

# A CAYLEY-TYPE IDENTITY FOR TREES

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ABSTRACT. We prove a weighted generalization of the formula for the number of plane vertex-labeled trees.

## 1. INTRODUCTION

It is well known that the number of vertex-labeled trees on  $n$  vertices is  $n^{n-2}$ . The formula was discovered by Carl Wilhelm Borchardt in 1860 [Borchardt(1860)] and was extended by Cayley in [Cayley(1889)]. Since then many proofs of this formula were given, and many extensions were found. A beautiful well-known extension is the following weighted Cayley formula.

**Theorem 1.1.** *Let  $T_n$  be the set of vertex-labeled trees with  $n$  vertices labeled by  $[n] = \{1, \dots, n\}$ . Associate a variable  $x_i$  to every  $i \in [n]$ , and associate the monomial  $\prod_{i \in [n]} x_i^{d_T(i)}$  to  $T \in T_n$ , where  $d_T(i)$  is the degree of  $i$  in  $T$ . Then*

$$\sum_{T \in T_n} \prod_{i \in [n]} x_i^{d_T(i)} = \prod_{i=1}^n x_i \left( \sum_{i=1}^n x_i \right)^{n-2}.$$

An identity closely related to Cayley's formula is the formula, due to Leroux and Miloudi, [Leroux and Miloudi(1992)] (see also [Callan(2014)] for a short proof) which says that for  $n \geq 2$  there are  $\binom{2n-3}{n-1}$  vertex-labeled plane trees on  $n$  vertices. By a plane tree we mean an abstract tree enriched with cyclic orders for the edges which emanate from each vertex.

In this note we prove a "weighted version" for this formula, namely

**Theorem 1.2.** *Associate a variable  $x_i$  to every  $i \in [n]$ , and for integers  $m \geq 1$  and  $a$  denote by  $\binom{x+m+a}{x+a}$  the polynomial*

$$\prod_{i=a+1}^{a+m} (x+i),$$

we extend the definition to  $m = 0$  by writing  $\binom{x+a}{x+a} = 1$ . For  $n \geq 2$  it holds that

$$\sum_{T \in T_n} \prod_{i \in [n]} \binom{x_i + d_T(i) - 1}{x_i - 1} = \prod_{i=1}^n x_i \binom{\sum_{i \in [n]} x_i + 2n - 3}{\sum_{i \in [n]} x_i + n - 1}.$$

For example, for  $n = 2$  the left and right hand sides of the formula give  $x_1 x_2$ . For  $n = 3$  the formula gives

$$x_1 x_2 x_3 (x_1 + x_2 + x_3 + 3).$$

Dividing both sides by  $\prod x_i$  and substituting  $x_1 = \dots = x_n = 0$  gives precisely the Leroux-Miloudi formula. The weighted Leroux-Miloudi formula was used in [Luria and Tessler(2016)] to calculate and prove the threshold for the appearance of spanning 2-spheres in the Linial-Meshulam model for random 2-complexes.

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## 2. PROOF OF THE FORMULA

When  $n = 2$  the formula trivially holds (in fact, correctly interpreted, the formula extends to  $n = 1$ ). Our proof will be inductive. By dividing both sides by  $\prod x_i$  the theorem is seen to be equivalent to proving, for  $n \geq 2$ ,

$$(1) \quad \sum_{T \in \mathcal{T}_n} \prod_{i \in [n]} \binom{x_i + d_T(i) - 1}{x_i} = \binom{\sum_{i \in [n]} x_i + 2n - 3}{\sum_{i \in [n]} x_i + n - 1}.$$

Substitute  $y_i = x_i + 1$ . (1) then translates to

$$(2) \quad L_n(y_1, \dots, y_n) := \sum_{T \in \mathcal{T}_n} \prod_{i \in [n]} \binom{y_i + d_T(i) - 2}{y_i - 1} = \binom{\sum_{i \in [n]} y_i + n - 3}{\sum_{i \in [n]} y_i - 1} =: R_n(y_1, \dots, y_n).$$

Both the left hand and right hand side are polynomials of degree  $n - 2$  in  $n$  variables. Thus, any monomial does not contain at least one of the variables. Hence, (2) will follow from proving that for each  $i = 1, \dots, n$

$$(3) \quad L_n(y_1, \dots, y_n)|_{y_i=0} = R_n(y_1, \dots, y_n)|_{y_i=0}.$$

Since  $L_n, R_n$  are in addition symmetric, it is enough to prove (3) for  $i = n$ . As

$$\binom{n}{\sum_{i=1}^n y_i} \Big|_{y_n=0} = \sum_{i=1}^{n-1} y_i$$

we have

$$(4) \quad R_n|_{y_n=0} = (n - 3 + \sum_{i=1}^{n-1} y_i) R_{n-1}.$$

The induction will therefore follow if we could show that

$$(5) \quad L_n|_{y_n=0} = (n - 3 + \sum_{i=1}^{n-1} y_i) L_{n-1}.$$

Denote by  $w(T)$  the summand in (2) which corresponds to the tree  $T$ ,

$$w(T) = \prod_{i \in [n]} \binom{y_i + d_T(i) - 2}{y_i - 1}.$$

Observe that if  $y_n = 0$  then  $w(T) = 0$  whenever  $d_T(n) > 1$ . Thus,

$$(6) \quad L_n(y_1, \dots, y_n)|_{y_i=0} = \sum_{T \in \mathcal{T}'_n} \prod_{i \in [n]} \binom{y_i + d_T(i) - 2}{y_i - 1},$$

where  $\mathcal{T}'_n \subseteq \mathcal{T}_n$  is the collection of trees in which  $n$  is a leaf. For  $t \in \mathcal{T}'_n$  let  $a(t) \in [n - 1]$  be the single neighbour of  $n$  and let  $t(T) \in \mathcal{T}_{n-1}$  be the tree obtained from erasing the vertex  $n$ . It holds that

$$(7) \quad w(T) = (y_{a(t)} + d_T(a(t)) - 2)w(t(T)) = (y_{a(t)} + d_{t(T)}(a(t)) - 1)w(t(T)).$$

Since the sum of degrees of vertices in a graph is twice the number of edges, and the number of edges in a tree on  $m$  vertices is  $m - 1$  (7) yields, for any  $T \in \mathcal{T}_{n-1}$ ,

$$(8) \quad \sum_{T' \in t^{-1}(T)} w(T') = \sum_{a \in [n-1]} (y_a + d_T(a) - 1)w(T) = \left( \sum_{a=1}^{n-1} y_a + n - 3 \right) w(T).$$

Putting (6),(8) and the definition of  $L_n$  together

$$L_n|_{y_i=0} = \sum_{T \in \mathcal{T}'_n} \prod_{i \in [n]} \binom{y_i + d_T(i) - 2}{y_i - 1} = \left( \sum_{a=1}^{n-1} y_a + n - 3 \right) \sum_{T \in \mathcal{T}_{n-1}} w(T) = L_{n-1}$$

which is precisely (5)

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