# Counting Consecutive Pattern Matches in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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**Abstract**

In this paper, we focus on consecutive pattern distributions in permutations that avoid classical pattern 123 and 132, that is, $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$. We first study generating functions of length-3 consecutive pattern matching in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ and then extend method to sets of patterns and some more general cases.

**Keywords:** permutations, classical pattern, consecutive patterns, Catalan number, Dyck paths

## 1 Introduction

Let $\mathcal{S}_n$ denote the set of the permutations of length $n$. In this paper, we use one-line notation for permutations. For example, $\sigma = 145623 \in \mathcal{S}_6$ is a permutation of length 6.

For a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathcal{S}_n$, we let $\mathrm{Des}(\sigma) := \{i : \sigma_i > \sigma_{i+1}\}$ and $\mathrm{des}(\sigma) := |\mathrm{Des}(\sigma)|$. If $i \in \mathrm{Des}(\sigma)$, we say that $\sigma$ has a *descent* at $i^{\text{th}}$ position. An *ascent* is an index $1 \leq i \leq n-1$ s.t. $i \notin \mathrm{Des}(\sigma)$. We use $\mathrm{asc}(\sigma)$ to denote the number of ascents in $\sigma \in \mathcal{S}_n$, and obviously $\mathrm{asc}(\sigma) + \mathrm{des}(\sigma) = n - 1$.

Now consider a sequence $\lambda = \lambda_1\lambda_2\ldots\lambda_k$ of distinct integers. We replace the $i^{\text{th}}$ smallest element in $\lambda$ by $i$ to get the *reduction* of $\lambda$, denoted by $\mathrm{red}(\lambda)$. For example,

$$\mathrm{red}(51) = 21, \quad \mathrm{red}(2638) = 1324.$$

For a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathcal{S}_n$ and a permutation $\lambda \in \mathcal{S}_k$ where $k \leq n$, if there exist integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

$$\mathrm{red}(\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}) = \lambda,$$

we say $\sigma$ *contains classical pattern* $\lambda$. Otherwise, we say $\sigma$ *avoids* classical pattern $\lambda$. If there exists an integer $i$ such that

$$\mathrm{red}(\sigma_i\sigma_{i+1}\sigma_{i+2}\ldots\sigma_{i+k-1}) = \lambda,$$

we say $\sigma$ *contains consecutive pattern* $\lambda$, or $\sigma$ has a consecutive $\lambda$-match at position $i$. If $\sigma$ doesn't contain consecutive pattern $\lambda$, we say $\sigma$ *avoids* $\lambda$ *consecutively.* We let $\lambda\text{-mch}(\sigma)$ denote the number of consecutive $\lambda$-matches in $\sigma$. Clearly, a descent is equivalent to consecutive pattern 21-match. Notions of pattern matching can be naturally extended to a set of patterns. For instance, the number of peaks in permutation $\sigma$ is equal to $\Gamma\text{-mch}(\sigma)$ where $\Gamma = \{132, 231\}$.

In short, a consecutive pattern requires the indices of the subsequence in the permutation to be adjacent while a classical pattern does not have this restriction. Taking pattern $\lambda = 132$ as an example, permutation $\sigma = 23541$ avoids consecutive pattern $\lambda$ but contains classical pattern $\lambda$ because $\text{red}(\sigma_1\sigma_3\sigma_4) = \text{red}(254) = \lambda$. We let $\mathcal{S}_n(\lambda)$ denote the set of permutations in $\mathcal{S}_n$ avoiding $\lambda$. In the example above, $\sigma \notin \mathcal{S}_5(\lambda)$ although $\lambda\text{-mch}(\sigma) = 0$.

Both classical permutation patterns and consecutive permutation patterns are popular topics in combinatorics, especially in enumerative combinatorics. One well-known result is that for any $\sigma \in \mathcal{S}_3$, the number of permutations in $\mathcal{S}_n$ avoiding classical pattern $\sigma$ is counted by $n^{\text{th}}$ Catalan number, that is,

$$|\mathcal{S}_n(\sigma)| = C_n = \frac{1}{n+1}\binom{2n}{n}, \quad \forall \sigma \in \mathcal{S}_3. \tag{1}$$

More generally, combinatorists are interested in finding (exponential) generating functions such as

$$F_\sigma(t) := \sum_{n\geq 0} |\mathcal{S}_n(\sigma)| t^n \tag{2}$$

and

$$G_\sigma(t, x) := \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\alpha\in\mathcal{S}_n} x^{\sigma\text{-mch}(\alpha)}. \tag{3}$$

For two permutation patterns $\sigma$ and $\lambda$, we say $\sigma$ and $\lambda$ are *Wilf equivalent* if

$$F_\sigma(t) \equiv F_\lambda(t). \tag{4}$$

Similarly, we say $\sigma$ and $\lambda$ are *c-Wilf equivalent* if

$$G_\sigma(t, 0) \equiv G_\lambda(t, 0), \tag{5}$$

where 'c' stands for 'consecutive'. We say $\sigma$ and $\lambda$ are *strongly c-Wilf equivalent* if

$$G_\sigma(t, x) \equiv G_\lambda(t, x). \tag{6}$$

Although both classical patterns and consecutive patterns have been studied separately for a long time (e.g. [1, 3]), there is not much research about the consecutive pattern matching in $\mathcal{S}_n(\sigma)$. Very recently, Lara Pudwell studied distribution of valleys and peaks in $\mathcal{S}_n(123)$ in [6]. In this paper, we shall study a more general situation. Our focus is to find the consecutive pattern matching in $\mathcal{S}_n(\sigma)$ for all $\sigma \in \mathcal{S}_3$ and keep track number of descents meanwhile. In other words, we want to study the following generating function,

$$A_\lambda^{\overline{\Gamma}}(t, y, x_1, \ldots, x_s) := \sum_{n\geq 0} t^n \sum_{\sigma\in\mathcal{S}_n(\lambda)} y^{\text{des}(\sigma)} x_1^{\gamma_1\text{-mch}(\sigma)} \cdots x_s^{\gamma_s\text{-mch}(\sigma)}, \tag{7}$$

where $\lambda \in \mathcal{S}_3$ and $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ is a set of consecutive patterns. As mentioned above, for $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)|$ is counted by Catalan numbers which also counts Dyck paths. We will mainly use bijections between $\mathcal{S}_n(\sigma)$ and Dyck paths to find formulas for $A_\lambda^{\overline{\Gamma}}$.

The outline of this paper is as follows. In Section 2, we will study strong c-Wilf equivalence by explicitly constructing the relevant generating functions. We will also review the bijections of Krattenthaler [4] and Elizalde and Deutsch [2] and three Dyck path recursions. In Section 3, we shall compute all generating functions of the form $A_{\lambda}^{\overline{\gamma}}(t,y,x)$ where $\lambda$ and $\gamma$ are in $\mathcal{S}_3$. In Section 4, we shall study the generating functions of $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ tracking multiple consecutive patterns. In particular, we compute $A_{123}^{\overline{132},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3)$ and $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$. In Section 5 and 6, we will study generating functions of the form

$$A_{\lambda}^{\overline{\gamma}}(t,x) := \sum_{n\geq 0} t^n \sum_{\sigma\in\mathcal{S}_n(\lambda)} x^{\gamma\text{-mch}(\sigma)}, \tag{8}$$

where $\lambda = 123$ or $132$, and $\gamma$ is of some special shapes.

# 2 Preliminaries

## 2.1 c-Wilf equivalence in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

We shall begin with studying c-Wilf equivalence in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ about length 3 patterns.

Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathcal{S}_n$, we let $\sigma^r$ be the *reverse* of $\sigma$ defined by $\sigma^r = \sigma_n\ldots\sigma_2\sigma_1$, and $\sigma^c$ be the *complement* of $\sigma$ defined by $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$. Further, we let $\sigma^{rc}$ be the *reverse-complement* of $\sigma$. For example, for $\sigma = 15324$, we have $\sigma^r = 42351$, $\sigma^c = 51342$, $\sigma^{rc} = 24315$.

It is clear that the distribution of consecutive pattern $\gamma$ in $\mathcal{S}_n(\lambda)$ is equal to the distribution of consecutive pattern $\gamma^\star$ in $\mathcal{S}_n(\lambda^\star)$ for $\star = r, c$ and $rc$. Since $123 = 321^r$, $132 = 231^r = 312^c = 213^{rc}$, we only need to study the distribution of patterns in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ to get the distribution of patterns in any set $\mathcal{S}_n(\lambda)$ where $\lambda$ has size 3. In other words, we need to study the 10 generating functions: $A_{123}^{\overline{132}}(t,y,x)$, $A_{123}^{\overline{213}}(t,y,x)$, $A_{123}^{\overline{231}}(t,y,x)$, $A_{123}^{\overline{312}}(t,y,x)$, $A_{123}^{\overline{321}}(t,y,x)$, $A_{132}^{\overline{123}}(t,y,x)$, $A_{132}^{\overline{213}}(t,y,x)$, $A_{132}^{\overline{231}}(t,y,x)$, $A_{132}^{\overline{312}}(t,y,x)$ and $A_{132}^{\overline{321}}(t,y,x)$, where

$$A_{\lambda}^{\overline{\gamma}}(t,y,x) := \sum_{n\geq 0} t^n \sum_{\sigma\in\mathcal{S}_n(\lambda)} y^{\mathrm{des}(\sigma)} x^{\gamma\text{-mch}(\sigma)}. \tag{9}$$

Since $\mathcal{S}_n(123)$ is closed under the action reverse-complement (since $123^{rc} = 123$), we have the following lemma.

**Lemma 1.** *For any pattern $\gamma$, we have*

$$A_{123}^{\overline{\gamma}}(t,y,x) = A_{123}^{\overline{\gamma^{rc}}}(t,y,x). \tag{10}$$

As a consequence of Lemma 1, we have

$$A_{123}^{\overline{132}}(t,y,x) = A_{123}^{\overline{213}}(t,y,x), \tag{11}$$

$$A_{123}^{\overline{231}}(t,y,x) = A_{123}^{\overline{312}}(t,y,x). \tag{12}$$

On the other hand, $\mathcal{S}_n(132)$ is not closed under reverse, complement or reverse-complement. We shall use the following bijection between $\mathcal{S}_n(312)$ and $\mathcal{S}_n(213)$ in [8].

**Theorem 1** ([8], Theorem 1)**.** *There is a bijection* $\phi_n : \mathcal{S}_n(312) \leftrightarrow \mathcal{S}_n(213)$ *such that the position of* 1 *in* $\sigma$ *and* $\phi_n(\sigma)$ *are the same. The bijection* $\phi_n$ *acts recursively on the numbers to the left and to the right of the number* 1*, and descent positions are not changed, i.e.* $\mathrm{Des}(\sigma) = \mathrm{Des}(\phi_n(\sigma))$*. Thus* $\mathrm{des}(\sigma) = \mathrm{des}(\phi_n(\sigma))$.

Since the bijection in Theorem 1 does not change the descent positions and the consecutive pattern $k \cdots 21$ can be seen as $k-1$ consecutive descents, we have

$$A_{312}^{\overline{k\cdots 21}}(t,y,x) = A_{213}^{\overline{k\cdots 21}}(t,y,x). \tag{13}$$

From the recursive definition of the bijection $\phi_n$ in [8], the number of consecutive patterns $1k\cdots 32$ does not change in each recursive step since this counts the number of $k-2$ consecutive descents after the smallest number. We have

$$A_{312}^{\overline{1k\cdots 32}}(t,y,x) = A_{213}^{\overline{1k\cdots 32}}(t,y,x). \tag{14}$$

Since $312 = 132^c$ and $213 = 132^{rc}$, and the action complement changes the number of descents to the number of ascents, we have the following lemma.
**Lemma 2.**

$$A_{132}^{\overline{12\cdots k}}(t,y,x) = 1 + \frac{A_{132}^{\overline{k\cdots 21}}(ty,\frac{1}{y},x) - 1}{y}, \tag{15}$$

$$A_{132}^{\overline{k12\cdots (k-1)}}(t,y,x) = 1 + \frac{A_{132}^{\overline{(k-1)\cdots 21k}}(ty,\frac{1}{y},x) - 1}{y}. \tag{16}$$

It follows immediately that

$$A_{132}^{\overline{123}}(t,y,x) = 1 + \frac{A_{132}^{\overline{321}}(ty,\frac{1}{y},x) - 1}{y}, \tag{17}$$

$$A_{132}^{\overline{312}}(t,y,x) = 1 + \frac{A_{132}^{\overline{213}}(ty,\frac{1}{y},x) - 1}{y}. \tag{18}$$

Setting $y=1$ in equations above, we have $A_{132}^{\overline{123}}(t,1,x) = A_{132}^{\overline{321}}(t,1,x)$ and $A_{132}^{\overline{312}}(t,1,x) = A_{132}^{\overline{213}}(t,1,x)$.

Therefore, we only need to compute the following six generating functions for patterns of length 3: $A_{123}^{\overline{132}}(t,y,x)$, $A_{123}^{\overline{231}}(t,y,x)$, $A_{123}^{\overline{321}}(t,y,x)$, $A_{132}^{\overline{123}}(t,y,x)$, $A_{132}^{\overline{213}}(t,y,x)$ and $A_{132}^{\overline{231}}(t,y,x)$.

## 2.2 Bijections from $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ to $(n,n)$-Dyck paths

We shall first review Dyck paths. In this paper, we use *down-right-Dyck paths*. Given an $n \times n$ square, we will label the coordinates of the columns from left to right and the coordinates of the rows from top to bottom with $0, 1, \ldots, n$. An $(n,n)$-*Dyck path* is a path made up of unit down-steps $D$ and unit right-steps $R$ which starts at $(0,0)$ and ends at $(n,n)$ and stays on or below the diagonal $y = x$. The set of $(n,n)$-Dyck paths is denoted by $\mathcal{D}_n$.

Given a Dyck path $P$, we let the first return of $P$, denoted by $\mathrm{ret}(P)$, be the smallest number $i > 0$ such that $P$ goes through the point $(i,i)$. For example, for $P = DDRDDRRRDDRDRDRRDR$ shown in Figure 1, $\mathrm{ret}(P) = 4$ since the leftmost point on the diagonal that $P$ goes through is $(4,4)$.

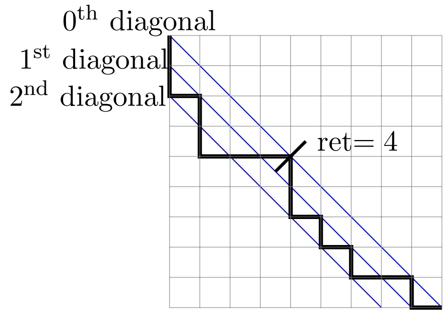
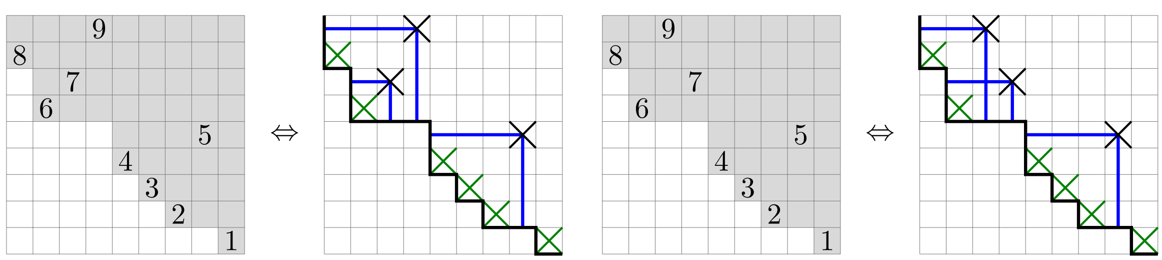



Figure 1: An $(n, n)$-Dyck path

We refer to positions $(i, i)$ where $P$ goes through as *return positions* of $P$. We shall also label the diagonals that go through corners of squares that are parallel to and below the main diagonal with $0, 1, 2, \ldots$ starting at the main diagonal, as shown in Figure 1. The *peaks* of a path $P$ are the positions of consecutive $DR$ steps.

As we have mentioned, $|\mathcal{S}_n(132)| = |\mathcal{S}_n(123)| = |\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n^{\text{th}}$ Catalan number. Many bijections are known between these Catalan objects (see [9]), and we use the bijection of Krattenthaler [4] between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ and the bijection of Elizalde and Deutsch [2] between $\mathcal{S}_n(123)$ and $\mathcal{D}_n$. The second and the last author of this paper also discussed the two bijections in [7, 8] with more details.

We shall first describe Krattenthaler's [4] bijection $\Phi$ between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$. Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$, we write it on an $n \times n$ table by placing $\sigma_i$ in the $i^{\text{th}}$ column counting from left to right and $\sigma_i^{\text{th}}$ row counting from bottom to top. Then, we shade the cells to the north-east of the cell that contains $\sigma_i$. $\Phi(\sigma)$ is the path that goes along the south-west boundary of the shaded cells. For example, this process is pictured in Figure 2(a) in the case where $\sigma = 867943251 \in \mathcal{S}_9(132)$. In this case, $\Phi(\sigma) = DDRDDRRRDDRDRDRRDR$.




(a) The map $\Phi$ (b) The map $\Psi$

Figure 2: $\mathcal{S}_n(132) \Leftrightarrow \mathcal{D}_n$, $\mathcal{S}_n(123) \Leftrightarrow \mathcal{D}_n$

Given $\sigma = \sigma_1 \ldots \sigma_n$, we say that $\sigma_j$ is a *left-to-right minimum* of $\sigma$ if $\sigma_i > \sigma_j$ for all $i < j$. It is easy to see that the left-to-right minima of $\sigma$ correspond to *peaks* of the path $\Phi(\sigma)$. For example, for the permutation $\sigma$ pictured in Figure 2(a), there are 6 left-to-right minima, $\{8, 6, 4, 3, 2, 1\}$.

The *horizontal segments* (or *segments*) of the path $\Phi(\sigma)$ are the maximal consecutive sequences of $R$ steps in $\Phi(\sigma)$. For example, in Figure 2(a), the lengths of the horizontal segments, reading from top to bottom, are $1, 3, 1, 1, 2, 1$, and $\{6, 7, 9\}$ is the set of numbers associated with the second horizontal segment of $\Phi(\sigma)$.

The map $\Phi$ is invertible since for each Dyck path $P$, the peaks of $P$ gives the left-to-right minima of the 132-avoiding permutation, and the rest numbers are uniquely determined by the left-to-right minima. More details about $\Phi$ can be found in [4]. We have the following properties for $\Phi$ from [7].

**Lemma 3** ([7], Lemma 3)**.** *Let $P \in \mathcal{D}_n$ and $\sigma = \Phi^{-1}(P)$. Then for each horizontal segment $H$ of $P$, the numbers associated to $H$ form a consecutive increasing sequence in $\sigma$ and the least number of the sequence sits immediately above the first right-step of $H$. Hence the descents in $\sigma$ only occur between two different horizontal segments of $P$. The number $n$ is in the column of last right-step before the first return.*

The bijection $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$ given by Elizalde and Deutsch [2] can be described in a similar way. Given any permutation $\sigma \in \mathcal{S}_n(123)$, $\Psi(\sigma)$ is constructed exactly as in the previous section. Figure 2(b) shows an example of this map, from $\sigma = 869743251 \in \mathcal{S}_9(123)$ to the Dyck path $DDRDDRRRDDRDRDRRDR$. The map $\Psi$ is invertible by the same reason that each 123-avoiding permutation has a unique left-to-right minima set. More details about $\Psi$ can be found in [2]. We then have the following lemma from [7].

**Lemma 4** ([7], Lemma 4)**.** *Let $P \in \mathcal{D}_n$ and $\sigma = \Psi^{-1}(P)$. Then for each horizontal segment $H$ of $P$, the least element of the set of numbers associated to $H$ sits directly above the first right-step of $H$ and the remaining numbers of the set form a consecutive decreasing sequence in $\sigma$.*

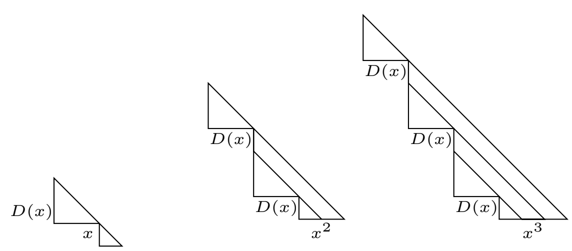
### 2.3 Recursions of Dyck paths

We review three Dyck path recursions for the generating function of Catalan numbers,

$$D(x) = \sum_{n \geq 0} |\mathcal{D}_n| x^n = \sum_{n \geq 0} C_n x^n, \tag{19}$$

which will help us to develop recursions to compute generating functions $A_\lambda^{\overline{\Gamma}}(t, y, x)$. The recursions give the same formula for $D(x)$, while they have different applications when tracking consecutive patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

*Recursion 1.* We can get a recursion by expanding the last horizontal segment. The contribution of empty path to $D(x)$ is 1. When the last horizontal segment is of length $k > 0$, the path ends with steps $DR^k$. The total weight of the last horizontal segment is $x^k$ as it has $k$ columns. Referring to Figure 3, when we trace back the Dyck path from the end, the steps before $DR^k$ can be decomposed into $k$ smaller Dyck paths: one path after the last $D$ step on the $(i-1)^{\text{th}}$ diagonal before the last $D$ step on the $i^{\text{th}}$ diagonal for $i = 0, \ldots, k-1$, each with weight $D(x)$.




(a) (b) (c)

Figure 3: The first recursion of $D(x)$ when the last segment is of size $k = 1, 2, 3$

Thus the contribution of the case when the last horizontal segment has size $k$ is $x^k D(x)^k$, and the recursion for $D(x)$ is

$$D(x) = 1 + \sum_{k>0} x^k D(x)^k. \tag{20}$$

*Recursion 2.* We can also get a recursion by breaking a path $P$ at the first return into 2 Dyck paths: $P = DP_1RP_2$. Here the path $P_1$ locates between the first $D$ step and the last $R$ step before the first return, and $P_2$ locates after the first return, both with weight $D(x)$; the $R$ step before the first return has weight $x$. Considering that the empty path has weight 1, we have

$$D(x) = 1 + xD(x)^2. \tag{21}$$

*Recursion 3.* The third recursion of Dyck path can be seen as the combination of the first recursion and the second recursion. We expand the last horizontal segment before the first return.

The contribution of the empty path to the generating function $D(x)$ is 1. When the last horizontal segment before the first return is of length $k > 0$, the path before the first return ends with steps $DR^k$, which has the weight $x^k$. Referring to Figure 4, the path after the first return forms a Dyck path with weight $D(x)$, and the path before the steps $DR^k$ contains $k-1$ smaller Dyck paths: one path after the last $D$ step on the $(i-1)^{\text{th}}$ diagonal before the last $D$ step on the $i^{\text{th}}$ diagonal for $i = 1, \ldots, k-1$, each with weight $D(x)$. The contribution of the case when the last horizontal segment has size $k$ is $D(x)x^kD(x)^{k-1}$, and the recursion for $D(x)$ is

$$D(x) = 1 + D(x)\sum_{k>0} x^k D(x)^{k-1}. \tag{22}$$

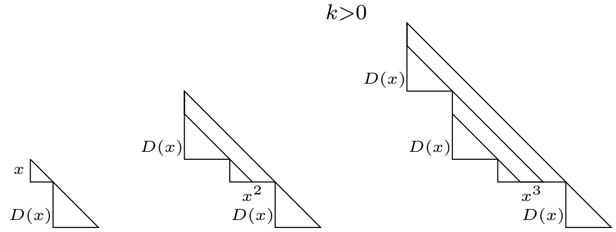



Figure 4: The third recursion of $D(x)$ when the last segment before first return is of size $k = 1, 2, 3$

# 3 Computation of generating functions $A_\lambda^{\overline{\gamma}}(t, y, x)$

In this section, we compute the generating function

$$A_\lambda^{\overline{\gamma}}(t, y, x) := \sum_{n\geq 0} t^n \sum_{\sigma\in\mathcal{S}_n(\lambda)} y^{\mathrm{des}(\sigma)} x^{\gamma\text{-}\mathrm{mch}(\sigma)} \tag{23}$$

for $\lambda$ and $\gamma$ in $\mathcal{S}_3$. By the classification in Section 2.1, we need to calculate 6 generating functions.

## 3.1 The function $A_{123}^{\overline{132}}(t, y, x)$

Given $\sigma \in \mathcal{S}_n(123)$, we can see from Elizalde and Deutsch's bijection and Lemma 4 that $\mathrm{des}(\sigma)$ is equal to the total number of patterns $RD$ and $RRR$ in the corresponding Dyck path $\Psi(\sigma)$. Further, 132-mch$(\sigma)$ is equal to the number of path pattern $DRRR$ in the Dyck path $\Psi(\sigma)$. In other words, we only need to study the distribution of pattern $DRRR$ (which looks like on the $n \times n$ square) in all Dyck paths of size $n$. We can directly get the recursion of $A_{123}^{\overline{132}}(t, y, x)$ using the first Dyck path recursion. We will always use the shorthand $A$ for $A_\lambda^{\overline{\gamma}}(t, y, x)$ in each computation.

**Theorem 2.** *$A_{123}^{\overline{132}}(t,y,x)$ satisfies the following recursion,*

$$A = 1 + t(y(A-1)+1)^2 + t^3(x-1)y^2(y(A-1)+1)^3, \tag{24}$$

*and*

$$A_{123}^{\overline{132}}(t,1,x)|_{t^n x^k} = \frac{1}{k}\sum_{i=k}^{\lfloor \frac{n}{3} \rfloor}(-1)^{i-k}\binom{2n-3i}{n-3i, n+1-i, k-1, i-k}. \tag{25}$$

*Proof.* By the Dyck path bijection $\Psi$, we can take $A$ as the generating function of Dyck paths, where for each path, $t$ tracks the size of the path, $y$ tracks the total number patterns $RD$ and $RRR$ in the path, and $x$ tracks the number of patterns $DRRR$.

We say that a horizontal segment is *interior* if it is neither the first horizontal segment nor the last horizontal segment. For each Dyck path $P$, we call the path $PD$ (adding an extra $D$ step at the end of path $P$) the extended path of $P$.

For our convenience, we define a generating function $A_1$ of Dyck paths similar to $A$:

$$A_1(t,y,x) := \sum_{n\geq 0} t^n \sum_{P\in\mathcal{D}_n} y^{\{RD,RRR\}\text{-mch}(PD)} x^{DRRR\text{-mch}(PD)}. \tag{26}$$

As we defined above, in $A_1$, $t$ tracks the size of $P$, $y$ tracks the total number patterns $RD$ and $RRR$ in the extended path $PD$, and $x$ tracks the number of patterns $DRRR$ in the extended path $PD$ for each Dyck path $P$. In other words, $A_1$ denotes the generating function of Dyck paths of which we suppose there are some lower horizontal segments after the last step, and the last segment is an interior segment.

The difference between $A$ and $A_1$ in this case is that, for any nonempty path, $A_1$ counts one more pattern $RD$ formed by the last $R$ step and the extended $D$ step. Thus we have the following relation between $A$ and $A_1$,

$$A = \frac{A_1 - 1}{y} + 1. \tag{27}$$

Next, we shall formulate a recursion for $A_1$. We shall use Recursion 1 described in Section 2.3, i.e. we expand the last horizontal segment of a Dyck path $P$. The empty path has contribution 1 to $A_1$. If the last horizontal segment is of length 1, then the path ends with steps $DR$, and there is a Dyck path structure before the last two steps with weight $A_1$, as shown in Figure 3(a). The weight of the last segment is $ty$ as the last $R$ step and the extended $D$ step form an $RD$ pattern contributing 1 to the power of $y$. Thus the contribution in this case is $tyA_1$.

If the last horizontal segment is of length 2, then the path ends with steps $DRR$, and there are 2 Dyck path structures before the last three steps both with weight $A_1$, as shown in Figure 3(b). The weight of the last segment is $t^2y$ as the last $R$ step and the extended $D$ step form an $RD$ pattern contributing 1 to the power of $y$. Thus the contribution in this case is $t^2yA_1^2$.

If the last horizontal segment is of length $k \geq 3$, then the path ends with steps $DR^k$, and there are $k$ Dyck path structures before the last $k+1$ steps all with weight $A_1$, as shown in Figure 3(c). The weight of the last segment is $t^kxy^{k-1}$ as the last $R$ step and the extended $D$ step form an $RD$ pattern, and the $R^k$ steps in the last segment contain $k-2$ $RRR$ patterns, contributing $k-1$ to the power of $y$. The $DRRR$ steps in the last segment contribute 1 to the power of $x$. Thus the contribution in this case is $t^kxy^{k-1}A_1^k$.

It follows that we have

$$\begin{aligned} A_1 &= 1 + tyA_1 + t^2yA_1^2 + t^3xy^2A_1^3 + t^4xy^3A_1^4 + \cdots \\ &= 1 + tyA_1 + t^2yA_1^2 + \frac{t^3xy^2A_1^3}{1 - tyA_1}. \end{aligned} \tag{28}$$

We can get the equation (24) from equation (27) and (28).

We shall prove formula (25) in Section 5.1. □

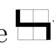
Setting $x = 0$ and $y = 1$ in the generating function, we get

$$A_{123}^{\overline{132}}(t, 1, 0) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2}, \tag{29}$$

which coincides with the generating function of Motzkin numbers (OEIS A001006 [5]). In other words, the number of permutations in $\mathcal{S}_n$ avoiding both classical pattern 123 and consecutive pattern 132 is equal to the number of Motzkin paths from $(0, 0)$ to $(n, 0)$.

We will generalize our method to compute $A_{123}^{\overline{1m\cdots 2}}(t, y, x)$ in Section 5.1.

## 3.2 The function $A_{123}^{\overline{231}}(t, y, x)$

For any permutation $\sigma \in \mathcal{S}_n(123)$, Lemma 4 implies that the number of consecutive patterns 231 matches in $\sigma$ is equal to the number of path patterns $DRRD$ (looks like ) in the Dyck path $\Psi(\sigma)$. As with the analysis of the generating function $A_{123}^{\overline{132}}(t, y, x)$, we only need to study the distribution of pattern $DRRD$ in all Dyck paths of size $n$ to get $A_{123}^{\overline{231}}(t, y, x)$.

**Theorem 3.** $A_{123}^{\overline{231}}(t, y, x)$ *satisfies the following recursion,*

$$\begin{aligned} A = t^2(y-1)^2 \big(y \left((A-1)^2x^2y + x \left((A-1)^3y^3 + (A-1)^2y^2 + (A-1)y + 2A - 1\right)\right. \\ -((A-1)y+1)(y((A-2)(A-1)y + 2A - 3) + 3)) + 1) + (A-1)y((y-3)y+3) + 1 \\ - t(y-1)^2((A-1)y+1)(y(A(x+y-2) - x - y + 3) - 1), \end{aligned} \tag{30}$$

*and*

$$A_{123}^{\overline{231}}(t, 1, x)|_{t^nx^k} = \frac{1}{n}\sum_{i=0}^{\lfloor \frac{3n-1}{5} \rfloor} (-1)^{4n+k+1}\binom{n}{i}\binom{n-2i}{3n-1}\binom{2n-2i}{k}. \tag{31}$$

*Proof.* By the Dyck path bijection $\Psi$, we can take $A$ as the generating function of Dyck paths where for each path, $t$ tracks the size of the path, $y$ tracks the total number patterns $RD$ and $RRR$ in the path, and $x$ tracks the number of patterns $DRRD$.

We define a generating function $A_1$ of to $A$ like what we do in Section 3.1. In $A_1$, for each Dyck path $P$, we let $t$ track the size of $P$, let $y$ track the total number patterns $RD$ and $RRR$ in $PD$, and let $x$ track the number of patterns $DRRD$ in $PD$.

The difference between $A$ and $A_1$ in this case is that, for any nonempty path $P$, $A_1$ counts one more pattern $RD$ that the last $R$ step and the extended $D$ step form; if the last segment of $P$ has length 2, then the last segment and the extended $D$ step form a $DRRD$ pattern contributing 1 to the power of $x$. Thus we have the following relation between $A$ and $A_1$,

$$A = 1 + \frac{A_1 - 1}{y} + t^2(1-x)A_1^2. \tag{32}$$

Next, we shall formulate a recursion for $A_1$ using the first recursion of a Dyck path $P$ by expanding the last horizontal segment. The empty path still has contribution 1. Let $k$ be the size of the last segment, then the case when $k = 1$ still has contribution $tyA_1$.

If $k = 2$, then the path ends with steps $DRR$, and there are 2 Dyck path structures before the last three steps both with weight $A_1$, as shown in Figure 3(b). The weight of the last segment is $t^2xy$ as the last $R$ step and the extended $D$ step form an $RD$ pattern contributing 1 to the power of $y$, and the last segment of $P$ and the extended $D$ step form a $DRRD$ pattern contributing 1 to the power of $x$. Thus the contribution in this case is $t^2xyA_1^2$.

If $k \geq 3$, there will be no pattern $DRRD$ appear in the last segment, thus the power of $x$ is always 0. Similar to the discussion in Section 3.1, the contribution in this case is $t^k y^{k-1} A_1^k$.

It follows that we have

$$\begin{aligned} A_1 &= 1 + tyA_1 + t^2xyA_1^2 + t^3y^2A_1^3 + \cdots \\ &= 1 + tyA_1 + t^2xyA_1^2 + \frac{t^3y^2A_1^3}{1-tyA_1}. \end{aligned} \tag{33}$$

By equation (32), we can get the recursion for $A$ from equation (33) to prove equation (30).

We shall prove formula (31) in Section 5.2. □

We will generalize our method to compute $A_{123}^{\overline{2m\cdots31}}(t,y,x)$ in Section 5.2.

### 3.3 The function $A_{123}^{\overline{321}}(t,y,x)$

By Lemma 4, the consecutive pattern 321 does not correspond to a single path pattern when mapped into Dyck path by $\Psi$. We will show the details in the proof of following theorem.

**Theorem 4.** *The generating function $A_{123}^{\overline{321}}(t,y,x)$ is given by*

$$\begin{aligned} A_{123}^{\overline{321}}(t,y,x) = {} & 1 - \frac{1}{2ty^2\left(x^2(-t)y+x+t\right)^3}\Big(2(x-1)^2t^6y^3\left(x^2y-1\right)^2 \\ & +t^2\left(-x^2(2x+5)y^2+(6x+2)y-1\right)+2t^4y\left((2-3x)x^4y^3+\left(3x^2+2x-2\right)x^2y^2\right. \\ & \left.-\left(3x^2+x-1\right)y+1\right)+2(x-1)t^5y^2\left(x^5y^3-3x^4y^2+4x^2y-xy-1\right) \\ & +2t^3y\left(3x^4y^2-x^3y-5x^2y+x+2\right)+t(4xy-2)-1 \\ & +(t(xy-1)-1)^2\sqrt{4x^2t^2y^2-4xty-4t^2y+1}\Big). \end{aligned} \tag{34}$$

*Proof.* We can count consecutive pattern 321 matches from the horizontal segments of a Dyck path. It is not hard to see from Lemma 4 that the contribution of a horizontal segment to 321 match is given by Table 1 if there are more than one horizontal segments in the path.

It is clear from the table that it matters whether a segment is interior or not when counting consecutive pattern 321 in the corresponding Dyck path. In this proof, we shall define 2 generating

Table 1: the contribution of a horizontal segment to the power of $x$ in $A_{123}^{\overline{321}}(t, y, x)$

| size | first segment | interior segment | last segment |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 |
| 3 | 1 | 1 | 0 |
| 4 | 2 | 2 | 1 |
| 5 | 3 | 3 | 2 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

functions, $A_0$ and $A_1$. For a permutation $\sigma \in \mathcal{S}_n$, we let $\sigma 0$ denote the length-$n+1$ word obtained by adding a 0 in the end of $\sigma$, and let $(n+1)\sigma 0$ denote the length-$n+2$ word obtained by adding an $(n+1)$ before $\sigma$ and a 0 in the end of $\sigma$. Then we define

$$A_0(t, y, x) := \sum_{n \geq 0} t^n \sum_{\sigma \in \mathcal{S}_n(123)} y^{\text{des}((n+1)\sigma 0)} x^{321\text{-mch}((n+1)\sigma 0)}, \tag{35}$$

$$A_1(t, y, x) := \sum_{n \geq 0} t^n \sum_{\sigma \in \mathcal{S}_n(123)} y^{\text{des}(\sigma 0)} x^{321\text{-mch}(\sigma 0)}. \tag{36}$$

Both $A_0$ and $A_1$ track the same statistics as $A$. $A_0$ is the generating function of Dyck paths where we take every segment as an interior segment, and $A_1$ is the generating function of Dyck paths where we suppose that the last segment is an interior segment.

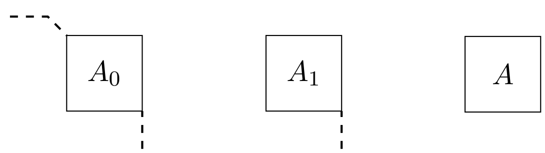



Figure 5: the generating functions $A_0$, $A_1$ and $A$

In other words, we suppose that there is some $R$ steps before the Dyck path structure and a $D$ step following the Dyck path structure that do not contribute to the size but contribute to the pattern matches in the generating function $A_0$; and there is a $D$ step after the Dyck path structure that do not contribute to the size but contribute to the pattern matches in the generating function $A_1$. Figure 5 gives sketches of $A_0$, $A_1$ and $A$.

We shall begin with computing a recursion for $A_0$ since it only consists of interior segments. For each segment in $A_0$, we can get its contribution to descents and 321 matches by Lemma 4, summarized in Table 2.

Table 2: the contribution of horizontal segments in $A_0$

| size | contribution to descents (power of $y$) | contribution to 321 match (power of $x$) |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 0 |
| 3 | 2 | 1 |
| 4 | 3 | 2 |
| $\cdots$ | $\cdots$ | $\cdots$ |

Then, by applying the first Dyck path recursion to expand the last horizontal segment of $A_0$, we can get a recursion for $A_0$ based on Table 2:

$$
\begin{aligned}
A_0 &= 1 + txyA_0 + t^2yA_0^2 + t^3xy^2A_0^3 + t^4x^2y^3A_0^4 + \cdots \\
&= 1 + txyA_0 + \frac{t^2yA_0^2}{1 - txyA_0}.
\end{aligned}
\tag{37}
$$

Next, we want to compute the recursion of $A_1$ in terms of $A_0$ and $A_1$. We use the third recursion for Dyck paths which expands the last horizontal segment before the first return that is of size $k$. When $k = 1$, there are only 2 steps, $DR$, before the first return. There is a Dyck path structure $P_1$ after the first return with weight $A_0$ since all the segments in $P_1$ are interior segments. The contribution of this case is $tyA_0$ as shown in Figure 6(a).

When $k = 2$, the path ends with steps $DRR$ before the first return. There is a Dyck path structure $P_1$ after the first return with weight $A_0$ for the same reason. By the third recursion of Dyck paths, there is another Dyck path structure $P_2$ above the last horizontal segment before the first return. The weight of $P_2$ is $A_1$ since it contains the first segment of the whole path. The contribution of this case is $t^2yA_0A_1$ as shown in Figure 6(b).

When $k = 3$, the path ends with steps $DRRR$ before the first return. There is a Dyck path structure $P_1$ after the first return with weight $A_0$. By the third recursion of Dyck paths, there are two Dyck path structures $P_2$, $P_3$ (counting from top to bottom) above the last horizontal segment before the first return. If $P_2$ is not empty, then it has weight $A_1 - 1$ since it contains the first segment, and the weight of $P_3$ is $A_0$. If $P_2$ is empty, then it has weight 1, and $P_3$ has weight $A_1$ since it contains the first segment of the whole path. The contribution of this case is $t^3xy^2A_0(A_0(A_1 - 1) + A_1)$ as shown in Figure 6(c).

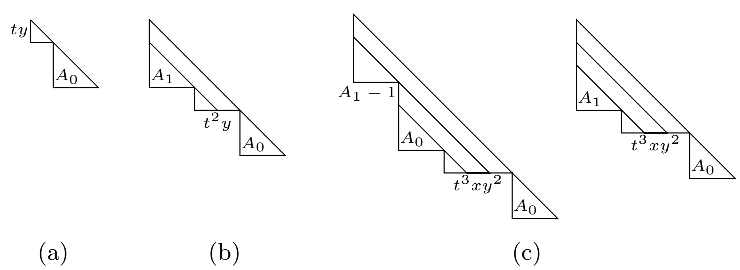



Figure 6: Contributions to $A_1$ when the last segment before the first return has size 1, 2, 3

Generally, if the last horizontal segment before the first return has size $k \geq 3$, the total contribution should be $t^kx^{k-2}y^{k-1}A_0(A_0^{k-2}(A_1 - 1) + \cdots + A_0(A_1 - 1) + A_1)$. It follows that

$$
\begin{aligned}
A_1 &= 1 + tyA_0 + t^2yA_0A_1 + t^3xy^2A_0(A_0(A_1 - 1) + A_1) + \cdots \\
&= 1 + tyA_0 + \frac{A_0 - A_1}{A_0 - 1}\frac{t^2yA_0}{1 - txy} + \frac{A_1 - 1}{A_0 - 1}\frac{t^2yA_0^2}{1 - txyA_0}.
\end{aligned}
\tag{38}
$$

Finally for $A$, we use the first Dyck path recursion by expanding the last horizontal segment. The last segment of size $k$ contains $k - 2$ pattern $RRR$ and $k - 3$ pattern $RRRR$. Thus it contributes nothing to the power of $y$ and $x$ when $k < 3$, and it has weight $t^kx^{k-3}y^{k-2}$ when $k \geq 3$.

Above the last segment, all the Dyck path structures have weight of either $A_0$ or $A_1$ as they only have the first segment and some interior segments. Similar to our analysis in the case when $k \geq 3$

in $A_1$, the contribution of everything above the last segment is

$$A_0^{k-1}(A_1-1)+A_0^{k-2}(A_1-1)+\cdots+A_0^2(A_1-1)+A_0(A_1-1)+A_1.$$

Now we can write $A$ in terms of $A_1$ and $A_0$ as follows,

$$\begin{aligned} A &= 1+tA_1+t^2(A_0(A_1-1)+A_1)+t^3y(A_0^2(A_1-1)+A_0(A_1-1)+A_1)+\cdots \\ &= 1+tA_1+t^2(A_0A_1-A_0+A_1)+\frac{A_0-A_1}{A_0-1}\frac{t^3y}{1-txy}+\frac{A_1-1}{A_0-1}\frac{t^3yA_0^3}{1-txyA_0}. \end{aligned} \tag{39}$$

Solving equations (37),(38),(39) for $A_0$, $A_1$ and $A$, one can get the formula for $A$ in this theorem. Notice that $A$ is a root of some quadratic equation. □

### 3.4 The function $A_{132}^{\overline{123}}(t,y,x)$

In this case, we are working on permutations in $\mathcal{S}_n(132)$. Similar to the case of $\mathcal{S}_n(123)$, we can apply the bijection $\Phi$ of Krattenthaler [4] to translate the consecutive patterns in $\mathcal{S}_n(132)$ into path patterns in Dyck paths. The cases in $\mathcal{S}_n(132)$ seem to be more straightforward, and we can find the generating function $A$ directly without the auxiliary functions $A_0$ or $A_1$.

Give any $\sigma\in\mathcal{S}_n(132)$, Lemma 3 implies that the number of descents of $\sigma$ is one less than the number of horizontal segments in the corresponding Dyck path $\Phi(\sigma)$, i.e. des$(\sigma)$ is equal to the number of occurrence of the path pattern $RD$ in $\Phi(\sigma)$. Further, the consecutive pattern 123 corresponds to the path pattern $RRR$. Now we are ready to prove the following theorem.

**Theorem 5** ($A_{132}^{\overline{123}}(t,y,x)$)**.** *The generating function $A_{132}^{\overline{123}}(t,y,x)$ is given by*

$$A_{132}^{\overline{123}}(t,y,x)= \frac{\sqrt{x^2t^2+2xt(ty-1)+t^2(y-4)y-2ty+1}+xt\left(2ty^2-2(t+1)y+1\right)-2t^2(y-1)y+ty-1}{2ty(x(ty-1)-ty)}, \tag{40}$$

*and*

$$A_{132}^{\overline{123}}(t,1,x)|_{t^nx^k}=\frac{1}{n+1}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{n+1-i}(-1)^{2j}\binom{n+1}{i}\binom{i+k-1}{k}\binom{2n-3i-2j-k}{n-i}. \tag{41}$$

*Proof.* By the last paragraph, the function $A$ is a generating function of Dyck paths, where for each path $P$, $t$ tracks the size of the path, $y$ tracks the number of patterns $RD$ in $P$, and $x$ tracks the number of patterns $RRR$ in $P$. Thus, a horizontal segment of size 1 and 2 contributes nothing to the power of $x$, and a horizontal segment of size $k\geq 3$ contributes $k-2$ to the power of $x$. This allows us to get a recursion for $A$ by expanding the last horizontal segment.

Now we suppose that the last horizontal segment is of size $k$. $k=0$ is the empty path which has contribution 1 to the function $A$. When $k=1$, the path ends with steps $DR$ and there is a Dyck path structure $P_1$ above the last horizontal segment with weight $(y(A-1)+1)$, since there will be an extra descent between $P_1$ and the last segment if $P_1$ is not empty. When $k\geq 2$, the path ends with steps $DR^k$. The last segment has weight $t^kx^{k-2}$ since it contains $k-2$ pattern $RRR$. There are $k$ Dyck path structures $P_1,\ldots,P_k$ above the last segment, each has weight $(y(A-1)+1)$, since

$P_i$ will create a new descent with the lower structures if $P_i$ is not empty. Thus the contribution of the case when $k \geq 2$ is $t^k x^{k-2}(y(A-1)+1)^k$. It follows that

$$\begin{aligned} A &= 1 + t(y(A-1)+1) + t^2(y(A-1)+1)^2 + t^3x(y(A-1)+1)^3 + \cdots \\ &= 1 + t(y(A-1)+1) + \frac{t^2(y(A-1)+1)^2}{1 - tx(y(A-1)+1)}, \end{aligned} \tag{42}$$

and one can solve $A$ from the quadratic equation above to prove (40).

We shall prove formula (41) in Section 6.1. □

Setting $x = 0$ and $y = 1$ in the generating function, we get $A_{132}^{\overline{123}}(t,1,0) = A_{123}^{\overline{132}}(t,1,0)$ which is again the generating function of Motzkin numbers (OEIS A001006 [5]). We will generalize our method to compute $A_{132}^{\overline{1\cdots m}}(t,y,x)$ in Section 6.1.

### 3.5 The function $A_{132}^{\overline{213}}(t,y,x)$

**Theorem 6.** *The generating function $A_{132}^{\overline{213}}(t,y,x)$ is given by*

$$A_{132}^{\overline{213}}(t,y,x) = \frac{-\sqrt{t\left(2y\left(x(t-1)t - t^2 - 1\right) + ty^2(-xt+t+1)^2 + t - 2\right) + 1} + (x-1)t^2y(2y-1) + t(y-1) + 1}{2ty((x-1)ty+1)}. \tag{43}$$

*Proof.* Similar to Section 3.4, we shall still use the first Dyck path recursion to compute $A_{132}^{\overline{213}}(t,y,x)$. For a Dyck path $P$, we expand the last horizontal segment which is of size $k$, as shown is Figure 7. By Lemma 3, there is a length $k$ increasing sequence lying above the last horizontal segment. There are $k$ Dyck path structures above the last segment. Other than inside the $k$ Dyck path structures, the only possible location of consecutive pattern 213 occurrence is the junction of the last Dyck path structure $P_k$ (when $P_k$ is nonempty) and the last segment (when $k \geq 2$) which is in the red circle area of Figure 7.

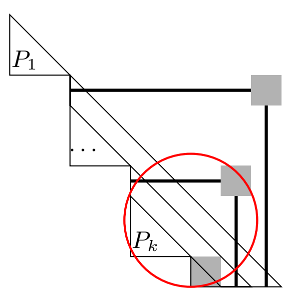



Figure 7: the first recursion of Dyck path

The total contribution of cases when $k = 0$ or $1$ is still $1 + t(y(A-1)+1)$. When $k \geq 2$, we may have an extra consecutive pattern 213 in the red circle area of Figure 7 as we discussed. The weight of each $P_i$ for $i = 1, \ldots, k-1$ is $(y(A-1)+1)$. If $P_k$ is not empty, then the weight of $P_k$ is $xy(A-1)$ and the whole contribution is $t^k xy(A-1)(y(A-1)+1)^{k-1}$; if $P_k$ is empty, then the weight of $P_k$ is 1 and the whole contribution is $t^k(y(A-1)+1)^{k-1}$. Thus, the contribution of the case when $k \geq 2$ is $t^k(xy(A-1)+1)(y(A-1)+1)^{k-1}$.

Summing over all the cases, we can get the following recursion for $A$:

$$\begin{aligned} A &= 1+t(y(A-1)+1)+t^2(y(A-1)+1)(xy(A-1)+1) \\ &\quad +t^3(y(A-1)+1)^2(xy(A-1)+1)+\cdots \\ &= 1+t(y(A-1)+1)+\frac{t^2(y(A-1)+1)(xy(A-1)+1)}{1-t(y(A-1)+1)}, \end{aligned} \tag{44}$$

which can be adjusted to a quadratic equation about $A$, and Theorem 6 follows. ☐

### 3.6 The function $A_{132}^{\overline{231}}(t,y,x)$

**Theorem 7.** *The generating function*

$$A_{132}^{\overline{231}}(t,y,x)=\frac{2txy-ty-t+1-\sqrt{-4t^2xy+t^2y^2+2t^2y+t^2-2ty-2t+1}}{2txy}, \tag{45}$$

*and*

$$A_{132}^{\overline{231}}(t,1,x)|_{t^nx^k}=\frac{1}{n}\binom{n}{k}\sum_{i=0}^{n-k}(-1)^{4k+i+n+1}\binom{2i-n}{n+1-4k}. \tag{46}$$

*Proof.* The easiest way of proving the theorem is to use the second Dyck path recursion by breaking the Dyck path at the first return into two smaller Dyck path structures. In other words, given $\sigma\in\mathcal{S}_n(132)$, suppose that the number $n$ is in the $k^{\text{th}}$ position. We break $\sigma$ at the position of number $n$. The numbers to the left of $n$ forms a 132-avoiding permutation $P_1$ of numbers $\{n-k+1,\ldots,n-1\}$, and the numbers to the right of $n$ forms a 132-avoiding permutation $P_2$ of numbers $\{1,\ldots,n-k\}$. This structure is shown in Figure 8.

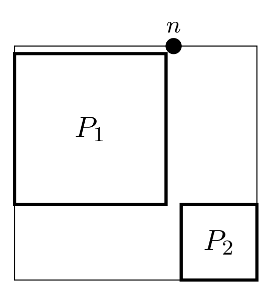



Figure 8: the structure of $\mathcal{S}_n(132)$

The case that $P_2$ is empty contributes $tA$ to the generating function $A$, and the case that the $P_1$ is empty while the second structure $A$ is not empty contributes $ty(A-1)$ to $A$ since there is a descent between $n$ and $P_2$. If both $P_1$ and $P_2$ are not empty, then a consecutive pattern 231 appears at the position of $n$, and the contribution of this case is $txy(A-1)^2$. Thus we have:

$$A=1+t(A+y(A-1))+txy(A-1)^2. \tag{47}$$

Equation (45) can be obtained by solving the quadratic equation (47).

We shall prove formula (46) in Section 6.4. ☐

Here we shall mention that the first Dyck path recursion also works well for the consecutive pattern 231, since the number of consecutive patterns 231 in $\sigma \in \mathcal{S}_n(132)$ is equal to the number of horizontal segments which are not the last segment and have size bigger than 1. This computation will help in Section 4.2 when we track multiple patterns. Let $A_1$ be the generating function that we suppose the last segment is an interior segment similar to (26). By expanding the last segment, we have the following recursion for $A_1$,

$$A_1 = 1 + tyA_1 + t^2yA_1^2 + t^3xyA_1^3 + \cdots = 1 + tyA_1 + t^2yA_1^2 + \frac{t^3xyA_1^3}{1 - tA_1}. \tag{48}$$

Again by expanding the last segment, we have

$$A = 1 + tyA_1 + t^2yA_1^2 + \cdots = 1 + \frac{tyA_1}{1 - tA_1}. \tag{49}$$

One can solve equation (48) to get the formula of $A_1$, and substitute the formula of $A_1$ in equation (49) to get the formula of $A$ and prove Theorem 7.

We will generalize our method to compute $A_{132}^{\overline{2\cdots m1}}(t,1,x)$ in Section 6.4.

# 4 Tracking multiple patterns — $A_{\lambda}^{\overline{\gamma_1},\overline{\gamma_2},\ldots,\overline{\gamma_r}}(t, y, x_1, x_2, \ldots, x_r)$

Since $|\mathcal{S}_3| = 6$, permutations in $\mathcal{S}_n(\lambda)$ for $\lambda \in \mathcal{S}_3$ only contain the 5 consecutive patterns of length 3 except $\lambda$. Suppose that $\mathcal{S}_3 = \{\lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ and $\sigma \in \mathcal{S}_n(\lambda)$. If $\sigma$ has $\alpha_i$ matches of consecutive pattern $\gamma_i$ for $i = 1, \ldots, 5$, then $\alpha_5 = n - 2 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ since any permutation of length $n$ has $n - 2$ consecutive patterns of length 3.

Thus if we can get a generating function for permutations avoiding $\lambda \in \mathcal{S}_3$ tracking 4 consecutive patterns of length 3, then we can modify the function to get a new generating function tracking all 5 patterns of length 3. By symmetry, we only need to consider the cases when $\lambda = 123$ or $\lambda = 132$.

When $\lambda = 132$, we are able to compute the generating function tracking all 5 patterns of length 3 and the descent statistic. When $\lambda = 123$, we have the recursion for the generating function tracking 3 patterns of length 3 and descent.

## 4.1 The function $A_{123}^{\overline{132},\overline{231},\overline{321}}(t, y, x_1, x_2, x_3)$

**Theorem 8.** $A_{123}^{\overline{132},\overline{231},\overline{321}}(t, y, x_1, x_2, x_3)$ *satisfies the recursion*

$$\begin{aligned} A = {}& 1/(x_3(x_1-x_2)(A_0x_3^2t^2y^2(A_0t^2y(x_1-x_2)-1)-(A_0+1)x_3ty(A_0t^2y(x_1-x_2)-1)+A_0x_1t^2y-1)) \\ &\cdot (x_1(x_3^2ty(A_0^2t^2(x_2t^2y^2 + y(3x_2t + 2x_2 + t + 1) + 1) + A_0((x_2 + 1)t^3y + t^2(2x_2y + y + 2) + t + 1) \\ &+ t + 1) + x_2t^2(A_0^2(-t)y + A_0 - 1) + A_0^2x_3^4t^5y^3 - A_0x_3^3t^2y^2(A_0(x_2 + 1)t^3y + t^2(A_0(2x_2y + y + 2) + 1) \\ &+ t + 1) + x_3(A_0x_2t^4y(A_0 - y) - A_0t^3y(A_0x_2 + x_2 + 1) - A_0t^2(x_2y + y + 1) - t - 1)) + x_2(-A_0^2x_3^4t^5y^3 \\ &- x_3^2ty(A_0^2(2t-1)t^2y + A_0(t^3y + t^2(y+2) + t + 1) + t + 1) + x_3(t^2(A_0^2(-y) + y + 1) + A_0(A_0+1)t^3y + t + 1) \\ &+ A_0x_3^3t^2y^2(A_0t^3y + (A_0 + 1)t^2 + t + 1) + (A_0 - 1)t) + x_3t(x_3ty - 1)(A_0^2x_3ty(x_3ty - 1) + A_0 - 1) \\ &+ A_0x_1^2t^2y(-A_0x_2t^2 + A_0x_3^3t^2y^2 - (A_0 + 1)x_3^2ty + x_3) + A_0x_2^2x_3t^3y(A_0x_3^2t(t + 1)y^2 \\ &- x_3y(A_0t^2y + 2A_0t + A_0 + t + 1) + (A_0 + 1)ty + 1)), \end{aligned} \tag{50}$$

*where*

$$A_0 = 1 + tx_3yA_0 + t^2x_1yA_0^2 + \frac{t^3x_2x_3y^2A_0^3}{1 - tx_3yA_0}. \tag{51}$$

The right hand side of (50) is a rational function of $A_0$, where both the numerator and denominator are degree 2 polynomials in $A_0$.

*Proof.* The proof of this theorem is similar to Theorem 4. It is not hard to see from Lemma 4 that the contribution of a horizontal segment to $132, 231, 321$ matches and descents is given by Table 3 if there is more than one horizontal segment in the path.

Table 3: the contribution of a horizontal segment to $132, 231, 321$ matches and descents

| size | first segment | interior segment | last segment |
|---|---|---|---|
| 1 | $0, 0, 0, 0$ | $0, 0, 1, 1$ | $0, 0, 0, 0$ |
| 2 | $0, 1, 0, 0$ | $0, 1, 0, 1$ | $0, 0, 0, 0$ |
| 3 | $1, 0, 1, 1$ | $1, 0, 1, 2$ | $1, 0, 0, 1$ |
| 4 | $1, 0, 2, 2$ | $1, 0, 2, 3$ | $1, 0, 1, 2$ |
| 5 | $1, 0, 3, 3$ | $1, 0, 3, 4$ | $1, 0, 2, 3$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Again, we shall define 2 generating functions, $A_0$ and $A_1$, similar to what we have defined in Section 3.3. Both $A_0$ and $A_1$ track the same statistics as $A$. $A_0$ is the generating function of Dyck paths where we take every segment as an interior segment, and $A_1$ is the generating function of Dyck paths where we suppose that the last segment is an interior segment.

We shall begin with computing a recursion for $A_0$ since it only consists of interior segments. By expanding the last horizontal segment of $A_0$, we can get the following recursion which proves equation (51).

$$\begin{aligned} A_0 &= 1 + tx_3yA_0 + t^2x_1yA_0^2 + t^3x_2x_3y^2A_0^3 + t^4x_2x_3^2y^3A_0^4 + \cdots \\ &= 1 + tx_3yA_0 + t^2x_1yA_0^2 + \frac{t^3x_2x_3y^2A_0^3}{1 - tx_3yA_0}. \end{aligned} \tag{52}$$

Next, we want to compute the recursion of $A_1$. We use the third recursion for a Dyck path $P$ which expands the last horizontal segment before the first return which is of size $k$. If $k = 1$, the only steps before the first return are $DR$. There is a Dyck path structure $P_1$ after the first return with weight $A_0$, and the contribution of this case is $tyA_0$ as shown in Figure 9(a).

When $k = 2$, the path ends with steps $DRR$ before the first return. There is a Dyck path structure $P_1$ after the first return with weight $A_0$. By the third recursion of Dyck paths, there is another Dyck path structure $P_2$ above the last horizontal segment before the first return. The weight of $P_2$ is $A_1$ since it contains the first segment of the whole path. The weight of the last segment before the first return is $t^2x_2y$, and the contribution of this case is $t^2x_2yA_0A_1$ as shown in Figure 9(b).

When $k = 3$, the path ends with steps $DRRR$ before the first return place. There is a Dyck path structure $P_1$ after the first return with weight $A_0$. There are two Dyck path structures $P_2$, $P_3$ (counting from top to bottom) above the last horizontal segment before the first return. If $P_2$ is not empty, then it has weight $A_1 - 1$ since it contains the first segment, and the weight of $P_3$ is $A_0$. If $P_2$ is empty, then it has weight 1, and $P_3$ has weight $A_1$ since it contains the first segment of the whole path. The weight of the last segment before the first return is $t^3x_1x_3y^2$, and the contribution of this case is $t^3x_1x_3y^2A_0(A_0(A_1-1)+A_1)$ as shown in Figure 9(c).

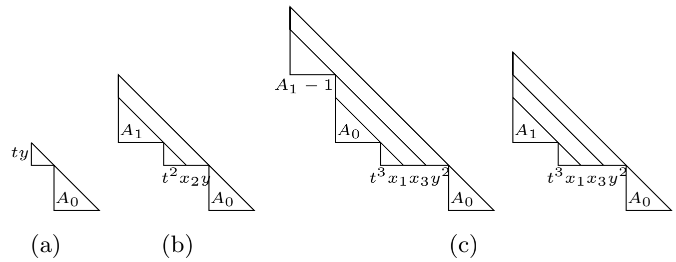



Figure 9: Case $A_{123}^{\overline{132},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3)$: contributions to $A_1$ when $k=1,2,3$

In general, if the last horizontal segment before the first return has size $k\geq 3$, the total contribution is

$$t^kx_1x_3^{k-2}y^{k-1}A_0(A_0^{k-2}(A_1-1)+\cdots+A_0(A_1-1)+A_1).$$

It follows that

$$\begin{aligned} A_1 &= 1+tyA_0+t^2x_2yA_0A_1+t^3x_1x_3y^2A_0(A_0(A_1-1)+A_1)+\cdots \\ &= 1+tyA_0+t^2x_2yA_0A_1-\frac{t^3x_1x_3y^3A_0(A_0-A_1-A_0A_1+tx_3yA_0A_1)}{(1-tx_3y)(1-tx_3yA_0)}. \end{aligned} \quad (53)$$

Finally for $A$, we use the first Dyck path recursion by expanding the last horizontal segment. Suppose that the last segment has size $k$. Similar to Section 3.3, the total contribution when $k<3$ is still $1+tA_1+t^2(A_0(A_1-1)+A_1)$. When $k\geq 3$, the weight of the last segment is $t^kx_1x_3^{k-3}y^{k-2}$, the weight of everything above the last segment is

$$A_0^{k-1}(A_1-1)+A_0^{k-2}(A_1-1)+\cdots+A_0^2(A_1-1)+A_0(A_1-1)+A_1,$$

and the contribution for case $k\geq 3$ is

$$t^kx_1x_3^{k-3}y^{k-2}(A_0^{k-1}(A_1-1)+\cdots+A_0(A_1-1)+A_1).$$

It follows that

$$\begin{aligned} A &= 1+tA_1+t^2(A_0(A_1-1)+A_1)+\sum_{k\geq 3}t^kx_1x_3^{k-3}y^{k-2}(A_0^{k-1}(A_1-1)+\cdots+A_0(A_1-1)+A_1) \\ &= 1+tA_1+t^2(A_0(A_1-1)+A_1)+\frac{t^3x_1y(-A_0+A_1+(tx_3y+1)A_0(A_0-A_1+A_0A_1))}{(1-tx_3y)(1-tx_3yA_0)}. \end{aligned} \quad (54)$$

Using equations (52),(53) and (54), one can write $A$ in terms of $A_0$ to prove this theorem. □

## 4.2 The function $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$

We have the following explicit generating function for $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$.

**Theorem 9.** *The generating function*

$$
\begin{aligned}
&A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)\\
&=(2x_1^2x_3x_4^2t^4y^2+2x_1^2x_3x_4^2t^3y^2-2x_1^2x_3x_4t^4y^2-2x_1^2x_3x_4t^3y-2x_1^2x_3x_4t^2y+x_1^2x_3t^3y-x_1^2x_4t^3y+x_1^2t^2\\
&-2x_1x_2x_3^2x_4t^4y^2-2x_1x_2x_3^2x_4t^3y^2+x_1x_2x_3^2t^4y^2+x_1x_2x_3x_4t^4y^2+(x_1t-x_3t-1)(x_3ty-x_4ty+1)\\
&\cdot\sqrt{(x_1t+ty((x_2-1)x_3t-x_4)+1)^2-4t(x_1(-x_4)ty+x_1+x_2x_3ty)}+x_1x_2x_3t^3y-2x_1x_3^2x_4t^3y^2\\
&+x_1x_3^2t^4y^2+x_1x_3^2t^3y+2x_1x_3^2t^2y-2x_1x_3x_4^2t^4y^2-2x_1x_3x_4^2t^3y^2-2x_1x_3x_4^2t^2y^2+x_1x_3x_4t^4y^2+3x_1x_3x_4t^3y^2\\
&+3x_1x_3x_4t^3y+2x_1x_3x_4t^2y+2x_1x_3x_4ty-x_1x_3t^3y-2x_1x_3t^2y-x_1x_3t^2-x_1x_4^2t^3y^2+3x_1x_4t^2y-2x_1t\\
&+x_2x_3^3t^4y^2+2x_2x_3^3t^3y^2+x_2x_3^2x_4t^4y^2+2x_2x_3^2x_4t^3y^2+2x_2x_3^2x_4t^2y^2-2x_2x_3^2t^4y^2-x_2x_3^2t^3y^2\\
&-x_2x_3^2t^3y-x_2x_3x_4t^3y^2-x_2x_3t^2y-x_3^3t^4y^2+x_3^2x_4t^4y^2-x_3^2x_4t^3y^2-x_3^2t^3y^2-x_3^2t^3y+x_3^2t^2y+x_3x_4^2t^3y^2\\
&+x_3x_4t^3y^2-x_3x_4t^2y^2-2x_3x_4t^2y-x_3t^2y+x_3ty+x_3t+x_4^2t^2y^2-2x_4ty+1)/(2x_3ty(-x_1x_4t+x_3t+x_4)\\
&\cdot(x_1(-x_4)ty+x_1+x_2x_3ty)). \qquad (55)
\end{aligned}
$$

*Proof.* The computation of $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$ is based on the computation of generating functions $A_{132}^{\overline{123}}(t,y,x)$, $A_{132}^{\overline{213}}(t,y,x)$ and $A_{132}^{\overline{231}}(t,y,x)$. However, we have not yet computed $A_{132}^{\overline{321}}(t,y,x)$ directly since we showed in Section 2.1 that $A_{132}^{\overline{321}}(t,y,x)$ and $A_{132}^{\overline{123}}(t,y,x)$ are the same.

To add the four consecutive patterns of length 3 simultaneously, we have to count the number of patterns 321 directly from the corresponding Dyck path. It is not hard to see that the number of consecutive patterns 321 is equal to the number of interior segment of size 1 in the corresponding Dyck path, which makes the computation possible.

Since we need to consider whether a horizontal segment of a Dyck path is interior or not, we shall define generating functions $A_0$ and $A_1$ similar to Section 3.3. Both $A_0$ and $A_1$ tracks the same statistics as $A$. $A_0$ is the generating function of Dyck paths while we take every segment as an interior segment, and $A_1$ is the generating function of Dyck paths while we suppose that the last segment is an interior segment.

We shall begin with computing a recursion for $A_0$ using the first Dyck path recursion. If the last segment of a path is of size $k$, then there are $k$ Dyck path structures $P_1,\ldots,P_k$ above the last segment. Each of the paths $P_1,\ldots,P_{k-1}$ has weight $A_0$, and the path $P_k$ has weight $(x_2(A_0-1)+1)$ since there will be a consecutive pattern 213 between $P_k$ and the last segment when $P_k$ is not empty. Based on our analysis of pattern $123, 213, 231$ and $321$, we can get the weight of the last segment and obtain the following recursion.

$$
\begin{aligned}
A_0 &= 1+tx_4yA_0+t^2x_3yA_0(x_2(A_0-1)+1)+t^3x_1x_3yA_0^2(x_2(A_0-1)+1)+\cdots\\
&= 1+tx_4yA_0+\frac{t^2x_3yA_0(x_2(A_0-1)+1)}{1-tx_1A_0}. \qquad (56)
\end{aligned}
$$

Next, we want to compute the recursion of $A_1$. We use the third recursion for Dyck path which expands the last horizontal segment before the first return that is of size $k$. Similar to Section 4.1, the contribution when $k=1$ is still $tyA_0$.

When $k\geq 2$, the path ends with steps $DR^k$ before the first return, and there are $k-1$ Dyck path structures $P_1,\ldots,P_{k-1}$ (counting from top to bottom) above the last segment before the first return. There is also a Dyck path structure after the first return with weight $A_0$. Among the paths

$P_1,\ldots,P_{k-1}$, we suppose that $P_i$ is the first nonempty path for $i=1,\ldots,k$ ($i=k$ means all of $P_1,\ldots,P_{k-1}$ are empty). If $i=k-1$, then $P_{k-1}$ is the only nonempty path among $P_1,\ldots,P_{k-1}$, and it has weight $x_2(A_1-1)$ since it contains the first segment and there is a consecutive pattern 213 between $P_{k-1}$ and the last segment. When $i<k-1$, $P_{k-1}$ does not contain the first segment and it has weight $x_2(A_0-1)+1$; $P_i$ has weight $(A_1-1)$; each of $P_{i+1},\ldots,P_{k-2}$ has weight $A_0$. Summing cases for $i$ from 1 to $k$, one can obtain that the total weight of the Structures above the last horizontal segment before the first return is

$$(A_0^{k-3}+A_0^{k-4}+\cdots+1)(x_2(A_0-1)+1)(A_1-1)+x_2(A_1-1)+1.$$

Thus by adding all the cases for $k$, we can obtain the following recursion,

$$\begin{aligned} A_1 &= 1+tyA_0+t^2x_3yA_0(x_2(A_1-1)+1) \\ &\quad +t^3x_1x_3yA_0((x_2(A_0-1)+1)(A_1-1)+x_2(A_1-1)+1) \\ &\quad +t^4x_1^2x_3yA_0((A_0+1)(x_2(A_0-1)+1)(A_1-1)+x_2(A_1-1)+1)+\cdots \\ &= 1+tyA_0+t^2x_3yA_0(x_2(A_1-1)+1)+ \\ &\quad \frac{t^3x_1x_3yA_0(A_1-x_2A_0+x_2A_0A_1-tx_1A_0+tx_1x_2A_0-tx_1x_2A_0A_1)}{(1-tx_1)(1-tx_1A_0)}. \end{aligned} \tag{57}$$

Finally for $A$, we expand the last horizontal segment like $A_0$ and analyze the weight of Dyck paths above the last segment like $A_1$ to get

$$\begin{aligned} A &= 1+tA_1+t^2((x_2(A_0-1)+1)(A_1-1)+x_2(A_1-1)+1) \\ &\quad +t^3x_1\,((A_0+1)(x_2(A_0-1)+1)(A_1-1)+x_2(A_1-1)+1)+\cdots \\ &= 1+tA_1+\frac{A_1-1-tyA_0-t^2x_3yA_0(x_2(A_1-1)+1)}{tx_1x_3yA_0}. \end{aligned} \tag{58}$$

One can solve the quadratic equation (56) to get a formula for $A_0$, and then solve $A_1$ from (57) and solve $A$ from (58) to obtain Theorem 9. $\square$

Theorem 9 gives the generating function with all information about length 2 or 3 consecutive pattern matches in $\mathcal{S}_n(132)$. The generating function $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$ is a root of a quadratic equation, and one can get its Taylor series about $t$ easily using mathematical softwares like Mathematica. Table 4 shows the coefficient of $t^n$ for $n$ from 1 to 5 in $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$.

Table 4: coefficient list of $A_{132}^{\overline{123},\overline{213},\overline{231},\overline{321}}(t,y,x_1,x_2,x_3,x_4)$

| $n$ | coefficient of $t^n$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | $y+1$ |
| 3 | $x_1+x_2y+x_3y+x_4y^2+y$ |
| 4 | $x_1^2+x_1x_2y+x_1x_3y+2x_1y+x_2x_3y^2+x_2x_3y+2x_2x_4y^2+x_3x_4y^2+x_3y^2+x_3y+x_4^2y^3+x_4y^2$ |
| 5 | $x_1^3+x_1^2x_2y+x_1^2x_3y+3x_1^2y+x_1x_2x_3y^2+2x_1x_2x_3y+3x_1x_2x_4y^2+x_1x_3x_4y^2+2x_1x_3y^2$ $+3x_1x_3y+3x_1x_4y^2+x_2^2x_3y^2+x_2x_3^2y^2+3x_2x_3x_4y^3+2x_2x_3x_4y^2+3x_2x_3y^2+3x_2x_4^2y^3$ $+x_3^2y^2+x_3x_4^2y^3+2x_3x_4y^3+x_3x_4y^2+x_3y^2+x_4^3y^4+x_4^2y^3$ |

## 5 General results about consecutive patterns in $\mathcal{S}_n(123)$

In this section, we shall compute 2 kinds of consecutive patterns, $1m(m-1)\cdots 2$ and $2m(m-1)\cdots 31$, for any positive integer $m$. We shall just track the length of the permutation and the number of consecutive pattern matches, and no longer track the number of descents. That is, for two permutations $\lambda, \gamma$, let

$$A_{\lambda}^{\overline{\gamma}}(t,x) := \sum_{n\geq 0} t^n \sum_{\sigma\in\mathcal{S}_n(\lambda)} x^{\gamma\text{-mch}(\sigma)}. \tag{59}$$

By the action reverse-complement, the distribution of consecutive patterns $1m(m-1)\cdots 2$ and $2m(m-1)\cdots 31$ in $\mathcal{S}_n(123)$ are equal to the distribution of consecutive patterns $(m-1)\cdots 21m$ and $m(m-2)\cdots 21(m-1)$, thus we have

$$\begin{aligned} A_{123}^{\overline{1m(m-1)\cdots 2}}(t,x) &= A_{123}^{\overline{(m-1)\cdots 21m}}(t,x), & (60)\\ A_{123}^{\overline{2m(m-1)\cdots 31}}(t,x) &= A_{123}^{\overline{m(m-2)\cdots 21(m-1)}}(t,x). & (61)\end{aligned}$$

We shall compute the recursions of the two generating functions above.

### 5.1 The function $A_{123}^{\overline{1m(m-1)\cdots 2}}(t,x)$

**Theorem 10.** $A_{123}^{\overline{1m(m-1)\cdots 2}}(t,x)$ *satisfies the recursion*

$$A = \frac{1+(x-1)t^m A^m}{1-tA}, \text{ and} \tag{62}$$

$$A_{123}^{\overline{1m(m-1)\cdots 2}}(t,x)|_{t^n x^k} = \frac{1}{k}\sum_{i=k}^{\lfloor \frac{n}{m}\rfloor}(-1)^{i-k}\binom{2n-mi}{n-mi, n+1-i, k-1, i-k}. \tag{63}$$

*Proof.* Referring to Lemma 4, the number of consecutive patterns $1m(m-1)\cdots 2$ in $\sigma\in\mathcal{S}_n(123)$ is equal to the number of path pattern $DR^m$ in the corresponding Dyck path $\Psi(\sigma)$. We shall compute the distribution of pattern $DR^m$ in Dyck paths to get the generating function $A$. Note that the number of patterns $DR^m$ is equal to the number of horizontal segments of length at least $m$.

We use the first Dyck path recursion by expanding the last horizontal segment. Let $k$ denote the size of the last horizontal segment, then there are $k$ Dyck path structures above the last segment, each having weight $A$. The weight of the last horizontal segment is $t^k$ when $k<m$ and $xt^k$ when $k\geq m$ since there will be a path pattern $DR^m$ appear in the last segment. Thus we have

$$\begin{aligned} A &= 1+tA+\cdots+t^{m-1}A^{m-1}+x(t^mA^m+\cdots)\\ &= \frac{1-t^mA^m}{1-tA}+x\frac{t^mA^m}{1-tA}\\ &= \frac{1+(x-1)t^mA^m}{1-tA}. & (64)\end{aligned}$$

We can multiply both sides of equation (62) with $t$ and substitute $tA$ with $F$ to get

$$F = t\frac{1+(x-1)F^m}{1-F}. \tag{65}$$

It follows from Lagrange Inversion Theorem that

$$F|_{t^n} = \frac{1}{n}\delta^n(z)|_{z^{n-1}}, \tag{66}$$

where $\delta(z) = \dfrac{1+(x-1)z^m}{1-z}$. Thus,

$$\begin{aligned}
A|_{t^{n-1}x^k} &= F|_{t^n x^k} \\
&= \frac{1}{n}(1+(x-1)z^m)^n(1-z)^{-n}|_{z^{n-1}x^k} \\
&= \frac{1}{n}\left(\sum_{i=0}^{n}\binom{n}{i}(x-1)^i z^{mi}\right)\left(\sum_{i\geq 0}\binom{n+i-1}{n-1}z^i\right)\Bigg|_{z^{n-1}x^k} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{n-1}{m}\rfloor}\binom{n}{i}(x-1)^i\binom{2n-mi-2}{n-1}\Bigg|_{x^k} \\
&= \frac{1}{n}\sum_{i=k}^{\lfloor\frac{n-1}{m}\rfloor}(-1)^{i-k}\binom{n}{n-i,k,i-k}\binom{2n-mi-2}{n-1} &(67)\\
&= \frac{1}{k}\sum_{i=k}^{\lfloor\frac{n-1}{m}\rfloor}(-1)^{i-k}\binom{2n-mi-2}{n-mi-1,n-i,k-1,i-k}, &(68)
\end{aligned}$$

and the formula (63) follows by substituting $n$ with $n+1$. ∎

Let $m=3$, we get a formula for the coefficient of $t^n x^k$ in $A_{123}^{\overline{132}}(t,1,x)$ which proves formula (25) in Theorem 2.

## 5.2 The function $A_{123}^{\overline{2m(m-1)\cdots 31}}(t,x)$

**Theorem 11.** $A_{123}^{\overline{2m(m-1)\cdots 31}}(t,x)$ *satisfies the following recursion,*

$$A = t\frac{A^{m+2}+(x-1)(A-1)^{m-1}}{A^{m-1}(A-1)}, \quad \textit{and} \tag{69}$$

$$A_{123}^{\overline{2m(m-1)\cdots 31}}(t,x)|_{t^n x^k} = \frac{1}{n}\sum_{i=0}^{\lfloor\frac{mn-1}{m+2}\rfloor}(-1)^{mn+n+k+1}\binom{n}{i}\binom{mn-mi-2n+i}{mn-1}\binom{mn-mi-n+i}{k}, \tag{70}$$

*here* $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ *is the generalized binomial coefficient.*

*Proof.* Referring to Lemma 4, the number of consecutive patterns $2m(m-1)\cdots 31$ in $\sigma\in\mathcal{S}_n(123)$ is equal to the number of path pattern $DR^{m-1}D$ in the corresponding Dyck path $\Psi(\sigma)$. We shall compute the distribution of pattern $DR^{m-1}D$ in Dyck paths to get the generating function $A$.

It is clear that the number of patterns $DR^{m-1}D$ is equal to the number of horizontal segments of length $m$ which are not the last segment, and we need to define generating function $A_1$ that tracks the same statistics as $A$ while we suppose the last segment is an interior segment.

We use the first Dyck path recursion for both $A_1$ and $A$. First for $A_1$, we let $k$ denote the size of the last horizontal segment, then there are $k$ Dyck path structures above the last segment, each having weight $A_1$. The weight of the last horizontal segment is $t^k$ when $k \neq m-1$ and $xt^k$ when $k = m-1$ since there will be a path pattern $DR^{m-1}D$ appear in the last segment. Thus we have

$$A_1 = 1 + tA_1 + \cdots + t^{m-2}A_1^{m-2} + xt^{m-1}A_1^{m-1} + t^mA_1^m + \cdots = \frac{1}{1-tA_1} + (x-1)t^{m-1}A_1^{m-1}. \quad (71)$$

Then for $A$, we expand the last segment which is of size $k$. There are $k$ Dyck path structures above the last segment, each having weight $A_1$. The weight of the last horizontal segment is $t^k$ for any $k$ since it does not contain pattern $DR^{m-1}D$. It follows that

$$A = 1 + tA_1 + \cdots = \frac{1}{1-tA_1}. \quad (72)$$

Equation (72) implies that $A_1 = \frac{A-1}{tA}$. Substitute $A_1$ with $\frac{A-1}{tA}$ in (71) and multiply both sides with $\frac{tA^2}{A-1}$ gives (69).

Next, we shall compute the coefficient using Lagrange Inversion Theorem. It follows that

$$A|_{t^n} = \frac{1}{n}\delta^n(z)|_{z^{n-1}}, \quad (73)$$

where $\delta(z) = \dfrac{z^{m+2} + (x-1)(z-1)^{m-1}}{z^{m-1}(z-1)}$. Thus,

$$\begin{aligned}
A|_{t^nx^k} &= \frac{1}{n}(z^{m+2} + (x-1)(z-1)^{m-1})^n(z^{m-1}(z-1))^{-n}|_{z^{n-1}x^k} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{mn-1}{m+2}\rfloor}\binom{n}{i}(x-1)^{(m-1)(n-i)}z^{(m+2)i}(z-1)^{(m-1)(n-i)-n}|_{z^{mn-1}x^k} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{mn-1}{m+2}\rfloor}\binom{n}{i}\left((x-1)^{(m-1)(n-i)}(z-1)^{(m-1)(n-i)-n}|_{z^{mn-1-mi-2i}x^k}\right) \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{mn-1}{m+2}\rfloor}(-1)^{mn+n+k+1}\binom{n}{i}\binom{mn-mi-2n+i}{mn-1}\binom{mn-mi-n+i}{k}. \Box \quad (74)
\end{aligned}$$

Let $m = 3$, we get a formula for the coefficient of $t^nx^k$ in $A_{123}^{\overline{231}}(t,1,x)$ which proves formula (31) in Theorem 3.

## 6 General results about consecutive patterns in $\mathcal{S}_n(132)$

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$, we let $\Phi'(\sigma)$ be the Dyck path obtained by removing the initial $n+1-\sigma_1$ $D$ steps from $\Phi(\sigma)$, and let $\Phi''(\sigma)$ be the Dyck path obtained by removing the last one $R$ step of $\Phi'(\sigma)$.

For example, the Dyck path corresponds to the permutation $\sigma = 42351 \in \mathcal{S}_5(132)$ is $\Phi(\sigma) = DDRDDRRRDR$, then $\Phi'(\sigma) = RDDRRRDR$ and $\Phi''(\sigma) = RDDRRRD$. Then by the bijection of Krattenthaler's [4], we have the following theorem.

**Theorem 12.** *Let $m$ be a positive integer and let $\gamma = \gamma_1 \cdots \gamma_m \in \mathcal{S}_m(132)$, then*

*(a) if $\gamma_m = m$, then the distribution of consecutive pattern $\gamma$ in $\mathcal{S}_n(132)$ is equal to the distribution of pattern $\Phi'(\gamma)$ in $\mathcal{D}_n$;*

*(b) if $\gamma_{m-1}\gamma_m = m1$, then the distribution of consecutive pattern $\gamma$ in $\mathcal{S}_n(132)$ is equal to the distribution of pattern $\Phi''(\gamma)$ in $\mathcal{D}_n$.*

Based on Theorem 12, we can prove several general results about the consecutive pattern distributions in $\mathcal{S}_n(132)$.

## 6.1 The function $A_{132}^{\overline{1\cdots m}}(t,x)$

**Theorem 13.** $A_{132}^{\overline{1\cdots m}}(t,x)$ *satisfies the following recursion,*

$$A = \frac{1 - t^{m-1}A^{m-1}}{1-tA} + \frac{t^{m-1}A^{m-1}}{1-txA}, \text{ and} \tag{75}$$

$$A_{132}^{\overline{1\cdots m}}(t,x)|_{t^n x^k} = \frac{1}{n+1} \sum_{i=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{j=0}^{n+1-i} (-1)^{mj-j} \binom{n+1}{i} \binom{i+k-1}{k} \binom{2n-mi-mj+j-k}{n-i}. \tag{76}$$

*Proof.* Referring to Theorem 12(a), the number of consecutive patterns $1\cdots m$ in $\sigma \in \mathcal{S}_n(132)$ is equal to the number of path pattern $\Phi'(1\cdots m) = R^m$ in the corresponding Dyck path $\Phi(\sigma)$. A horizontal segment of size $k \geq m$ contains $k-m+1$ pattern $R^m$, and we can expand the last segment of a Dyck path to get a recursion for $A$.

Let $k$ denote the size of the last horizontal segment, then there are $k$ Dyck path structures above the last segment, each having weight $A$. The weight of the last horizontal segment is $t^k$ when $k \leq m-1$ and $x^{k-m+1}t^k$ when $k \geq m$. Thus we have

$$A = 1 + tA + \cdots + t^{m-1}A^{m-1} + t^m x A^m + t^{m+1}x^2A^{m+1} + \cdots = \frac{1-t^{m-1}A^{m-1}}{1-tA} + \frac{t^{m-1}A^{m-1}}{1-txA}. \tag{77}$$

We can multiply both sides of equation (77) with $t$ and substitute $tA$ with $F$ to get

$$F = t\left(\frac{1-F^{m-1}}{1-F} + \frac{F^{m-1}}{1-xF}\right). \tag{78}$$

It follows from Lagrange Inversion Theorem that

$$F|_{t^n} = \frac{1}{n}\delta^n(z)|_{z^{n-1}}, \tag{79}$$

where $\delta(z) = \dfrac{1 - z^{m-1}}{1 - z} + \dfrac{z^{m-1}}{1 - xz}$. Thus,

$$\begin{aligned}
A|_{t^{n-1}x^k} &= F|_{t^n x^k} \\
&= \frac{1}{n}\left(\frac{1 - z^{m-1}}{1 - z} + \frac{z^{m-1}}{1 - xz}\right)^n\Bigg|_{z^{n-1}x^k} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{n-1}{m-1}\rfloor}\binom{n}{i} z^{(m-1)i}\left(\sum_{a\geq 0}\binom{i+a-1}{a}x^a z^a\right)\left(\sum_{j=0}^{n-i}(-z)^{(m-1)j}\right) \\
&\quad\cdot\left(\sum_{b\geq 0}\binom{n-i+b-1}{b}z^b\right)\Bigg|_{z^{n-1}x^k} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{n-1}{m-1}\rfloor}\binom{n}{i}\binom{i+k-1}{k} z^{mi-i+k}\left(\sum_{j=0}^{n-i}(-z)^{(m-1)j}\right)\left(\sum_{b\geq 0}\binom{n-i+b-1}{b}z^b\right)\Bigg|_{z^{n-1}} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{n-1}{m-1}\rfloor}\sum_{j=0}^{n-i}(-1)^{mj-j}\binom{n}{i}\binom{i+k-1}{k}\binom{2n-mi-mj+j-k-2}{n-i-1}, \qquad (80)
\end{aligned}$$

and the formula (76) follows by substituting $n$ with $n+1$. □

Let $m = 3$, we get a formula for the coefficient of $t^n x^k$ in $A_{132}^{\overline{123}}(t, 1, x)$ which proves formula (41) in Theorem 5.

### 6.2 The function $A_{132}^{\overline{a12\cdots(a-1)(a+1)\cdots m}}(t, x)$

**Theorem 14.** $A_{132}^{\overline{a12\cdots(a-1)(a+1)\cdots m}}(t, x)$ *satisfies the following recursion,*

$$A = \frac{1 + t^{m-1}A^{m-a}(x-1)(A-1)}{1 - tA}. \qquad (81)$$

*Proof.* By Theorem 12(a), the number of consecutive pattern $\gamma = a12\cdots(a-1)(a+1)\cdots m$ in $\sigma \in \mathcal{S}_n(132)$ is equal to the number of path pattern $\Phi'(\gamma) = RD^{a-1}R^{m-1}$ in the corresponding Dyck path $\Phi(\sigma)$. We shall expand last segment of a Dyck path $P$ using the first Dyck path recursion. Suppose the size of the last segment is $k$, then there are $k$ Dyck paths above the last segment with weight $A$.

The weight of the last segment is $t^k$, except when the last segment is part of a pattern $RD^{a-1}R^{m-1}$, which means that the Dyck path $P$ ends with $RD^{a-1}R^{m-1+i}$. $\gamma_1 = a$ means that the path is at the $m - a + i^{\text{th}}$ diagonal and there are $m + 1 - a + i$ Dyck path structures, $P_1, \ldots, P_{m+1-a+i}$, before the second step of $RD^{a-1}R^{m-1+i}$, of which $P_{m+1-a+i}$ must be nonempty. Thus the weight of the $m + 1 - a + i$ Dyck path structures are $A^{m-a+i}(A-1)$. The weight of the path $RD^{a-1}R^{m-1+i}$ from the second step is $t^{m-1+i}x$, and the case when when the last segment is part of a pattern $RD^{a-1}R^{m-1+i}$ gives correction weight

$$t^{m-1}A^{m-a}(x-1)(A-1) + t^m A^{m-a+1}(x-1)(A-1) + \cdots = \frac{t^{m-1}A^{m-a}(x-1)(A-1)}{1 - tA}$$

to the generating function $A$, and the recursion for $A$ is

$$A = \frac{1}{1-tA} + \frac{t^{m-1}A^{m-a}(x-1)(A-1)}{1-tA} = \frac{1+t^{m-1}A^{m-a}(x-1)(A-1)}{1-tA}. \qquad \square \tag{82}$$

## 6.3 The function $A_{132}^{\overline{\gamma}}(t,x)$ when $\gamma_1 = m-1$ and $\gamma_m = m$

**Theorem 15.** *For any $\gamma \in \mathcal{S}_m(132)$ with $\gamma_1 = m-1$ and $\gamma_m = m$, $A_{132}^{\overline{\gamma}}(t,x)$ satisfies the following recursion,*

$$A = \frac{1+t^{m-1}(x-1)(A^2-A)}{1-tA}. \tag{83}$$

*Proof.* By Theorem 12(a), the distribution of such a consecutive pattern $\gamma$ in $\mathcal{S}_n(132)$ is equal to the distribution of $\Phi'(\gamma)$ in $\mathcal{D}_n$. In this case, $\Phi'(\gamma) = RP_0R$, where $P_0 = \Phi(\gamma_2 \cdots \gamma_{m-1})$ is a Dyck path of size $m-2$.

We use the first Dyck path recursion for $A$. By expanding the last segment of a Dyck path $P$ when the last segment is of size $k$, there are $k$ Dyck paths above the last segment with weight $A$. The weight of the last segment is $t^k$, except when the last segment is part of a pattern $RP_0R$. Like section 6.2, the correction weight is

$$t^{m-1}A(x-1)(A-1) + t^m A^2(x-1)(A-1) + \cdots = \frac{t^{m-1}(x-1)(A^2-A)}{1-tA},$$

and the recursion for $A$ is

$$A = \frac{1}{1-tA} + \frac{t^{m-1}(x-1)(A^2-A)}{1-tA}, \tag{84}$$

which proves the theorem. $\square$

## 6.4 The function $A_{132}^{\overline{2\cdots m1}}(t,x)$

**Theorem 16.** $A_{132}^{\overline{2\cdots m1}}(t,x)$ *satisfies the following recursion,*

$$A = t\frac{A^{m-1}+(x-1)(A-1)^{m-1}}{A^{m-4}(A-1)}, \text{ and} \tag{85}$$

$$A_{132}^{\overline{2\cdots m1}}(t,x)|_{t^n x^k} = \frac{1}{n}\binom{n}{k}\sum_{i=0}^{n-k}(-1)^{mk+k+i+n+1}\binom{mi-i-n}{n+1-mk-k}. \tag{86}$$

*Proof.* Referring to Theorem 12(b), the number of consecutive patterns $\gamma = 2\cdots m1$ in $\sigma \in \mathcal{S}_n(132)$ is equal to the number of path pattern $\Phi''(\gamma) = R^{m-1}D$ in the corresponding Dyck path $\Phi(\sigma)$, which is equal to the number of horizontal segments of size $k \geq m-1$ that are not the last segment. We shall define generating function $A_1$ where we track the same statistics as $A$ and we suppose the last segment is an interior segment.

We begin with computing the recursion of $A_1$. Let $k$ denote the size of the last horizontal segment, then there are $k$ Dyck path structures above the last segment, each having weight $A_1$. The weight of the last horizontal segment is $t^k$ when $k < m-1$ and $xt^k$ when $k \geq m-1$. Thus we have

$$A_1 = 1 + tA_1 + \cdots + t^{m-2}A_1^{m-2} + t^{m-1}xA_1^{m-1} + t^m xA_1^m + \cdots = \frac{1+(x-1)t^{m-1}A_1^{m-1}}{1-tA_1}. \tag{87}$$

Then for $A$, we expand the last segment which is of size $k$. There are $k$ Dyck path structures above the last segment, each having weight $A_1$. The weight of the last horizontal segment is $t^k$ for any $k$ since it does not contain pattern $R^{m-1}D$. It follows that

$$A = 1 + tA_1 + \cdots = \frac{1}{1 - tA_1}. \tag{88}$$

Equation (88) implies that $A_1 = \frac{A-1}{tA}$. Substitute $A_1$ with $\frac{A-1}{tA}$ in (87) and multiply both sides with $\frac{tA^2}{A-1}$ gives (85).

Next, we shall compute the coefficient using Lagrange Inversion Theorem. It follows that

$$A|_{t^n} = \frac{1}{n}\delta^n(z)|_{z^{n-1}}, \tag{89}$$

where $\delta(z) = \dfrac{z^{m-1} + (x-1)(z-1)^{m-1}}{z^{m-4}(z-1)}$. Thus,

$$\begin{aligned}
A|_{t^n x^k} &= \frac{1}{n}(z^{m-1} + (x-1)(z-1)^{m-1})^n (z^{m-4}(z-1))^{-n}|_{z^{n-1}x^k} \\
&= \frac{1}{n}((z-1)^{m-1}x + (z^{m-1} - (z-1)^{m-1}))^n (z-1)^{-n}|_{z^{mn-3n-1}x^k} \\
&= \frac{1}{n}\binom{n}{k}(z^{m-1} - (z-1)^{m-1})^{n-k}(z-1)^{-n}\Big|_{z^{mn-3n-1}} \\
&= \frac{1}{n}\binom{n}{k}\sum_{i=0}^{n-k}(-1)^i (z-1)^{mi-i-n} z^{(m-1)(n-k-i)}\Big|_{z^{mn-3n-1}} \\
&= \frac{1}{n}\binom{n}{k}\sum_{i=0}^{n-k}(-1)^{mk+k+i+n+1}\binom{mi-i-n}{n+1-mk-k}. \qquad \square
\end{aligned} \tag{90}$$

Let $m = 3$, we get a formula for the coefficient of $t^n x^k$ in $A_{132}^{\overline{231}}(t, 1, x)$ which proves formula (46) in Theorem 7.

### 6.5 The function $A_{132}^{\overline{a23\cdots(a-1)(a+1)\cdots m1}}(t, x)$

**Theorem 17.** $A_{132}^{\overline{a23\cdots(a-1)(a+1)\cdots m1}}(t, x)$ *satisfies the following recursion,*

$$A = \frac{tA^{m-a+2} + t^{a-2}(x-1)(A-1-tA)(A-1)^{m-a}}{(A-1)A^{m-a-1}}. \tag{91}$$

*Proof.* By Theorem 12(b), the number of consecutive pattern $\gamma = a23\cdots(a-1)(a+1)\cdots m1$ in $\sigma \in \mathcal{S}_n(132)$ is equal to the number of path pattern $\Phi''(\gamma) = RD^{a-2}R^{m-2}D$ in the corresponding Dyck path $\Phi(\sigma)$. We define $A_1$ to be the generating function of Dyck paths tracking the same statistics as $A$ and we suppose the last segment is an interior segment, then we can compute a recursion for $A_1$ by expanding the last segment. Suppose the size of the last segment is $k$, then there are $k$ Dyck paths above the last segment with weight $A$.

The weight of the last segment is $t^k$, except when the last segment is part of a pattern $\Phi''(\gamma) = RD^{a-2}R^{m-2}D$, which means that the Dyck path $P$ ends with $RD^{a-2}R^{m-2}$, and the extended path

$PD$ contains $RD^{a-2}R^{m-2}D$. $\gamma_1 = a$ means that the path is at the $m-a^{\text{th}}$ diagonal and there are $m+1-a$ Dyck path structures, $P_1, \ldots, P_{m+1-a}$, before the second step of $\Phi''(\gamma)$, of which $P_{m+1-a}$ must be nonempty. Thus the weight of the $m+1-a$ Dyck path structures are $A_1^{m-a}(A_1-1)$. The weight of the path $RD^{a-2}R^{m-2}$ from the second step is $t^{m-2}x$, and the case when when the last segment is part of a pattern $RD^{a-2}R^{m-2}D$ gives correction weight

$$t^{m-2}A_1^{m-a}(x-1)(A_1-1)$$

to the generating function $A_1$, and the recursion for $A_1$ is

$$A_1 = \frac{1}{1-tA_1} + t^{m-2}A_1^{m-a}(x-1)(A_1-1). \tag{92}$$

Since the last horizontal segment of a Dyck path does not contribute to pattern $RD^{a-2}R^{m-2}D$, we expand the last segment of a Dyck path to get

$$A = \frac{1}{1-tA_1}. \tag{93}$$

(92) and (93) together proves (91). $\square$

### 6.6 The function $A_{132}^{\overline{\gamma}}(t,x)$ when $\gamma_1 = m-1$, $\gamma_{m-1} = m$ and $\gamma_m = 1$

**Theorem 18.** *For any $\gamma \in \mathcal{S}_m(132)$ with $\gamma_1 = m-1$, $\gamma_{m-1} = m$ and $\gamma_m = 1$, $A_{132}^{\overline{\gamma}}(t,x)$ satisfies the following recursion,*

$$A = \frac{tA^3}{A-1} + t^{m-3}(x-1)(A-1-tA). \tag{94}$$

*Proof.* This theorem is analogous to Theorem 15. By Theorem 12(b), the distribution of such a consecutive pattern $\gamma$ in $\mathcal{S}_n(132)$ is equal to the distribution of $\Phi''(\gamma)$ in $\mathcal{D}_n$. In this case, $\Phi''(\gamma) = RP_0RD$, where $P_0 = \Phi(\gamma_2 \cdots \gamma_{m-2})$ is a Dyck path of size $m-3$.

We define $A_1$ to be the generating function of Dyck paths tracking the same statistics as $A$ and we suppose the last segment is an interior segment. Using the first Dyck path recursion, there are $k$ Dyck paths above the last segment of size $k$ with weight $A$.

The weight of the last segment is $t^k$, except when the last segment is part of a pattern $\Phi''(\gamma) = RP_0RD$. Like section 6.5, the correction weight is $t^{m-2}(x-1)(A_1^2 - A_1)$, which implies that

$$A_1 = \frac{1}{1-tA_1} + t^{m-2}(x-1)(A_1^2 - A_1). \tag{95}$$

Similarly, we have $A_1 = \dfrac{A-1}{tA}$, and the theorem can be proved by this substitution. $\square$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