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ON PRACTICAL NUMBERS OF SOME SPECIAL FORMS

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Abstract. In this paper we study practical numbers of some special forms. For any integers $b \geqslant 0$ and $c > 0$, we show that if $n^2 + bn + c$ is practical for some integer $n > 1$, then there are infinitely many nonnegative integers n with $n^2 + bn + c$ practical. We also prove that there are infinitely many practical numbers of the form $q^4 + 2$ with q practical, and that there are infinitely many practical Pythagorean triples (a, b, c) with $\gcd(a, b, c) = 6$ (or $\gcd(a, b, c) = 4$).

1. Introduction

A positive integer m is called a *practical number* if each $n = 1, \ldots, m$ can be written as the sum of some distinct divisors of n. This concept was introduced by Srinivasan [3] who noted that any practical number greater than 2 must be divisible by 4 or 6. In 1954, B. M. Stewart [4] obtained the following structure theorem for practical numbers.

Theorem 1.1. *Let $p_1 < \ldots < p_k$ be distinct primes and let $a_1, \ldots, a_k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is practical if and only if $p_1 = 2$ and*

$$p_j - 1 \leqslant \sigma(p_1^{a_1} p_2^{a_2} \cdots p_{j-1}^{a_{j-1}}) \quad \text{for all } 1 < j \leqslant k,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n.

It is interesting to compare practical numbers with primes. All practical numbers are even except 1 while all primes are odd except 2. Moreover, if $P(x)$ denotes the number of practical numbers not exceeding x, then there is a positive constant c such that

$$P(x) \sim \frac{cx}{\log x} \quad \text{as } x \to \infty, \tag{1.1}$$

which was established by Weingartner [7]. This is quite similar to the Prime Number Theorem.

Inspired by the famous Goldbach's conjecture and twin prime conjecture, Margenstern [1] conjectured that every positive even integer is the sum of two practical numbers and that there are infinitely many practical numbers m with $m - 2$ and $m + 2$ also practical. Both conjectures were confirmed by Melfi [2] in 1996.

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Whether there are infinitely many primes of the form x^2+1 with $x\in\mathbb{Z}$ is a famous unsolved problem in number theory. Motivated by this, in 2017 the second author [6, A294225] conjectured that there are infinitely many positive integers q such that q, $q+2$ and q^2+2 are all practical, which looks quite challenging. Thus, it is natural to study for what $a,b,c\in\mathbb{Z}^+$ there are infinitely many practical numbers of the form an^2+bn+c. Note that if $a\equiv b \pmod 2$ and $2\nmid c$ then an^2+bn+c is odd for any $n\in\mathbb{N}$ and hence an^2+bn+c cannot take practical values for infinitely many $n\in\mathbb{N}$.

Based on our computation we formulate the following conjecture.

Conjecture 1.1. *Let a,b,c be positive integers with $2\nmid ab$ and $2\mid c$. Then there are infinitely many $n\in\mathbb{N}$ with an^2+bn+c practical. Moreover, in the case $a=1$, there is an integer n with $1<n\leqslant\max\{b,c\}$ such that n^2+bn+c is practical.*

Though we are unable to prove this conjecture fully, we make the following progress.

Theorem 1.2. *Let $b\in\mathbb{N}$ and $c\in\mathbb{Z}^+$. If n^2+bn+c is practical for some integer $n>1$, then there are infinitely many $n\in\mathbb{N}$ with n^2+bn+c practical.*

If $1\leqslant b\leqslant 100$ and $1\leqslant c\leqslant 100$ with $2\nmid b$ and $2\mid c$, then we can easily find $1<n\leqslant\max\{b,c\}$ with n^2+bn+c practical. For example, n^2+n+2 with $n=2$ is practical. For each positive even number $b\leqslant 20$ we make the set

$$S_b:=\{1\leqslant c\leqslant 100:\ n^2+bn+c \text{ is practical for some } n=2,\ldots,20000\}$$

explicit:

$$\begin{aligned}
S_0=&\{1\leqslant c\leqslant 100:\ c\not\equiv 1,10 \pmod{12} \text{ and } c\neq 43,67,93\},\\
S_2=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,11 \pmod{12} \text{ and } c\neq 44,68,94\},\\
S_4=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,5 \pmod{12} \text{ and } c\neq 47,71,97\},\\
S_6=&\{1\leqslant c\leqslant 100:\ c\not\equiv 7,10 \pmod{12} \text{ and } c\neq 52,76\},\\
S_8=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,5 \pmod{12} \text{ and } c\neq 59,83\},\\
S_{10}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,11 \pmod{12} \text{ and } c\neq 68,92\},\\
S_{12}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 1,10 \pmod{12} \text{ and } c\neq 79\},\\
S_{14}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,11 \pmod{12} \text{ and } c\neq 92\},\\
S_{16}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,5 \pmod{12}\},\\
S_{18}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 7,10 \pmod{12}\},\\
S_{20}=&\{1\leqslant c\leqslant 100:\ c\not\equiv 2,5 \pmod{12}\}.
\end{aligned}$$

For example, applying Theorem 1.2 with $b=20$, we see that for any $c=1,\ldots,100$ with $c\not\equiv 2,5 \pmod{12}$ there are infinitely many $n\in\mathbb{N}$ with $n^2+20n+c$ practical. It is easy to see that if c is congruent to 2 or 5 modulo 12 then $n^2+20n+c$ is not practical for any integer $n\geqslant 2$.

By Theorem 1.2 and the fact $2 \in S_0$, there are infinitely many $n \in \mathbb{N}$ with n^2+2 practical. Moreover, we have the following stronger result.

Theorem 1.3. $2^{35\times 3^k+1}+2$ *is practical for every* $k=0,1,2,\ldots$. *Hence there are infinitely many practical numbers* q *with* q^4+2 *also practical.*

We prove Theorem 1.3 by modifying Melfi's cyclotomic method in [2].

We now turn to Pythagorean triples involving practical numbers, and call a Pythagorean triple (a,b,c) with a,b,c all practical a *practical Pythagorean triple*. Obviously, there are infinitely many practical Pythagorean triples. In fact, if $a^2+b^2=c^2$ with a,b,c positive integers then $(2^ka)^2+(2^kb)^2=(2^kc)^2$ for all $k=0,1,2,\ldots$. By Theorem 1.1, 2^ka, 2^kb and 2^kc are all practical if k is large enough.

Our following theorem was originally conjectured by the second author [5].

Theorem 1.4. *Let* d *be* 4 *or* 6. *Then there are infinitely many practical Pythagorean triples* (a,b,c) *with* $\gcd(a,b,c)=d$.

We are going to show Theorems 1.2-1.4 in the next section.

2. Proofs of our theorems

Lemma 2.1. *Let* m *be any practical number. Then* mn *is practical for every* $n=1,\ldots,\sigma(m)+1$. *In particular,* mn *is practical for every* $1\leqslant n\leqslant 2m$.

This lemma follows easily from Theorem 1.1; see [2] for details. Note that if $m>1$ is practical then $m-1$ can be written as the sum of some divisors of m and hence $(m-1)+m\leqslant\sigma(m)$.

Proof of Theorem 1.2. Set $f(n)=n^2+bn+c$. It is easy to verify that

$$f(n+f(n))=f(n)(f(n)+2n+b+1).$$

Note that

$$f(n)-(2n+b+1)=n(n-2)+b(n-1)+c-1\geqslant 0.$$

If $n\geqslant 2$ is an integer with $f(n)$ practical, then $f(n+f(n))=f(n)(f(n)+2n+b+1)$ is also practical by Lemma 2.1 and the inequality

$$f(n)+2n+b+1\leqslant 2f(n).$$

So the desired result follows. □

For a positive integer m, the cyclotomic polynomial $\Phi_m(x)$ is defined by

$$\Phi_m(x):=\prod_{\substack{a=1\\ \gcd(a,m)=1}}^{m}\left(x-e^{2\pi ia/m}\right).$$

Clearly,

$$x^n-1=\prod_{d\mid n}\Phi_d(x)\quad\text{for all } n=1,2,3,\ldots. \tag{2.2}$$

Proof of Theorem 1.3. Write $m_k = 2^{35\times 3^k+1} + 2$ for $k = 0, 1, 2, \ldots$. Note that $m_{2k} = q_k^4 + 2$ with $q_k = 2^{(35\times 9^k+1)/4}$ practical. So it suffices to prove that m_k is practical for every $k = 0, 1, 2, \ldots$.

Via a computer we find that

$$m_0 = 2^{36} + 2,\ m_1 = 2^{106} + 2,\ m_2 = 2^{316} + 2$$

are all practical.

Now assume that m_k is practical for a fixed integer $k \geqslant 2$. For convenience, we write x for 2^{3^k}. Then

$$x \geqslant 2^9 = 512,\ m_k = 2(x^{35} + 1) \text{ and } m_{k+1} = 2(x^{105} + 1).$$

In view of (2.2),

$$\frac{x^{210} - 1}{x^{105} - 1} = \frac{x^{70} - 1}{x^{35} - 1}\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{210}(x). \tag{2.3}$$

Since $x \geqslant 512$, we have

$$\frac{x^2}{2} < \Phi_6(x) = x^2 - x + 1 < x^2. \tag{2.4}$$

Clearly,

$$x^7 > x^3\frac{x^3 - 1}{x - 1} = x^5 + x^4 + x^3$$

and

$$x^8 > 2x^7 \geqslant x^7 + x + 1.$$

Thus

$$x^8 < \Phi_{30}(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1 < 2x^8 \tag{2.5}$$

Similarly, for

$$\Phi_{42}(x) = x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1$$

and

$$\begin{aligned}\Phi_{210}(x) =& x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} + x^{39} + x^{36}\\ &- x^{35} + x^{34} - x^{33} + x^{32} - x^{31} - x^{28} - x^{26} - x^{24} - x^{22}\\ &- x^{20} - x^{17} + x^{16} - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8\\ &+ 2x^7 - x^6 + x^5 + x^2 - x + 1,\end{aligned}$$

we can prove that

$$x^{12} < \Phi_{42}(x) < 2x^{12} \text{ and } \Phi_{210}(x) < x^{48}. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we get

$$\frac{x^{22}}{2} < \Phi_6(x)\Phi_{30}(x)\Phi_{42}(x) < 4x^{22} \tag{2.7}$$

and hence $\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x) < 4(x^{35} + 1)$. Thus, by Lemma 2.1 and the induction hypothesis we obtain that

$$2(x^{35} + 1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)$$

is practical.

By (2.7),

$$2(x^{35}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x) > x^{57} > x^{48}.$$

So, applying (2.6) and Lemma 2.1, we conclude that

$$2(x^{35}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{210}(x)$$

is practical. In view of (2.3), this indicates that m_{k+1} is practical. This completes the proof. □

Lemma 2.2. [2] *For every $k \in \mathbb{N}$, both $2(3^{3^k\cdot 70}-1)$ and $2(3^{3^k\cdot 70}+1)$ are practical numbers.*

Proof of Theorem 1.4. (i) We first consider the case $d = 4$. For each $k = 0, 1, 2, \ldots$, define

$$a_k = 2(3^{3^k\cdot 70}-1),\ b_k = 4\cdot 3^{3^k\cdot 35},\ \text{and } c_k = 2(3^{3^k\cdot 70}+1).$$

It is easy to see that $a_k^2 + b_k^2 = c_k^2$ and $\gcd(a_k, b_k, c_k) = 4$. By Lemma 2.2, a_k and c_k are both practical. Theorem 2.1 implies that b_k is practical. This proves Theorem 1.4 for $d = 4$.

(ii) Now we handle the case $d = 6$. For any $k = 0, 1, 2, \ldots$, define

$$x_k = 3(3^{3^k\cdot 70}-1),\ y_k = 6\cdot 3^{3^k\cdot 35},\ \text{and } z_k = 3(3^{3^k\cdot 70}+1).$$

Then $x_k^2 + y_k^2 = z_k^2$ and $\gcd(x_k, y_k, z_k) = 6$. Note that y_k is practical for any $k = 0, 1, 2, \ldots$ by Theorem 2.1.

Now it remains to show by induction that x_k and z_k are practical for all $k = 0, 1, 2, \ldots$. Via a computer, we see that $x_0 = 3^{71} - 3$ and $z_0 = 3^{71} + 3$ are practical numbers. Suppose that x_k and z_k are practical for some nonnegative integer k. Then

$$x_{k+1} = 3(3^{3^{k+1}\cdot 70}-1) = x_k(3^{3^k\cdot 70} - 3^{3^k\cdot 35}+1)(3^{3^k\cdot 70}+3^{3^k\cdot 35}+1) \quad (2.8)$$

and

$$z_{k+1} = 3(3^{3^{k+1}\cdot 70}+1) = z_k\Phi_{12}(3^{3^k})\Phi_{60}(3^{3^k})\Phi_{84}(3^{3^k})\Phi_{420}(3^{3^k}). \quad (2.9)$$

In view of (2.8), by applying Lemma 2.1 twice, we see that x_{k+1} is practical. It is easy to check that

$$\Phi_{12}(3^{3^k}) \leqslant 2z_k,\ \Phi_{60}(3^{3^k}) \leqslant 2z_k\Phi_{12}(3^{3^k}),$$
$$\Phi_{84}(3^{3^k}) \leqslant 2z_k\Phi_{12}(3^{3^k})\Phi_{60}(3^{3^k}),\ \Phi_{420}(3^{3^k}) \leqslant 2z_k\Phi_{12}(3^{3^k})\Phi_{60}(3^{3^k})\Phi_{84}(3^{3^k}).$$

In light of these and (2.9), by applying Lemma 2.1 four times, we see that z_{k+1} is practical. This concludes the induction step.

The proof of Theorem 1.4 is now complete. □

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