Generalized Beatty sequences and complementary triples

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Abstract

A generalized Beatty sequence is a sequence V defined by $V(n) = p\lfloor n\alpha \rfloor + qn + r$, for $n = 1, 2, \ldots$, where α, p, q, r are real numbers. These occur in several problems, as for instance in homomorphic embeddings of Sturmian languages in the integers. Our results are for the case that α is the golden mean, but some would generalise to arbitrary quadratic irrationals. We mainly consider the following question: For which sixtuples of integers p, q, r, s, t, u are the two sequences $V = (p\lfloor n\alpha \rfloor + qn + r)$ and $W = (s\lfloor n\alpha \rfloor + tn + u)$ complementary sequences?

We also study complementary triples, i.e., three sequences $V_i = (p_i\lfloor n\alpha \rfloor + q_i n + r_i)$, $i = 1, 2, 3$, with the property that the sets they determine are disjoint with union the positive integers.

1 Introduction

A Beatty sequence is the sequence $A = (A(n))_{n\geq 1}$, with $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$, where α is a positive real number. What Beatty observed is that when $B = (B(n))_{n\geq 1}$ is the sequence defined by $B(n) = \lfloor n\beta \rfloor$, with α and β satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1}$$

then A and B are *complementary* sequences, that is, the sets $\{A(n) : n \geq 1\}$ and $\{B(n) : n \geq 1\}$ are disjoint and their union is the set of positive integers. In particular if $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, this gives that the sequences $(\lfloor n\varphi \rfloor)_{n\geq 1}$ and $(\lfloor n\varphi^2 \rfloor)_{n\geq 1}$ are complementary.

Among the numerous results on Beatty sequences, a paper of Carlitz, Scoville and Hoggatt [3, Theorem 13, p. 20] studies the monoid generated by $A = (A(n))_{n\geq 1}$ and $B = (B(n))_{n\geq 1}$ for the composition of sequences in the case where α is equal to $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

Theorem 1 (Carlitz-Scoville-Hoggatt) *Let $U = (U(n))_{n\geq 1}$ be a composition of the sequences $A = (\lfloor n\varphi \rfloor)_{n\geq 1}$ and $B = (\lfloor n\varphi^2 \rfloor)_{n\geq 1}$, containing i occurrences of A and j occurrences of B, then for all $n \geq 1$*

$$U(n) = F_{i+2j}A(n) + F_{i+2j-1}n - \lambda_U,$$

where F_k are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) and λ_U is a constant.

The sequences U are examples of what we call generalized Beatty sequences, and as an extension of Beatty's observation the following natural questions can be asked.

Question 1 Let α be an irrational number, and let A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ be the Beatty sequence of α. Let Id defined by $\mathrm{Id}(n) = n$ be the identity map on the integers. For which sixtuples of integers p, q, r, s, t, u are the two sequences

$$V = pA + q\,\mathrm{Id} + r \quad \text{and} \quad W = sA + t\,\mathrm{Id} + u$$

complementary sequences?

Question 2 For which nonuples of integers $(p_1, q_1, r_1, p_2, q_2, r_2, p_3, q_3, r_3)$ the three sequences

$$V_i = p_i A + q_i \,\mathrm{Id} + r_i, \; i = 1, 2, 3$$

are a complementary triple, i.e., the sets they determine are disjoint with union the positive integers[1].

Remark 2 The theorem of Carlitz, Scoville and Hoggatt above was rediscovered by Kimberling [15, Theorem 5, p. 3]: it is thus attributed to Kimberling in, e.g., [11, p. 575], [12, p. 647], [17, p. 20–21]. This was corrected in [2, Theorem 2, p. 2] .

Remark 3 One can ask whether the monoid generated by other complementary sequences by composition can be written as a subset of the set of linear combinations of a finite number of elements. Some answers for Beatty sequences can be found in the rich paper of Fraenkel [10] (see, e.g., p. 645). Another, possibly unexpected, example is given by the Thue-Morse sequence. Namely call odious (resp. evil) the integers whose binary expansion contains an odd (resp. even) number of 1's, then it was proved in [1, Corollaries 1 and 3] that the sequences $(A(n))_{n\geq 0}$ and $(B(n))_{n\geq 0}$ of odious and evil numbers satisfy for all n

$$\begin{array}{llll} A(n) = 2n + 1 - t(n), & B(n) = 2n + t(n), & A(n) - B(n) = 1 - 2t(n) & \\ A(A(n)) = 2A(n), & B(B(n)) = 2B(n), & A(B(n)) = 2B(n) + 1, & B(A(n)) = 2A(n) + 1. \end{array}$$

where $(t(n))_{n\geq 0}$ is the Thue-Morse sequence, i.e., the characteristic function of odious integers. (This sequence can be defined by $t(0) = 0$ and for all $n \geq 0$, $t(2n) = t(n)$ and $t(2n + 1) = 1 - t(n)$.) This easily implies that any finite composition of $(A(n))_{n\geq 0}$ and $(B(n))_{n\geq 0}$ can be written as $(\alpha A(n) + \beta B(n) + \gamma)_{n\geq 0}$, since $t(A(n)) = 1$ and $t(B(n)) = 0$ for all n.

2 Complementary pairs

Let α be an irrational number, and let A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ be the Beatty sequence of α. Let Id defined by $\mathrm{Id}(n) = n$ be the identity map on $\mathbb{N}$. Here we consider the question:

Complementary pair problem: *for which sixtuples of integers (p, q, r, s, t, u) the two sequences $V = (V(n))_{n\geq 1}$ and $W = (W(n))_{n\geq 1}$ defined by*

$$V = pA + q\,\mathrm{Id} + r \;\text{ and }\; W = sA + t\,\mathrm{Id} + u \tag{2}$$

are a complementary pair*—meaning that as subsets of* $\mathbb{N}$*, V and W are disjoint and their union is* $\mathbb{N}$*?*

In the sequel we will require that as a function $A : \mathbb{N} \to \mathbb{N}$ is injective, since we then have a 1-to-1 correspondence between sequences and subsets of $\mathbb{N}$. (See [14] for non-injective Beatty sequences.)

In the case that V and W are increasing, we will also require, without loss of generality, that $V(1) = 1$.

The homogeneous Sturmian sequence generated by a real number α is the sequence $(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)_{n\geq 1}$. It is well known that the homogeneous Sturmian sequence generated by the golden mean φ is

$$x_{\mathrm{F}} = 2\;1\;2\;2\;1\;2\;1\;2\;2\;1\;2\;2\;1\;2\;1\;2\;2\;1\;2\;1\;\ldots,$$

obtained by replacing 0 by 2 in the unique fixed point of the Fibonacci morphism $0 \to 01$, $1 \to 0$. The following lemma is thus implied trivially by

$$V = pA + q\,\mathrm{Id} + r \quad \Rightarrow \quad V(n+1) - V(n) = p(A(n+1) - A(n)) + q.$$

Lemma 4 *Let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\varphi \rfloor) + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the Fibonacci sequence on the alphabet $\{2p+q, p+q\}$.*

[1]And when is this partition "nice"? (in Fraenkel's terminology [10] we look for a "nice" integer DCS –Disjoint Covering System).

Another observation is that the $q\,\mathrm{Id} + r$ part in the generalized Beatty sequence generates arithmetic sequences. The following lemma, which will be useful in proving Theorem 7, shows that in some weak sense the Wythoff part pA of a generalized Beatty sequence is orthogonal to this arithmetic sequence part.

Lemma 5 *Let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\varphi \rfloor) + qn + r$ with $p \neq 0$, then neither $(V(1), V(2), V(3))$, nor*[2] *$(V(2), V(3), V(4))$ can be an arithmetic sequence of length 3.*

Proof: We have by Lemma 1

$$(V(2) - V(1)) \;\; (V(3) - V(2)) \;\; (V(4) - V(3)) \;\; \ldots = \Delta V = (2p + q) \;\; (p + q) \;\; (2p + q) \;\; \ldots$$

Since $p \neq 0$ we thus have $V(2) - V(1) \neq V(3) - V(2)$ and $V(3) - V(2) \neq V(4) - V(3)$. □

Remark 6 We note for further use that the proof of Lemma 5 yields

$$\begin{cases} p &= -V(1) + 2V(2) - V(3) \\ q &= \;\; V(1) - 3V(2) + 2V(3) \\ r &= \;\; V(1) + V(2) - V(3). \end{cases}$$

Let $\alpha = \varphi$, the golden mean. Then the classical solution is $(p, q, r) = (1, 0, 0)$ and $(s, t, u) = (1, 1, 0)$, which corresponds to the Beatty pair $([n\varphi]), ([n\varphi^2])$. Another solution is given by

$$(p, q, r) = (-1, 3, -1), \quad (s, t, u) = (1, 2, 0),$$

which corresponds to the Beatty pair $([n(5 - \sqrt{5})/2]), ([n(5 + \sqrt{5})/2])$, which is equal to

$$([n(3 - \varphi)]), ([n(\varphi + 2)]).$$

Theorem 7 *Let $\alpha = \varphi$. Then there are no more than two increasing solutions to the complementary pair problem: $(p, q, r, s, t, u) = (1, 0, 0, 1, 1, 0)$ and $(p, q, r, s, t, u) = (-1, 3, -1, 1, 2, 0)$.*

Proof: Recall that $V(1) = 1$. We first note that $V(2) < 5$, since otherwise $(W(1), W(2), W(3)) = (2, 3, 4)$, which is not allowed by Lemma 5. There are therefore three cases to consider, according to the value of $V(2)$.

- I. $V(1) = 1, V(2) = 2$. Then by Lemma 5, $V(3) = 3$ is not possible.
  - If $V(3) = 4$, then, by Remark 6, $p = -1$, $q = 3$, $r = -1$, which is one of the two solutions.
  - If $V(3) = 5$, then, by Remark 6, $p = -2$, $q = 5$, $r = -2$, which implies that $V(4) = 6$, $V(5) = 7$, $V(6) = 10$. So $W(1) = 3$, $W(2) = 4$, $W(3) = 8$, which gives $s = -3$, $t = 7$, $u = -1$ (Remark 6 applied to W), implying $W(5) = 10$, which contradicts complementarity.
  - If $V(3) = m$ with $m > 5$, then $W(1) = 3$, $W(2) = 4$, $W(3) = 5$, which contradicts Lemma 5.
- II. $V(1) = 1$, $V(2) = 3$.
  - If $V(3) = 4$, then, by Remark 6, $p = 1$, $q = 0$, $r = 0$, which is one of the two solutions.
  - If $V(3) = 5$, then we obtain a contradiction with Lemma 5.
  - If $V(3) = 6$, then, by Remark 6, $p = -1$, $q = 4$, $r = -2$, which implies $V(5) = 10$. But we must then have $W(1) = 2, W(2) = 4, W(3) = 5$, so (Remark 6 applied to W), $s = 1$, $t = 0$, $u = 1$, which implies $W(6) = 10$, a contradiction with complementarity.
  - If $V(3) = m$ with $m > 6$, then we obtain a contradiction with Lemma 5, since then $W(2) = 4$, $W(3) = 5$, $W(4) = 6$.
- III. $V(1) = 1$, $V(2) = 4$.

[2] This does not hold for $(V(3), V(4), V(5))$!

– If $V(3) = 5$, then, by Remark 6, $p = 2$, $q = -1$, $r = 0$, thus $V(4) = 8$; hence $W(1) = 2$, $W(2) = 3$, $W(3) = 6$. Hence, by Remark 6 applied to W, $s = -2$, $t = 5$, $u = -1$, so that $W(5) = 8 = V(4)$, which contradicts complementarity.

– If $V(3) = 6$, then $W(1) = 2$, $W(2) = 3$, $W(3) = 5$. Thus, by Remark 6 applied to W, $s = -1$, $t = 3$, $u = 0$. Hence $W(4) = 6 = V(3)$, which contradicts complementarity.

– If $V(3) = 7$, then we obtain a contradiction with Lemma 5.

– If $V(3) = m$ with $m > 7$, then it follows that $V(3) = 8$, since we have $W(1) = 2$, $W(2) = 3$, $W(3) = 5$, yielding, by Remark 6 applied to W, $W(n) = (-A(n) + 3n) = 2, 3, 5, 6, 7, 9, 10, 12, 13, 14, \ldots$ With $V(3) = 8$, one obtains (by Remark 6) that $V(n) = -A(n) + 5n - 3$, but then $V(5) = 14 = W(10)$, i.e., V and W are not complementary. □

2.1 Generalized Pell equations

If V and W are not increasing, then an analysis as in the proof of Theorem 7 is still possible, but very lengthy. We therefore consider another approach in this subsection. Considering the densities of V and W in $\mathbb{N}$, one sees that a *necessary* condition for $(pA + q\,\mathrm{Id} + r, sA + t\,\mathrm{Id} + u)$ to be a complementary pair is that

$$\frac{1}{p\alpha + q} + \frac{1}{s\alpha + t} = 1 \tag{3}$$

In the sequel we concentrate on the case $\alpha =: \varphi = (1 + \sqrt{5})/2$, but our arguments would easily generalise to the case of arbitrary quadratic irrationals.

Proposition 8 *A necessary condition for the pair $v = pA + q\,\mathrm{Id} + r$ and $w = sA + t\,\mathrm{Id} + u$ to be a complementary pair is that $p \neq 0$ is a solution to the generalized Pell equation*

$$5p^2x^2 - 4x = y^2, \quad x, y \in \mathbb{Z}$$

Proof: Using $\varphi^2 = 1 + \varphi$, a straightforward manipulation shows that (3) implies

$$(ps + pt + qs - p - s)\varphi = q + t - ps - qt.$$

But since φ is irrational, this can only hold if

$$ps + pt + qs - p - s = 0, \quad q + t - ps - qt = 0. \tag{4}$$

The first equation gives $pt = p - (p + q - 1)s$. Eliminating pt from $p^2s + (q - 1)pt - pq = 0$, we obtain $p^2s + (p - (p + q - 1)s)(q - 1) - pq = 0$. This gives the quadratic equation

$$s\,q^2 + (p - 2)s\,q - (p^2 + p - 1)s + p = 0.$$

Since q is an integer, $\Delta := (p-2)^2s^2 + 4s((p^2+p-1)s - p)$ has to be an integer squared. Trivial manipulations yield that

$$\Delta = 5p^2s^2 - 4ps. \tag{5}$$

Since p divides the square Δ, $5p^2s^2 - 4ps = p^2y^2$ for some integer y, and hence p also divides s. If we put $s = px$, we obtain $5p^3x^2 - 4p^2x = p^2y^2$, which finishes the proof of the proposition. □

Actually there is a simple characterization of the integers p such that the Diophantine equation above has a solution.

Proposition 9 *The generalized Pell equation*

$$5p^2x^2 - 4x = y^2, \quad x, y \in \mathbb{Z}$$

has a solution for $p > 0$ if and only if p divides some Fibonacci number of odd index, i.e., if and only p divides some number in the set $\{1, 2, 5, 13, 34, \ldots\}$.

Proof: First suppose that there are integers $p > 0$ and $x, y \in \mathbb{Z}$ such that $5p^2x^2 - 4x = y^2$. Let $d := \gcd(x, y)$ and $x' = x/d$, $y' = y/d$, so that $\gcd(x', y') = 1$. We thus have

$$5p^2dx'^2 - 4x' = dy'^2.$$

Thus x' divides dy'^2, but it is prime to y', hence x' divides d. Since clearly d divides $4x'$, we have $d = \alpha x'$ for some α dividing 4, hence α belongs to $\{1, 2, 4\}$. This yields $\alpha(5p^2x'^2 - y'^2) = 4$. We distinguish three cases.

- If $\alpha = 1$, then we have $5p^2x'^2 - y'^2 = 4$. But the equation $5X^2 - 4 = Y^2$ has an integer solution if and only if X is a Fibonacci number with odd index [18, p. 91]. Hence px' must be a Fibonacci number with odd index, thus p divides a Fibonacci number with odd index.
- If $\alpha = 2$, then we have $5p^2x'^2 - y'^2 = 2$. Note that x' must be odd, otherwise x' and y' would be even, which contradicts $\gcd(x', y') = 1$. Thus $5p^2x'^2 \equiv p^2 \bmod 4$, hence $p^2 - 2 \equiv y'^2 \bmod 4$. If p is even, this yields $y'^2 \equiv 2 \bmod 4$, while if p is odd, this gives $y'^2 \equiv 3 \bmod 4$. There is no such y' in both cases.
- If $\alpha = 4$, then we have $5p^2x'^2 - y'^2 = 1$, thus $5(2px')^2 - (2y')^2 = 4$, then $2px'$ must be a Fibonacci number with odd index, thus p divides a Fibonacci number with odd index.

Now suppose that p divides some Fibonacci number with odd index, say there exists a k with $F_{2k+1} = p\beta$. We will construct an integer solutions in (x, y) to the equation $5p^2x^2 - 4x = y^2$. We know (again [18, p. 91]) that there exists some integer γ with $5F_{2k+1}^2 - 4 = \gamma^2$ thus $5p^2\beta^2 - 4 = \gamma^2$. Let $x = \beta^2$ and $y = \beta\gamma$. Then

$$5p^2x^2 - 4x = 5p^2\beta^4 - 4\beta^2 = \beta^2(5p^2\beta^2 - 4) = \beta^2\gamma^2 = y^2. \quad \Box$$

Corollary 10 *There are no solutions to the complementary pair problem if -1 is not a square modulo p, i.e., if p does not belong to the sequence* $1, 2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, \ldots$ *(sequence A008784 in [20]). This is in particular the case if p has a prime divisor congruent to* 3 *modulo* 4.

Proof: We will prove that if there are solutions to the complementary problem for p, thus if p divides an odd-indexed Fibonacci number (Propositions 8 and 9), then -1 is a square modulo p. Using again the characterization in [18, p. 91], there exist two integers x, y with $5p^2x^2 - 4 = y^2$. We distinguish two cases.

- If p is odd, we have $y^2 \equiv -4 \bmod p$ and $2^2 \equiv 4 \bmod p$. But 2 is invertible modulo p, hence, by taking the quotient of the two relations, we obtain that -1 is a square modulo p.
- If p is even, remembering that $px = F_{2k+1}$ for some k, we claim that p must be congruent to 2 modulo 4 and that x must be odd. Namely the sequence of odd-indexed Fibonacci numbers, reduced modulo 4, is easily seen to be the periodic sequence $(1\ 2\ 1)^\infty$. Hence it never takes the value 0 modulo 4. The equality $5p^2x^2 - 4 = y^2$ implies that y must be even, thus we have $5(p/2)^2x^2 - 1 = (y/2)^2$, say $(y/2)^2 = -1 + z(p/2)$. Up to replacing $(y/2)$ with $(y + p)/2$, we may suppose that $(y/2)$ is even (recall that $p/2$ is odd). Thus $z(p/2)$ is even, hence z is even, say $z = 2z'$. This gives $(y/2)^2 = -1 + z'p$, thus -1 is a square modulo p.

Remark 11 We have just seen that if the integer p divides some odd-indexed Fibonacci number then -1 is a square modulo p (sequence A008784 in [20]). A natural question is then whether it is true that if -1 is a square modulo p, then p must divide some odd-indexed Fibonacci number. The answer is negative, since on one hand $12^2 \equiv -1 \bmod 29$, and, on the other hand, the sequence of odd-indexed Fibonacci numbers modulo 29 is the periodic sequence $(1, 2, 5, 13, 5, 2, 1)^\infty$ which is never zero.

Let us look at examples of solutions to the Diophantine equation for values of p that divide some Fibonacci number with odd index. Consider, for example, the case where $p = s$. Then Equation (5) becomes $\Delta = 5p^4 - 4p^2$, so the Diophantine equation is

$$5x^2 - 4 = y^2, \quad x, y \in \mathbb{Z}.$$

For $p = F_1 = 1$ we obtain the two sequences $V = A + r$ and $W = A + \mathrm{Id} + u$. These are complementary only when $r = u = 0$, and we obtain the classical Beatty pair $(A, A + \mathrm{Id})$.

For $p = F_3 = 2$ we obtain the two sequences $V = 2A + 2\,\mathrm{Id} + r$ and $W = 2A - 2\,\mathrm{Id} + u$. These cannot be complementary for any r and u, since for $u = 0$ we have $W(n) = 2(\lfloor n\varphi \rfloor) - 2n = 2(\lfloor n(\varphi - 1)\rfloor$, which gives all even numbers, since $\varphi - 1 < 1$. This an example where Equation (3) does not apply, since W as a function is not injective.

For $p = F_5 = 5$ we obtain the two sequences $V = 5A + 4\,\mathrm{Id} + r$ and $W = 5A - 7\,\mathrm{Id} + u$. To make these complementary we are forced to choose $r = u = 3$, and we obtain

$V = (12, 26, 35, 49, 63, 72, 86, 95, 109, 123, 132, 146, 160, 169, 183, 192, 206, 220, 229, 243, 252, 266, \dots)$,

$W = (1, 4, 2, 5, 8, 6, 9, 7, 10, 13, 11, 14, 17, 15, 18, 16, 19, 22, 20, 23, 21, 24, 27, 25, 28, 31, 29, 32, 30, \dots)$.

Now a proof that V and W form a complementary pair is much harder, when we let V start with $V(0) = 3$, to include 3 in the union. We can perform the following trick. We split W into $(W(A(n)))_{n\geq 1}$, and $(W(B(n)))_{n\geq 1}$ (cf. Proposition 12). The two sequences WA and WB are increasing, and we can prove that $(V(n))_{n\geq 0}$, $(W(A(n)))_{n\geq 1}$, and $(W(B(n)))_{n\geq 1}$ form a partition of the positive integers by exhibiting a three-letter sequence such that the preimages of the letters are precisely these three sequences.

For $p = F_{2m+1} \geq 13$ it seems that we can always choose r and u for in such a way that we get almost complementary sequences: namely, e.g., for $p = 13$ we find $q = 9$ and $t = -20$. If we take $r = u = 9$, then we almost get a complementary pair. One finds $V = 9, 31, 66, 88, 123, 158, 180, 215, \dots$ and $W = 2, 8, 1, 7, 13, 6, 12, 5, 11, 17, 10 \dots$. So 3 and 4 are missing. It might be that for all $F_{2m+1} > 5$ the two sequences are complementary, excluding finitely many values. Possibly this can be proved using the Lambek-Moser Theorem ([16]).

3 Complementary triples

Here we will find several complementary triples consisting of sequences

$$V_i = p_i A + q_i\,\mathrm{Id} + r_i, \;\; i = 1, 2, 3.$$

It is interesting that the case $p_1 = p_2 = p_3 = 1$ cannot be realized. This was proved by Uspensky in 1927, see [9]. Also see [23] for the inhomogeneous Beatty case $(V_i(n))_n = ([n\alpha_i + \beta_i])_n, \; i = 1, 2, 3$.

There is one triple in which we will be particularly interested (see Theorem 19):

$$((p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3)) = ((2, -1, 0), (4, 3, 2), (2, -1, 2)).$$

We will allow that the sequences (V_i) are each indexed either by $\{0, 1, 2, \dots\}$ or by $\{1, 2, \dots\}$.

3.1 Two classical triples

Once more let $A(n) = \lfloor n\varphi \rfloor$ for $n \geq 1$ be the terms of the lower Wythoff sequence, and let B given by $B(n) = \lfloor n\varphi^2 \rfloor$ for $n \geq 1$ be the upper Wythoff sequence. Then we have the disjoint union

$$A(\mathbb{N}) \cup B(\mathbb{N}) = \mathbb{N}. \tag{6}$$

Since $B = A + \mathrm{Id}$, this is the classical complementary pair $((1, 0, 0), (1, 1, 0))$.

Here is a way to create complementary triples from complementary pairs.

Proposition 12 *Let (V, W) be a complementary pair $V = pA + q\,\mathrm{Id} + r$ and $W = sA + t\,\mathrm{Id} + u$. Then (V_1, V_2, V_3) is a complementary triple, where the three parameters of V_1 are $(p + q, p, r - p)$, those of V_2 are $(2p + q, p + q, r)$, and $V_3 = W$.*

Proof: From Theorem 1 we obtain that for $n = 1, 2, \dots$

$$AA(n) = A(n) + n - 1,\ AB(n) = 2A(n) + n. \tag{7}$$

Substituting Equation (6) in $V(\mathbb{N}) \cup W(\mathbb{N}) = \mathbb{N}$ we obtain the disjoint union

$$V(A(\mathbb{N})) \cup V(B(\mathbb{N})) \cup W(\mathbb{N}) = \mathbb{N}. \tag{8}$$

For $n = 1, 2, \dots$ we have by Equation (7)

$$\begin{aligned} V(A(n)) &= pA(A(n)) + qA(n) + r = p[A(n) + n - 1] + qA(n) + r = (p+q)A(n) + pn + r - p, \\ V(B(n)) &= pA(B(n)) + qB(n) + r = p[2A(n) + n] + q[A(n) + n] + r = (2p+q)A(n) + (p+q)n + r. \end{aligned}$$

This, combined with Equation (8) implies the statement of the proposition. □

Applying Proposition 12 to the basic complementary pair $((1,0,0),(1,1,0))$ gives that $((1,1,-1),(2,1,0),(1,1,0))$ and $((1,0,0),(2,1,-1),(3,2,0))$ are complementary triples[3], which we will call classical triples.

Let $w = 1231212312312\dots$ be the fixed point of the morphism

$$1 \to 12,\ 2 \to 3,\ 3 \to 12.$$

Then $w^{-1}(1) = AA, w^{-1}(2) = B$ and $w^{-1}(3) = AB$ give the three sequences V_1, V_3 and V_2 of the first classical triple (see [6]).

The question arises: is there also a morphism generating the second triple? The answer is positive.

Proposition 13 *Let* $(V_1, V_2, V_3) = (A,\ 2A + \mathrm{Id} - 1,\ 3A + 2\mathrm{Id}) = (A,\ BA,\ BB)$. *Then* (V_1, V_2, V_3) *is a complementary triple. Let* μ *be the morphism on* $\{1, 2, 3\}$ *given by*

$$1 \to 121,\ 2 \to 13,\ 3 \to 13,$$

with fixed point z. *Then* $z^{-1}(1) = V_1$, $z^{-1}(2) = V_2$ *and* $z^{-1}(3) = V_3$.

Proof: The four words of length 4 occurring in the infinite Fibonacci word x_{F} are $010, 100, 001, 101$. Coding these with the alphabet $\{1, 2, 3, 4\}$ in the given order, they generate the 3-block morphism $\hat{f}_3$ that describes the successive occurrences of the words of length 3 in x_{F} (cf. [6]). It is given by

$$\hat{f}_3(1) = 12, \quad \hat{f}_3(2) = 3, \quad \hat{f}_3(3) = 14, \quad \hat{f}_3(4) = 3.$$

It has just one fixed point, which is

$$z' := 1, 2, 3, 1, 4, 1, 2, 3, 1, 2, 3, 1, 4, 1, 2, 3, \dots.$$

It is not difficult to see, applying Equation (6) at various levels, that

$$z'^{-1}(1) = AA,\ z'^{-1}(2) = BA, \quad z'^{-1}(3) = AB, \quad z'^{-1}(4) = BB.$$

Again by Equation (6), we see that we have to merge the letters 1 and 3 to obtain the sequence A. This is not possible with $\hat{f}_3$. However the square of this 3-block substitution is given by

$$1 \to 123,\ 2 \to 14,\ 3 \to 123,\ 4 \to 14,$$

and now we *can* consistently merge 1 and 3 to the single letter 1, obtaining the substitution μ, after mapping 4 to 3. Under this projection the sequence z' maps to z. □

[3] In [20] these are (A003623, A003622, A001950) and (A000201, A035336, A101864).

3.2 Non-classical triples

Let $\mathcal{L}$ be a language, i.e., a sub-semigroup of the free semigroup generated by a finite alphabet under the concatenation operation. A homomorphism of $\mathcal{L}$ into the natural numbers is a map $\mathrm{S} : \mathcal{L} \to \mathbb{N}$ satisfying $\mathrm{S}(vw) = \mathrm{S}(v) + \mathrm{S}(w)$, for all $v, w \in \mathcal{L}$.

Let $\mathrm{S}(\mathcal{L}_\mathrm{F})$ be the Fibonacci language, i.e., the set of all words occurring in x_F. The following result is proved in [7].

Theorem 14 ([7]) *Let $\mathrm{S} : \mathcal{L}_\mathrm{F} \to \mathbb{N}$ be a homomorphism. Define $a = \mathrm{S}(0), b = \mathrm{S}(1)$. Then $\mathrm{S}(\mathcal{L}_\mathrm{F})$ is the union of the two generalized Beatty sequences $\big((a-b)\lfloor n\varphi \rfloor + (2b-a)n\big)$ and $\big((a-b)\lfloor n\varphi \rfloor + (2b-a)n + a - b\big)$.*

For a few choices of a and b, the two sequences in $\mathrm{S}(\mathcal{L}_\mathrm{F})$ and the sequence $\mathbb{N} \setminus \mathrm{S}(\mathcal{L})$ form a complementary triple of generalized Beatty sequences. The goal of this section is to prove this for $a = 3$, $b = 1$. It turns out that the three sequences

$$(2\lfloor n\varphi \rfloor - n)_{n\ge 1},\ (2\lfloor n\varphi \rfloor - n + 2)_{n\ge 1},\ (4\lfloor n\varphi \rfloor + 3n + 2)_{n\ge 0},$$

form a complementary triple.

Remark 15 Note that the indices for $(4\lfloor n\varphi \rfloor) + 3n + 2)_{n\ge 0}$ are $(n \ge 0)$, not $(n \ge 1)$

Recall that the binary Fibonacci sequence is defined as the iterative fixed point of the morphism f defined on $\{0,1\}^*$ by $f(0) = 01,\ f(1) = 0$. We let $x_\mathrm{F} = (x_\mathrm{F}(n))_{n\ge 1}$ denote this sequence. It is easy to see that x_F can be obtained as an infinite concatenation of two kinds of blocks, namely 01 and 001 (part (i) of Lemma 16 below). Kimberling introduced in the OEIS [20] the sequence A284749 obtained by replacing in this concatenation every block 001 by 2. We let $x_\mathrm{K} = A284749$ denote this sequence.

Lemma 16 *Let g, h, k be the morphisms defined on $\{0,1\}^*$ by*

$$g(0) = 01,\ g(1) = 011; \quad h(0) = 01,\ h(1) = 001; \quad k(0) = 01,\ k(1) = 2.$$

Let furthermore i be the morphism defined on $\{0,1,2\}^*$ by

$$i(0) = 01,\ i(1) = 2,\ i(2) = 0122.$$

Then (i) $x_\mathrm{F} = f^\infty(0) = hg^\infty(0)$, (ii) $x_\mathrm{K} = k(g^\infty(0))$, (iii) $x_\mathrm{K} = i^\infty(0)$.

Proof:

(i) An easy induction proves that for all $n \ge 0$ one has $hg^k = f^{2k}h$. (Note that it suffices to prove that the values of both sides are equal when applied to 0 and to 1.) By letting n tend to infinity this implies $hg^\infty(0) = f^\infty(0)$.

(ii) Assertion (i) clearly implies that x_F is an infinite concatenation of blocks $h(0)$ and $h(1)$, thus of blocks 01 and 001, thus that $kg^\infty(0) = x_\mathrm{K}$.

(iii) An easy induction shows that $kg^n = i^{n+1}$. Hence the result by letting n tend to infinity. □

Lemma 17 *Define the morphism ℓ from $\{0,1\}^*$ to $\{0,1,2\}^*$ by $\ell(0) = 012$, $\ell(1) = 0022$. Then the sequence $v = (v_n)_{n\ge 1} = \ell g^\infty(0)$ is obtained from $x_\mathrm{K} = i^\infty(0)$ by replacing 1 by 0 in all blocks 0122 (but not in 0120). The positions of 2 in v are obtained by adding 2 to the positions of 0.*

Proof: The relation $x_\mathrm{K} = i^\infty(0) = k(g^\infty(0))$ shows that x_K is the concatenation of two types of blocks, the blocks 012 and the blocks 0122. The two assertions follow. □

Lemma 18 *Let w be the sequence obtained from v by replacing all 2's by 1's. Let m be the morphism defined on $\{0,1\}^*$ by $m(0) = 011$, $m(1) = 0011$. Then $w = m(g^\infty(0))$.*

Proof: Letting q the morphism defined by $q(0) = 0$, $q(1) = 1$, $q(2) = 1$, one has $w = q(v) = q(\ell(g^\infty(0))) = m(g^\infty(0))$ since, clearly, $q\ell = m$. □

Theorem 19 *Let v be the sequence defined above, i.e., $v = \ell(g^\infty(0))$, where $g(0) = 01$, $g(1) = 011$ and $\ell(0) = 012$, $\ell(1) = 0022$. Then the increasing sequences of integers defined by $v^{-1}(0)$, $v^{-1}(1)$, $v^{-1}(2)$ form a partition of the set of positive integers $\mathbb{N}^*$. Furthermore*

- $v^{-1}(0) = \{1, 4, 5, 8, 11, 12, 15, 16, 19, 22, \ldots\}$ *is equal to the sequence of integers* $(2\lfloor n\varphi\rfloor - n)_{n\geq 1}$, *where* φ *is the golden ratio* $\frac{1+\sqrt{5}}{2}$ *(sequence* A050140 *in* [20]*),*
- $v^{-1}(1) = \{2, 9, 20, 27, \ldots\}$ *is equal to the sequence of integers* $(4\lfloor n\varphi\rfloor + 3n + 2)_{n\geq 0}$.
- $v^{-1}(2) = \{3, 6, 7, 10, 13, 14, 17, 18, 21, 24, \ldots\}$ *is equal to the sequence of integers* $((2\lfloor n\varphi\rfloor - n + 2)_{n\geq 1})$ *(i.e.,* 2+A050140*).*

Proof: Since $v^{-1}(0) = w^{-1}(0)$, by the definition of w, in order to prove the assertion on $v^{-1}(0)$ it suffices to prove that $w^{-1}(0)$ is the sequence $(2\lfloor n\varphi\rfloor - n)_{n\geq 1}$. According to Lemma 4 the first difference of the latter is the Fibonacci binary sequence on the alphabet $\{3, 1\}$. It thus suffices to prove that the first difference of $w^{-1}(0)$ is equal to Δ. Recall that $w = mg^\infty(0)$ from Lemma 18. Define the words $a_k = mg^k(0)$ and $b_k = mg^k(1)$. Then $a_{k+1} = mg^k(g(0)) = mg^k(01) = a_k b_k$ and $b_{k+1} = mg^k(g(1)) = mg^k(011) = a_k b_k b_k$. Note that x_k is a prefix of x_{k+1} and of y_{k+1}, and that x_k and y_k both converge to w. Since the runlengths of 0's and 1's in a_k and b_k are equal to 1 or 2, we can write each a_k under the form $01^{x_0}01^{x_1}01^{x_2}\ldots$ with $x_i \in \{0, 2\}$ where no two consecutive x_i's can be equal to 0, and each b_k under the form $01^{y_0}01^{y_1}01^{y_2}\ldots$ with $y_i \in \{0, 2\}$ where no two consecutive y_j's can be equal to 0. We associate with a_k the word $A_k = x_0 x_1 \ldots$ and with b_k the word $B_k = y_0 y_1 \ldots$: $a_0 = 011$, $b_0 = 0011$ hence $A_0 = 2$ and $B_0 = 02$; the recurrence relations for a_k and b_k give easily $A_{k+1} = A_k B_k$ and $B_{k+1} = A_k B_k B_k$. Defining the morphism r on $\{0, 2\}^*$ by $r(2) = 20$, $r(0) = 200$, a straightforward induction shows that $A_k = r^k(0)$ and $B_k = r^k(1)$. Hence A_k and B_k both converge to the iterative fixed point of r. It is well known and easy to prove that this iterative fixed point deprived of its first symbol, i.e., $020020200200\ldots$ is the binary Fibonacci sequence on $\{2, 0\}$. To finish the proof of the fist assertion of our theorem, we note that the first differences of the indexes of occurrences of 0 in w (i.e., the first differences of the terms of $w^{-1}(0)$) are exactly 1+the number of 1's separating these occurrences in w.

The proof of the second assertion in the theorem is similar to the proof of the first one. Namely define z to be the sequence obtained from v by replacing all 2's by 0's. It is clear that the positions of 1 in v and z are the same. It is also clear that $z = \ell'(g^\infty(0))$, where ℓ' is the morphism defined on $\{0, 1\}^*$ by $\ell'(0) = 010$, $\ell'(1) = 0000$. Reasoning as in the proof of the first assertion above, it suffices to prove that 1+the lengths of runs of 0's in z is the first difference of the sequence $(4\lfloor n\varphi\rfloor) + 3n + 2)_{n\geq 0}$. But this last sequence is the binary Fibonacci sequence on the alphabet $\{7, 11\}$. Define $x_n = \ell'(g^n(0))$ and $y_n = \ell'(g^n(1))$. Then one obtains easily that $x_{n+1} = x_n y_n$ and $y_{n+1} = x_n y_n y_n$. Now note that x_n and y_n begin with 0, and define x'_n, y'_n by $x_n 0 = 0x'_n$ and $y_n 0 = 0y'_n$ so that $x'_{n+1} = x'_n y'_n$ and $y'_{n+1} = x'_n y'_n y'_n$. Note that both x'_n and y'_n begin with 1. Write as above $x'_n = 10^{c_1}10^{c_2}\ldots$ and $y'_n = 10^{d_1}10^{d_2}\ldots$. Associate with x_k and y_k respectively the words $X_k = c_1 c_2 \ldots$ and $Y_k = d_1 d_2 \ldots$. We obtain $X_1 = 6$, $Y_1 = 10$, and $X_{k+1} = X_k Y_k$, $Y_{k+1} = X_k Y_k Y_k$. We conclude as above.

The third assertion of our theorem is a consequence of the last assertion of Lemma 17. □

Remark 20 Some of the sequences above are images of Sturmian sequences by a morphism. Namely $v = \ell(g^\infty(0))$, $w = m(g^\infty(0))$, $x_{\rm K} = k(g^\infty(0))$. Such sequences are examples of sequences called *quasi-Sturmian* in [4]. Their block complexity is of the form $n + C$ for n large enough ($C = 1$ for Sturmian sequences). This was studied in [21], [5], and [4].

4 Generalized Beatty sequences and return words

In this section we show that generalized Beatty sequences are closely related to return words.

Theorem 21 *Let $x_{\rm F}$ be the Fibonacci word, and let w be any word in the Fibonacci language $\mathcal{L}_{\rm F}$. Let Y be the sequence of positions of the occurrences of w in $x_{\rm F}$. Then Y is a generalized Beatty sequence, i.e., for all $n \geq 0$ $Y(n+1) = p\lfloor n\varphi \rfloor + qn + r$ with parameters p, q, r, which can be explicitly computed.*

Proof: Let $x_{\rm F} = r_0(w)r_1(w)r_2(w)r_3(w)\ldots$, written as a concatenation of return words of the word w (cf. [13], Lemma 1.2). According to Theorem 2.11 in [13], if we skip $r_0(w)$, then the return words occur as the Fibonacci word on the alphabet $\{r_1(w), r_2(w)\}$. Thus the distances between occurrences of w in $x_{\rm F}$ are equal to $l_1 := |r_1(w)|$ and $l_2 := |r_2(w)|$. We can apply the converse of Lemma 4: solving the equations

$$2p + q = l_1, \quad p + q = l_2$$

gives $p = l_1 - l_2,\ q = 2l_2 - l_1$. Inserting $n = 0$, we find that $r = |r_0(w)| + 1$, as the first occurrence of w is at the beginning of $r_1(w)$. $\square$

4.1 The Kimberling transform

Here we will obtain non-classical triples appearing in another way, namely as the three indicator functions $x^{-1}(0), x^{-1}(1)$ and $x^{-1}(2)$, of a sequence x on an alphabet $\{0, 1, 2\}$ of three symbols. In our examples the sequence x is a 'transform' $\mathcal{T}(x_{\rm F})$ of the Fibonacci sequence $x_{\rm F} = 0, 1, 0, 0, 1, 0, 1, ...$ These transforms have been introduced by Kimberling in the OEIS [20]. Our main example is: $\mathcal{T}$:[001 $\rightarrow$ 2]. As a word, $x_{\rm F} = 01001010010010100\ldots$, and replacing each 001 by 2 gives $x_{\rm K} = 01201220120\ldots$.

For the transform method $\mathcal{T}$ we can derive a 'general' result similar to Theorem 21. However, since Kimberling applies the StringReplace procedure from Mathematica, which replaces occurrences of w consecutively from left to right, we do not obtain a sequence of return words in the case that w has overlaps in $x_{\rm F}$. This restricts the number of words w to which the following theorem applies considerably.

Theorem 22 *Let $x_{\rm F}$ be the Fibonacci word, and let w be any overlap free[4] word in the Fibonacci language $\mathcal{L}_{\rm F}$. Consider the transform $\mathcal{T}(x_{\rm F})$, which replaces every occurrence of the word w in $x_{\rm F}$ by the letter 2. Let Y be the sequence $(\mathcal{T}(x_{\rm F}))^{-1}(2)$, i.e., the positions of 2's in $\mathcal{T}(x_{\rm F})$. Then Y is a generalized Beatty sequence (i.e., for all $n \geq 1$ $Y(n) = p\lfloor n\varphi \rfloor + qn + r$) with parameters p, q, r, which can be explicitly computed.*

Proof: As in the proof of Theorem 21, let $x_{\rm F} = r_0(w)r_1(w)r_2(w)\ldots$, written as a concatenation of return words of the word w. Now the distances between 2's in $\mathcal{T}(x_{\rm F})$ are equal to $l_1 := |r_1(w)| - |w| + 1$ and $l_2 := |r_2(w)| - |w| + 1$. We can apply the converse of Lemma 4: solving the equations

$$2p + q = l_1, \quad p + q = l_2$$

gives $p = l_1 - l_2,\ q = 2l_2 - l_1$. Inserting $n = 1$, we find that $r = |r_0(w)| - l_2 + 1$. $\square$

Example 23 We take $\mathcal{T}$: [001 $\rightarrow$ 2], with image $\mathcal{T}(x_{\rm F}) = 01201220120\ldots$, so $Y = (3, 6, 7, 10, \ldots)$. Here $r_0(w) = 01, r_1(w) = 00101, r_2(w) = 001$. This gives $l_1 = 5 - 3 + 1 = 3, l_2 = 3 - 3 + 1 = 1$, so $p = 2$ and $q = -1$ and $r = 2 + 1 - 1 = 2$. So Y is the generalized Beatty sequence $(Y_n)_{n\geq 1} = (2\lfloor n\varphi \rfloor - n + 2)_{n \geq 1}$.

The question arises whether not only $\mathcal{T}(x_{\rm F})^{-1}(2)$, but also $\mathcal{T}(x_{\rm F})^{-1}(0)$ and $\mathcal{T}(x_{\rm F})^{-1}(1)$ are generalized Beatty sequences. In general this will not be true. However, this holds for $\mathcal{T}$: [001 $\rightarrow$ 2]. Here it suffices to prove this for $\mathcal{T}(x_{\rm F})^{-1}(1)$, since clearly $\mathcal{T}(x_{\rm F})^{-1}(0) = \mathcal{T}(x_{\rm F})^{-1}(1) - 1$.

Theorem 24 *Let $\mathcal{T}$: [001 $\rightarrow$ 2], and let $Z = (Z(n))_{n\geq 0}$ be the sequence[5] $Z = \mathcal{T}(x_{\rm F})^{-1}(1) = x_{\rm K}^{-1}(1)$. Then, for all $n \geq 0$, one has $Z(n) = \lfloor n\varphi \rfloor + 2n + 2$.*

[4] This means that there are no 'overlapping' occurrences of w in $x_{\rm F}$, as, e.g., for $w = 010$

[5] Z is the sequence $A284624$ with offset 0

Proof: Since $x_{\rm K} = 0\ 1\ 2\ 0\ 1\ 2\ 2\ 0\ 1\ 2\ 0\ \dots$ is the sequence obtained by replacing each word $w = 001$ by 2 in $x_{\rm F}$, we have by Theorem 21 that the positions of 2 in $x_{\rm K}$ are given by $V^{-1}(2) = (2\lfloor n\varphi\rfloor - n + 2)_{n\ge1}$. By Lemma 4, the difference sequence of $V^{-1}(2)$ equals the Fibonacci word on the alphabet $\{3, 1\}$. The return word structure of $w = 001$ is given by

$$r_0(w) = 01, \quad r_1(w) = 00101, \quad r_2(w) = 001.$$

Let $(Z(n))_{n\ge0}$ be the sequence of positions of 1 in the transformed Fibonacci word. Note that $Z(0) = 2$, the 1 coming from $r_0(w)$. This is exactly the reason why it is convenient to start Z from index 0: the other 1's are coming from the $r_1(w)$'s—note that $r_2(w)$ is mapped to 2. Since the distance between occurrences of 2 in $x_{\rm K}$ are given by the Fibonacci word $3\ 1\ 3\ 3\ 1\ 3\ 1\ 3\ 3\ 1\ \dots$, which codes the appearance of the words $r_1(w)$ and $r_2(w)$, we have to map the word $w' = 13$ to 4 to obtain the distances between occurrences of 1 in $x_{\rm K}$, obtaining the word $u = 3\ 4\ 3\ 4\ 4\ 3\ \dots$ To obtain a description of u, we apply Theorem 21 a second time with $w' = 13$. We have $r_0(w') = 3$, $r_1(w') = 133$, $r_2(w') = 13$. Solving $2p + q = l_1 = l_1 := |r_1(w')| - |w'| + 1 = 2, p + q = l_2 = |r_2(w')| - |w'| + 1 = 1$ yields $p = 1, q = 0$. The conclusion is that positions of 4 in u are given by the generalized Beatty sequence $(\lfloor n\varphi\rfloor + 1)_{n\ge1}$. This forces that u is nothing else than the Fibonacci word on $\{4, 3\}$, preceded by 3. But then Z is a generalized Beatty sequence with parameters p and q as solutions of $2p + q = 4, p + q = 3$, which gives $p = 1$, $q = 2$. Since $Z(1) = 5$, we must have $r = 2$, which fits perfectly with the value $Z(0) = 2$. $\Box$

Here is an example where $\mathcal{T}(x_{\rm F})^{-1}(0)$ and $\mathcal{T}(x_{\rm F})^{-1}(1)$ are *not* generalized Beatty sequences.

Example 25 We take $\mathcal{T} : [00100 \to 2]$, with image $\mathcal{T}(x_{\rm F}) = 010010121010010121012\dots$, so $Y = (8, 17, 21\dots)$. Here $r_0(w) = 0100101, r_1(w) = 0010010100101, r_2(w) = 00100101$. This gives $l_1 = 9, l_2 = 4$, so $p = 5$ and $q = -1$ and $r = 4$. So Y is the generalized Beatty sequence $(Y_n)_{n\ge1} = (5\lfloor n\varphi\rfloor - n + 4)_{n\ge1}$. The positions of 0 are given by $(\mathcal{T}(x_{\rm F}))^{-1}(0) = 1, 3, 4, 6, 10, 12, 13, \dots$, with difference sequence $2, 1, 2, 4, 2, 1, \dots$, so by Lemma 4 this sequence is not a generalized Beatty sequence. However, it can be shown that $(\mathcal{T}(x_{\rm F}))^{-1}(0)$ is a union of 4 generalized Beatty sequences, and the same holds for $(\mathcal{T}(x_{\rm F}))^{-1}(1)$.

Example 25 raises the question whether the sequences $\mathcal{T}(x_{\rm F})^{-1}(0)$ and $\mathcal{T}(x_{\rm F})^{-1}(1)$ are always finite unions of generalized Beatty sequences. This can be proved—generalizing the proof of Theorem 24—under the condition that

$$|r_0(w)| \le |r_1(w)| - |w| \qquad \text{(SR0)}.$$

For this generalization one needs the following proposition.

Proposition 26 *Let w be a word from the Fibonacci language, and let $r_0(w)r_1(w)r_2(w)\dots$ be the return sequence of w in the Fibonacci word $x_{\rm F}$. Then (1) $r_0(w)$ is a suffix of $r_1(w)$, and (2) if $r_2(w) = wt_2(w)$, then $t_2(w)$ is a suffix of $r_1(w)$.*

Proof: Let $s_0 = 1, s_1 = 00, s_2 = 101, s_3 = 00100, \dots$ be the singular words introduced in [25]. According to [13, Theorem 1.9.] there is a unique largest singular word s_k occurring in w, so we can write $w = \mu_1 s_k \mu_2$, for two words μ_1, μ_2 from the Fibonacci language. It is known—see [25] and the remarks after [13, Proposition 1.6.]— that the two return words of the singular word s_k are

$$r_1(s_k) = s_k s_{k+1}, \qquad r_2(s_k) = s_k s_{k-1}.$$

According to [13, Lemma 3.1], the two return words of w are given by

$$r_1(w) = \mu_1 r_1(s_k)\mu_1^{-1}, \quad r_2(w) = \mu_1 r_2(s_k)\mu_1^{-1}.$$

Substituting the first equation in the second, we obtain the key equation

$$r_1(w) = \mu_1 s_k s_{k+1}\mu_1^{-1}, \quad r_2(w) = \mu_1 s_k s_{k-1}\mu_1^{-1}. \tag{9}$$

Proof of (1): We compare the return word decompositions of $x_{\rm F}$ by s_k and by w:

$$r_0(s_k)r_1(s_k)r_2(s_k)r_1(s_k)\dots = r_0(w)r_1(w)r_2(w)r_1(w)\dots = r_0(w)\mu_1 r_1(s_k)\mu_1^{-1}\mu_1 r_2(s_k)\mu_1^{-1}\mu_1 r_1(s_k)\mu_1^{-1}\dots$$

It follows that we must have $r_0(s_k) = r_0(w)\mu_1$, and so $r_0(w) = r_0(s_k)\mu_1^{-1}$. By [13, Lemma 2.3], $r_0(s_k)$ equals s_{k+1}, with the first letter deleted. Thus we obtain from Equation (9) that $r_0(w)$ is a suffix of $r_1(w)$.

Proof of (2): Since $s_{k+1} = s_{k-1}s_{k-3}s_{k-1}$, by [25, Property 2], we can do the following computation, starting from Equation (9):

$$r_1(w) = \mu_1 s_k s_{k+1}\mu_1^{-1} = w\mu_2^{-1}s_{k+1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}s_{k-3}s_{k-1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}s_{k-3}\mu_2\mu_2^{-1}s_{k-1}\mu_1^{-1}.$$

For $r_2(w)$ we have

$$r_2(w) = \mu_1 s_k s_{k-1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}\mu_1^{-1}.$$

Now note that in this concatenation μ_2^{-1} cancels against a suffix of w. We claim that it also cancels against a prefix of s_{k-1}. This follows, since by [13, Proposition 2.5] *any* occurrence of s_k in $x_{\rm F}$ is directly followed by a $s_{k+1} = s_{k-1}s_{k-3}s_{k-1}$ with the last letter deleted. It now follows that $t_2(w) = \mu_2^{-1}s_{k-1}\mu_1^{-1}$, and we see that this word is a suffix of $r_1(w)$. □

Here is an example where the (SR0) condition is not satisfied.

Example 27 We take $\mathcal{T} : [10100 \to 2]$, with image $\mathcal{T}(x_{\rm F}) = 01002100221002\ldots$, so $Y = (5, 9, 10\ldots)$. Here $r_0(w) = 0100, r_1(w) = 10100100, r_2(w) = 10100$. The positions of 0 are given by $(\mathcal{T}(x_{\rm F}))^{-1}(0) = 1, 3, 4, 7, 8\ldots$, which can be written as a union of two generalized Beatty sequences, *except* that the 1 from the first 0 will not be in this union.

With Equation (9) we can deduce an equivalent simple formulation of condition (SR0). If $w = \mu_1 s_k \mu_2$, then $r_0(w)$ equals $s_{k+1}\mu_1^{-1}$ with the first letter removed, and $r_1(w) = \mu_1 s_k s_{k+1}\mu_1^{-1}$, so

$$|w| = |\mu_1| + F_k + |\mu_2|, \quad |r_0(w)| = F_{k+1} - |\mu_1| - 1, \quad |r_1(w)| = F_{k+1} + F_k.$$

Filling this into condition (SR0) we obtain

$$|\mu_2| \le 1 \qquad (\text{SR}0').$$

Using (SR0′), together with Theorem 6 in [25], one can show that the generalization of Theorem 24 does apply to at most 3 words w of length m, for all $m \ge 2$ (in fact, only 2, if m is not a Fibonacci number).

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