ . Thus  $-C \in \mathcal{C}_\mu$ , so  $(\mathcal{C}2)$  holds.

— We give an indirect argument for  $(\mathcal{C}3)$ : Let  $X, Y \in \mathcal{C}_\mu$  with  $X_\pm = Y_\pm$  and  $Y \notin \{X, -X\}$ , such that  $X_\pm$  is minimal in  $\mathcal{C}_\mu$  with respect to set-inclusion  $\subseteq$ . There is an element  $f \in (X_+ \cap Y_+) \cup (X_- \cap Y_-)$  and an element  $f' \in \text{sep}(X, Y)$ . Now let  $\alpha, \alpha' \in \mathbb{R}^E$  with  $\sum_{e \in E} \alpha(e) \cdot \mu_e = \sum_{e \in E} \alpha'(e) \cdot \mu_e = 0$  such that  $E_\alpha = X$  and  $E_{\alpha'} = Y$ . Let  $\beta \in \mathbb{R}^E$  where

$$\beta(e) = \alpha(e) - \frac{\alpha(f)}{\alpha'(f)} \alpha'(e)$$

for all  $e \in E$ . Then

$$\begin{aligned} \sum_{e \in E} \beta(e) \cdot \mu_e &= \sum_{e \in E} \left( \alpha(e) - \frac{\alpha(f)}{\alpha'(f)} \alpha'(e) \right) \cdot \mu_e \\ &= \left( \sum_{e \in E} \alpha(e) \cdot \mu_e \right) - \frac{\alpha(f)}{\alpha'(f)} \left( \sum_{e \in E} \alpha'(e) \cdot \mu_e \right) \\ &= 0, \end{aligned}$$

with  $\beta(f) = 0$  and

$$\beta(f') = \alpha(f') - \frac{\alpha(f)}{\alpha'(f)} \cdot \alpha'(f) \neq 0$$

since  $f' \in \text{sep}(E_\alpha, E_{\alpha'})$ , therefore  $\emptyset \neq (E_\beta)_\pm \subsetneq (E_\alpha)_\pm$  which contradicts the minimality of  $X_\pm$ . Therefore our assumption must be wrong and  $Y \in \{X, -X\}$ .

— The proof of  $(\mathcal{C}4')$  is similar: Let  $X, Y \in \mathcal{C}_\mu$  with  $X \neq -Y$ , and let  $f \in X_+ \cap Y_-$ . Again let  $\alpha, \alpha' \in \mathbb{R}^E$  with  $\sum_{e \in E} \alpha(e) \cdot \mu_e = \sum_{e \in E} \alpha'(e) \cdot \mu_e = 0$  such that  $E_\alpha = X$  and  $E_{\alpha'} = Y$ . Define  $\beta \in \mathbb{R}^E$  as above, then again  $\sum_{e \in E} \beta(e) \cdot \mu_e = 0$  and  $\beta(f) = 0$ . Since  $X \neq -Y$ , and obviously  $X \neq Y$ , we obtain that  $X_\pm \neq Y_\pm$  from  $(\mathcal{C}3)$ . Thus there is an element  $g \in (X_\pm \cap Y_0) \cup (X_0 \cap Y_\pm)$ . Since either  $\alpha(g) = 0$  or  $\alpha'(g) = 0$ , we obtain that

$$\beta(g) = \alpha(g) - \frac{\alpha(f)}{\alpha'(f)} \alpha'(g) \neq 0.$$

So  $\emptyset_{\sigma E} \neq E_\beta \in \mathcal{D}_\mu$ . Furthermore, for all  $g \in (X_\pm \cap Y_0)$ , we have  $\beta(g) = \alpha(g)$ , and thus  $E_\beta(g) = X(g)$ . Also, for all  $g \in (X_0 \cap Y_\pm)$  we have

$$\beta(g) = -\frac{\alpha(f)}{\alpha'(f)} \alpha'(g)$$

and since  $\operatorname{sgn}\left(-\frac{\alpha(f)}{\alpha'(f)}\right) = 1$ , we have  $E_\beta(g) = Y(g)$ . Finally, for all  $g \in X_0 \cap Y_0$ , we clearly have  $\beta(g) = 0$ . Thus we found  $Z = E_\beta \in \mathcal{D}_\mu$  with  $Z(f) = 0$ ,  $Z_+ \subseteq X_+ \cup Y_+$ , and  $Z_- \subseteq X_- \cup Y_-$ . We claim that there is some  $Z' \in \mathcal{C}_\mu$  with  $Z' \subseteq_\sigma Z$ , yielding the desired signed circuit for  $(\mathcal{C}_4')$ . We give a constructive argument for this claim. Assume that  $Z \notin \mathcal{C}_\mu$ , then there is an  $Z' \in \mathcal{C}_\mu$  such that  $Z'_\pm \subsetneq Z_\pm$ . Let  $\gamma \in \mathbb{R}^E$  with  $\sum_{e \in E} \gamma(e) \cdot \mu_e = 0$  such that  $E_\gamma = Z'$ . Let  $f \in Z'_\pm \cap Z_\pm$  such that  $\left|\frac{\beta(f)}{\gamma(f)}\right|$  is minimal, thus for all  $f' \in Z'_\pm \cap Z_\pm$  we have

$$|\beta(f')| \geq \left| \frac{\beta(f)}{\gamma(f)} \gamma(f') \right|.$$

Therefore if we let  $\delta \in \mathbb{R}^E$  such that  $\delta(e) = \beta(e) - \frac{\beta(f)}{\gamma(f)} \gamma(e)$ , we have  $\sum_{e \in E} \delta(e) \cdot \mu_e = 0$  and  $\emptyset_{\sigma E} \neq E_\delta \subsetneq_\sigma E_\beta$ , guaranteed by the choice of  $f$ . So we found some  $Z'' = E_\delta \in \mathcal{D}_\mu$  with  $Z'' \subsetneq_\sigma Z$ . If  $Z'' \in \mathcal{C}_\mu$  we are done, otherwise we continue the last construction where  $Z''$  takes on the role of  $Z$ . Since  $E$  is finite and our construction strictly reduces the cardinality of the support of the signed subset in question, we finally construct a signed subset  $Z^{(2n)'}$  with minimal possible support, and therefore we eventually find some  $Z^{(2n)'}$  with  $Z^{(2n)'} \subseteq_\sigma Z$ .  $\square$

**Corollary 3.1.17.** *Let  $E, C$  be finite sets, and  $\mu \in \mathbb{R}^{E \times C}$  a real matrix. Then*

$$M(\mathcal{O}(\mu)) = M(\mu),$$

*i.e. the underlying matroid of the oriented matroid represented by  $\mu$  is the matroid represented by  $\mu$ .*

*Proof.* Obvious from Definition 1.2.55 and Definition 3.1.15.  $\square$

**Definition 3.1.18.** *Let  $M = (E, \mathcal{I})$  be a matroid. Then  $M$  shall be called **orientable matroid**, if there is an oriented matroid  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$ , such that  $M = M(\mathcal{O})$ , or equivalently  $\{C_\pm \mid C \in \mathcal{C}\} = \mathcal{C}(M)$ . Furthermore, every oriented matroid  $\mathcal{O}$  with this property shall be called **orientation of  $M$** .  $\blacksquare$*

**Corollary 3.1.19.** *Every matroid that can be represented over  $\mathbb{R}$  is orientable.*

*Proof.* Follows from Lemma 3.1.16 and Corollary 3.1.17.  $\square$

Thus every gammoid is orientable (Lemma 3.4.1).

**Definition 3.1.20.** *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. We say that  $\mathcal{O}$  is **realizable**, if there is a finite set  $Q$  and a matrix  $\mu \in \mathbb{R}^{E \times Q}$ , such that  $\mathcal{O} = \mathcal{O}(\mu)$ . If there is no such matrix, we shall call  $\mathcal{O}$  **non-realizable**.  $\blacksquare$*



**Remark 3.1.21.** Of course, not every oriented matroid arises in this way from a matrix over a linearly ordered field. H. Miyata, S. Moriyama, and K. Fukuda published a listing of all non-realizable oriented matroids<sup>1</sup> of rank 4 with  $|E| = 8$ , and of rank 3 and rank 6 with  $|E| = 9$ ; the results are based on the oriented matroid database<sup>2</sup> by L. Finschi and K. Fukuda. •

**Example 3.1.22** ([BLS<sup>+</sup>99], p.20). The oriented matroid we want to present now has been named  $\mathbf{RS}(8)$ . It is an orientation of the rank 4 uniform matroid with 8 elements, and therefore clearly an orientation of a gammoid. It has  $2 \cdot \binom{8}{5} = 112$  signed circuits as well as 112 signed cocircuits. Since these signed subsets come in pairs  $X$  and  $-X$ , we only have to list half of them. Let  $E = \{1, 2, \dots, 8\}$ . Then  $\mathbf{RS}(8) = (E, \mathcal{C}, \mathcal{C}^*)$  where

$$\begin{aligned} \mathcal{C} = \pm\{ & \{ 1, -2, -3, 4, -5\}, \{ 1, 2, -3, 4, -6\}, \{ 1, -2, 3, 4, -7\}, \\ & \{ 1, -2, -3, 4, -8\}, \{-1, 2, -3, 5, -6\}, \{-1, 2, 3, 5, -7\}, \\ & \{-1, 2, 3, 5, -8\}, \{ 1, -2, 3, 6, -7\}, \{-1, -2, 3, 6, -8\}, \\ & \{-1, 2, -3, 7, -8\}, \{-1, 2, -4, 5, -6\}, \{ 1, -2, 4, -5, -7\}, \\ & \{-1, -2, 4, 5, -8\}, \{ 1, 2, 4, -6, -7\}, \{-1, -2, 4, 6, -8\}, \\ & \{-1, 2, -4, 7, 8\}, \{-1, 2, 5, -6, -7\}, \{-1, 2, 5, -6, -8\}, \\ & \{ 1, -2, -5, -7, 8\}, \{ 1, 2, -6, -7, 8\}, \{-1, 3, -4, 5, 6\}, \\ & \{ 1, -3, -4, -5, 7\}, \{ 1, -3, -4, -5, 8\}, \{ 1, 3, 4, -6, -7\}, \\ & \{ 1, -3, 4, -6, -8\}, \{-1, 3, -4, -7, 8\}, \{-1, 3, 5, 6, -7\}, \\ & \{-1, 3, 5, 6, -8\}, \{ 1, -3, -5, 7, -8\}, \{ 1, 3, -6, -7, 8\}, \\ & \{ 1, 4, -5, -6, -7\}, \{-1, 4, 5, -6, -8\}, \{ 1, -4, -5, -7, 8\}, \\ & \{ 1, 4, -6, -7, -8\}, \{-1, 5, 6, 7, -8\}, \{-2, 3, -4, -5, 6\}, \\ & \{ 2, -3, -4, -5, 7\}, \{ 2, 3, -4, -5, 8\}, \{ 2, -3, 4, -6, 7\}, \\ & \{ 2, 3, -4, -6, 8\}, \{-2, 3, -4, -7, 8\}, \{-2, 3, 5, 6, -7\}, \\ & \{-2, 3, -5, 6, -8\}, \{ 2, -3, -5, 7, -8\}, \{ 2, -3, -6, 7, 8\}, \\ & \{ 2, 4, 5, -6, -7\}, \{-2, 4, -5, 6, -8\}, \{ 2, -4, -5, 7, 8\}, \\ & \{ 2, -4, -6, 7, 8\}, \{-2, -5, 6, 7, -8\}, \{ 3, 4, 5, -6, -7\}, \\ & \{-3, 4, 5, -6, -8\}, \{ 3, -4, -5, -7, 8\}, \{ 3, -4, -6, -7, 8\}, \\ & \{-3, -5, 6, 7, -8\}, \{-4, -5, 6, 7, 8\} \end{aligned}$$

<sup>1</sup>See: [http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented\\_matroids/](http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/) [MMF]

<sup>2</sup>See: <http://www.om.math.ethz.ch/> [Fin]

and where  $\mathcal{C}^*$  is uniquely determined by  $\mathcal{C}$ .<sup>3</sup> Proposition 1.5.1 in [BLS<sup>+</sup>99] states that the oriented matroid  $\mathbf{RS}(8)$  is a non-realizable orientation of the underlying uniform matroid, thus we may expect gammoids to have non-realizable orientations. For the full proof, we refer the reader to p. 23 in [BLS<sup>+</sup>99]. The idea of the proof is the following: Assume that  $\mathbf{RS}(8)$  is realizable, then there is a matrix  $\mu \in \mathbb{R}^{\{1,2,\dots,8\} \times \{1,2,3,4\}}$  such that  $\mu|_{\{1,2,3,4\} \times \{1,2,3,4\}}$  is an identity matrix. This leaves us with a variable matrix  $\mu|_{\{5,6,7,8\} \times \{1,2,3,4\}}$ , for which we would have to find values that yield the correct signed circuits of  $\mathbf{RS}(8)$ . The signed circuits of  $\mathbf{RS}(8)$  can be translated to strict inequalities that  $\mu$  must obey. For instance, the signed circuit  $\{1, -2, -3, 4, -5\}$  states that  $\mu_5 = \alpha\mu_1 - \beta\mu_2 - \gamma\mu_3 + \delta\mu_4$  must have a solution with  $\alpha, \beta, \gamma, \delta > 0$ . In this particularly easy case, we obtain the inequalities  $\mu(5, 1) > 0$ ,  $\mu(5, 2) < 0$ ,  $\mu(5, 3) < 0$ ,  $\mu(5, 4) > 0$ .<sup>4</sup> The proof of non-realizability is completed by the observation that the constructed system of inequalities has no solutions, therefore there is no matrix  $\mu$  with  $\mathcal{O}(\mu) = \mathbf{RS}(8)$ , and  $\mathbf{RS}(8)$  is non-realizable.  $\blacktriangle$

For every realizable oriented matroid of the form  $\mathcal{O}(\mu)$ , we obtain another realizable oriented matroid  $\mathcal{O}(\mu')$  where  $\mu'$  is obtained from  $\mu$  by multiplying an arbitrary set of rows with  $-1$ . Clearly, for the underlying matroids we have  $M(\mu) = M(\mu')$ . It is

<sup>3</sup>There has not been a complete example of an oriented matroid in this work so far, thus we present  $\mathcal{C}^*$  in order for the reader to check their understanding of the signed cocircuits of an oriented matroid with a non-trivial example.

$$\begin{aligned} \mathcal{C}^* = \pm\{ & \{-1, 2, 3, 4, -5\}, \{-1, -2, -3, -4, -6\}, \{1, 2, 3, 4, 7\}, \{-1, -2, -3, 4, 8\}, \\ & \{-1, -2, -3, -5, -6\}, \{-1, 2, -3, -5, -7\}, \{-1, 2, 3, -5, -8\}, \{1, -2, 3, -6, 7\}, \\ & \{-1, -2, -3, -6, 8\}, \{1, 2, 3, 7, -8\}, \{1, 2, -4, 5, 6\}, \{1, -2, -4, 5, 7\}, \\ & \{1, 2, -4, 5, -8\}, \{-1, -2, -4, -6, 7\}, \{-1, -2, -4, -6, -8\}, \{1, 2, -4, -7, -8\}, \\ & \{-1, -2, -5, -6, 7\}, \{1, 2, 5, 6, 8\}, \{1, -2, 5, 7, 8\}, \{1, 2, 6, -7, -8\}, \\ & \{-1, 3, 4, -5, -6\}, \{1, 3, 4, 5, 7\}, \{-1, 3, 4, -5, 8\}, \{1, 3, 4, -6, 7\}, \\ & \{-1, -3, 4, 6, 8\}, \{1, -3, -4, -7, -8\}, \{-1, -3, -5, -6, -7\}, \{-1, 3, -5, -6, -8\}, \\ & \{1, 3, 5, 7, -8\}, \{-1, -3, 6, -7, 8\}, \{1, -4, 5, 6, 7\}, \{-1, 4, -5, 6, 8\}, \\ & \{1, -4, 5, 7, -8\}, \{1, -4, -6, -7, -8\}, \{1, 5, 6, 7, 8\}, \{-2, -3, -4, 5, -6\}, \\ & \{2, 3, 4, -5, 7\}, \{-2, -3, 4, 5, 8\}, \{-2, 3, -4, -6, 7\}, \{2, -3, 4, 6, 8\}, \\ & \{2, 3, 4, 7, 8\}, \{-2, 3, -5, -6, 7\}, \{-2, -3, 5, -6, 8\}, \{-2, -3, 5, -7, 8\}, \\ & \{2, -3, 6, -7, -8\}, \{-2, -4, 5, -6, 7\}, \{2, 4, -5, 6, 8\}, \{2, -4, -5, -7, -8\}, \\ & \{2, 4, 6, 7, 8\}, \{-2, 5, -6, 7, 8\}, \{-3, -4, 5, 6, -7\}, \{-3, 4, 5, 6, 8\}, \\ & \{-3, -4, -5, -7, -8\}, \{-3, -4, 6, -7, -8\}, \{3, -5, -6, 7, -8\}, \{-4, -5, -6, -7, -8\}. \end{aligned}$$

<sup>4</sup>In general, the strict inequalities derived are not linear, as Cramer's rule yields polynomial terms eliminating the coefficients of the non-trivial linear combinations of the zero that correspond to signed circuits of  $\mathcal{O}$ .

easy to see that the next definition carries this operation over to all oriented matroids ([BLS<sup>+</sup>99], p.3).

**Definition 3.1.23.** Let  $E$  be a set,  $X \subseteq E$ , and  $C \in \sigma E$  a signed subset of  $E$ . The  **$X$ -flip of  $C$**  is defined to be the signed subset

$$C_{-X}: E \longrightarrow \{-1, 0, 1\}, \quad e \mapsto \begin{cases} -C(e) & \text{if } e \in X, \\ C(e) & \text{otherwise.} \end{cases}$$

■

**Definition 3.1.24.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid, and let  $X \subseteq E$ . The  **$X$ -flip reorientation of  $\mathcal{O}$**  is the triple  $\mathcal{O}_{-X} = (E, \mathcal{C}_{-X}, \mathcal{C}_{-X}^*)$  where

$$\mathcal{C}_{-X} = \{C_{-X} \mid C \in \mathcal{C}\} \quad \text{and} \quad \mathcal{C}_{-X}^* = \{C'_{-X} \mid C' \in \mathcal{C}^*\}.$$

Let  $\mathcal{O}'$  also be an oriented matroid. We say that  $\mathcal{O}'$  is a **reorientation of  $\mathcal{O}$** , if there is a subset  $X \subseteq E$  with  $\mathcal{O}' = \mathcal{O}_{-X}$ . ■

**Lemma 3.1.25.** Let  $\mathcal{O} = (E, \mathcal{C})$  be an oriented matroid, and let  $X \subseteq E$ . Then  $\mathcal{O}_{-X}$  is an oriented matroid.

*Proof.* For every  $X \subseteq E$  and  $C \in \sigma E$ , it is clear from Definition 3.1.23, that  $(C_{-X})_{-X} = C$ , therefore the map  $\varphi_X: \sigma E \longrightarrow \sigma E, C \mapsto C_{-X}$  is an involution on  $\sigma E$ . Since  $(\emptyset_{\sigma E})_{-X} = \emptyset_{\sigma E}$ , we obtain that  $\emptyset_{\sigma E} \notin \mathcal{C}_{-X}$  from  $\emptyset_{\sigma E} \notin \mathcal{C}$ , thus (C1) holds. Furthermore, for any  $C \in \sigma E$  we have  $(C_{-X})_+ = (C_+ \setminus X) \cup (C_- \cap X)$  and  $(C_{-X})_- = (C_- \setminus X) \cup (C_+ \cap X)$ . In particular we have for  $C, D \in \sigma E$  that  $C = -D$  if and only if  $C_{-X} = -D_{-X}$ . We also have  $C \perp D$  if and only if  $C_{-X} \perp D_{-X}$ : Case (ii) of Definition 3.1.5 is oblivious of any sign flips in  $C$  and  $D$  since  $C_0 = (C_{-X})_0$  and  $D_0 = (D_{-X})_0$ ; whereas in case (i) the passage from  $C$  and  $D$  to  $C_{-X}$  and  $D_{-X}$ , respectively, introduces an even amount of sign flips. So  $C(e)D(e) = C_{-X}(e)D_{-X}(e)$  and  $C(f)D(f) = C_{-X}(f)D_{-X}(f)$ . Therefore (C2), (C3), and (C4) carry over from  $\mathcal{C}$  to  $\mathcal{C}_{-X}$ . Since  $\{C_{\pm} \mid C \in \mathcal{C}\} = \{C_{\pm} \mid C \in \mathcal{C}_{-X}\}$ ,  $\{C'_{\pm} \mid C' \in \mathcal{C}^*\} = \{C'_{\pm} \mid C' \in \mathcal{C}_{-X}^*\}$ , and for all  $C \in \mathcal{C}_{-X}$  and  $D \in \mathcal{C}_{-X}^*$ , we have  $C \perp D$ ; we obtain that  $\mathcal{C}_{-X}^*$  is indeed the unique family of signed cocircuits of the oriented matroid on  $E$  with the family of signed circuits  $\mathcal{C}_{-X}$ , thus  $\mathcal{O}_{-X}$  is an oriented matroid (Remark 3.1.9). □

**Corollary 3.1.26.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid, and let  $X \subseteq E$ . Then  $M(\mathcal{O}_{-X}) = M(\mathcal{O})$ .

*Proof.* For  $C \in \sigma E$ , we have  $C_{\pm} = (C_{-X})_{\pm}$ .  $\square$

**Definition 3.1.27.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. The **reorientation class** of  $\mathcal{O}$  is defined to be  $[\mathcal{O}] = \{\mathcal{O}_{-X} \mid X \subseteq E\}$ .  $\blacksquare$

**Example 3.1.28.** According to L. Finschi's database [Fin], the gammoid from Example 2.2.17 has exactly one equivalence class with respect to relabeling and reorientation of oriented matroids. Thus it is easy to check that it has two reorientation classes,  $\mathcal{O}_1 = (E, \mathcal{C}_1, \mathcal{C}_1^*)$  and  $\mathcal{O}_2 = (E, \mathcal{C}_2, \mathcal{C}_2^*)$  where

$$\begin{aligned} \mathcal{C}_1 = \pm\{ & \{a, b, -c, e\}, \{a, b, -d, -f\}, \{b, -c, -d, g\}, \{d, e, f, -g\}, \\ & \{-a, b, -c, f, g\}, \{-a, -c, d, e, f\}, \{-a, -c, d, f, g\}, \\ & \{a, b, d, e, -g\}, \{a, b, e, -f, -g\}, \{a, c, d, e, -g\}, \\ & \{a, c, e, -f, -g\}, \{b, -c, d, e, f\}, \{b, -c, e, f, g\} \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_2 = \pm\{ & \{-a, b, c, e\}, \{a, -b, d, -f\}, \{b, c, -d, -g\}, \{d, e, -f, g\}, \\ & \{-a, b, -c, f, g\}, \{-a, b, d, e, g\}, \{-a, b, e, f, g\}, \\ & \{-a, -c, d, e, g\}, \{-a, -c, d, f, g\}, \{a, c, d, e, -f\}, \\ & \{a, c, e, -f, -g\}, \{b, c, d, e, -f\}, \{b, c, e, -f, -g\} \} \end{aligned}$$

Here,  $[\mathcal{O}_2] = [\varphi[\mathcal{O}_1]]$  where  $\varphi = (ac)(fg)$  is a corresponding relabeling.  $\blacktriangle$

**Lemma 3.1.29.** Let  $E, Y$  be finite sets,  $\mu \in \mathbb{R}^{E \times Y}$ , and  $X \subseteq E$ . Let  $\nu \in \mathbb{R}^{E \times Y}$  be the matrix where for every  $e \in E$  and  $y \in Y$ ,

$$\nu(e, y) = \begin{cases} -\mu(e, y) & \text{if } e \in X, \\ \mu(e, y) & \text{otherwise.} \end{cases}$$

Then  $(\mathcal{O}(\mu))_{-X} = \mathcal{O}(\nu)$ .

*Proof.* Let  $\alpha \in \mathbb{R}^E$ . Let  $\beta \in \mathbb{R}^E$  be defined such that

$$\beta(e) = \begin{cases} -\alpha(e) & \text{if } e \in X, \\ \alpha(e) & \text{otherwise.} \end{cases}$$

Clearly,  $(E_\alpha)_{-X} = E_\beta$  and  $\sum_{e \in E} \beta(e) \cdot \nu(e) = \sum_{e \in E} \alpha(e) \cdot \mu(e)$ . Thus  $\sum_{e \in E} \alpha(e) \cdot \mu(e) = 0$  if and only if  $\sum_{e \in E} \beta(e) \cdot \nu(e) = 0$ , and consequently  $E_\alpha \in \mathcal{C}_\mu$  if and only if  $E_\beta \in \mathcal{C}_\nu$ . Therefore  $(\mathcal{O}(\mu))_{-X} = \mathcal{O}(\nu)$ .  $\square$

**Corollary 3.1.30.** *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid, and let  $X \subseteq E$ . Then  $\mathcal{O}$  is realizable if and only if  $\mathcal{O}_{-X}$  is realizable.*

**Definition 3.1.31.** *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be a oriented matroid,  $R \subseteq E$ . The **restriction of  $\mathcal{O}$  to  $R$**  is the triple  $\mathcal{O}|R = (R, \mathcal{C}_R, \mathcal{C}_R^*)$  where*

$$\mathcal{C}_R = \{C \in \sigma R \mid \exists D \in \mathcal{C}: D_\pm \subseteq R \text{ s.t. } D|_R = C\}$$

and

$$\mathcal{C}_R^* = \{C' \in \mathcal{D}_R^* \mid \nexists D' \in \mathcal{D}_R^*: D'_\pm \subsetneq C'_\pm\},$$

and where

$$\mathcal{D}_R^* = \{C' \in \sigma R \setminus \{\emptyset_{\sigma R}\} \mid \exists D' \in \mathcal{C}^*: D'|_R = C'\}.$$

Let  $Q \subseteq E$ . The **contraction of  $\mathcal{O}$  to  $Q$**  is the triple  $\mathcal{O}.Q = (Q, \mathcal{C}_{i_Q}, \mathcal{C}_{i_Q}^*)$  where

$$\mathcal{C}_{i_Q}^* = \{C' \in \sigma R \mid \exists D' \in \mathcal{C}^*: D'_\pm \subseteq R \text{ s.t. } D'|_R = C'\}$$

and

$$\mathcal{C}_{i_Q} = \{C \in \mathcal{D}_{i_Q} \mid \nexists D \in \mathcal{D}_{i_Q}: D_\pm \subsetneq C_\pm\},$$

and where

$$\mathcal{D}_{i_Q} = \{C \in \sigma R \setminus \{\emptyset_{\sigma R}\} \mid \exists D \in \mathcal{C}: D|_R = C\}. \quad \blacksquare$$

**Lemma 3.1.32.** *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be a oriented matroid,  $X \subseteq E$ . Then  $\mathcal{O}|X$  and  $\mathcal{O}.X$  are oriented matroids, and further*

$$(\mathcal{O}^*|X)^* = \mathcal{O}.X \quad \text{as well as} \quad (\mathcal{O}^*.X)^* = \mathcal{O}|X$$

holds.

For a proof, please refer to Propositions 3.3.1 and 3.3.2 ([BLS<sup>+</sup>99], p.110) in *Oriented Matroids* by A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler.

## 3.2 Colorings

The notion of colorings used in this work originates from *Antisymmetric Flows in Matroids* by J. Nešetřil and W. Hochstättler [HN06]. A recommended source for the properties and bearings of this notion is R. Nickel's thesis *Flows and Colorings in Oriented Matroids* [Nic12].

**Remark 3.2.1.** The first appearance of a notion of a coloring of general oriented matroids can be tracked down to the paper *On  $(k, d)$ -Colorings and Fractional Nowhere-Zero Flows* by L.A. Goddyn, M. Tarsi, and C.-Q. Zhang [GTZ98], who define the *star flow index* of an reorientation class  $[\mathcal{O}']$  of oriented matroids to be

$$\xi^*([\mathcal{O}']) = \min_{\mathcal{O} \in [\mathcal{O}']} \max_{D \in \mathcal{C}_\mathcal{O}^*} \frac{|D_\pm|}{|D_+|}$$

where  $\mathcal{C}_\mathcal{O}$  denotes the family of signed cocircuits of  $\mathcal{O}$ . The star flow index is closely related to the chromatic number of a graph  $G$  through a result of J.G. Minty [Min62]: If  $G = (V, E)$  is a graph, and  $\mu \in \mathbb{R}^{E \times V}$  is the signed edge-vertex-incidence matrix of an orientation of the edges of  $G$ , then for  $\mathcal{O}' = (\mathcal{O}(\mu))^*$  we have  $\lceil \xi^*([\mathcal{O}']) \rceil = \chi(G)$  where  $\chi(G)$  is the well-known chromatic number of  $G$ . *We would like to point out that this is **not the chromatic number** of oriented matroids that **we are concerned with** in this work.*<sup>5</sup> •

**Definition 3.2.2.** Let  $E$  be a set. A **signed multi-subset of  $E$**  – or shorter: **signed multiset** – is a map  $S: E \rightarrow \mathbb{Z}$ . The **family of signed multi-subsets of  $E$**  shall be denoted by  $\mathbb{Z}.E$ . Since  $\{-1, 0, 1\} \subseteq \mathbb{Z}$ , we shall identify the signed subsets with the corresponding signed multisets, i.e.  $\sigma E \equiv \{F \in \mathbb{Z}.E \mid \forall e \in E: F(e) \in \{-1, 0, 1\}\}$ . The **empty signed multi-subset of  $E$**  is the map

$$\emptyset_{\mathbb{Z}.E}: E \rightarrow \mathbb{Z}, e \mapsto 0. \quad \blacksquare$$

<sup>5</sup>For further details on the differences between the oriented flow number of L.A. Goddyn, M. Tarsi, and C.-Q. Zhang [GTZ98] and the chromatic number of J. Nešetřil and W. Hochstättler [HN06], see [Nic12], pp. 98f.

**Definition 3.2.3** (Dual of Definition 1, [HN06]). Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. The **coflow lattice of  $\mathcal{O}$**  shall consist of all integral linear combinations of cocircuits of  $\mathcal{O}$ , i.e.

$$\mathbb{Z}\mathcal{C}^* = \left\{ F \in \mathbb{Z}E \mid \exists \alpha \in \mathbb{Z}^{\mathcal{C}^*} : \forall e \in E : F(e) = \sum_{C' \in \mathcal{C}^*} \alpha(C') \cdot C'(e) \right\}.$$

Each element  $F \in \mathbb{Z}\mathcal{C}^*$  shall be called a **coflow of  $\mathcal{O}$** . A **nowhere-zero coflow of  $\mathcal{O}$**  is a coflow  $F \in \mathbb{Z}\mathcal{C}^*$  where  $F(e) \neq 0$  for all  $e \in E$ . ■

**Definition 3.2.4.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. We define the **chromatic number of  $\mathcal{O}$**  to be

$$\chi(\mathcal{O}) = \min \left\{ \max \{ |F(e)| + 1 \mid e \in E \} \mid F \in \mathbb{Z}\mathcal{C}^*, \forall e \in E : F(e) \neq 0 \right\}.$$

By convention, we set  $\chi(\mathcal{O}) = \infty$  if there is no nowhere-zero coflow in  $\mathbb{Z}\mathcal{C}^*$ . ■

The only oriented matroid  $\mathcal{O}$  with  $\chi(\mathcal{O}) = 1$  is the trivial oriented matroid,<sup>6</sup> i.e. the oriented matroid  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  where  $E = \mathcal{C} = \mathcal{C}^* = \emptyset$ .

**Remark 3.2.5.** Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid. We have  $\chi(\mathcal{O}) = \infty$  if and only if there is an element  $e \in E$  such that  $D(e) = 0$  for all  $D \in \mathcal{C}^*$ : Let  $\mathcal{C}^* = \pm\{D_1, D_2, \dots, D_k\}_{\neq}$ , i.e. for each pair  $D \in \mathcal{C}^*$  we choose precisely one element from  $\{D, -D\}$ . Then there is no cancellation of non-zero summands in

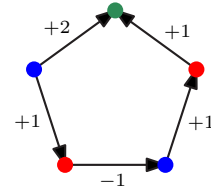
$$F(e) = \sum_{i=1}^k 2^{i-1} \cdot D_i(e)$$

for  $e \in E$ , thus  $F$  is a nowhere-zero coflow of  $\mathcal{O}$  if and only if for every  $e \in E$  there is some  $D \in \mathcal{C}^*$  with  $e \in D_{\pm}$ . This is the case if and only if  $M(\mathcal{O})$  has no loop. •

We give a quick tour justifying why the name *chromatic number* is appropriate in this context. For a more detailed introduction, we refer the reader to Chapter 4 in [Nic12].

<sup>6</sup>That is, if we set  $\min \{ \max \{ |F(e)| + 1 \mid e \in \emptyset \} \mid F \in \mathbb{Z}\emptyset \} = 1$ . The rationale behind this is that a matroid  $M = (E, \mathcal{I})$  with  $E = \emptyset$  is the graphical matroid of every edge-less graph; and the trivial oriented matroid is an orientation of  $M$ .

**Example 3.2.6.** Consider the undirected graph  $G = (V, E)$  where  $V = \{1, 2, 3, 4, 5\}$  and  $E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ . A proper coloring of  $G$  is a map  $\varphi: V \rightarrow \mathbb{Z}$  such that  $\varphi(v) \neq \varphi(w)$  whenever  $\{v, w\} \in E$ . The chromatic number of  $G$  is defined as  $\chi(G) = \min \{|\varphi[V]| \mid \varphi \text{ proper coloring of } G\}$ , in this case  $\chi(G) = 3$  and  $\varphi(1) = \varphi(3) = 1$ ,  $\varphi(2) = \varphi(4) = 2$ ,  $\varphi(5) = 3$  is a corresponding proper coloring. An orientation of  $G$  is a digraph  $D = (V, A)$  such that for every  $\{u, v\} \in E$  we have the equivalency  $(u, v) \in A \Leftrightarrow (v, u) \notin A$ , and such that  $E = \{\{u, v\} \mid (u, v) \in A\}$ . Every orientation  $D$  of  $G$  gives rise to a map  $\sigma: V \times V \rightarrow \{-1, 0, 1\}$  where  $\sigma(u, v) = +1$  if  $(u, v) \in A$ ,  $\sigma(u, v) = -1$  if  $(v, u) \in A$ , and  $\sigma(u, v) = 0$  if  $\{u, v\} \notin E$ . A nowhere-zero-coflow on  $G$  with respect to the orientation  $D$  is a map  $f: E \rightarrow \mathbb{Z}$ , such that  $f(e) \neq 0$  for all  $e \in E$ , and such that for every closed walk  $v_1 v_2 \dots v_k$  in  $G$ , i.e.  $v_1 = v_k$ , we have



$$\sum_{i=1}^{k-1} (\sigma(v_i, v_{i+1}) \cdot f(\{v_i, v_{i+1}\})) = 0.$$

Every coloring  $\varphi: V \rightarrow \mathbb{Z}$  induces a coflow  $\hat{\varphi}: E \rightarrow \mathbb{Z}$  on  $G$  with respect to an orientation  $D$  by setting  $\hat{\varphi}(\{u, v\}) = \sigma(u, v) \cdot (\varphi(v) - \varphi(u))$ . Furthermore,  $\hat{\varphi}$  is a nowhere-zero-coflow if and only if  $\varphi$  is a proper coloring of  $G$ . Conversely, if  $f$  is a nowhere-zero-coflow on  $G$  with respect to  $D$ , then we may reconstruct a proper coloring  $\tilde{f}: V \rightarrow \mathbb{Z}$  from it by choosing a vertex  $v$  per component of  $G$  and setting  $\tilde{f}(v) = 0$ . For every other vertex  $x$ , let  $w_1 w_2 \dots w_k$  be a walk from the chosen vertex  $v = w_1$  of the component containing  $x$  to  $x = w_k$  in  $G$ . We set

$$\tilde{f}(x) = \left( \sum_{i=1}^{k-1} (\sigma(w_i, w_{i+1}) \cdot f(\{w_i, w_{i+1}\})) \right) \bmod_{\max\{|f(e)|+1 \mid e \in E\}}.$$

Then  $\tilde{f}(x)$  is a proper coloring of  $G$  which uses at most  $\max\{|f(e)|+1 \mid e \in E\}$  colors. Furthermore, every graph  $G = (V, E)$  gives rise to a cycle matroid  $M(G) = (E, \mathcal{I})$  where a set of edges  $X \subseteq E$  is independent, if and only if  $(V, X)$  does not contain a cycle walk<sup>7</sup> (see [Wel76], p.10). The cycle matroid associated with the above graph  $G$  is  $M(G) = (E, \{X \subseteq E \mid |X| \leq 4\})$ , the uniform matroid of rank 4 on  $E$ . The cocircuits of cycle matroids are the  $\subseteq$ -minimal subsets  $D \subseteq E$ , such that the graph  $G \setminus X = (V, E \setminus X)$  has more components than  $G = (V, E)$ . Furthermore, every orientation  $D$  of  $G$  yields an oriented matroid  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  with  $M(\mathcal{O}) = M(G)$  by

<sup>7</sup>Remember that the trivial walk  $v$  is not a cycle walk.



the following construction:  $C \in \mathcal{C}$  if and only if there is a cycle walk  $x_1x_2\dots x_k$  in  $G$  with  $C_{\pm} = \{x_1, x_2, \dots, x_k\}$  and  $C(\{x_i, x_{i+1}\}) = \sigma(x_i, x_{i+1})$  for all  $i \in \{1, 2, \dots, k-1\}$ . In other words, an edge in the support of a signed circuit  $C$  of  $\mathcal{O}$  is assigned  $+1$  if its orientation agrees with the corresponding arc in the cycle walk, and  $-1$  otherwise. Dually, we have  $D \in \mathcal{C}^*$  if and only if there is a minimal edge-cut  $X \subseteq E$  and a partition  $L, R$  of  $V$  such that every edge  $e \in X$  has the property  $|e \cap L| = 1$ ; with  $D_{\pm} = X$  and  $D(\{l, r\}) = \sigma(l, r)$  for all  $l \in L$  and  $r \in R$  with  $\{l, r\} \in X$ . Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be the oriented matroid that corresponds to the orientation  $D$  of  $G$  as above. We have  $\mathcal{C} = \pm \left\{ \left\{ \{1, 2\}, -\{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\} \right\} \right\}$  and

$$\begin{aligned} \mathcal{C}^* = \pm \left\{ \right. & \left\{ \{1, 2\}, \{1, 5\} \right\}, \left\{ \{1, 2\}, -\{2, 3\} \right\}, \left\{ \{1, 2\}, -\{3, 4\} \right\}, \left\{ \{1, 2\}, -\{4, 5\} \right\}, \\ & \left\{ \{2, 3\}, \{1, 5\} \right\}, \left\{ \{2, 3\}, -\{3, 4\} \right\}, \left\{ \{2, 3\}, -\{4, 5\} \right\}, \\ & \left. \left\{ \{3, 4\}, \{1, 5\} \right\}, \left\{ \{3, 4\}, -\{4, 5\} \right\}, \left\{ \{4, 5\}, \{1, 5\} \right\} \right\}. \end{aligned}$$

Furthermore, all coflows of  $G$  with respect to  $D$  are integral linear combinations of the signed cocircuits of the oriented matroid  $\mathcal{O}$  corresponding to the orientation  $D$  of  $G$ ; the coflow  $\hat{\varphi}$  may be written as the a linear combination of cocircuits of  $\mathcal{O}$   $\hat{\varphi} = \left\{ \{1, 2\}, -\{2, 3\} \right\} + \left\{ \{3, 4\}, -\{4, 5\} \right\} + 2 \cdot \left\{ \{4, 5\}, \{1, 5\} \right\}$ . For all graphs  $G = (V, A)$ , the equation  $\chi(G) = \chi(\mathcal{O})$  holds, where  $\mathcal{O}$  corresponds to an orientation  $D$  of the cycle matroid of  $G$ . Thus the chromatic number of oriented matroids is a generalization of the chromatic number of graphs.  $\blacktriangle$

**Example 3.2.7.** Consider the oriented matroids  $\mathcal{O}_1$  and  $\mathcal{O}_2$  given in Example 3.1.28.  $M(\mathcal{O}_1) = M(\mathcal{O}_2) = (E, \mathcal{I})$  is the matroid given in Example 2.2.17. For both orientations, the corresponding coflow lattice is the free integer module  $\mathbb{Z}^E$ . Therefore we have  $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 2$ .  $\blacktriangle$

Let  $\mathcal{O}$  and  $\mathcal{O}'$  be two oriented matroids such that  $M(\mathcal{O}) = M(\mathcal{O}')$ . We would like to mention that it is still an open problem whether in this case the equation  $\chi(\mathcal{O}) = \chi(\mathcal{O}')$  holds in general ([Nic12] (Q4), p.69). This question clearly is beyond the scope of this work. However, if  $\mathcal{O}' = \mathcal{O}_{-X}$  is the reorientation of  $\mathcal{O}$  with respect to some set  $X \subseteq E$ , then

$$\mathbb{Z}\mathcal{C}_{-X}^* = \left\{ F \in \mathbb{Z}^E \mid \exists F' \in \mathbb{Z}\mathcal{C}^* : \forall e \in E : F(e) = (-1)^{\chi_X(e)} F'(e) \right\}$$

where  $\chi_X$  is the characteristic function of  $X \subseteq E$ , i.e.  $\chi_X(e) = 1$  if  $e \in X$  and  $\chi_X(e) = 0$  if  $e \notin X$ . Therefore the nowhere-zero coflows of  $\mathcal{O}$  are in a  $|\cdot|$ -preserving one-to-one

correspondence with the nowhere-zero coflows of  $\mathcal{O}'$ , thus  $\chi(\mathcal{O}) = \chi(\mathcal{O}')$  whenever  $\mathcal{O}'$  is a reorientation of  $\mathcal{O}$ .

**Theorem 3.2.8** ([HN06], Theorem 1). *Let  $r \in \mathbb{N}$  and  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid such that  $M(\mathcal{O}) = (E, \{X \subseteq E \mid |X| \leq r\})$  is the uniform matroid of rank  $r$  on  $E$ . Then*

$$\chi(\mathcal{O}) = \begin{cases} 2 & \text{if } n \cdot (n-r) \text{ is even,} \\ 3 & \text{if } n \cdot (n-r) \text{ is odd.} \end{cases}$$

See [HN06] for the proof. Theorem 3.2.8 is the generalization of the fact that the chromatic number of a cycle graph – with at least 3 vertices – is 2 if the cycle graph consists of an even number of vertices, and 3 if the cycle graph consists of an odd number of vertices.

**Theorem 3.2.9** ([HN08], Theorem 3). *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be an oriented matroid such that  $M(\mathcal{O})$  has no loops, no parallel edges, and  $\text{rk}_{M(\mathcal{O})}(E) \geq 3$ . Then*

$$\chi(\mathcal{O}) \leq \text{rk}_{M(\mathcal{O})}(E) + 1$$

where equality holds if and only if  $M(\mathcal{O})$  is isomorphic to the cycle matroid  $M(K)$  of the complete graph  $K = (V, \binom{V}{2})$  with  $|V| = \text{rk}_{M(\mathcal{O})}(E) + 1$  vertices.

See [HN08] for the proof. If  $M(\mathcal{O})$  has no loops but it has two parallel edges, i.e. some  $e, f \in E$  with  $e \neq f$  and  $\text{rk}_{M(\mathcal{O})}(\{e, f\}) = 1$ , then  $\{e, f\}$  is a circuit of  $M(\mathcal{O})$ . By Lemma 1.2.35 we obtain that the equality  $e \in D_{\pm} \Leftrightarrow f \in D_{\pm}$  holds for every  $D \in \mathcal{C}^*$ . Let  $C \in \mathcal{C}$  be the signed circuit with  $C_{\pm} = \{e, f\}$ , and let  $D_1, D_2 \in \mathcal{C}^*$  with  $e \in (D_{1\pm} \cap D_{2\pm})$ . Since  $C \perp D_1$  and  $C \perp D_2$ , we obtain that  $D_1(e)D_2(e) = D_1(f)D_2(f)$ , i.e. the sign of  $e$  uniquely determines the sign of  $f$  in any cocircuit of  $\mathcal{O}$ . In the proof of Lemma 1.2.35 we established that cocircuits are the complements of hyperplanes, therefore every signed cocircuits  $D'$  of the restriction  $\mathcal{O}|(E \setminus \{f\})$  corresponds to a signed cocircuit  $D$  of  $\mathcal{O}$  where  $\{e, f\} \subseteq D_{\pm}$  if and only if  $e \in D'_{\pm}$ . Thus a nowhere-zero coflow  $\varphi'$  of  $\mathcal{O}|(E \setminus \{f\})$  extends naturally to a nowhere-zero coflow  $\varphi$  of  $\mathcal{O}$  with  $\varphi'(f) \in \{-\varphi(e), \varphi(e)\}$  by taking any integer linear combination of cocircuits of the restriction  $\mathcal{O}|(E \setminus \{f\})$  with respect to the corresponding cocircuits of  $\mathcal{O}$ . Consequently,  $\chi(\mathcal{O}) = \text{rk}_{M(\mathcal{O})}(E) + 1$  if and only if  $M(\mathcal{O})$  is isomorphic to the cycle matroid of a multi-graph on  $\text{rk}_{M(\mathcal{O})}(E) + 1$  vertices that has at least one edge between every pair of distinct vertices.

### 3.3 Lattice Path Matroids are 3-Colorable

The results presented in this section have been presented in the technical report *Lattice Path Matroids are 3-Colorable* by I. Albrecht and W. Hochstättler [AH15].

**Definition 3.3.1** ([GHN16], Definition 4). Let  $M = (E, \mathcal{I})$  be a matroid. A flat  $X \in \mathcal{F}(M)$  is called **coline of  $M$** , if  $\text{rk}_M(X) = \text{rk}_M(E) - 2$ . A flat  $Y \in \mathcal{F}(M)$  is called **copoint of  $M$  on  $X$** , if  $X \subseteq Y$  and  $\text{rk}_M(Y) = \text{rk}_M(E) - 1$ . If further  $|Y \setminus X| = 1$ , we say that  $Y$  is a **simple copoint on  $X$** . If otherwise  $|Y \setminus X| > 1$ , we say that  $Y$  is a **multiple copoint on  $X$** <sup>8</sup>. A **quite simple coline**<sup>9</sup> is a coline  $X \in \mathcal{F}(M)$ , such that there are more simple copoints on  $X$  than there are multiple copoints on  $X$ . ■

The following definitions are basically those found in J.E. Bonin and A. deMier's paper *Lattice path matroids: Structural properties* [BdM06].

**Definition 3.3.2.** Let  $n \in \mathbb{N}$ . A **lattice path** of length  $n$  is a tuple  $(p_i)_{i=1}^n \in \{\text{N}, \text{E}\}^n$ . We say that the  **$i$ -th step** of  $(p_i)_{i=1}^n$  is towards the North if  $p_i = \text{N}$ , and towards the East if  $p_i = \text{E}$ . ■

**Definition 3.3.3.** Let  $n \in \mathbb{N}$ , and let  $p = (p_i)_{i=1}^n$  and  $q = (q_i)_{i=1}^n$  be lattice paths of length  $n$ . We say that  $p$  is **south of  $q$**  if for all  $k \in \{1, 2, \dots, n\}$ ,

$$\left| \left\{ i \in \mathbb{N} \setminus \{0\} \mid i \leq k \text{ and } p_i = \text{N} \right\} \right| \leq \left| \left\{ i \in \mathbb{N} \setminus \{0\} \mid i \leq k \text{ and } q_i = \text{N} \right\} \right|.$$

We say that  $p$  and  $q$  have **common endpoints**, if

$$\left| \left\{ i \in \mathbb{N} \setminus \{0\} \mid i \leq n \text{ and } p_i = \text{N} \right\} \right| = \left| \left\{ i \in \mathbb{N} \setminus \{0\} \mid i \leq n \text{ and } q_i = \text{N} \right\} \right|$$

holds. We say that the **lattice path  $p$  is south of  $q$  with common endpoints**, if  $p$  and  $q$  have common endpoints and  $p$  is south of  $q$ . In this case, we write  $p \preceq q$ . ■

**Definition 3.3.4.** Let  $n \in \mathbb{N}$ , and let  $p, q \in \{\text{E}, \text{N}\}^n$  be lattice paths such that  $p \preceq q$ . We define the set of **lattice paths between  $p$  and  $q$**  to be

$$P[p, q] = \left\{ r \in \{\text{N}, \text{E}\}^n \mid p \preceq r \preceq q \right\}. \quad \blacksquare$$

<sup>8</sup>In [GHN16] and [AH15], multiple copoints are called *fat copoints*.

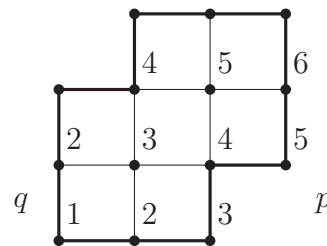
<sup>9</sup>In [GHN16] and [AH15], quite simple colines are called *positive colines*.

**Definition 3.3.5.** A matroid  $M = (E, \mathcal{I})$  is called **strong lattice path matroid**, if its ground set has the property  $E = \{1, 2, \dots, |E|\}$  and if there are lattice paths  $p, q \in \{E, N\}^{|E|}$  with  $p \preceq q$ , such that  $M = M[p, q]$ , where  $M[p, q]$  denotes the transversal matroid presented by the family  $\mathcal{A}_{[p, q]} = (A_i)_{i=1}^{\text{rk}_M(E)} \subseteq E$  with

$$A_i = \left\{ j \in E \mid \exists (r_j)_{j=1}^{|E|} \in P[p, q]: r_j = N \text{ and } |\{k \in E \mid k \leq j, r_k = N\}| = i \right\},$$

i.e. each  $A_i$  consists of those  $j \in E$ , such that there is a lattice path  $r$  between  $p$  and  $q$  such that the  $j$ -th step of  $r$  is towards the North for the  $i$ -th time in total. Furthermore, a matroid  $M = (E, \mathcal{I})$  is called **lattice path matroid**, if there is a bijection  $\varphi: E \rightarrow \{1, 2, \dots, |E|\}$  such that  $\varphi[M] = (\varphi[E], \{\varphi[X] \mid X \in \mathcal{I}\})$  is a strong lattice path matroid. ■

**Example 3.3.6.** Let us consider the two lattice paths  $p = (E, E, N, E, N, N)$  and  $q = (N, N, E, N, E, E)$ . We have  $p \preceq q$  and the strong lattice path matroid  $M[p, q]$  is the transversal matroid  $M(\mathcal{A})$  presented by the family  $\mathcal{A} = (A_i)_{i=1}^3$  of subsets of  $\{1, 2, \dots, 6\}$  where  $A_1 = \{1, 2, 3\}$ ,  $A_2 = \{2, 3, 4, 5\}$ , and  $A_3 = \{4, 5, 6\}$ . ▲



**Theorem 3.3.7** ([BdM06], Theorem 2.1). Let  $p, q$  be lattice paths of length  $n$ , such that  $p \preceq q$ . Let  $\mathcal{B} \subseteq 2^{\{1, 2, \dots, n\}}$  consist of the bases of the strong lattice path matroid  $M = M[p, q]$  on the ground set  $\{1, 2, \dots, n\}$ . Let

$$\varphi: P[p, q] \rightarrow \mathcal{B}, \quad (r_i)_{i=1}^n \mapsto \{j \in \mathbb{N} \mid 1 \leq j \leq n, r_j = N\}.$$

Then  $\varphi$  is a bijection between the family of lattice paths  $P[p, q]$  between  $p$  and  $q$  and the family of bases of  $M$ .

*Proof.* Clearly,  $\varphi$  is well-defined: let  $r = (r_i)_{i=1}^n \in P[p, q]$ , and let  $m = \text{rk}_M(\{1, 2, \dots, n\})$ , then there are  $j_1 < j_2 < \dots < j_m$  such that  $r_i = N$  if and only if  $i \in \{j_1, j_2, \dots, j_m\}$ . Thus the map

$$\iota_r: \varphi(r) \rightarrow \{1, 2, \dots, m\},$$

where  $\iota_r(i) = k$  for  $k$  such that  $i = j_k$ , witnesses that the set  $\varphi(r) \subseteq \{1, 2, \dots, n\}$  is indeed a transversal of  $\mathcal{A}_{[p, q]}$ , and therefore a base of  $M[p, q]$ . It is clear from Definition 3.3.5 that  $\varphi$  is surjective. It is obvious that if we consider only lattice paths of a fixed given length  $n$ , then the indexes of the steps towards the North uniquely determine such a lattice path. Thus  $\varphi$  is also injective. □

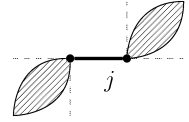
**Proposition 3.3.8.** *Let  $p = (p_i)_{i=1}^n$ ,  $q = (q_i)_{i=1}^n$  be lattice paths of length  $n$  such that  $p \preceq q$ . Let  $j \in E = \{1, 2, \dots, n\}$  and  $M = M[p, q]$ . Then*

(i)  $\text{rk}_M(\{1, 2, \dots, j\}) = |\{i \in \{1, 2, \dots, j\} \mid q_i = N\}|$ .

(ii) *The element  $j$  is a loop in  $M$  if and only if*

$$|\{i \in \{1, 2, \dots, j-1\} \mid p_i = N\}| = |\{i \in \{1, 2, \dots, j\} \mid q_i = N\}|,$$

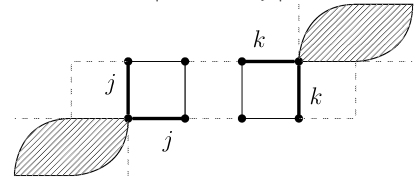
*i.e. the  $j$ -th step is forced to go towards East for all  $r \in P[p, q]$ .*



(iii) *For all  $k \in E$  with  $j < k$ ,  $j$  and  $k$  are parallel edges in  $M$  if and only if*

$$\begin{aligned} |\{i \in \{1, 2, \dots, j-1\} \mid p_i = N\}| &= |\{i \in \{1, 2, \dots, k-1\} \mid p_i = N\}| \\ &= |\{i \in \{1, 2, \dots, j\} \mid q_i = N\}| - 1 \\ &= |\{i \in \{1, 2, \dots, k\} \mid q_i = N\}| - 1, \end{aligned}$$

*i.e. the  $j$ -th and  $k$ -th steps of any  $r \in P[p, q]$  are in a common corridor towards the East that is one step wide towards the North.*



*Proof.* For every  $r \in P[p, q]$ , we have  $r \preceq q$ , therefore  $r$  is south of  $q$ , thus for all  $k \in E$ ,  $|\{j \in \{1, 2, \dots, k\} \mid r_k = N\}| \leq |\{j \in \{1, 2, \dots, k\} \mid q_k = N\}|$ . Consequently,  $\{i \in \{1, 2, \dots, j\} \mid q_i = N\}$  is a maximal independent subset of  $\{1, 2, \dots, j\}$  and so statement (i) holds. An element  $j \in E$  is a loop in  $M$ , if and only if  $\text{rk}_M(\{j\}) = 0$ , which is the case if and only if  $\{j\}$  is not independent in  $M$ . This is the case if and only if for all bases  $B$  of  $M$ ,  $j \notin B$  holds, because every independent set is a subset of a base (Lemma 1.2.7). The latter holds if and only if for all  $(r_i)_{i=1}^n \in P[p, q]$  the  $j$ -th step is towards the East, i.e.  $r_j = E$ . This, in turn, is the case if and only if  $|\{i \in \{1, 2, \dots, j-1\} \mid p_i = N\}| = |\{i \in \{1, 2, \dots, j\} \mid q_i = N\}|$ . Thus statement (ii) holds, too. Let  $j, k \in E$  with  $j < k$ . It is easy to see that if  $j$  and  $k$  are in a common corridor, then every lattice path  $r = (r_i)_{i=1}^n$  of length  $n$  with  $r_j = r_k = N$  cannot be between  $p$  and  $q$ , i.e.  $p \preceq r \preceq q$  cannot hold: a lattice path  $r$  with  $r_j = r_k = N$  is either below  $p$  at  $j-1$  or above  $q$  at  $k$ . Thus  $\{j, k\}$  cannot be independent in  $M$ . By (i), neither  $j$  nor  $k$  can be a loop in  $M$ , thus  $j$  and  $k$  must be parallel edges in  $M$ . Conversely, let  $j < k$  be parallel edges in  $M$ . Then  $j$  is not a loop in  $M$ , so there is a path  $r^1 = (r_i^1)_{i=1}^n \in P[p, q]$  with  $r_j^1 = N$  which is minimal with regard to  $\preceq$ , and then

$$|\{i \in \{1, 2, \dots, j-1\} \mid r_i^1 = N\}| = |\{i \in \{1, 2, \dots, j-1\} \mid p_i = N\}|.$$

Since  $j$  and  $k$  are parallel edges,  $\{j, k\} \not\subseteq B$  for all bases  $B$  of  $M$ . Therefore there is no  $r = (r_i)_{i=1}^n \in P[p, q]$  such that  $r_i = r_k = N$ . This yields the equation

$$\begin{aligned} |\{i \in \{1, 2, \dots, k\} \mid q_i = N\}| &= |\{i \in \{1, 2, \dots, j\} \mid r_i^1 = N\}| \\ &= |\{i \in \{1, 2, \dots, j-1\} \mid r_i^1 = N\}| + 1. \end{aligned}$$

Since  $k$  is not a loop in  $M$ , it follows that

$$|\{i \in \{1, 2, \dots, j-1\} \mid p_i = N\}| = |\{i \in \{1, 2, \dots, j\} \mid q_i = N\}| - 1.$$

Thus (iii) holds. □

**Lemma 3.3.9.** *Let  $p = (p_i)_{i=1}^n$  and  $q = (q_i)_{i=1}^n$  be lattice paths of length  $n$ , such that  $p \preceq q$ , and such that  $M = M[p, q]$  is a strong lattice path matroid on  $E = \{1, 2, \dots, n\}$  which has no loops. Let  $j \in E$  such that  $q_j = N$ . Then*

$$\{1, 2, \dots, j-1\} = \text{cl}_M(\{1, 2, \dots, j-1\}).$$

Furthermore, for all  $k \in E$  with  $k \geq j$ ,

$$\text{rk}_M(\{1, 2, \dots, j-1\} \cup \{k\}) = \text{rk}_M(\{1, 2, \dots, j-1\}) + 1.$$

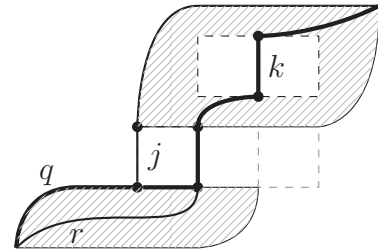
*Proof.* By Proposition 3.3.8 (i), we have

$$\text{rk}_M(\{1, 2, \dots, j-1\}) = |\{i \in \{1, 2, \dots, j-1\} \mid q_i = N\}|.$$

Now fix some  $k \in E$  with  $k \geq j$ . Since  $M$  has no loop, there is a base  $B$  of  $M$  with  $k \in B$  and thus a lattice path  $r = (r_i)_{i=1}^n \in P[p, q]$  with  $r_k = N$  (Theorem 3.3.7).

We can construct a lattice path  $s = (s_i)_{i=1}^n \in P[p, q]$  that follows  $q$  for the first  $j-1$  steps, then goes towards the East until it meets  $r$ , and then goes on as  $r$  does. The base  $B_s = \{i \in E \mid s_i = N\}$  that corresponds to the constructed path yields

$$\begin{aligned} \text{rk}_M(\{1, 2, \dots, j-1\} \cup \{k\}) &\geq |(\{1, 2, \dots, j-1\} \cup \{k\}) \cap B_s| \\ &= 1 + |\{i \in \{1, 2, \dots, j-1\} \mid q_i = N\}| \\ &= 1 + \text{rk}_M(\{1, 2, \dots, j-1\}). \end{aligned}$$



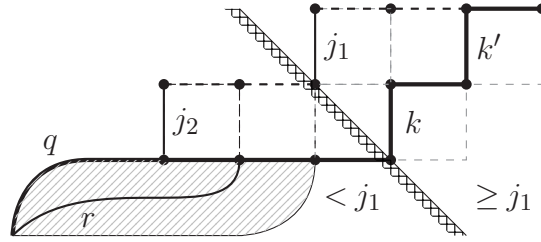


Fig. 3.1 Construction of the lattice path  $s$  in the proof of Theorem 3.3.10.

Since  $\text{rk}_M$  is unit increasing (Theorem 1.2.21,  $(R2')$ ), adding a single element to a set can increase the rank by at most one, thus the inequality in the above formula is indeed an equality. This implies that  $k \notin \text{cl}_M(\{1, 2, \dots, j-1\})$  (Lemma 1.2.18). Since  $k$  was arbitrarily chosen with  $k \geq j$ , we obtain  $\{1, 2, \dots, j-1\} = \text{cl}_M(\{1, 2, \dots, j-1\})$ .  $\square$

**Theorem 3.3.10.** *Let  $p = (p_i)_{i=1}^n$ ,  $q = (q_i)_{i=1}^n$  be lattice paths, such that  $p \preceq q$  and such that  $M = M[p, q] = (E, \mathcal{I})$  has no loop and no parallel edges, and such that  $\text{rk}_M(E) \geq 2$ . Let  $j_1 = \max\{i \in E \mid q_i = N\}$  and  $j_2 = \max\{i \in E \mid i \neq j_1 \text{ and } q_i = N\}$ . Then the following holds*

- (i)  $\{1, 2, \dots, j_2 - 1\}$  is a coline of  $M$ , we shall call it the **Western coline of  $M$** .
- (ii)  $\{1, 2, \dots, j_1 - 1\}$  is a copoint on the Western coline of  $M$ , which is a multiple copoint whenever  $j_1 - j_2 \geq 2$ .
- (iii) For every  $k \geq j_1$  the set  $\{1, 2, \dots, j_2 - 1\} \cup \{k\}$  is a simple copoint on the Western coline of  $M$ .

*Proof.* Lemma 3.3.9 provides that the set  $W = \{1, 2, \dots, j_2 - 1\}$  as well as the set  $X = \{1, 2, \dots, j_1 - 1\}$  is a flat of  $M$ . By construction of  $j_1$  and  $j_2$  we have that  $\text{rk}(W) = \text{rk}(E) - 2$  and  $\text{rk}(X) = \text{rk}(E) - 1$ . Thus  $W$  is a coline of  $M$  — so (i) holds — and  $X$  is a copoint of  $M$ , which follows from and the construction of  $j_2$  and  $j_1$ . Since  $|X \setminus W| = |\{j_2, j_2 + 1, \dots, j_1 - 1\}| = j_1 - j_2$  we obtain statement (ii). Let  $k \geq j_1$ , and let  $X_k = \{1, 2, \dots, j_2 - 1\} \cup \{k\}$ . Lemma 3.3.9 yields that  $\text{rk}(X_k) = \text{rk}(E) - 1$ , thus  $\text{cl}(X_k)$  is a copoint on the Western coline  $W$ . It remains to show that  $\text{cl}(X_k) = X_k$ , which implies that  $X_k$  is indeed a simple copoint on  $W$ . We prove this fact by showing that for all  $k' \geq j_1$ ,  $\text{rk}(X_k \cup \{k'\}) = \text{rk}(E)$  by constructing a lattice path. Without loss of generality we may assume that  $k < k'$ . Since  $M$  has no loops and no parallel edges, there is a lattice path  $r = (r_i)_{i=1}^n \in P[p, q]$  with  $r_k = r_{k'} = N$ . There is a lattice path  $s = (s_i)_{i=1}^n \in P[p, q]$  that follows  $q$  for the first  $j_2 - 1$  steps, then goes towards the East

until it meets  $r$ , and then goes on as  $r$  does. The constructed path  $s$  yields that

$$\begin{aligned} \operatorname{rk}(X_k \cup \{k'\}) &\geq \left| (W \cup \{k, k'\}) \cap \{i \in E \mid s_i = N\} \right| \\ &= 2 + |W \cap \{i \in E \mid q_i = N\}| \\ &= 2 + \operatorname{rk}(W) = 1 + \operatorname{rk}(X_k) = 1 + \operatorname{rk}(X_{k'}), \end{aligned}$$

where  $X'_k = W \cup \{k'\}$ . Thus  $k' \notin \operatorname{cl}(X_k)$  and  $k \notin \operatorname{cl}(X'_k)$ . This completes the proof of statement (iii).  $\square$

**Theorem 3.3.11.** *Let  $M = (E, \mathcal{I})$  be a strong lattice path matroid with  $\operatorname{rk}_M(E) \geq 2$  such that  $|E| = n$  and such that  $M$  has neither a loop nor a pair of parallel edges. Then either the Western coline is quite simple, or the element  $n \in E$  is a coloop, and in the latter case there is either another coloop or  $\operatorname{rk}_M(E) \geq 3$ .*

*Proof.* If  $j_1 \leq n - 1$  as defined in Theorem 3.3.10,  $W = \{1, 2, \dots, j_2 - 1\}$  has at most a single multiple copoint and at least two simple copoints, therefore it is quite simple. Otherwise  $j_1 = n$  is a coloop. If there is another coloop  $e_1$ , then  $\{1, 2, \dots, n - 1\} \setminus \{e_1\}$  is a quite simple coline with two simple copoints. If  $n$  is the only coloop, the rank of  $M$  is 2, and there is no other coloop, then this would imply that there are parallel edges — a contradiction to the assumption that  $M$  is a simple matroid.  $\square$

**Corollary 3.3.12.** *Every simple lattice path matroid  $M = (E, \mathcal{I})$  with  $\operatorname{rk}_M(E) \geq 2$  has a quite simple coline.*

*Proof.* Without loss of generality, we may assume that  $M$  is a strong lattice path matroid on  $E = \{1, 2, \dots, n\}$ , and we may use  $j_1$  and  $j_2$  as defined in Theorem 3.3.10. From Theorem 3.3.11, we obtain the following: If  $j_1 < n$ , the Western coline is quite simple. Otherwise, if  $j_1 = n$ , then  $n$  is a coloop. If there is another coloop  $e_1$ , then  $\{1, 2, \dots, n - 1\} \setminus \{e_1\}$  is a quite simple coline. If there is no other coloop, then we have  $\operatorname{rk}_M(E) \geq 3$ , and the contraction  $M' = M.E \setminus \{n\}$  is a strong lattice path matroid without loops, without parallel edges, and without coloops, such that  $\operatorname{rk}_{M'}(E \setminus \{n\}) = \operatorname{rk}_M(E) - 1 \geq 2$ . Thus the corresponding  $j'_1 < n - 1$  and the Western coline  $W'$  of  $M'$  is quite simple in  $M'$  (Theorem 3.3.11). But then  $\tilde{W} = W' \cup \{n\}$  is a coline of  $M$ , and  $\tilde{X}$  is a copoint on  $\tilde{W}$  with respect to  $M$  if and only if  $X' = \tilde{X} \setminus \{n\}$  is a copoint on  $W'$  with respect to  $M'$ . Since  $|\tilde{W} \setminus \tilde{X}| = |W' \setminus X'|$ , we obtain that  $\tilde{W}$  is a quite simple coline of  $M$ .  $\square$



**Definition 3.3.13** ([GHN16], Definition 2). *Let  $\mathcal{O}$  be an oriented matroid. We say that  $\mathcal{O}$  is **generalized series-parallel**, if every non-trivial minor  $\mathcal{O}'$  of  $\mathcal{O}$  with a simple underlying matroid  $M(\mathcal{O}')$  has a  $\{0, \pm 1\}$ -valued coflow which has exactly one or two nonzero-entries. ■*

**Lemma 3.3.14** ([GHN16], Lemma 5). *If an orientable matroid  $M$  has a quite simple coline, then every orientation  $\mathcal{O}$  of  $M$  has a  $\{0, \pm 1\}$ -valued coflow which has exactly one or two nonzero-entries.*

For a proof, see [GHN16].

**Remark 3.3.15.** A simple matroid of rank 1 has only one element, no circuit and a single cocircuit consisting of the sole element of the matroid; so every rank-1 oriented matroid is generalized series-parallel. Observe that every simple matroid  $M = (E, \mathcal{I})$  with  $\text{rk}_M(E) = 2$  is a lattice path matroid, as it is isomorphic to the strong lattice path matroid  $M[p, q]$  where  $p = (p_i)_{i=1}^{|E|}$  with

$$p_i = \begin{cases} E & \text{if } i < |E| - 2, \\ N & \text{otherwise,} \end{cases}$$

and where  $q = (q_i)_{i=1}^{|E|}$  with

$$q_i = \begin{cases} N & \text{if } i \leq 2, \\ E & \text{otherwise.} \end{cases}$$

Therefore Lemma 3.3.14 and Corollary 3.3.12 yield that  $\mathcal{O}$  has a  $\{0, \pm 1\}$ -valued coflow which has exactly one or two nonzero-entries. Consequently, every oriented matroid  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  with  $\text{rk}_{M(\mathcal{O})}(E) \leq 2$  is generalized series-parallel. •

**Theorem 3.3.16** ([BdM06], Theorem 3.1). *The class of lattice path matroids is closed under minors, duals and direct sums.*

For a proof, see [BdM06], pp. 5ff; furthermore see Figure 3.2 for examples of the constructions involved. The main observation is that a base  $B$  of a strong lattice path matroid  $M$  corresponds to a lattice path  $r$  for which the  $k$ th step is to the North if and only if  $k \in B$ . So the base  $E \setminus B$  of  $M^*$  corresponds to a lattice path  $r^*$  for which the  $k$ th step is to the East if and only if  $k \in B$ , but this lattice path  $r^*$  is the path  $r$  mirrored at South-West-to-North-East line through the origin. So the dual of a lattice path matroid  $M$  is the lattice path matroid  $M^*$  where the upper bound lattice path of

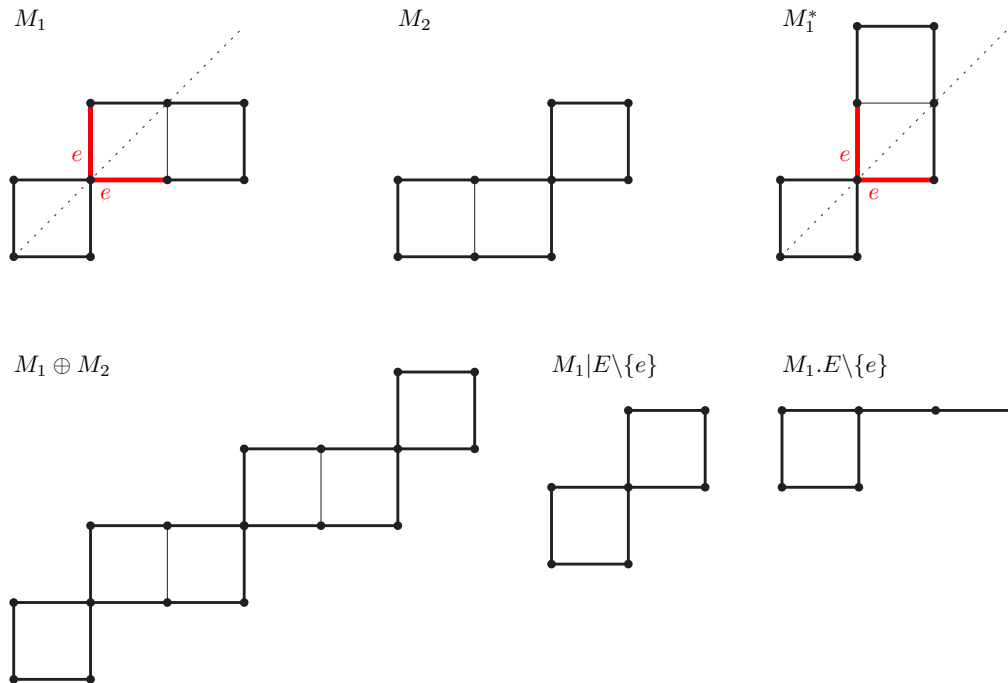


Fig. 3.2 Constructions on Lattice Path Matroids

$M^*$  is the SW-NE-mirror image of the lower bound lattice path of  $M$ , and the lower bound lattice path of  $M^*$  is the SW-NE-mirror image of the upper bound lattice path of  $M$ . The direct sum of lattice path matroids has lower and upper bound lattice paths that correspond to the concatenation of the lower bound lattice paths – and upper bound lattice paths, respectively – of the summand lattice path matroids. In order to obtain the restriction  $N = M|E \setminus \{e\}$  of a lattice path matroid  $M = (E, \mathcal{I})$  with  $e \in E$ , we have to take all lattice paths between the lower and upper bound lattice path of  $M$ , and remove the step corresponding to  $e$ . Thus the lower and upper bound lattice paths of  $N$  arise from the step- $e$ -omissions of lattice paths that may differ from the upper and lower bound lattice paths of  $M$  only at the step corresponding to  $e$  and at most one other step.

**Corollary 3.3.17.** *All orientations of lattice path matroids are generalized series-parallel.*

*Proof.* Lemma 3.3.14, Remark 3.3.15, Theorem 3.3.16 and Corollary 3.3.12.  $\square$

**Theorem 3.3.18** ([GHN16], Theorem 3). *Let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  be a generalized series-parallel oriented matroid such that  $M(\mathcal{O})$  has no loops. Then there is a nowhere-zero coflow  $F \in \mathbb{Z}\mathcal{C}^*$  such that  $|F(e)| < 3$  for all  $e \in E$ . Thus  $\chi(\mathcal{O}) \leq 3$ .*

For a proof, see [GHN16].

**Corollary 3.3.19.** *Let  $\mathcal{O}$  be an oriented matroid such that  $M(\mathcal{O})$  is a lattice path matroid without loops. Then  $\chi(\mathcal{O}) \leq 3$ .*

*Proof.* Theorem 3.3.18 and Corollary 3.3.17. □

### 3.4 Oriented Gammoids

In this section, we examine the class of oriented matroids whose underlying matroids are gammoids.

**Lemma 3.4.1.** *Let  $M = (E, \mathcal{I})$  be a gammoid. Then  $M$  is orientable.*

*Proof.* By Theorem 2.7.13 there is a set  $T$  with  $|T| = \text{rk}_M(E)$  and there is a matrix  $\mu \in \mathbb{R}^{E \times T}$ , such that  $M = M(\mu)$ . Then the oriented matroid  $\mathcal{O}(\mu)$  is an orientation of  $M(\mu) = M$  (Corollary 3.1.17).  $\square$

Given a gammoid  $M = \Gamma(D, T, E)$ , the oriented matroid, whose existence is guaranteed by the previous lemma, depends on the actual values of the indeterminate weighting  $w: A \rightarrow \mathbb{R}$  of  $D$ , and obtaining the signatures of the circuits from the matrix  $\mu$  requires some computational effort. The same applies to the integer-valued representation obtained from the probabilistic method described in Proposition 2.7.17, where it is possible to use E.H. Bareiss's variant of Gaussian Elimination [Bar68] that works without division, and which has polynomial computational complexity and a polynomial bound on the absolute of the intermediate values that may occur during the calculation. We further point out that Example 3.1.22 indicates that there are orientations of gammoids which cannot be represented by a real matrix.

**Lemma 3.4.2.** *Let  $E$  and  $T$  be finite sets, and let  $\mu \in \mathbb{R}^{E \times T}$  be a matrix, and  $M = M(\mu)$  be the matroid represented by  $\mu$  over  $\mathbb{R}$ . Further, let  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*) = \mathcal{O}(\mu)$  be the oriented matroid obtained from  $\mu$ , let  $C \in \mathcal{C}(M)$  be a circuit of  $M$  and let  $c \in C$  be an arbitrary element of that circuit. Let  $T_0 \subseteq T$  such that  $\text{idet}(\mu|(C \setminus \{c\}) \times T_0) = 1$ . Consider the signed subset  $C_c \in \sigma E$  with*

$$C_c(e) = \begin{cases} 0 & \text{if } e \notin C, \\ -1 & \text{if } e = c, \\ \text{sgn}\left(\frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)}\right) & \text{otherwise} \end{cases}$$

where

$$\nu_e: C \setminus \{c\} \times T_0 \rightarrow \mathbb{R}, \quad (x, t) \mapsto \begin{cases} \mu(c, t) & \text{if } x = e, \\ \mu(x, t) & \text{otherwise.} \end{cases}$$

Then  $C_c \in \mathcal{C}$ .

*Proof.* By CRAMER's rule we obtain that

$$\mu_c = \sum_{e \in C \setminus \{c\}} \frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)} \cdot \mu_e.$$

Therefore,

$$-\mu_c + \sum_{e \in C \setminus \{c\}} \frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)} \cdot \mu_e = 0$$

is a non-trivial linear combination of the zero vector. Clearly  $C_c$  consists of the signs of the corresponding coefficients and therefore  $C_c \in \mathcal{C}$  is an orientation of  $C$  with respect to  $\mathcal{O}(\mu)$ .  $\square$

### 3.4.1 Heavy Arc Orientations

In this section, we develop a notion of orientations of gammoids which stem from indeterminate weightings with a special property, that allows us to determine the signed circuits of the orientation without carrying out any computations in  $\mathbb{R}$ . Instead, we only have to inspect a given representation  $(D, T, E)$  with respect to the given linear order on  $A$  and the given signs of the arc weighting.

**Definition 3.4.3.** Let  $D = (V, A)$  be a digraph, let  $\sigma: A \rightarrow \{-1, 1\}$  be a map and let  $\ll$  be a binary relation on  $A$ . We shall call  $(\sigma, \ll)$  a **heavy arc signature of  $D$** , if  $\ll$  is a linear order on  $A$ .  $\blacksquare$

**Definition 3.4.4.** Let  $D = (V, A)$  be a digraph and  $(\sigma, \ll)$  be a heavy arc signature of  $D$ . The  **$(\sigma, \ll)$ -induced routing order of  $D$**  shall be the linear order  $\lll$  on the family of routings of  $D$ , where  $Q \lll R$  holds if and only if the  $\ll$ -maximal element  $x$  of the symmetric difference  $Q_A \triangle R_A$  has the property  $x \in R_A$ , where  $Q_A = \bigcup_{p \in Q} |p|_A$  and  $R_A = \bigcup_{p \in R} |p|_A$ .  $\blacksquare$

**Remark 3.4.5.** Clearly,  $\lll$  is a linear order on all routings in  $D$ , because every routing  $R$  in  $D$  is uniquely determined by its set of traversed arcs  $R_A$  (Lemmas 1.5.22 and 1.5.23).  $\bullet$

**Definition 3.4.6.** Let  $D = (V, A)$  be a digraph, and let  $(\sigma, \ll)$  be a heavy arc signature of  $D$ . Let  $R: X \rightrightarrows Y$  be a routing in  $D$  where  $X = \{x_1, x_2, \dots, x_n\}_{\neq}$  and  $Y = \{y_1, y_2, \dots, y_m\}_{\neq}$  are implicitly ordered. The **sign of  $R$  with respect to  $(\sigma, \ll)$**  shall be

$$\text{sgn}_{\sigma}(R) = \text{sgn}(\varphi) \cdot \left( \prod_{p \in R, a \in |p|_A} \sigma(a) \right)$$

where  $\varphi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$  is the unique map such that for all  $i \in \{1, 2, \dots, n\}$  there is a path  $p \in R$  with  $p_1 = x_i$  and  $p_{-1} = y_{\varphi(x)}$ ; and where

$$\text{sgn}(\varphi) = (-1)^{|\{(i,j) \mid i,j \in \{1,2,\dots,n\}: i < j \text{ and } \varphi(i) > \varphi(j)\}|}. \quad \blacksquare$$

**Definition 3.4.7.** Let  $D = (V, A)$  be a digraph such that  $V = \{v_1, v_2, \dots, v_n\}_{\neq}$  is implicitly ordered,  $(\sigma, \ll)$  be a heavy arc signature of  $D$ , and let  $T, E \subseteq V$  be subsets that inherit the implicit order of  $V$ . Furthermore, let  $M = \Gamma(D, T, E)$  be the corresponding gammoid, and let  $C \in \mathcal{C}(M)$  be a circuit of  $M$  such that  $C = \{c_1, c_2, \dots, c_m\}_{\neq}$  inherits its implicit order from  $V$ ; and let  $i \in \{1, 2, \dots, m\}$ . The **signature of  $C$  with respect to  $M$ ,  $i$ , and  $(\sigma, \ll)$**  shall be the signed subset  $C_{(\sigma, \ll)}^{(i)}$  of  $E$  where

$$C_{(\sigma, \ll)}^{(i)}(e) = \begin{cases} 0 & \text{if } e \notin C, \\ -\text{sgn}_{\sigma}(R_i) & \text{if } e = c_i, \\ (-1)^{i-j+1} \cdot \text{sgn}_{\sigma}(R_j) & \text{if } e = c_j \neq c_i, \end{cases}$$

and where for all  $k \in \{1, 2, \dots, m\}$

$$R_k = \max_{\ll\ll} \{R \mid R: C \setminus \{c_k\} \rightrightarrows T \text{ in } D\}$$

denotes the unique  $\ll\ll$ -maximal routing from  $C \setminus \{c_k\}$  to  $T$  in  $D$ . \blacksquare

**Remark 3.4.8.** The factors  $(-1)^{i-j+1}$  in Definition 3.4.7 do not appear explicitly in Lemma 3.4.2, where  $\nu_e$  is obtained from the restriction  $\mu|(C \setminus \{c\}) \times T_0$  by replacing the values in row  $e$  with the values of  $\mu_c$ . We have to account for the number of row transpositions that are needed to turn  $\nu_e$  into the restriction  $\mu|(C \setminus \{e\}) \times T_0$ , which depends on the position of  $e = c_j$  relative to  $c = c_i$  with respect to the implicit order of  $V$ . \bullet

**Definition 3.4.9.** Let  $D = (V, A)$  be a digraph and  $(\sigma, \ll)$  a heavy arc signature of  $D$ , and let  $w: A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$ . We say that  $w$  is a  **$(\sigma, \ll)$ -weighting of  $D$**  if, for all  $a \in A$ , the inequality  $|w(a)| \geq 1$ , the strict inequality

$$\sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} |w(x)| \right) < |w(a)|,$$

and the equality  $\text{sgn}(w(a)) = \sigma(a)$  hold. \blacksquare

**Lemma 3.4.10.** *Let  $D = (V, A)$  be a digraph and  $(\sigma, \ll)$  be a heavy arc signature of  $D$ . There is a  $(\sigma, \ll)$ -weighting of  $D$ .*

*Proof.* Let  $w: A \rightarrow \mathbb{R}$  be an indeterminate weighting of  $D$ , which exists due to Lemma 1.1.11. It is clear from Definition 1.1.10 that for every  $\zeta \in \mathbb{Z}^A$  and every  $\tau \in \{-1, 1\}^A$ , the map  $w_{\zeta, \tau}: A \rightarrow \mathbb{R}$ , which has

$$w_{\zeta, \tau}(a) = \tau(a) \cdot \frac{w(a)}{\operatorname{sgn}(w(a))} + \tau(a) \cdot \zeta(a)$$

for all  $a \in A$ , is an indeterminate weighting of  $D$ , too. Now, let  $\zeta \in \mathbb{Z}^A$ , such that for all  $a \in A$  we have the following recurrence relation

$$\zeta(a) = \left[ \sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} (|w(x)| + \zeta(x)) \right) \right].$$

The map  $\zeta$  is well-defined by this recurrence relation because  $|A| < \infty$  and therefore there is a  $\ll$ -minimal element  $a_0$  in  $A$ , and we have  $\zeta(a_0) = \prod_{x \in \emptyset} (|w(x)| + \zeta(x)) = 1$ . Then  $w_{\zeta, \sigma}$  is a  $(\sigma, \ll)$ -weighting of  $D$ . Clearly,

$$\begin{aligned} \operatorname{sgn}(w_{\zeta, \sigma}(a)) &= \operatorname{sgn}\left(\sigma(a) \cdot \frac{w(a)}{\operatorname{sgn}(w(a))} + \sigma(a) \cdot \zeta(a)\right) \\ &= \operatorname{sgn}(\sigma(a)) \cdot \operatorname{sgn}\left(\frac{w(a)}{\operatorname{sgn}(w(a))} + \zeta(a)\right) \\ &= \sigma(a) \cdot 1 = \sigma(a) \end{aligned}$$

holds for all  $a \in A$ . Furthermore, we have

$$\begin{aligned} |w_{\zeta, \sigma}(a)| &= \left| \sigma(a) \cdot \frac{w(a)}{\operatorname{sgn}(w(a))} + \sigma(a) \cdot \zeta(a) \right| \\ &> |\zeta(a)| = \left[ \sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} (|w(x)| + \zeta(x)) \right) \right] \\ &\geq \sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} |w_{\zeta, \sigma}(x)| \right). \quad \square \end{aligned}$$

**Lemma 3.4.11.** *Let  $D = (V, A)$  be a digraph,  $(\sigma, \ll)$  be a heavy arc weighting of  $D$ ,  $E, T \subseteq V$ ,  $C \in \mathcal{C}(\Gamma(D, T, E))$  be a circuit in the corresponding gammoid, and let  $c, d \in C$ . Furthermore, let  $R_c: C \setminus \{c\} \rightrightarrows T$  and  $R_d: C \setminus \{d\} \rightrightarrows T$  be the  $\ll\ll$ -maximal routings in  $D$ . Then*

$$\{p_{-1} \mid p \in R_c\} = \{p_{-1} \mid p \in R_d\}$$

*holds.*

*Proof.* Let  $S$  be a  $C$ - $T$ -separator of minimal cardinality in  $D$ , i.e. a  $C$ - $T$ -separator with  $|S| = |C| - 1$ . Since  $R_c$  and  $R_d$  are both  $C$ - $T$ -connectors with maximal cardinality, we obtain that for every  $s \in S$  there is  $p_c^s \in R_c$  and a  $p_d^s \in R_d$  such that  $s \in |p_c^s|$  and  $s \in |p_d^s|$  (Corollary 1.5.29), thus there are paths  $l_c^s, l_d^s, r_c^s, r_d^s \in \mathbf{P}(D)$  such that  $p_c^s = l_c^s.r_c^s$  and  $p_d^s = l_d^s.r_d^s$  with  $(r_c^s)_1 = (r_d^s)_1 = s$ . Now let  $R_c^S = \{r_c^s \mid s \in S\}$  and  $R_d^S = \{r_d^s \mid s \in S\}$ , clearly both  $R_c^S$  and  $R_d^S$  are routings from  $S$  to  $T$  in  $D$ . Assume that  $R_c^S \neq R_d^S$ , then we have  $R_c^S \ll\ll R_d^S$  — without loss of generality, by possibly switching names for  $c$  and  $d$ . Then  $Q = \{l_c^s.r_d^s \mid s \in S\}$  is a routing from  $C \setminus \{c\}$  to  $T$  in  $D$ . But for the symmetric differences we have the equality

$$\left( \bigcup_{p \in Q} |p|_A \right) \Delta \left( \bigcup_{p \in R_c} |p|_A \right) = \left( \bigcup_{p \in R_d^S} |p|_A \right) \Delta \left( \bigcup_{p \in R_c^S} |p|_A \right),$$

which implies  $R_c \ll\ll Q$ , a contradiction to the assumption that  $R_c$  is the  $\ll\ll$ -maximal routing from  $C \setminus \{c\}$  to  $T$ . Thus  $R_c^S = R_d^S$  and the claim of the lemma follows.  $\square$

Now we have amassed all ingredients that we need in order to show that every heavy arc signature of  $D$  corresponds to an orientation of a gammoid whenever  $D$  is an acyclic digraph. Thus heavy arc signatures yield orientations of cascade matroids.

**Lemma 3.4.12.** *Let  $D = (V, A)$  be an acyclic digraph where  $V$  is implicitly ordered,  $(\sigma, \ll)$  be a heavy arc signature of  $D$ , and  $T, E \subseteq V$ . Then there is a unique oriented matroid  $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$  where*

$$\mathcal{C} = \left\{ \pm C_{(\sigma, \ll)}^{(1)} \mid C \in \mathcal{C}(\Gamma(D, T, E)) \right\}.$$

*Proof.* Let  $M = \Gamma(D, T, E)$ , and let  $w: A \rightarrow \mathbb{R}$  be a  $(\sigma, \ll)$ -weighting of  $D$  which exists due to Lemma 3.4.10. Furthermore, let  $\mu \in \mathbb{R}^{E \times T}$  be the matrix defined as in the Lindström Lemma 2.7.4, with respect to the  $(\sigma, \ll)$ -weighting  $w$  and the implicit order on  $V$ . Theorem 2.7.13 along with its proof yields that we have  $M = M(\mu)$ . Let  $\mathcal{O} = \mathcal{O}(\mu) = (E, \mathcal{C}_\mu, \mathcal{C}_\mu^*)$  be the oriented matroid that arises from  $\mu$ , so  $M(\mathcal{O}) = M(\mu)$



holds (Corollary 3.1.17). We show that  $\mathcal{C}_\mu = \mathcal{C}$ . It suffices to prove that for all  $C \in \mathcal{C}(M)$ , all  $D \in \mathcal{C}_\mu$  with  $D_\pm = C$ , and all  $D' \in \mathcal{C}$  with  $D'_\pm = C$  we have  $D \in \{D', -D'\}$ . Now, let  $C \in \mathcal{C}(M)$  and let  $C = \{c_1, c_2, \dots, c_k\}_\neq$  implicitly ordered respecting the implicit order of  $V$ . The claim follows if  $D(c_1)D(c_j) = D'(c_1)D'(c_j)$  holds for all  $j \in \{2, 3, \dots, k\}$ . Let  $T_0 \subseteq T$  be the target vertices onto which the  $\lll$ -maximal and  $|\cdot|$ -maximal  $C$ - $T$ -connectors link in  $D$  (Lemma 3.4.11). From Lemma 3.4.2 we obtain that

$$\begin{aligned} D(c_1)D(c_j) &= -1 \cdot \operatorname{sgn} \left( \frac{\det(\nu_j)}{\det(\mu|(C \setminus \{c_1\}) \times T_0)} \right) \\ &= -\operatorname{sgn}(\det(\nu_j)) \cdot \operatorname{sgn}(\mu|(C \setminus \{c_1\}) \times T_0) \end{aligned}$$

where

$$\nu_j: C \setminus \{c_1\} \times T_0 \longrightarrow \mathbb{R}, \quad (x, t) \mapsto \begin{cases} \mu(c_1, t) & \text{if } x = c_j, \\ \mu(x, t) & \text{otherwise.} \end{cases}$$

Observe that  $\nu_j$  arises from the restriction  $\mu|C \setminus \{c_j\} \times T_0$  by a row-permutation, which has at most one non-trivial cycle, and this cycle then has the length  $j - 1$ , therefore

$$\det(\nu_j) = (-1)^{j-2} \det(\mu|C \setminus \{c_j\} \times T_0)$$

holds, so

$$D(c_1)D(c_j) = (-1)^{1-j} \operatorname{sgn}(\det(\mu|C \setminus \{c_j\} \times T_0)) \cdot \operatorname{sgn}(\mu|(C \setminus \{c_1\}) \times T_0).$$

We further have

$$D'(c_1)D'(c_j) = (-1)^{j+1} \cdot \operatorname{sgn}_\sigma(R_1) \cdot \operatorname{sgn}_\sigma(R_j)$$

where for all  $i \in \{1, 2, \dots, k\}$

$$R_i = \max_{\lll} \{R \mid R: C \setminus \{c_i\} \rightrightarrows T \text{ in } D\}$$

denotes the unique  $\lll$ -maximal routing from  $C \setminus \{c_i\}$  to  $T$  in  $D$ . By the Lindström Lemma 2.7.4 we obtain that for all  $i \in \{1, 2, \dots, k\}$  the equation

$$\det(\mu|C \setminus \{c_i\} \times T_0) = \sum_{R: C \setminus \{c_i\} \rightrightarrows T_0} \left( \operatorname{sgn}(R) \prod_{p \in R} \left( \prod_{a \in |p|_A} w(a) \right) \right)$$

holds, where  $\text{sgn}(R)$  is the sign of the permutation implicitly given by the start and end vertices of the paths in  $R$ , both with respect to the implicit order on  $V$ . Since  $w$  is a  $(\sigma, \ll)$ -weighting, we have

$$\left| \sum_{R: C \setminus \{c_i\} \xrightarrow{\neq} T_0, R \neq R_i} \left( \text{sgn}(R) \prod_{p \in R} \left( \prod_{a \in |p|_A} w(a) \right) \right) \right| < |w(a_i)|$$

where  $a_i \in \bigcup_{p \in R_i} |p|_A$  is the  $\ll$ -maximal arc in the  $\ll\ll$ -maximal routing  $R_i$  from  $C \setminus \{c_i\}$  to  $T_0$  in  $D$ . Therefore the sign of  $\det(\mu|_{C \setminus \{c_i\}} \times T_0)$  is determined by the sign of the summand that contains  $w(a_i)$  as a factor, which is the summand that corresponds to  $R = R_i$ . Therefore

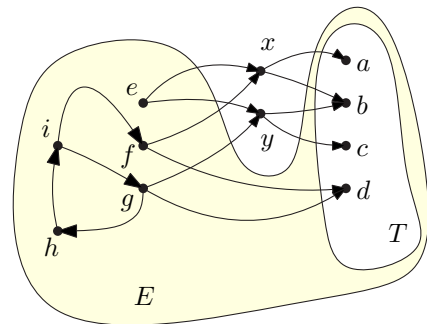
$$\begin{aligned} \text{sgn}(\det(\mu|_{C \setminus \{c_i\}} \times T_0)) &= \text{sgn} \left( \text{sgn}(R_i) \prod_{p \in R_i, a \in |p|_A} w(a) \right) \\ &= \text{sgn}(R_i) \prod_{p \in R_i, a \in |p|_A} \text{sgn}(w(a)) \\ &= \text{sgn}(R_i) \prod_{p \in R_i, a \in |p|_A} \sigma(a) \\ &= \text{sgn}_\sigma(R_i). \end{aligned}$$

So we obtain

$$\begin{aligned} D(c_1)D(c_j) &= (-1)^{1-j} \text{sgn}_\sigma(R_1) \cdot \text{sgn}_\sigma(R_j) \\ &= (-1)^{j+1} \cdot \text{sgn}_\sigma(R_1) \cdot \text{sgn}_\sigma(R_j) = D'(c_1)D'(c_j). \end{aligned} \quad \square$$

Unfortunately, we cannot omit the assumption that  $D$  is an acyclic digraph.

**Example 3.4.13.** We consider the digraph  $D = (V, A)$  with the implicitly ordered vertex set  $V = \{a, b, c, d, e, f, g, h, i, x, y\}_{\neq}$ , and  $A$  as depicted on the right. Let  $T = \{a, b, c, d\}$ . Clearly,  $\mathbf{W}(D)$  contains the cycle walk  $ghig$ . Let  $(\sigma, \ll)$  be the heavy arc signature of  $D$  where  $\sigma(a) = 1$  for all  $a \in A$ , and where  $a_1 \ll a_2$  if the tuple  $a_1$  is less than the tuple  $a_2$  with respect to the lexicographic order on  $V \times V$  derived from the implicit order of the vertex set.



Let  $C_1 = \{f, g, i\}$ ,  $C_2 = \{d, e, f, i\}$ ,  $C_f = \{d, e, g, i\}$ . Clearly  $C_1, C_2, C_f \in \mathcal{C}(\Gamma(D, T, E))$ . The following

routings are  $\lll$ -maximal among all routings in  $D$  with the same set of initial vertices and with targets in  $T$ .

$$\begin{array}{ll}
R_{\{f,g\}} = \{fxb, gyc\} & \text{sgn}_\sigma(R_{\{f,g\}}) = +1 \\
R_{\{f,i\}} = \{fxb, igyc\} & \text{sgn}_\sigma(R_{\{f,i\}}) = +1 \\
R_{\{g,i\}} = \{gyc, ifxb\} & \text{sgn}_\sigma(R_{\{g,i\}}) = -1 \\
R_{\{d,e,f\}} = \{d, eyc, fxb\} & \text{sgn}_\sigma(R_{\{d,e,f\}}) = -1 \\
R_{\{d,e,i\}} = \{d, exb, igyc\} & \text{sgn}_\sigma(R_{\{d,e,i\}}) = +1 \\
R_{\{d,f,i\}} = \{d, fxb, igyc\} & \text{sgn}_\sigma(R_{\{d,f,i\}}) = +1 \\
R_{\{e,f,i\}} = \{exb, fd, igyc\} & \text{sgn}_\sigma(R_{\{e,f,i\}}) = -1 \\
R_{\{d,e,g\}} = \{d, eyc, ghifxb\} & \text{sgn}_\sigma(R_{\{d,e,g\}}) = -1 \\
R_{\{d,g,i\}} = \{d, gyc, ifxb\} & \text{sgn}_\sigma(R_{\{d,g,i\}}) = -1 \\
R_{\{e,g,i\}} = \{exb, gyc, ifd\} & \text{sgn}_\sigma(R_{\{e,g,i\}}) = +1
\end{array}$$

Now let us calculate the signatures of  $C_1$ ,  $C_2$ , and  $C_f$  according to Definition 3.4.7. We obtain

$$(C_1)_{(\sigma, \lll)}^{(1)} = \{f, g, -i\}, (C_2)_{(\sigma, \lll)}^{(1)} = \{d, e, -f, -i\}, \text{ and } (C_f)_{(\sigma, \lll)}^{(1)} = \{-d, -e, -g, -i\}.$$

This clearly violates axiom  $(\mathcal{C}_4)$ : if we eliminate  $f$  from  $(C_1)_{(\sigma, \lll)}^{(1)}$  and  $(C_2)_{(\sigma, \lll)}^{(1)}$ , then the resulting signed circuit must have opposite signs for  $d$  and  $i$ , but  $d$  and  $i$  have the same sign with respect to  $(C_f)_{(\sigma, \lll)}^{(1)}$ . Therefore we see that the assumption, that  $D$  is acyclic, cannot be dropped from Lemma 3.4.12.  $\blacktriangle$

We can still use the construction involved in Lemma 3.4.12 for every representation  $(D, T, E)$ , but we first have to construct a complete lifting of  $D$  (Lemma 2.7.10). We then may use Lemma 3.4.12 together with a heavy arc orientation of the lifted digraph in order to obtain an orientation of the lifted representation, and then use the contraction formula from Lemma 2.7.11 in order to obtain the orientation of  $(D, T, E)$  (Lemma 3.1.32). Thus we have found a purely combinatorial way to determine an orientation of a gammoid from its representation, and the proof of Lemma 3.4.12 yields that every orientation obtained in this way is realizable.



# Chapter 4

## Conclusions and Open Problems

In this section, we demonstrate the significance of this work by summing up and contextualizing the main new results and concepts presented in this work. We introduced the notion of *duality respecting representations* of gammoids and proved that every gammoid has such a representation. For a long time, it has been a well-known fact that the class of gammoids is closed under duality, and the classical proofs of this property employ the important insight that strict gammoids are precisely the duals of transversal matroids. By shifting our focus away from strict gammoids, we were able to reveal that a gammoid and its dual are tightly related to each other: they are represented by special pairs of opposite digraphs<sup>1</sup>, and these special pairs are easily obtained from any representation. This discovery led us to the concept of a *standard representation*, which allowed us to define complexity measures for gammoids as minimal complexity measures of the digraphs that may appear in a standard representation of a gammoid. We defined the *arc-complexity* and *vertex-complexity* of gammoids and showed that the derived classes of gammoids with bounded arc- or vertex-complexity are closed under duality and minors, and that these classes are characterized by finitely many excluded minors. But in general, these classes are not closed under direct sums. In order to derive subclasses of gammoids that are closed under direct sums, we defined the *f-width* of gammoids. We were able to show that the subclasses of gammoids with bounded *f-width* are closed under direct sums for super-additive functions  $f$ .

---

<sup>1</sup>We would like to call such directed graphs dual to each other. This may be justified by observing that for lattices and partial orders, which may be considered special binary relations, the concept of duality merely swaps the first and the second component of that relation. Since directed graphs may be considered binary relations as well, it is quite natural to call the opposite digraph dual. Unfortunately, there are other notions of duality with respect to directed graphs that have equally good justifications.

Regarding our investigation into the problem of deciding whether a given matroid is a gammoid, the starting point was Mason’s  $\alpha$ -criterion for strict gammoids. Naturally we were more interested in the situation where the matroid under consideration is not a strict gammoid, and therefore we defined the concept of an  $\alpha$ -violation that captures minimal situations in a matroid that are not “strictly gammoidal”. Unfortunately, we showed that it is not possible to classify  $\alpha$ -violations into violations that correspond to gammoids and violations that correspond to non-gammoids — we saw that there are non-gammoids that have two copies of a violation, that may occur in a gammoid as its unique violation. We condensed our gathered experience with the recognition problem of gammoids into the notion of a *matroid tableau* and the corresponding 13-step directions for the derivation of a decisive matroid tableau.

We introduced the concept of *lifting* cycles in digraphs of representations of gammoids, in order to find acyclic representations of gammoids that have the original gammoid as a contraction minor. This concept may be used to avoid technicalities with the non-acyclic generalizations of the Lindström Lemma, but it may generally be applied in situations where the presence of cycles in digraphs complicates matters. One such situation arises when we try to use *heavy arc signatures* in order to orient a gammoid. We provided a way to determine orientations of gammoids without having to carry out actual calculations in  $\mathbb{Q}$  or  $\mathbb{R}$  as long as the corresponding digraph of the representation of the gammoid has no cycle walk. We also gave an example that this condition may not be dropped. Apart from that, we were able to show that the class of *lattice path matroids* is generalized series-parallel, and therefore 3-colorable.

In the following sections, we give some starting points for further research in the field of gammoids.

## 4.1 Other Complexity Measures

Let  $\mu$  be a measure that assigns every digraph  $D = (V, A)$  a value  $\mu(D) \in \mathbb{R}$ . Analogously to the definitions of the arc-complexity and vertex-complexity of a gammoid, we may define the  $\mu$ -complexity of a gammoid  $M$  to be

$$\hat{\mu}(M) = \min \left\{ \mu(D) \mid (D, T, E) \text{ is a standard representation of } M \right\}.$$

If  $\mu(D) = \mu(D^{\text{opp}})$  for all digraphs  $D$ , then the  $\mu$ -complexity has the property that  $\hat{\mu}(M) = \hat{\mu}(M^*)$ . Obviously, all complexity measures for directed graphs have this property as soon as they are obtained from measures for undirected graphs by ignoring

the orientation of the arcs. This yields a variety of new research questions about the properties of subclasses of gammoids with bounded  $\mu$ -complexity: for which measures  $\mu$  are the the classes consisting of gammoids  $M$  with  $\hat{\mu}(M) \leq k$  closed under minors, under duality, and under direct-sums? If such a class is closed under minors, then what are the excluded minors for that class? Which of these excluded minors are gammoids? Is the class characterized by finitely many excluded minors? Interesting choices of  $\mu$  include arboricity, star-arboricity, thickness, degeneracy, girth, tree number, DAG-width, and many more. Furthermore, we should consider the same questions with respect to the  $f$ - $\mu$ -width which may be defined as

$$\hat{\mu}_f(M) = \max \left\{ \frac{\hat{\mu}((M.Y)|X)}{f(|X|)} \mid X \subseteq Y \subseteq E \right\}$$

where  $M = (E, \mathcal{I})$ .

## 4.2 Arc Complexity of Uniform Matroids

The following is the most fundamental open problem that we encountered in the course of this work. It is most promising to be answered positively in the next few years – possibly by utilizing some results from the theory of digraphs – but the solution of this problem is unfortunately out of reach to the author within the schedule of this work. We are not able to show the following conjecture, but we are convinced that it is true.

**Conjecture 4.2.1.** *Let  $r \in \mathbb{N}$ ,  $U = (E, \mathcal{I})$  be a uniform matroid of rank  $r$  on the ground set  $E$ , i.e.  $\mathcal{I} = \{X \subseteq E \mid |X| \leq r\}$ . Then  $C_A(U) = r \cdot (|E| - r)$ .*

There is the following reformulation with respect to directed graphs.

**Conjecture 4.2.2.** *Let  $D = (V, A)$  be a digraph, and let  $X, Y \subseteq V$  with  $X \cap Y = \emptyset$ . If for every  $X' \subseteq X$  and every  $Y' \subseteq Y$  with  $|X'| = |Y'|$  there is a routing  $R: X' \rightrightarrows Y'$  in  $D$  with*

$$Y \cap \bigcup_{p \in R} |p| = Y',$$

then

$$|A| \geq |X| \cdot |Y|.$$

*Relative proof.* Let  $M = \Gamma(D, Y, X \cup Y)$ , and let  $Q \subseteq X \cup Y$  with  $|Q| \leq |Y|$ . Clearly  $|Q| = |Q_X| + |Q_Y|$  where  $Q_X = Q \setminus Y$  and  $Q_Y = Q \cap Y$ . Thus  $|Q| \leq |Y|$  implies  $|Q_X| \leq |Y \setminus Q_Y|$ . Therefore there is a subset  $Q'_Y \subseteq Y \setminus Q$  with  $|Q'_Y| = |Q_X|$ . By hypothesis we obtain a routing  $R: Q_X \rightrightarrows Q'_Y$  in  $D$  which avoids  $Q_Y \subseteq Y \setminus Q'_Y$ , thus

$R \cup \{q \mid q \in Q_Y\}$  is a routing from  $Q$  to  $Y$  in  $D$ . Consequently,  $Q$  is independent in  $M$ . We showed that every independent subset of  $X \cup Y$  with at most  $|Y|$  elements is independent in  $M$ , and it is clear that no subset of  $X \cup Y$  with more than  $|Y|$  elements is independent in  $M$ , therefore  $M$  is a uniform matroid of rank  $|Y|$  with  $|X \cup Y| = |X| + |Y|$  elements. The statement of this conjecture then follows from Conjecture 4.2.1.  $\square$

Closely related is the following conjecture which would be implied by the previous two conjectures.

**Conjecture 4.2.3.** *For all  $k \in \mathbb{N}$  there is a gammoid  $G = (E, \mathcal{I})$  such that*

$$C_A(G) \geq k \cdot |E|.$$

This conjecture might be easier to prove, and it still would imply that the classes  $\mathcal{W}^k$  of gammoids  $G$  with  $W^k(G) \leq 1$  for  $k \in \mathbb{N}$  contain an infinite sequence of strictly bigger subclasses of gammoids.

### 4.3 $\alpha$ -Violations

In Example 2.5.22 and Remark 2.5.23 we saw that  $\alpha$ -violations, which may be resolved into a strict gammoid by extension, may overlap in a common matroid. It is possible that this common matroid may not be resolved into a strict gammoid, although each restriction that encompasses only a single violation may be extended to a strict gammoid. It is an interesting open research problem to investigate in what ways  $\alpha$ -violations may overlap, and to determine under which circumstances overlapping obstructs the simultaneous resolution of the respective  $\alpha$ -violations into strict gammoid extensions of the matroid exhibiting the overlapping  $\alpha$ -violations.

### 4.4 Excluded Minors

Since every matroid of rank  $\leq 2$  is a gammoid, and every gammoid of rank 3 is a strict gammoid, we may use the formulas from Section 2.5.4 in order to compute all small<sup>2</sup> excluded minors of rank 3 for the class of gammoids, as well as the number of isomorphism classes of small gammoids of rank 3 with  $n$ -elementary ground sets [OEI]. For  $n \leq 10$ , this takes less than 4 hours on modern hardware. The following statements

---

<sup>2</sup>In this case: up to 10 elements.



are up to isomorphy: The smallest excluded minor of rank 3 is  $M(K_4)$  with 6 elements, the second smallest excluded minor is  $P_7$  with 7 elements. There are 3 excluded minors with 8 elements, 11 excluded minors with 9 elements, and 96 excluded minors with 10 elements. The difficult part in obtaining excluded minors with rank and corank greater than 3 is to prove, that the considered matroid is not a gammoid, and apart from  $P_8^-$ , we do not know any excluded minor with rank and corank greater than 3, that is representable over  $\mathbb{R}$  and strongly base-orderable. Therefore we ask: Are there other  $\mathbb{R}$ -representable and strongly base-orderable excluded minors for the class of gammoids with rank and corank greater than 3? Are there infinitely many such excluded minors? Furthermore, we do not know the excluded minors for  $\mathcal{W}^k$ , the classes of gammoids  $G$  with  $W^k(G) \leq 1$ . Is  $\mathcal{W}^k$  characterized by finitely many excluded minors? Moreover, let  $\mathcal{W}_f$  be the class of gammoids  $G$  with  $W_f(G) \leq 1$  for a super-additive function  $f$ . What is the smallest growing behavior of  $f$ , such that  $\mathcal{W}_f$  has infinitely many excluded minors? And, conversely, what is the biggest growing behavior of  $f$ , such that  $\mathcal{W}_f$  has finitely many excluded minors?

## 4.5 Complexity Class of Recognition Problems

V. Chandru, C.R. Coullard, and D.K. Wagner showed in [CCW85] that the problem of deciding, whether a given matroid  $M$  is a bicircular matroid, is NP-hard. The proof involves deciding, whether the frame matroid constructed from a given gain-network is a bicircular matroid or not, in order to answer an instance of the *Subset Product Problem*. Clearly, there are frame matroids which are not gammoids, for instance all Dowling geometries of rank  $\geq 3$  ([Oxl11], p.663) as well as all graphical matroids with an  $M(K_4)$  minor. On the other hand, every bicircular matroid is a gammoid, and thus it is possible that the additional information, that a given frame matroid is a gammoid, helps to decide whether the given matroid is bicircular within polynomial time. If this is the case, then ruling out that a given matroid is a gammoid must be NP-hard — which is the most likely scenario considering D. Mayhew’s result that every gammoid is a minor of an excluded minor for the class of gammoids [May16].

Closely related to the open problem of the complexity class of recognizing gammoids are the open problems regarding the complexity classes of finding a representation, finding a representation with the minimal number of arcs when this number is already known, and determining the minimal number of arcs needed to represent a gammoid; and all of the above for each of the subclasses  $\mathcal{W}^k$  for  $k \in \mathbb{N}$ , too.

## 4.6 Coloring

Every gammoid, that is also a binary matroid, is the polygon matroid of a series-parallel network [Ing77], therefore graphic gammoids are 3-colorable. The following conjecture motivated our studies of gammoids in the first place.

**Conjecture 4.6.1** ([GHN16], Conjecture 14). *Every simple gammoid of rank 2 or greater has a quite simple coline.*

If the conjecture holds, then all gammoids are generalized series-parallel matroids and therefore 3-colorable. Although we were not able to resolve this conjecture at this point, we are convinced that the newly developed theory in this work will prove helpful for future approaches. A related open problem is the question, whether every gammoid is generalized series-parallel, which may still be the case even if the Conjecture 4.6.1 is wrong: although no coline of the non-gammoid  $P_7$  is quite simple, all of its orientations are still generalized series-parallel.

We provided a method of obtaining an orientation of a gammoid from its representation by combinatorial means, but all orientations obtained in this way are representable. It has been known for long that there are oriented matroids whose orientations are not representable. One example of a non-representable orientation is the orientation  $\mathbf{RS}(8)$  of the uniform matroid  $U_{4,8}$  — which is a strict gammoid. Is there a way to obtain some or all non-representable orientations of gammoids in a purely combinatorial way from their representations, possibly by generalizing the notion of *heavy arc signatures*? And finally, is there a way to deal with cycle walks in the digraph of a given representation of a gammoid other than first lifting all the cycle walks, then orienting using an extended heavy arc signature, and then contracting the oriented extension?

*Who questions much, shall learn much, and retain much.*

— *Sir Francis Bacon.*

# Listings

## 5.1 Digraph Backtracking Algorithm

The routine “`isGammoid`” performs a backtracking search in the domain of all digraphs with a fixed number of vertices in order to determine whether a given input matroid  $M$  is a gammoid (Algorithm 2.5.17) – or, optionally, whether  $M$  is a gammoid representable with certain upper bounds on the number of arcs and vertices occurring. It has been tested with `SageMath` version 8 running on macOS 10.13.3. We present some runtime measurements of various inputs in order to convey a sense of how slow this algorithm actually is. We measured the performance using three matroids, one is a non-gammoid, one is a strict gammoid, and one is a non-strict gammoid, each with different upper bounds for the number of vertices allowed in a representing digraph candidate. We stopped the each measurement once a time-limit of 48 hours was reached. Here are the results:

(Example 2.5.25, $M(K_4)$ )	<code>sage: time(isGammoid(MK4()))</code> CPU time total: 27.4 ms
( $M(K_4)$ , $ V  =  E  + 1$ )	<code>sage: time(isGammoid(MK4(),7))</code> CPU time total: 1.95 s
( $M(K_4)$ , $ V  =  E  + 2$ )	<code>sage: time(isGammoid(MK4(),8))</code> CPU time total: 8min 3s
( $M(K_4)$ , $ V  =  E  + 3$ )	<code>sage: time(isGammoid(MK4(),9))</code> CPU time total: > 48h
(Example 2.2.17, $\Gamma(D,T,V)$ , $ V  = 9$ )	<code>sage: time(isGammoid(strictG,9))</code> CPU time total: 46min 57s
(Example 2.2.17, $\Gamma(D,T,V)$ , $ V  = 10$ )	<code>sage: time(isGammoid(strictG,10))</code> CPU time total: > 48h
(Example 2.2.17, $\Gamma(D,T,E)$ , $ V  = 9$ )	<code>sage: time(isGammoid(G),9)</code> CPU time total: 4h 28min 52s

```
(Example 2.2.17,  $\Gamma(D,T,E)$ ,  $|V| = 10$ )  sage: time(isGammoid(G,10))
                                         CPU time total: > 48h
```

Those times suggest that the digraph backtracking method is not suitable for deciding the value of  $\Gamma_{\mathcal{M}}(M)$  for  $M$  defined on ground sets larger than a few elements within a reasonable time frame. The generic bound derived from Remark 2.1.14 is useless in practice, for instance, the matroid defined in Example 2.2.17 has an upper bound of at most 123 vertices in a representing digraph. We might achieve some slight improvement in performance by utilizing a tree structure to store the set of essential paths and the family of maximal essential routings, but this measure would not have any influence on the rapid growth of number of digraph candidates that have to be traversed by the backtracking method (Remark 2.5.18).

```
from itertools import chain, combinations
from sage.all import DiGraph

def vertexBound(n,r):
    if r <= 3: # either strict gammoid or no gammoid
        return n
    if r >= n-3: # either transversal matroid or no gammoid
        return n + r
    return r*r*n + r + n

def arcBoundStrict(n,r):
    return (n-r)*(n-1) # Mason '72 Corollary 2.5

def augmentList(L,n):
    L = list(L)
    i = int(1)
    for j in range(n):
        while i in L:
            i += 1
        L.append(i)
    return L

class NonLoopArcIterator:
    def __init__(self, V):
        self.V = list(V)
        self.lenV = len(self.V)
        self.x = 0
        self.y = 0
```

```

def move2(self, x,y):
    self.x = x
    self.y = y
def next(self):
    if self.x < self.y-1:
        self.x += 1
    elif self.x == self.y -1:
        self.x = self.y
        self.y = 0
    elif self.y < self.x-1:
        self.y += 1
    else: # y == x-1 or y == x
        if self.lenV <= self.x+1:
            raise StopIteration
        self.y = self.x+1
        self.x = 0
    return (self.V[self.x], self.V[self.y])
def done(self):
    if self.x < self.y-1:
        return False
    elif self.x == self.y -1:
        return False
    elif self.y < self.x-1:
        return False
    else: # y == x-1 or y == x
        if self.lenV <= self.x+1:
            return True
        return False

def isGammoid(M, vertexLimit=None, arcLimit=None):
    T = frozenset(M.bases()[0])
    BM = frozenset(M.bases())
    if vertexLimit is None:
        vertexLimit = vertexBound(len(M.groundset()),M.rank())
    if arcLimit is None:
        arcLimit = arcBoundStrict(vertexLimit,M.rank())
    V = sorted(T) + sorted(M.groundset().difference(T))
    E = frozenset(V)
    if len(V) > vertexLimit:
        return False
    if len(V) < vertexLimit:
        V = augmentList(V,vertexLimit - len(V))
    sink_routing = frozenset([(t,t,frozenset([t])) for t in T])
    essentialMaximalRoutings = set([(T,sink_routing,T)])

```

```

newEssentialMaximalRoutings = []
rollbackIndex = [-1]
noLongerEssentialMaximalRoutings = []
rollbackIndex1 = [0]
independentBases = [T]
rollbackIndex2 = [0]
newEssentialPaths = []
rollbackIndex3 = [0]
noLongerEssentialPaths = []
rollbackIndex4 = [0]
rollbackArc = [None]
rollbackArcIdx = [(0,0)]
Vmax = len(E)
rollbackVmax = [0]
requestRollback = False
nbrOfBases = len(M.bases())
arcCount = 0
D = DiGraph()
D.add_vertices(E)
arcs = NonLoopArcIterator(V)
essentialPaths = {}
for u in V:
    for v in V:
        essentialPaths[(u,v)] = set()
for v in V:
    essentialPaths[(v,v)] = set([frozenset([v])])
while 1:
    if len(independentBases) == nbrOfBases and not requestRollback:
        return D, T, sorted(M.groundset()) # 'return 1'
    if arcs.done() or arcCount >= arcLimit or requestRollback:
        # pop state from stack
        idx = rollbackIndex.pop()
        idx1 = rollbackIndex1.pop()
        idx2 = rollbackIndex2.pop()
        idx3 = rollbackIndex3.pop()
        idx4 = rollbackIndex4.pop()
        arc = rollbackArc.pop()
        Vmax = rollbackVmax.pop()
        arcx, arcy = rollbackArcIdx.pop()
        if idx < 0: # 'd = 0'
            return False # 'return 0'
        essentialMaximalRoutings.difference_update(
            newEssentialMaximalRoutings[idx:])
        essentialMaximalRoutings.update(

```

```

        noLongerEssentialMaximalRoutings [idx1 :])
    del newEssentialMaximalRoutings [idx :]
    del noLongerEssentialMaximalRoutings [idx1 :]
    del independentBases [idx2 :]
    for v,w,p in newEssentialPaths [idx3 :]:
        essentialPaths [(v,w)].discard(p)
    for v,w,p in noLongerEssentialPaths [idx4 :]:
        essentialPaths [(v,w)].add(p)
    del newEssentialPaths [idx3 :]
    del noLongerEssentialPaths [idx4 :]
    arcCount -= 1
    D.delete_edge(arc[0], arc[1])
    arcs.move2(arcx, arcy)
    requestRollback = False
else:
    arc = arcs.next()
    if arc[0] in T: # skip arcs with sink-tails
        continue
    if arcs.y > Vmax:
        requestRollback = True
        continue
    # push state to stack
    rollbackArc.append(arc)
    rollbackArcIdx.append((arcs.x, arcs.y))
    rollbackIndex.append(len(newEssentialMaximalRoutings))
    rollbackIndex1.append(len(noLongerEssentialMaximalRoutings))
    rollbackIndex2.append(len(independentBases))
    rollbackIndex3.append(len(newEssentialPaths))
    rollbackIndex4.append(len(noLongerEssentialPaths))
    rollbackVmax.append(Vmax)
    # update digraph
    if arcs.y == Vmax:
        Vmax += 1
    left_part = []
    right_part = []
    for v in V:
        for p in essentialPaths [(v, arc[0])]:
            left_part.append( (v, p) )
        for p in essentialPaths [(arc[1], v)]:
            right_part.append( (v, p) )
    new_paths = set()
    for l0, l in left_part:
        for r0, r in right_part:
            if l.intersection(r):

```

```

        continue
    non_essential_path = False
    for x in l:
        for y in r:
            if D.has_edge(x,y):
                non_essential_path = True
                break
        if non_essential_path:
            break
    if non_essential_path:
        continue
    p = l.union(r)
    new_paths.add((l0 , r0 , p))
idx4 = len(noLongerEssentialPaths)
for v,w,p in new_paths:
    superseeded = []
    for q in essentialPaths[(v,w)]:
        if p.issubset(q):
            superseeded.append(q)
            noLongerEssentialPaths.append((v,w,q))
    newEssentialPaths.append((v,w,p))
    essentialPaths[(v,w)].difference_update(superseeded)
    essentialPaths[(v,w)].add(p)
criterion = [x for x in noLongerEssentialPaths[idx4:]
              if x[0] in E and x[1] in T]
idx = len(newEssentialMaximalRoutings)
idx1 = len(noLongerEssentialMaximalRoutings)
new_E_paths = [x for x in new_paths if x[1] in T and x[0] in E]
for X,R,P in essentialMaximalRoutings:
    if R.intersection(criterion):
        noLongerEssentialMaximalRoutings.append((X,R,P))
    else:
        for v,w,p in new_E_paths:
            if not w in X:
                continue
            if len(p.intersection(P)) > 1:
                continue
            X1 = X.difference([w]).union([v])
            if not X1 in independentBases:
                if not X1 in BM:
                    # found excess base
                    requestRollback = True
                    break
            independentBases.append(X1)

```



```
        R1 = R.difference([(w,w, frozenset([w]))]).union(
                                                    [(v,w,p)])
        if R1.intersection(sink_routing):
            newEssentialMaximalRoutings.append(
                (X1,R1,P.union(p)))
        if requestRollback:
            break
    essentialMaximalRoutings.difference_update(
        noLongerEssentialMaximalRoutings[idx1:])
    essentialMaximalRoutings.update(
        newEssentialMaximalRoutings[idx:])
D.add_edge(arc[0], arc[1])
arcCount += 1
```

## 5.2 Calculating $\alpha_N$ for $N \in \mathcal{X}(M, e)$

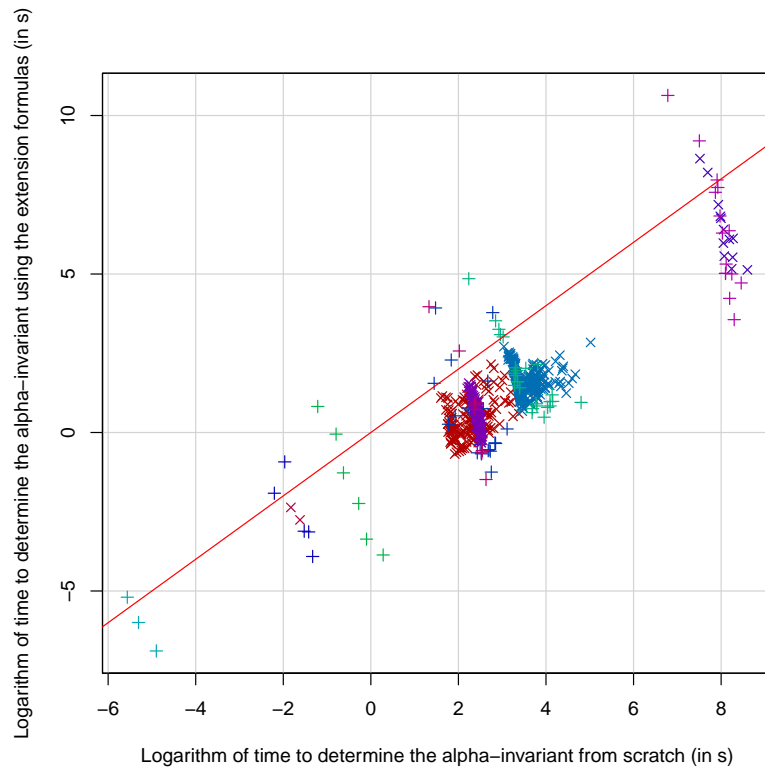


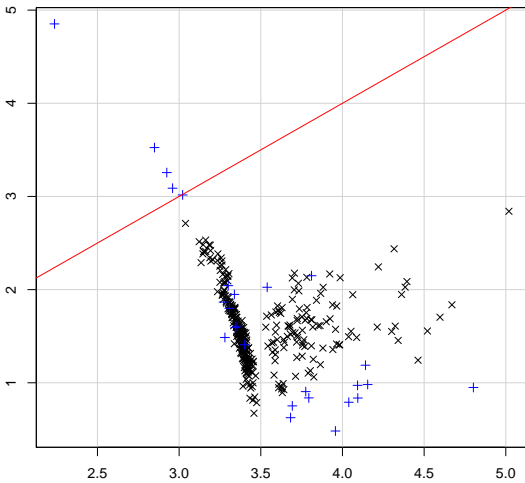
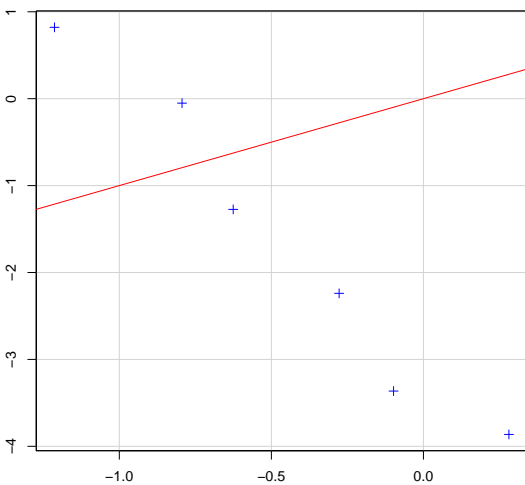
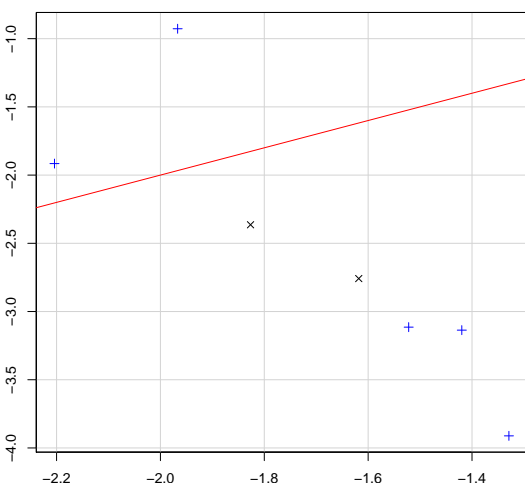
Fig. 5.1 Scatter plot of runtime measurements for Listing 5.2. Each  $+$ -mark corresponds to a principal extension, each  $\times$ -mark to a non-principal extension. The red line indicates equal runtime. The logarithms are dyadic.

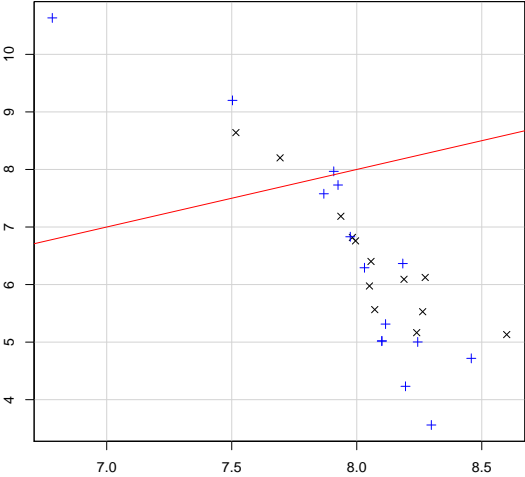
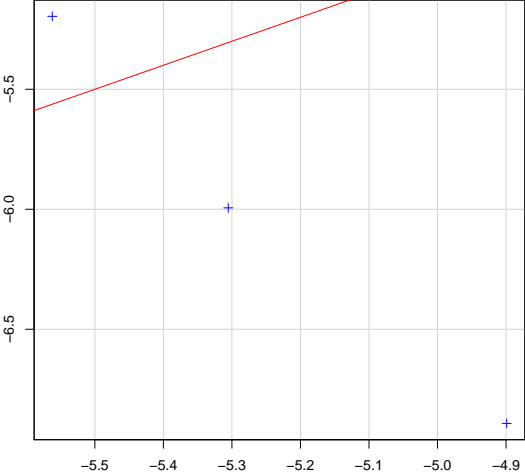
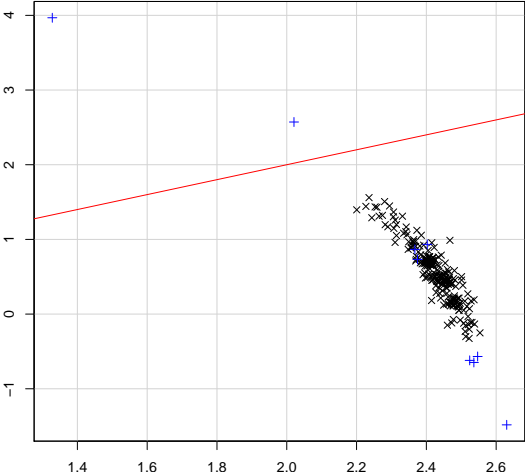
In this section, compare the performance of determining the  $\alpha_N$ -vector of a single element extension  $N \in \mathcal{X}(M, e)$  with the same rank as  $M$  obtained through the formulas derived in Section 2.5.4 with the performance of determining the  $\alpha_N$ -vector from scratch. The performance has been measured with `SageMath` version 8 running on macOS 10.13.3. The program code is listed at the end of this section, the code has been compiled with the built-in Cython compiler of `SageMath` prior to the measurements. For a fixed initial matroid  $M = (E, \mathcal{I})$ , we measured one representative  $N$  of each isomorphism class of the single element extensions of  $M$  that have the same rank as  $N$ . The runtime for each representative has been measured with 3 repetitions per method. In total we performed runtime measurements on 822 different single-element extensions, the median of the extension-formula-runtime to from-scratch-runtime ratio is approximately 0.2635, at least 95% of the measured ratios are smaller than 0.5972.

Therefore we expect the method using the formulas from Section 2.5.4 to be almost four times faster on average, and to be at least  $1\frac{1}{2}$  times faster in the usual case, than computation of  $\alpha_N$  from scratch (Figure 5.1 on p.234).

We give an overview over the measurements with respect to the specific initial matroids in the following table, where  $M_0$  is the initial matroid,  $k$  is the number of non-isomorphic same-rank single element extensions of  $M_0$ ,  $k_1$  is the number of non-isomorphic same-rank single element extensions of  $M_0$  that have a principal modular cut,  $r_{.5}$  is the median extension-formula-runtime to from-scratch-runtime ratio, and  $r_{.95}$  is the 95th percentile extension-formula-runtime to from-scratch-runtime ratio. The scatter plot depicts the dyadic logarithm of the run-time in seconds using the extension-formula (vertical axis) versus the dyadic logarithm of the run-time in seconds calculating from scratch (horizontal axis). The blue  $+$ -marks correspond to single element extensions of  $M_0$  with principal modular cuts, and the black  $\times$ -marks correspond to single element extensions of  $M_0$  that have no principal modular cuts. The red line indicates the locations that correspond to equal run-time for both methods.

$M_0$	$k$	$k_1$	$r_{.5}$	$r_{.95}$
Ex. 2.2.17, $\Gamma(D, T, E)$	177	22	.24	.65

<p>Ex. 2.2.17, <math>\Gamma(D, T, V)</math></p>	367	26	.26	.55	
<p><math>(E, 2^E)</math> <math> E  = 5</math></p>	with 6	6	.45	3.49	
<p><math>M(K_4)</math>, Ex. 2.5.25</p>	7	5	.45	1.81	

<p><math>M(K_4) \oplus M(K_4)</math>, Ex. 2.5.25</p>	<p>28</p>	<p>15</p>	<p>.26</p>	<p>2.87</p>	
<p><math>U_{4,2}</math>, [Oxl11], p.639</p>	<p>3</p>	<p>3</p>	<p>.62</p>	<p>1.22</p>	
<p><math>P_8^-</math>, [Oxl11], p.651</p>	<p>234</p>	<p>9</p>	<p>.27</p>	<p>.52</p>	

```

from itertools import chain, combinations
from sage.all import IncidenceStructure, Matrix, binomial
import timeit, sys
def powerset(s):
    return chain.from_iterable(combinations(s, r)
                               for r in xrange(len(s)+1))
def powersetSubsetIndex(S, X):
    if type(X) != list:
        X = sorted(X)
    idx0, x0, s0 = 0, 0, 0
    for i in range(len(S)):
        idx0 += binomial(len(X), i)
    idxs = sorted((X.index(s) for s in S))
    for i in idxs:
        for j in range(x0, i):
            idx0 += binomial(len(X) - j - 1, len(S) - s0 - 1)
        s0 += 1
        x0 = i + 1
    return idx0
def allFlats(M):
    return chain.from_iterable(M.flats(r) for r in xrange(0, M.rank()+1))
def leqWrtFlats(l, r, flats):
    l = frozenset(l)
    r = frozenset(r)
    if l == r:
        return True
    if not l.issubset(r):
        return False
    if not l in flats:
        return False
    return True
def downsetWrtFlats(r, flats):
    r = frozenset(r)
    return frozenset([r] + [l for l in flats if l.issubset(r)])
def alphaPosetLeq(M):
    return lambda l, r, f=frozenset(allFlats(M)): leqWrtFlats(l, r, f)
def alphaPosetDownsets(M):
    return lambda r, f=frozenset(allFlats(M)): downsetWrtFlats(r, f)
def moebiusMatrixForAlphaPoset(M):
    G = sorted(M.groundset())
    nG = len(G)

```

```

mu = Matrix(2**nG,2**nG, sparse=True)
idx1 = -1
leq = alphaPosetLeq(M)
downset = alphaPosetDownsets(M)
for L in powerset(G):
    idx1 += 1
    idx2 = -1
    for R in powerset(G):
        idx2 += 1
        if L == R:
            mu[idx1, idx2] = 1
        elif leq(L,R):
            s = 0
            for P in downset(R):
                if P == R:
                    continue
                s += mu[idx1, powersetSubsetIndex(P,G)]
            mu[idx1, idx2] = -s
return mu
def alphaVector(M):
    G = sorted(M. groundset ())
    nG = len(G)
    alpha = Matrix(1,2**nG, sparse=True)
    idx1 = -1
    leq = alphaPosetLeq(M)
    downset = alphaPosetDownsets(M)
    for L in powerset(G):
        idx1 += 1
        aval = len(L) - M.rank(L)
        for R in downset(L):
            if R == L:
                continue
            aval -= alpha[0, powersetSubsetIndex(R,G)]
        if aval:
            alpha[0, idx1] = aval
    return alpha
def calculateAlphaOfExtension(e, C, M, alpha=None, mu=None):
    G0 = sorted(M. groundset ())
    G1 = sorted(G0+[e])
    if alpha is None:
        alpha = alphaVector(M)
    if mu is None:
        mu = moebiusMatrixForAlphaPoset(M)
    alphaE = Matrix(1,2**len(G1), sparse=True)

```

```

idxE = -1
deltaAlphaE = {}
for X in powerset(G1):
    X = frozenset(X)
    idxE += 1
    val = 0
    if e in X:
        X0 = X.difference([e])
        clX0 = M.closure(X0)
        if not clX0 in C:
            if clX0 == X0:
                val = 0
            else:
                val = alpha[0, powersetSubsetIndex(X0, G0)]
        else:
            d = 1
            for Z in C:
                if Z.issubset(X0):
                    if Z == X0:
                        continue
                    #print setStr(Z), "<=", setStr(X0)
                    d -= deltaAlphaE[Z]
            deltaAlphaE[X0] = d
            val = alpha[0, powersetSubsetIndex(X0, G0)] + d
    else:
        s = 0
        for Y in powerset(X):
            Y = frozenset(Y)
            if Y == X:
                continue
            nuY = len(Y) - M.rank(Y)
            for Z in C:
                Z = frozenset(Z)
                if not Y.issubset(Z):
                    continue
                if not Z.issubset(X):
                    continue
                if Z == X:
                    continue
                s += nuY * mu[powersetSubsetIndex(Y, G0),
                             powersetSubsetIndex(Z, G0)]
            val = alpha[0, powersetSubsetIndex(X, G0)] + s
    if val:
        alphaE[0, idxE] = val

```



```

    return alphaE
def canonicalMatroidLabel(M):
    IS = IncidenceStructure(M.bases())
    phi = IS.canonical_label()
    return (frozenset((frozenset((phi[x] for x in B))
                           for B in M.bases())), len(M.groundset()))
def measureAlphaMRunTime(M, count=-1, repeats=3, f=sys.stdout, name="M"):
    head = [(M, alphaVector(M), moebiusMatrixForAlphaPoset(M))]
    visited = set([canonicalMatroidLabel(M)])
    eid = 1
    while eid in M.groundset():
        eid += 1
    while (head):
        new_head = []
        for M, alpha, mu in head:
            for M0 in M.extensions(element=eid):
                B0 = canonicalMatroidLabel(M0)
                if B0 in visited:
                    continue
                visited.add(B0)
                C0 = frozenset([F for F in allFlats(M)
                                if eid in M0.closure(F)])
                isPrincipal = False
                F0 = set(M.groundset())
                for F in C0:
                    F0.intersection_update(F)
                    if not F0 in C0:
                        break
                isPrincipal = F0 in C0
                alphas = []
                def f0():
                    alphas.append(calculateAlphaOfExtension
                                   (eid, C0, M, alpha, mu))
                def f1():
                    alphas.append(alphaVector(M0))
                t1 = timeit.timeit(f1, number=repeats)
                t0 = timeit.timeit(f0, number=repeats)
                for a0, a1 in zip(alphas, alpha[1:]):
                    if a0 != a1:
                        raise Exception("Calculation Error!")
                print >>f, ", ".join([str(name),
                                       str(1 if isPrincipal else 0), \
                                       "%.02f"%((t0*100)/t1), str(t0), str(t1)])
                f.flush()

```

```
        new_head.append((M0, alphas[0],
                        moebiusMatrixForAlphaPoset(M0)))
        count -= 1
        if count == 0:
            return
    if count < 0:
        return
    head = new_head
    eidx += 1
    raise Exception("ERROR! out of extensions!")
```

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# Index of Symbols and Notation

$\{\dots\}_\neq$	_____	set consisting of distinct elements given in list, p.16.
$\subseteq_\sigma$	_____	denotes the signed subset relation, p.185.
$\emptyset_{\sigma E}$	_____	denotes the empty signed subset of $E$ , p.186.
$\emptyset_{\mathbb{Z}.E}$	_____	denotes the empty signed multi-subset of $E$ , p.198.
$\vec{\bullet}_D$	_____	outer-extension operator in $D$ , p.62.
$\partial_D \bullet$	_____	outer-margin operator in $D$ , p.62.
$\prod p$	_____	product of the weights of all traversed arcs of $p$ , p.173.
$\lll$	_____	$(\sigma, \lll)$ -induced order on the routings of $D$ , p.213.
$\alpha_M: 2^E \rightarrow \mathbb{Z}$	_____	$\alpha$ -invariant of $M$ , p.108.
$(\mathbf{A}_M, \sqsubseteq_M)$	_____	$\alpha_M$ -poset, p.151.
$(\mathbf{B}_M^C, \sqsubseteq_M^C)$	_____	extension poset of $C \in \mathcal{M}(M)$ , p.157.
$\Delta \alpha_M$	_____	$\Delta \alpha$ -invariant of $M$ , p.154.
$\tilde{\Delta} \alpha_M$	_____	$\tilde{\Delta} \alpha$ -invariant of $M$ , p.157.
$\delta_D(X, T)$	_____	barrier between $X$ and $T$ in $D$ , p.104.
$\Gamma(D, T, E)$	_____	gammoid represented by $(D, T, E)$ , p.74.
$\Gamma(D, T, V)$	_____	strict gammoid, p.103.
$\Gamma_{\mathcal{M}}$	_____	class map for recognizing gammoids in $\mathcal{M}$ , p.124.
$\mu \in K^{R \times C}$	_____	$R \times C$ -matrix over $K$ , p.17.
$\mu^\top \in K^{C \times R}$	_____	transpose of $\mu$ , p.18.
$\mu_c^\top$	_____	$c$ -th column of $\mu$ , p.17.
$\mu_{(P, \leq)}$	_____	Möbius-function of the poset $(P, \leq)$ , p.21.
$\mu_r$	_____	$r$ -th row of $\mu$ , p.17.
$\mu _{R_0}$	_____	restriction of $\mu$ to the rows $R_0$ , p.18.
$\mu _{R_0 \times C_0}$	_____	restriction of $\mu$ to the rows $R_0$ and columns $C_0$ , p.18.
$(\sigma, \lll)$	_____	heavy arc signature of $D = (V, A)$ , p.213.
$\sigma E$	_____	class of signed subsets of $E$ , p.185.
$\zeta_{(P, \leq)}$	_____	zeta-matrix of the poset $(P, \leq)$ , p.21.
$\bigcap \mathcal{A}$	_____	denotes the intersection $\bigcap_{A \in \mathcal{A}} A$ , p.16.

$\bigcup \mathcal{A}$	denotes the union $\bigcup_{A \in \mathcal{A}} A$ , p.16.
$\mathcal{A} = (A_i)_{i \in I} \subseteq E$	family of subsets of $E$ indexed by $I$ , p.55.
$\mathcal{A}_\Delta = (A_i)_{i \in B} \subseteq A$	arc system of $(A \dot{\cup} B, \Delta)$ , p.58.
$\mathcal{A}_{D,T} = (A_i^{(D,T)})_{i \in V \setminus T}$	linkage system of $D$ to $T$ , p.67.
$\text{AC}(D) = (V_D, A_D)$	denotes the arc-cut digraph for $D$ , p.93.
$\text{AC}(D, T, E)$	denotes the arc-cut matroid for $(D, T, E)$ , p.93.
$\mathcal{A}_M = (A_i)_{i \in I} \subseteq E$	denotes the $\alpha$ -system of $M = (E, \mathcal{I})$ , p.112.
$(B1)$	(axiom) a base of $M$ exists, p.35.
$(B2)$	(axiom) equicardinality of bases, p.35.
$(B3)$	(axiom) base exchange, p.35.
$(B3')$	(axiom) strong base exchange, p.25.
$(B, \rho)$	denotes an $M$ -black box, p.98.
$\text{b}(M) = (\text{b}(M, i))_{i=1}^N$	binary encoding of $M$ , p.125.
$\mathcal{B}(M)$	family of all bases of $M = (E, \mathcal{I})$ , p.24.
$\mathcal{B}_M(F)$	family of all bases of $F$ in $M$ , p.24.
$(\mathcal{C}1)$	(o.m. axiom) $\emptyset_{\sigma E}$ is not a circuit, p.188.
$(\mathcal{C}2)$	(o.m. axiom) circuits closed under negation, p.188.
$(\mathcal{C}3)$	(o.m. axiom) incomparability of circuits, p.188.
$(\mathcal{C}4)$	(o.m. axiom) strong circuit elimination, p.188.
$(\mathcal{C}4')$	(o.m. axiom) weak circuit elimination, p.188.
$(\mathcal{C}^*1)$	(o.m. axiom) $\emptyset_{\sigma E}$ is not a cocircuit, p.188.
$(\mathcal{C}^*2)$	(o.m. axiom) cocircuits closed under negation, p.188.
$(\mathcal{C}^*3)$	(o.m. axiom) incomparability of cocircuits, p.188.
$(\mathcal{C}^*4)$	(o.m. axiom) strong cocircuit elimination, p.188.
$C_A(M)$	arc-complexity of the gammoid $M$ , p.83.
$C \perp D$	orthogonality of signed subsets $C, D \in \sigma E$ , p.187.
$\text{cl}_M$	closure operator of $M$ , p.26.
$\mathcal{C}(M)$	circuit set of the matroid $M$ , p.25.
$C_V(M)$	vertex-complexity of the gammoid $M$ , p.83.
$C_{-X} \in \sigma E$	the $X$ -flip of $C \in \sigma E$ , p.195.
$D = (A \dot{\cup} B, \Delta)$	directed bipartite graph for $\Delta$ from $A$ to $B$ , p.58.
$D = (V, A)$	directed graph, p.61.
$D^{\text{opp}} = (V, A^{\text{opp}})$	opposite digraph, p.61.
$D_{r \leftarrow s} = (V, A_{r \leftarrow s})$	$r$ - $s$ -pivot of the digraph $D = (V, A)$ , p.76.
$\mathcal{F}(\alpha)$	family of $\alpha$ -flats for $\alpha: 2^E \rightarrow \mathbb{Z}$ , p.108.
$\mathcal{F}(M)$	family of all flats of $M$ , p.26.



$\mathcal{F}(M, X)$	flats of $M$ that are proper subsets of $X$ , p.107.
$f[X]$	set of images of $x \in X$ under $f$ , p.16.
$f _{X'}$	restriction of the map $f: X \rightarrow Y$ to $X' \subseteq X$ , p.16.
$\mathcal{I}_\alpha$	zero-family of $\alpha: 2^E \rightarrow \mathbb{Z}$ , p.108.
(I1)	(axiom) $\emptyset$ is independent, p.23.
(I2)	(axiom) independence carries over to subsets, p.23.
(I3)	(axiom) augmentation of independent sets, p.23.
idet $\mu$	determinant-indicator of $\mu$ , p.19.
$I(D, T, E)$	matroid on $E$ induced by $D$ from $T = (T_0, \mathcal{T})$ , p.122.
$K^{m \times n}$	class of all $m \times n$ -matrices over $K$ , p.17.
$K^{R \times C}$	class of all $R \times C$ -matrices over $K$ , p.17.
kth( $n, r$ )	bijection that enumerates all $r$ -element subsets of $\{1, 2, \dots, n\}$ , p.124.
$M(\alpha) = (E, \mathcal{I}_\alpha)$	matroid corresponding to the matroid invariant $\alpha$ , p.108.
$M(\mu)$	matroid on $E$ represented by $\mu \in \mathbb{K}^{E \times C}$ , p.43
$M(\mathcal{A}) = (E, \mathcal{I}_\mathcal{A})$	transversal matroid presented by $\mathcal{A}$ , p.60.
$M.C$	contraction of $M$ to $C$ , p.39.
$M(\Delta, M_0)$	matroid induced by $\Delta \subseteq D \times E$ from $M_0$ , p.60.
$M = (E, \mathcal{I})$	(independence) matroid, p.23.
$M^* = (E, \mathcal{I}^*)$	dual matroid of $M$ , p.35.
$\mathcal{M}(M)$	class of all modular cuts of $M$ , p.48.
$M(K_4)$	polygon matroid of the complete graph on 4 vertices, p.149.
$M(\mathcal{O})$	underlying matroid of the oriented matroid $\mathcal{O}$ , p.188.
$M R$	restriction of $M$ to $R$ , p.38.
$\mathbf{N}(M)$	encoding length of $M$ , p.125.
$\mathbb{N}^X$	multi-sets over $X$ , p.17.
$\mathbb{N}^{(X)}$	finite multi-sets over $X$ , p.17.
(O1)	(o.m. axiom) orthogonality, p.188.
(O2)	(o.m. axiom) underlying matroid, p.188.
$\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$	oriented matroid, p.188.
$\mathcal{O}^* = (E, \mathcal{C}^*, \mathcal{C})$	dual oriented matroid of $\mathcal{O}$ , p.190.
$[\mathcal{O}]$	reorientation class of $\mathcal{O}$ , p.196.
$\mathcal{O}(\mu) = (E, \mathcal{C}_\mu, \mathcal{C}_\mu^*)$	oriented matroid represented by $\mu \in \mathbb{R}^{E \times C}$ , p.190.
$\mathcal{O} R$	restriction of $\mathcal{O}$ to $R$ , p.197.
$\mathcal{O}.Q$	contraction of $\mathcal{O}$ to $Q$ , p.197.

$(p_i)_{i=1}^n \in \{N, E\}^n$	lattice path on an $\{N, E\}$ -grid, p.203.
$ p $	set of vertices visited by $p$ , p.63.
$ p _A$	set of arcs traversed by $p$ , p.63.
$\mathbf{P}(D)$	set of paths in $D$ , p.63.
$\mathbf{P}(D; u, v)$	set of paths from $u$ to $v$ in $D$ , p.63.
$p \preceq q$	$p$ is never above $q$ with common endpoints, p.203.
$\mathbf{P}[p, q]$	lattice walks between $p$ and $q$ , p.203.
$\downarrow_{(P, \leq)} y$	the down-set of $y \in P$ with respect to the poset $(P, \leq)$ , p.20.
$(R1')$	(axiom) $\text{rk}(\emptyset) = 0$ , p.28.
$(R2')$	(axiom) $\text{rk}$ is unit-increasing, p.28.
$(R3')$	(axiom) if two points are dependent, so is their line, p.28.
$(R2'')$	unit-increment propagates to subsets, p.29.
$(R1)$	(axiom) $\text{rk}$ is non-negative and subcardinal, p.28.
$(R2)$	(axiom) $\text{rk}$ is non-decreasing, p.28.
$(R3)$	(axiom) $\text{rk}$ is submodular, p.28.
$(R4)$	there is a unique maximal superset of same rank, p.30.
$(R5)$	there are rank-cardinality independent subsets, p.31.
$\text{Rec}\Gamma_{\mathcal{M}}$	gammoid recognition problem for $\mathcal{M}$ , p.124.
$\text{rk}_M$	rank function of $M$ , p.26.
$\mathbb{R}[X]$	polynomials over commutative $\mathbb{R}$ with variables $X$ , p.19.
$R: X \rightrightarrows Y \subseteq \mathbf{W}(D)$	routing from $X$ to $Y$ in $D$ , p.65.
$\text{sep}(C, D)$	separator of signed subsets $C, D \in \sigma E$ , p.186.
$\text{sgn}_{\sigma}(R)$	sign of routing w.r.t. $(\sigma, \ll)$ , p.213.
$\mathbf{T} = (G, \mathcal{G}, \mathcal{M}, \mathcal{X}, \simeq)$	matroid tableau, p.160.
$\bigcup_{i=1}^n \mathbf{T}_i$	joint matroid tableau, p.161.
$[\mathbf{T}]_{\simeq}$	expansion matroid tableau, p.162.
$[\mathbf{T}]_{\equiv}$	extended matroid tableau, p.163.
$\mathbf{T}!$	conclusion matroid tableau, p.163.
$\mathbf{T}(M_1 \simeq M_2)$	identified matroid tableau, p.163.
$w = (w_i)_{i=1}^n \in V^n$	walk in $D = (V, A)$ , p.63.
$\mathbf{W}(D)$	set of walks in $D$ , p.63.
$\mathbf{W}(D; u, v)$	set of walks from $u$ to $v$ in $D$ , p.63.
$\mathbf{W}_f(M)$	$f$ -width of the gammoid $M$ , p.88.
$(w_1 w_2 \dots w_n)^i$	shorthand for $i$ -iterations of $w_1 w_2 \dots w_n \in V^n$ , p.63.
$w.q$	concatenated walk $w_1 w_2 \dots w_n q_2 q_3 \dots q_m$ , p.64.

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$X: E \longrightarrow \{-1, 0, 1\}$	_____	signed subset of $E$ , p.185.
$X_+$	_____	positive elements of $X$ , p.185.
$X_-$	_____	negative elements of $X$ , p.185.
$X_{\pm}$	_____	support of $X$ , p.185.
$X_0$	_____	zero-set of $X$ , p.185.
$-X$	_____	negation of $X$ , p.185.
$\mathcal{X}(M, e)$	_____	class of single-element extensions of $M$ by $e$ , p.48.
$\mathcal{V}(M)$	_____	family of all $\alpha_M$ -violations, p.145.
$\mathbb{Z}.E$	_____	family of signed multisets $S: E \longrightarrow \mathbb{Z}$ , p.198.



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