New Lower Bounds for the Number of Pseudoline Arrangements

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Abstract

Arrangements of lines and pseudolines are fundamental objects in discrete and computational geometry. They also appear in other areas of computer science, such as the study of sorting networks. Let B_n be the number of arrangements of n pseudolines and let $b_n = \log B_n$. The problem of estimating B_n was posed by Knuth in 1992. Knuth conjectured that $b_n \leq \binom{n}{2} + o(n^2)$ and also derived the first upper and lower bounds: $b_n \leq 0.7924(n^2 + n)$ and $b_n \geq n^2/6 - O(n)$. The upper bound underwent several improvements, $b_n \leq 0.6988n^2$ (Felsner, 1997), and $b_n \leq 0.6571n^2$ (Felsner and Valtr, 2011), for large n. Here we show that $b_n \geq cn^2 - O(n \log n)$ for some constant $c > 0.2083$. In particular, $b_n \geq 0.2083\, n^2$ for large n. This improves the previous best lower bound, $b_n \geq 0.1887n^2$, due to Felsner and Valtr (2011). Our arguments are elementary and geometric in nature. Further, our constructions are likely to spur new developments and improved lower bounds for related problems, such as in topological graph drawings.

Keywords: counting, pseudoline arrangement, recursive construction.

1 Introduction

Arrangement of Pseudolines. A *pseudoline* in the Euclidean plane is a curve extending from negative infinity to positive infinity. An *arrangement of pseudolines* is a family of pseudolines where each pair of pseudolines has a unique point of intersection (called '*vertex*'). An arrangement is *simple* if no three pseudolines have a common point of intersection, see Fig. 1 (left). Here the term arrangement always means simple arrangement if not specified otherwise. Two arrangements are *isomorphic* if they can be mapped onto each other by a homeomorphism of the plane; see Fig. 1.

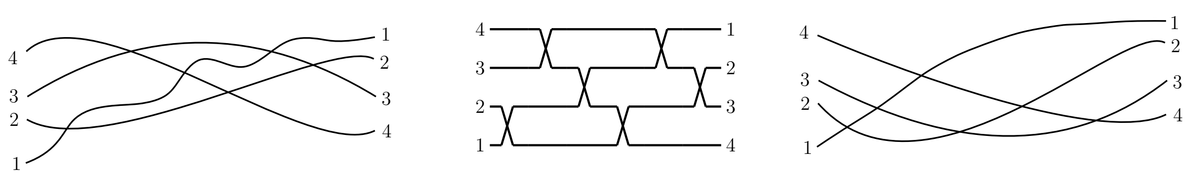

Figure 1: Left: A simple arrangement $\mathcal{A}$. Center: Wiring diagram of $\mathcal{A}$. Right: An arrangement $\mathcal{A}'$ that is not isomorphic to the arrangement on the left.

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There are several representations and encodings of pseudoline arrangements. These representations help one count the number of arrangements. Three classic representations are: *allowable sequences* (introduced by Goodman and Pollack, see, e.g., [9, 10]), *wiring diagrams* (see for instance [7]), and *zonotopal tilings* (see for instance [6]). A *wiring diagram* is an Euclidean arrangement of pseudolines consisting of piece-wise linear 'wires', each horizontal except for a short segment where it crosses another wire. Each pair of wires cross only once. The wiring diagram in Fig. 1 (center) represents the arrangement $\mathcal{A}$. The above representations have been shown to be equivalent; bijective proofs to this effect can be found in [6]. Wiring diagrams are also known as *reflection networks*,i.e., networks the bring n wires labeled from 1 to n into their reflection by means of performing switches of adjacent wires; see [13, p. 35]. Lastly, they are also known under the name of *primitive sorting networks*; see [14, Ch. 5.3.4]. The number of such networks is denoted by A_n (a closed formula is given later in this section).

Let the number of *nonisomorphic* arrangements of n pseudolines be denoted by B_n. We are interested in the growth rate of B_n; so let $b_n = \log_2 B_n$. Knuth [13] conjectured that $b_n \leq \binom{n}{2} + o(n^2)$; see also [7, p. 147] and [5, p. 259]. This conjecture is still open.

Upper bounds on the number of pseudoline arrangements. Felsner [5] used a horizontal encoding of an arrangement in order to estimate B_n. An arrangement can be represented by a sequence of horizontal cuts. The ith cut is the list of the pseudolines crossing the ith pseudoline in the order of the crossings. Using this approach, Felsner [5, Thm. 1] obtained the upper bound $b_n \leq 0.7213(n^2 - n)$. The author [5, Thm. 2] refined this bound by using *replace matrices*. A replace matrix M is a $n \times n$ binary matrix with the properties $\sum_{j=1}^{n} m_{ij} = n - i$ for all i and $m_{ij} \geq m_{ji}$ for all $i < j$. Using this technique, he established the upper bound $b_n < 0.6974\, n^2$.

In his seminal paper on the topic, Knuth [13] took a vertical approach for encoding. Let $\mathcal{A}$ be an arrangement of n pseudolines. Adding a $(n+1)$th pseudoline to $\mathcal{A}$, we get $\mathcal{A}'$, an arrangement of $(n+1)$ pseudolines. The course of the $(n+1)$th pseudoline describes a vertical cutpath from top to bottom. The number of arrangements of $\mathcal{A}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{A}' \setminus \{n+1\}$ is exactly the number of cutpaths in $\mathcal{A}$. Let γ_n denote the number of cutpaths in an arrangement of n pseudolines. Therefore, one has $B_{n+1} \leq \gamma_n \cdot B_n$; and $B_3 = 2$. Knuth [13] proved that $\gamma_n \leq 3^n$, concluding that $B_n \leq 3^{\binom{n+1}{2}}$ and thus $b_n \leq 0.5\,(n^2+n)\log_2 3 \leq 0.7924\,(n^2+n)$; this computation can be streamlined so that it yields $b_n \leq 0.7924\, n^2$, see [8]. Knuth also conjectured that $\gamma_n \leq n \times 2^n$, but this was refuted by Ondřej Bílka in 2010 [8]; see also [7, p. 147]. Felsner and Valtr [8] proved a refined result, $\gamma_n \leq 4n \times 2.48698^n$, by a careful analysis. This yields $b_n \leq 0.6571\, n^2$, which is the current best upper bound.

Lower bounds on the number of pseudoline arrangements. Knuth [13, p. 37] gave a recursive construction in the setting of reflection networks. The number of nonisomorphic arrangements in his construction, thus also B_n, obeys the recurrence

$$B_n \geq 2^{n^2/4 - n/4} B_{n/2}.$$

By induction this yields $B_n \geq 2^{n^2/6 - 5n/2}$.

Matoušek sketched a simple—still recursive—grid construction in his book [17, Sec. 6.2], see Fig 2. Let n be a multiple of 3 and $m = \frac{n}{3}$ (assume that m is odd). The $2m$ lines in the two extreme bundles form a regular grid of m^2 points. The lines in the central bundle are incident to $\frac{3m^2+1}{4}$ of these grid points. At each such point, there are 2 choices; going below it or above it, thus creating at least $\frac{3m^2}{4} = \frac{3(n/3)^2}{4} = \frac{n^2}{12}$ binary choices. Thus B_n obeys the recurrence

$$B_n \geq 2^{n^2/12} B_{n/3}^3,$$

which by induction yields $B_n \geq 2^{n^2/8}$.

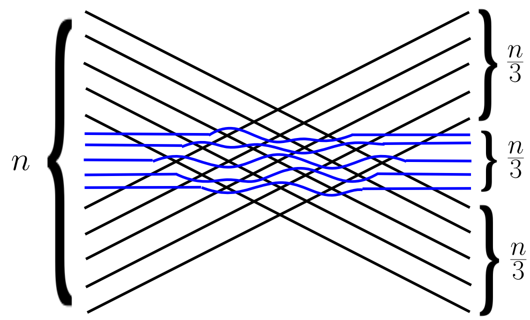


Figure 2: Grid construction for a lower bound on B_n.

Felsner and Valtr [8] used rhombic tilings of a centrally symmetric hexagon in an elegant recursive construction for a lower bound on B_n. Consider a set of $i+j+k$ pseudolines partitioned into the following three parts: $\{1,\ldots,i\}$, $\{i+1,\ldots,i+j\}$, $\{i+j+1,\ldots,i+j+k\}$, see Fig. 3[1]. A partial arrangement is called *consistent* if any two pseudolines from two different parts always cross but any two pseudolines from the same part never cross.

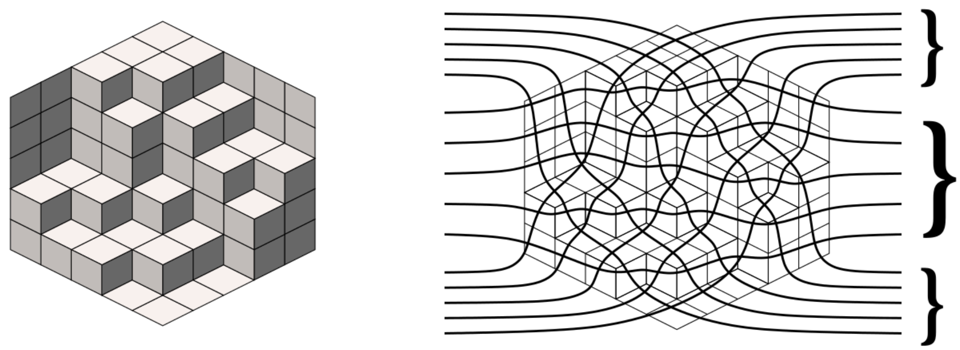


Figure 3: The hexagon $H(5,5,5)$ with one of its rhombic tilings and a consistent partial arrangement corresponding to the tiling.

The zonotopal duals of consistent partial arrangements are rhombic tilings of the centrally symmetric hexagon $H(i,j,k)$ with side lengths i,j,k. The enumeration of rhombic tilings of $H(i,j,k)$ was solved by MacMahon [16] (see also [4]), who showed that the number of tilings is

$$P(i,j,k) = \prod_{a=0}^{i-1}\prod_{b=0}^{j-1}\prod_{c=0}^{k-1} \frac{a+b+c+2}{a+b+c+1}. \tag{1}$$

A nontrivial (and quite involved) derivation using integral calculus shows that

$$\ln P(n,n,n) = \left(\frac{9}{2}\ln 3 - 6\ln 2\right) n^2 + O(n\log n) = 1.1323\ldots n^2 + O(n\log n).$$

[1]Figure reproduced from [8].

Assuming n to be a multiple of 3 in the recursion step, the construction yields the lower bound recurrence

$$B_n \geq P\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right) B_{n/3}^3. \tag{2}$$

By induction, the analytic solution to formula (1) together with the recurrence (2) yield the lower bound $B_n \geq 2^{cn^2 - O(n \log^2 n)}$, where $c = \frac{3}{4} \ln 3 - \ln 2 = 0.1887\ldots$ In particular, $b_n \geq 0.1887n^2$ for large n; this is the previous best lower bound.

Let A_n denote the total number of wiring diagrams with n wires (also known as *reflection networks*), and then B_n is also the corresponding number of equivalence classes (see [13, p. 35]). Stanley [19] established the following closed formula for A_n:

$$A_n = \frac{\binom{n}{2}!}{\prod_{k=1}^{n-1}(2n - 2k - 1)^k}$$

Table 1 shows the exact values of A_n and B_n, and their growth rate (up to four digits after the decimal point) with respect to n, for small values of n. The values of B_n for $n = 1$ to 9 are from [13, p. 35] and the values of B_{10}, B_{11}, and B_{12} are from [5], [20] and [18], respectively; the values of B_{13}, B_{14}, and B_{15} have been added recently, see [12, 18]. Observe that A_n grows much faster than B_n.

n	A_n	$\frac{\log_2 A_n}{n^2}$	B_n	$\frac{\log_2 B_n}{n^2}$
1	1	0	1	0
2	1	0	1	0
3	2	0.1111	2	0.1111
4	16	0.25	8	0.1875
5	768	0.3833	62	0.2381
6	292,864	0.5044	908	0.2729
7	1,100,742,656	0.6129	24,698	0.2977
8	48,608,795,688,960	0.7104	1,232,944	0.3161
9	29,258,366,996,258,488,320	0.7983	112,018,190	0.3301
10			18,410,581,880	0.3409
11			5,449,192,389,984	0.3496
12			2,894,710,651,370,536	0.3566
13			2,752,596,959,306,389,652	0.3624

Table 1: Values of A_n and B_n for small n.

Our results. Here we extend the method used by Matoušek in his grid construction; observe that it uses lines of 3 slopes. In Sections 2 (the 2nd part) and 3, we use lines of 6 and 12 different slopes in hexagonal type constructions; yielding lower bounds $b_n \geq 0.1981n^2$ and $b_n \geq 0.2083n^2$ for large n, respectively. In Sections A and B, we use lines of 8 and 12 different slopes in rectangular type constructions; yielding the lower bounds $b_n \geq 0.2n^2$ and $b_n \geq 0.2053n^2$ for large n, respectively. While the construction in Section 3 gives a better bound, the one in Section 2 is easier to analyze. For each of the two styles, rectangular and hexagonal, the constructions are presented in increasing order of complexity. Our main result is summarized in the following.

Theorem 1. *Let B_n be the number of arrangements of n pseudolines. Then $B_n \geq 2^{cn^2 - O(n \log n)}$, for some constant $c > 0.2083$. In particular, $B_n \geq 2^{0.2083n^2}$ for large n.*

Outline of the proof. We construct a line arrangement using lines of k different slopes (for a small k). Let $m = \lfloor n/k \rfloor$ or $m = \lfloor n/k \rfloor - 1$ (whichever is odd). Each bundle consists of m equidistant lines in the corresponding parallel strip; remaining lines are discarded, or not used in the counting. A crossing point is an i-wise crossing if it is incident to exactly i lines. Let $\lambda_i(m)$ denote the number i-wise crossings where each bundle consists of m lines. Our goal is to estimate $\lambda_i(m)$ for each i. Then we can locally replace the lines around each i-wise crossing with any of the B_i nonisomorphic pseudoline arrangements; and further apply recursively this construction to each of the k bundles of parallel lines exiting this junction. This yields a simple pseudoline arrangement for each possible replacement choice. Consequently, the number of nonisomorphic pseudoline arrangements in this construction, say, $T(n)$, satisfies the recurrence:

$$T(n) \geq F(n) \left[T\left(\frac{n}{k} \right) \right]^k, \tag{3}$$

where $F(n)$ is a multiplicative factor counting the number of choices in this junction.

Related work. In a comprehensive recent paper, Kynčl [15] obtained estimates on the number of isomorphism classes $T(G)$ of simple topological graphs that realize various graphs G. The author remarks that it is probably hard to obtain tight estimates on this quantity, "given that even for pseudoline arrangements, the best known lower and upper bounds on their number differ significantly". While our improvements aren't spectacular, it seems however likely that some of the techniques we used here can be employed to obtain sharper lower bounds for topological graph drawings too.

Notations and formulas used. For two similar figures F, F', let $\rho(F, F')$ denote their similarity ratio. For a planar region R, let $\text{area}(R)$ denote its area. By slightly abusing notation, let $\text{area}(i, j, k)$ denote the area of the triangle made by three lines ℓ_i, ℓ_j and ℓ_k. Assume that the equations of the three lines are $\alpha_s x + \beta_s y + \gamma_s = 0$, for $s = 1, 2, 3$, respectively. Then

$$\begin{aligned}
\text{area}(i, j, k) &= \frac{A^2}{2|C_1 C_2 C_3|}, \text{ where} \\
A &= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}, \\
C_1 &= (\alpha_2 \beta_3 - \beta_2 \alpha_3), \\
C_2 &= -(\alpha_1 \beta_3 - \beta_1 \alpha_3), \\
C_3 &= (\alpha_1 \beta_2 - \beta_1 \alpha_2).
\end{aligned}$$

Let $P(i, j, g, h)$ denote the parallelogram made by the pairs of parallel lines $\ell_i \parallel \ell_j$ and $\ell_g \parallel \ell_h$. Throughout this paper, $\log x$ is the logarithm in base 2 of x.

2 Preliminary constructions

Warm-up: a rectangular construction with 4 slopes. We start with a simple rectangular construction with 4 bundles of parallel lines whose slopes are $0, \infty, \pm 1$; see Fig. 4. Let $U = [0, 1]^2$ be the unit square we work with. The axes of all parallel strips are all incident to the center of U. The final construction is obtained by a small clockwise rotation, so that there are no vertical lines.

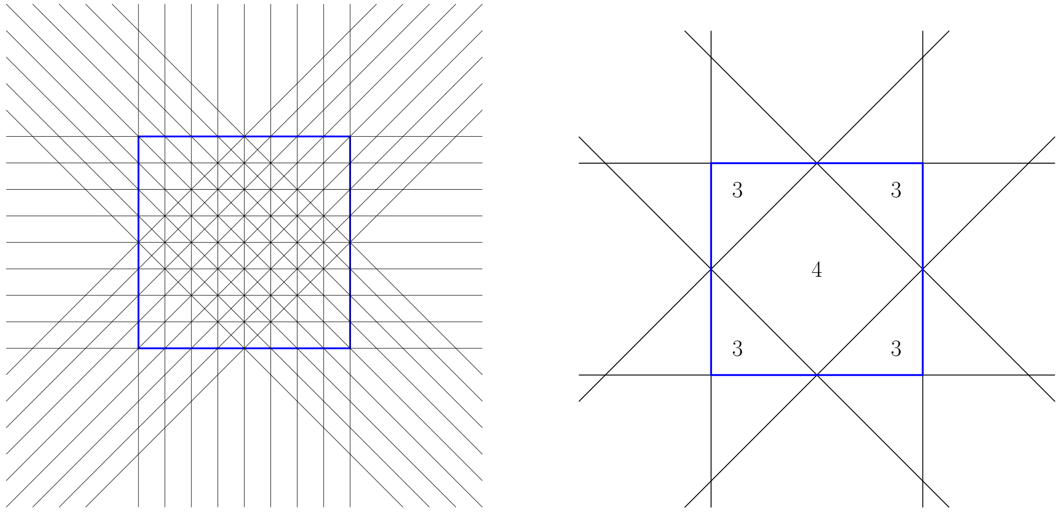


Figure 4: Construction with 4 slopes; here $m = 9$. The unit square $U = [0, 1]^2$ is shown in blue.

Let a_i, for $i = 3, 4$, denote the area of the region covered by exactly i of the 4 strips. It is easy to see that $a_3 = a_4 = 1/2$, and obviously $a_3 + a_4 = \text{area}(U) = 1$. Observe that $\lambda_i(m)$ is proportional to a_i, for $i = 3, 4$; taking the boundary effect into account, we have

$$\lambda_3(m) = a_3\, m^2 - O(m) = \frac{m^2}{2} - O(m), \text{ and } \quad \lambda_4(m) = a_4\, m^2 - O(m) = \frac{m^2}{2} - O(m).$$

Now we can derive the multiplicative factor in Equation (3) as follows:

$$F(n) \geq \prod_{i=3}^{4} B_i^{\lambda_i(n)} \geq 2^{m^2/2 - O(m)} \cdot 8^{m^2/2 - O(m)} = 2^{2m^2 - O(m)} = 2^{n^2/8 - O(n)}.$$

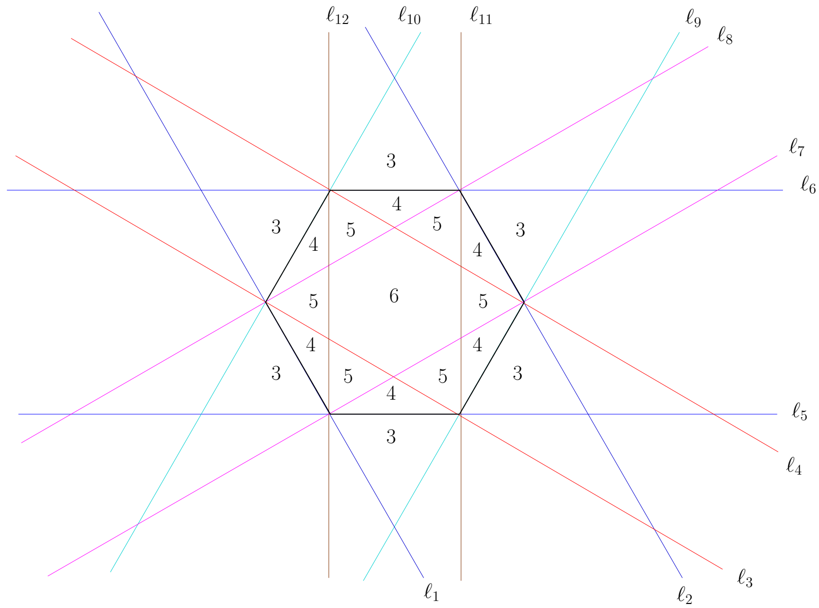
By induction on n, the resulting lower bound is $T(n) \geq 2^{n^2/6 - O(n \log n)}$; thereby this matches the constant, 1/6, in Knuth's lower bound described in Section 1.

Hexagonal construction with 6 **slopes.** This construction yields the lower bound $b_n \geq 0.1981 n^2$ for large n. Let H be a regular hexagon of unit side. Consider 6 bundles of parallel lines whose slopes are $0, \infty, \pm 1/\sqrt{3}, \pm\sqrt{3}$; see Fig. 5 (left). Three parallel strips are bounded by the pairs of lines supporting opposite sides of H, while the other three parallel strips are bounded by the pairs of lines supporting opposite short diagonals of H. The axes of all six parallel strips are all incident to the center of the circle.

Assume a coordinate system where the lower left corner of H is at the origin, and the lower side of H lies along the x-axis. Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_6$ be the partition of $\mathcal{L}$ into six bundles of parallel lines. The m lines in $\mathcal{L}_i$ are contained in the parallel strip bounded by the two lines ℓ_{2i-1} and ℓ_{2i}, for $i = 1, \ldots, 6$. The equation of line ℓ_i is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, $i = 1, \ldots, 12$, given in Fig 5 (right).

We refer to lines in $\mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5$ as the *primary* lines, and to lines in $\mathcal{L}_2 \cup \mathcal{L}_4 \cup \mathcal{L}_6$ as *secondary* lines. The final construction is obtained by a small clockwise rotation, so that there are no vertical lines. Observe that

- the distance between consecutive lines in any of the bundles of primary lines is $\frac{\sqrt{3}}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in any of the bundles of secondary lines is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$.



i	α_i	β_i	γ_i
1	$\sqrt{3}$	1	0
2	$\sqrt{3}$	1	$-2\sqrt{3}$
3	1	$\sqrt{3}$	-1
4	1	$\sqrt{3}$	-3
5	0	1	0
6	0	1	$-\sqrt{3}$
7	1	$-\sqrt{3}$	0
8	1	$-\sqrt{3}$	2
9	$\sqrt{3}$	-1	$-\sqrt{3}$
10	$\sqrt{3}$	-1	$\sqrt{3}$
11	1	0	-1
12	1	0	0

Figure 5: Left: The six parallel strips and corresponding covering multiplicities. These numbers only show incidences at the 3-wise crossings made by primary lines. Right: Coefficients of the lines $\ell_i \ : \ i = 1, \ldots, 12$.

Let $\sigma_0 = \sigma_0(m)$ and $\delta_0 = \delta_0(m)$ denote the basic parallelogram (resp., triangle) determined by the lines in $\mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5$; the side length of δ_0 is $\frac{2}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$. Let H' be the smaller regular hexagon made by the short diagonals of H; the similarity ratio $\rho(H', H)$ is equal to $\frac{1}{\sqrt{3}}$. Recall that (i) the area of an equilateral triangle of side s is $\frac{s^2\sqrt{3}}{4}$; and (ii) the area of a regular hexagon of side s is $\frac{s^2 3\sqrt{3}}{2}$; as such, we have

$$\begin{aligned}
\text{area}(H) &= \frac{3\sqrt{3}}{2}, \\
\text{area}(H') &= \frac{\text{area}(H)}{3} = \frac{\sqrt{3}}{2}, \\
\text{area}(\delta_0) &= \frac{4}{m^2}\frac{\sqrt{3}}{4}\left(1 - O\left(\frac{1}{m}\right)\right) = \frac{\sqrt{3}}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right), \\
\text{area}(\sigma_0) &= 2 \cdot \text{area}(\delta_0) = \frac{2\sqrt{3}}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right).
\end{aligned}$$

Let a_i, for $i = 3, 4, 5, 6$, denote the area of the region covered by exactly i of the 6 strips. The following observations are in order: (i) the six isosceles triangles based on the sides of H inside H have unit base and height $\frac{1}{2\sqrt{3}}$; (ii) the six smaller equilateral triangles incident to the vertices of H have side-length $\frac{1}{\sqrt{3}}$. These observations imply

$$\begin{aligned}
a_3 &= \text{area}(H) = \frac{3\sqrt{3}}{2}, \\
a_4 &= 6 \cdot \text{area}(3, 5, 7) = 6 \cdot \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{2}, \\
a_5 &= 6 \cdot \text{area}(3, 7, 11) = 6 \cdot \frac{1}{3}\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2},
\end{aligned}$$

$$a_6 = \text{area}(H') = \frac{\sqrt{3}}{2}.$$

Observe that $a_4 + a_5 + a_6 = \text{area}(H)$. Recall that $\lambda_i(m)$ denote the number i-wise crossings where each bundle consists of m lines. Observe that $\lambda_i(m)$ is proportional to a_i, for $i = 4, 5, 6$. Indeed, $\lambda_i(m)$ is equal to the number of 3-wise crossings of the primary lines that lie in a region covered by i parallel strips, which is roughly equal to the ratio $\frac{a_i}{\sigma_i}$, for $i = 4, 5, 6$. More precisely, taking also the boundary effect of the relevant regions into account, we obtain

$$\begin{aligned}
\lambda_4(m) &= \frac{a_4}{\text{area}(\sigma_0)} - O(m) = \frac{\sqrt{3}}{2}\frac{m^2}{2\sqrt{3}} - O(m) = \frac{m^2}{4} - O(m),\\
\lambda_5(m) &= \frac{a_5}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{4} - O(m),\\
\lambda_6(m) &= \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{4} - O(m).
\end{aligned}$$

For estimating $\lambda_3(m)$, the situation is little bit different, namely, in addition to considering 3-wise crossings of the primary lines, we also observe 3-wise crossings of the secondary lines at the centers of the small equilateral triangles contained in H'. See Fig. 6. It follows that

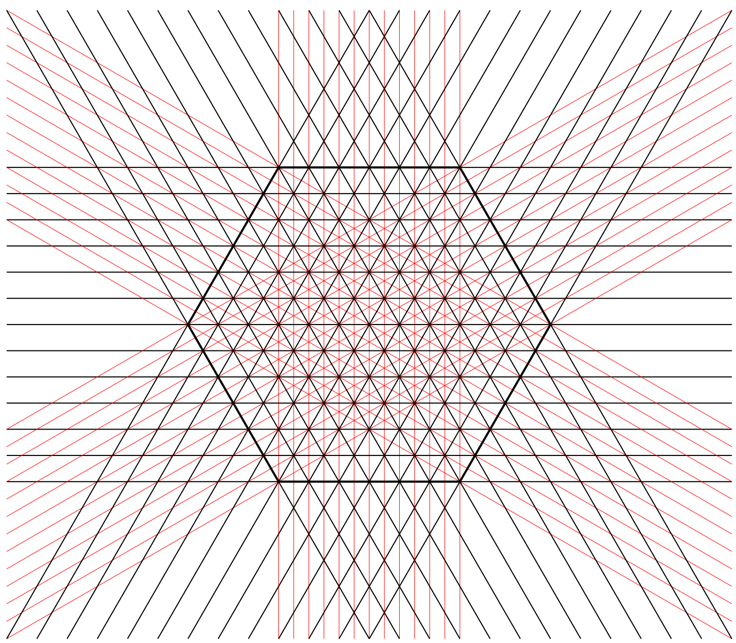


Figure 6: Triple incidences of secondary lines (drawn in red).

$$\lambda_3(m) = \frac{a_3}{\text{area}(\sigma_0)} + \frac{\text{area}(H')}{\text{area}(\delta_0)} - O(m) = \frac{3m^2}{4} + \frac{m^2}{2} - O(m) = \frac{5m^2}{4} - O(m).$$

The values of $\lambda_i(m)$, for $i = 3, \ldots, 6$, are summarized in Table 2; for convenience the linear terms are omitted. Since $m = n/6$, λ_i can be also viewed as a function of n.

i	3	4	5	6
$\lambda_i(m)$	$\frac{5m^2}{4}$	$\frac{m^2}{4}$	$\frac{m^2}{4}$	$\frac{m^2}{4}$
$\lambda_i(n)$	$\frac{5n^2}{4\cdot 36}$	$\frac{n^2}{4\cdot 36}$	$\frac{n^2}{4\cdot 36}$	$\frac{n^2}{4\cdot 36}$

Table 2: The values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, 4, 5, 6$.

Now we can derive the multiplicative factor in Equation (3) as follows:

$$F(n) \geq \prod_{i=3}^{6} B_i^{\lambda_i(n)} \geq 2^{5n^2/144} \cdot 8^{n^2/144} \cdot 62^{n^2/144} \cdot 908^{n^2/144} \cdot 2^{-O(n)}.$$

We prove by induction on n that $T(n) \geq 2^{cn^2 - O(n \log n)}$ for a suitable constant $c > 0$. It suffices to choose c (using the values of B_i for $i = 3, \ldots, 8$ in Table 1) so that

$$\frac{8 + \log 62 + \log 908}{144} \geq \frac{5c}{6}.$$

The above inequality holds if we set $c = \dfrac{\log(256 \cdot 62 \cdot 908)}{120} > 0.1981$, and the lower bound follows.

3 Hexagonal construction with 12 slopes

This construction yields the lower bound $b_n \geq 0.2083\, n^2$ for large n, which is our main result in Theorem 1. Let H be a regular hexagon of unit side. Consider 12 bundles of parallel lines whose slopes are $0, \infty$, $\pm\sqrt{3}/5$, $\pm 1/\sqrt{3}$, $\pm\sqrt{3}/2$, $\pm\sqrt{3}$, $\pm 3\sqrt{3}$. The axes of all 12 parallel strips are all incident to the center of the circle created by the vertices of H; see Figs. 7 and 8.

Assume a coordinate system where the lower left corner of H is at the origin, and the lower side of H lies along the x-axis. Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{12}$ be the partition of $\mathcal{L}$ into 12 bundles of parallel lines. The m lines in $\mathcal{L}_i$ are contained in the parallel strip Γ_i bounded by the two lines ℓ_{2i-1} and ℓ_{2i}, for $i = 1, \ldots, 24$. The equation of line ℓ_i is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, $i = 1, \ldots, 24$, given in Table 3.

i	α_i	β_i	γ_i
1	$3\sqrt{3}$	1	$-\sqrt{3}$
2	$3\sqrt{3}$	1	$-3\sqrt{3}$
3	$\sqrt{3}$	1	0
4	$\sqrt{3}$	1	$-2\sqrt{3}$
5	$\sqrt{3}$	2	$-\sqrt{3}$
6	$\sqrt{3}$	2	$-2\sqrt{3}$
7	1	$\sqrt{3}$	-1
8	1	$\sqrt{3}$	-3

i	α_i	β_i	γ_i
9	$\sqrt{3}$	5	$-2\sqrt{3}$
10	$\sqrt{3}$	5	$-4\sqrt{3}$
11	0	1	0
12	0	1	$-\sqrt{3}$
13	$\sqrt{3}$	-5	$\sqrt{3}$
14	$\sqrt{3}$	-5	$3\sqrt{3}$
15	1	$-\sqrt{3}$	0
16	1	$-\sqrt{3}$	2

i	α_i	β_i	γ_i
17	$\sqrt{3}$	-2	0
18	$\sqrt{3}$	-2	$\sqrt{3}$
19	$\sqrt{3}$	-1	$-\sqrt{3}$
20	$\sqrt{3}$	-1	$\sqrt{3}$
21	$3\sqrt{3}$	-1	$-2\sqrt{3}$
22	$3\sqrt{3}$	-1	0
23	1	0	-1
24	1	0	0

Table 3: Coefficients of the 24 lines.

Γ_2, Γ_6 and Γ_{10} are bounded by the pairs of lines supporting opposite sides of H, while Γ_4, Γ_8 and Γ_{12} are bounded by the pairs of lines supporting opposite short diagonals of H. Therefore $H = \Gamma_2 \cap \Gamma_6 \cap \Gamma_{10}$.

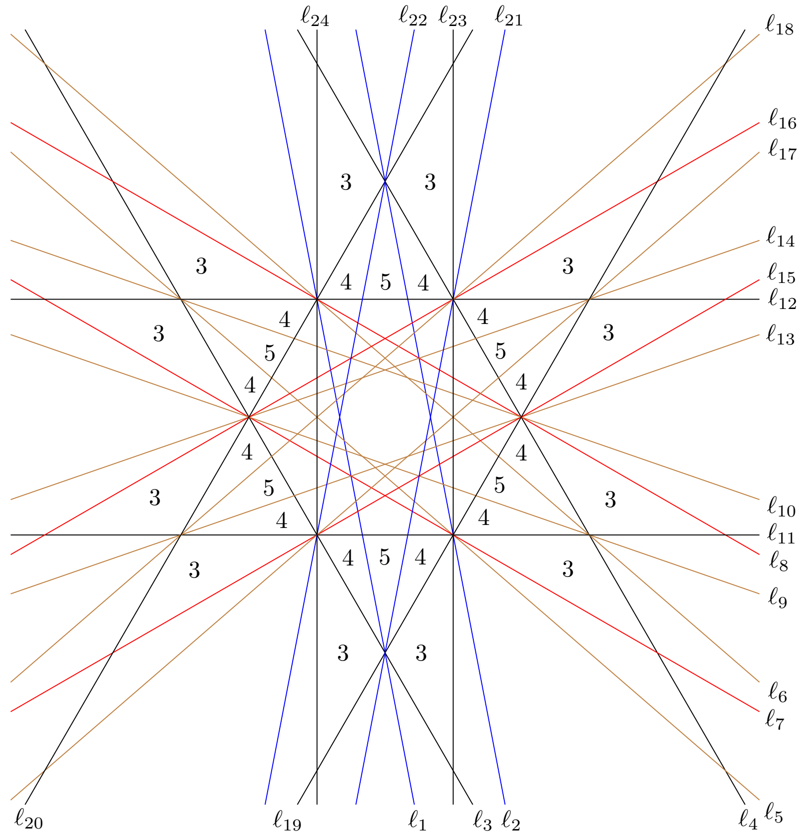


Figure 7: Construction with 12 slopes. The twelve parallel strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by the primary lines. The numbers inside H are shown in Fig. 8

We refer to lines in $\mathcal{L}_2 \cup \mathcal{L}_6 \cup \mathcal{L}_{10}$ as the *primary* lines, to lines in $\mathcal{L}_4 \cup \mathcal{L}_8 \cup \mathcal{L}_{12}$ as the *secondary* lines, and to the rest of the lines in $\mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5 \cup \mathcal{L}_7 \cup \mathcal{L}_9 \cup \mathcal{L}_{11}$ as the *tertiary* lines. The final construction is obtained by a small clockwise rotation, so that there are no vertical lines. Observe that the distance between consecutive lines in any of the bundles of

- primary lines is $\frac{\sqrt{3}}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- secondary lines is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- tertiary lines is $\sqrt{\frac{3}{7}}\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$.

We refer to the intersection points of the primary lines as *grid vertices*. There are two types of grid vertices: the grid vertices in H are intersection of 3 primary lines and the ones outside H are intersection of 2 primary lines.

Let $\sigma_0 = \sigma_0(m)$ and $\delta_0 = \delta_0(m)$ denote the basic parallelogram (resp., triangle) determined by the primary lines (i.e., lines in $\mathcal{L}_2 \cup \mathcal{L}_6 \cup \mathcal{L}_{10}$); the side length of δ_0 is $\frac{2}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$. We refer to these basic parallelograms as *grid cell*. Recall that (i) the area of an equilateral triangle of side s is $\frac{s^2\sqrt{3}}{4}$; and (ii) the area of a regular hexagon of side s is $\frac{s^2 3\sqrt{3}}{2}$; as such, we have

$$\mathrm{area}(H) = \frac{3\sqrt{3}}{2},$$

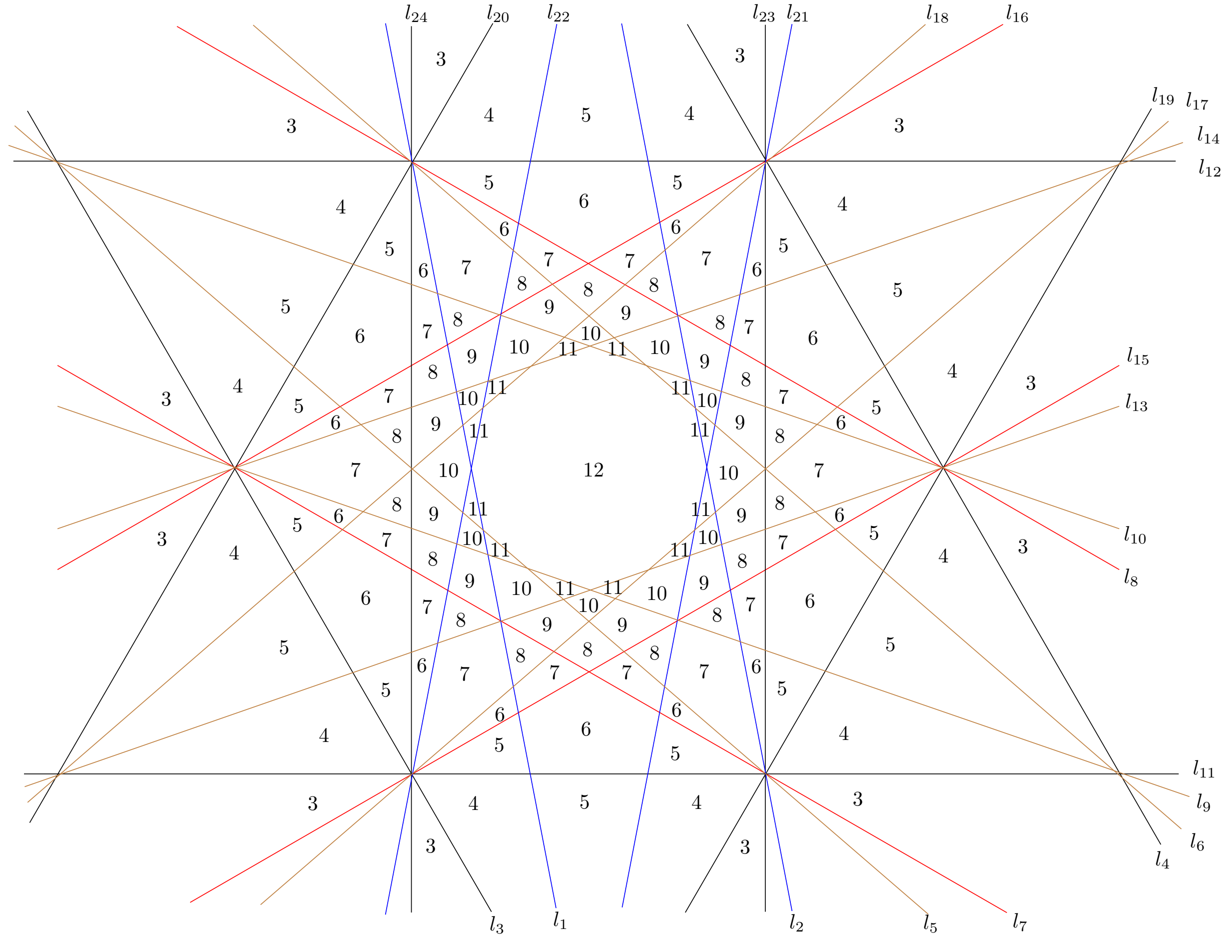


Figure 8: Construction with 12 slopes. The twelve parallel strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by the primary lines.

$$\operatorname{area}(\delta_0) = \frac{4}{m^2}\frac{\sqrt{3}}{4}\left(1 - O\left(\frac{1}{m}\right)\right) = \frac{\sqrt{3}}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right),$$
$$\operatorname{area}(\sigma_0) = 2 \cdot \operatorname{area}(\delta_0) = \frac{2\sqrt{3}}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right).$$

Let a_i, for $i = 3, \ldots, 12$, denote the area of the region covered by exactly i of the 6 strips. Recall that $\operatorname{area}(i, j, k)$ denotes the area of the triangle made by ℓ_i, ℓ_j and ℓ_k.

Observe that a_{12} is the area of the 12-gon $\bigcap_{i=1}^{12} \Gamma_i$. This 12-gon is not regular, since consecutive vertices lie on two concentric cycles of radii $\frac{1}{3}$ and $\frac{\sqrt{3}}{5}$ centered at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. So a_{12} is the sum of the areas of 12 congruent triangles; each with one vertex at the center of H and other two as the two consecutive vertices of the 12-gon. Each of these triangles have area $\frac{\sqrt{3}}{60}$. Therefore,

$$a_{12} = 12 \cdot \frac{\sqrt{3}}{60} = \frac{\sqrt{3}}{5},$$
$$a_{11} = 12 \cdot \operatorname{area}(1, 5, 9) = 12 \cdot \frac{1}{140\sqrt{3}} = \frac{\sqrt{3}}{35},$$
$$a_{10} = 6 \cdot (\operatorname{area}(1, 5, 13) - \operatorname{area}(1, 5, 9)) + 6 \cdot (\operatorname{area}(5, 9, 22) - \operatorname{area}(1, 5, 9))$$

$$
\begin{aligned}
&= 6 \cdot \left(\frac{\sqrt{3}}{70} - \frac{1}{140\sqrt{3}} \right) + 6 \cdot \left(\frac{1}{56\sqrt{3}} - \frac{1}{140\sqrt{3}} \right) = \frac{13\sqrt{3}}{140}, \\
a_9 &= 12 \cdot (\text{area}(1,7,22) - \text{area}(1,9,22)) = 12 \cdot \left(\frac{1}{20\sqrt{3}} - \frac{1}{56\sqrt{3}} \right) = \frac{9\sqrt{3}}{70}, \\
a_8 &= 6 \cdot (\text{area}(9,22,24) - \text{area}(7,22,24)) + 12 \cdot \text{area}(7,13,22) \\
&= 6 \cdot \left(\frac{\sqrt{3}}{40} - \frac{1}{20\sqrt{3}} \right) + 12 \cdot \frac{\sqrt{3}}{140} = \frac{19\sqrt{3}}{140}, \\
a_7 &= 12 \cdot (\text{area}(7,22,24) - \text{area}(13,22,24)) + 6 \cdot (\text{area}(1,17,22) - \text{area}(1,13,22)) \\
&= 12 \cdot \left(\frac{1}{20\sqrt{3}} - \frac{\sqrt{3}}{140} \right) + 6 \cdot \left(\frac{5}{28\sqrt{3}} - \frac{1}{14\sqrt{3}} \right) = \frac{23\sqrt{3}}{70}, \\
a_6 &= 12 \cdot (\text{area}(13,22,24)) + 6 \cdot (\text{area}(7,11,15) - 2 \cdot \text{area}(1,11,15)) \\
&= 12 \cdot \frac{\sqrt{3}}{140} + 6 \cdot \left(\frac{1}{4\sqrt{3}} - 2 \cdot \frac{1}{20\sqrt{3}} \right) = \frac{27\sqrt{3}}{70}, \\
a_5 &= 12 \cdot (\text{area}(1,11,15)) + 6 \cdot (\text{area}(1,11,21)) = 12 \cdot \frac{1}{20\sqrt{3}} + 6 \cdot \frac{1}{4\sqrt{3}} = \frac{7\sqrt{3}}{10}, \\
a_4 &= 12 \cdot (\text{area}(1,3,11)) = 12 \cdot \frac{1}{4\sqrt{3}} = \sqrt{3}, \\
a_3 &= 12 \cdot (\text{area}(4,7,11)) = 12 \cdot \frac{\sqrt{3}}{4} = 3\sqrt{3}.
\end{aligned}
$$

Observe that the region whose area is $\sum_{i=5}^{12} a_i$ consists of the hexagon H and 6 triangles outside H. Therefore,

$$
\sum_{i=5}^{12} a_i = \text{area}(H) + 6 \cdot \text{area}(1,11,21) = \frac{3\sqrt{3}}{2} + 6 \cdot \frac{1}{4\sqrt{3}} = 2\sqrt{3}.
$$

Recall that $\lambda_i(m)$ denotes the number i-wise crossings where each bundle consists of m lines. Observe that $\lambda_i(m)$ is proportional to a_i, for $i = 5, \ldots, 12$; indeed, $\lambda_i(m)$ is equal to the number of grid vertices that lie in a region covered by i parallel strips, which is roughly equal to the ratio $\frac{a_i}{\text{area}(\sigma_0)}$, for $i = 5, 6, \ldots, 12$. More precisely, taking also the boundary effect of the relevant regions into account, we obtain

$$
\begin{aligned}
\lambda_{12}(m) &= \frac{a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{\sqrt{3}}{5} \frac{m^2}{2\sqrt{3}} - O(m) = \frac{m^2}{10} - O(m), \\
\lambda_{11}(m) &= \frac{a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{70} - O(m), \\
\lambda_{10}(m) &= \frac{a_{10}}{\text{area}(\sigma_0)} - O(m) = \frac{13m^2}{280} - O(m), \\
\lambda_9(m) &= \frac{a_9}{\text{area}(\sigma_0)} - O(m) = \frac{9m^2}{140} - O(m), \\
\lambda_8(m) &= \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{19m^2}{280} - O(m), \\
\lambda_7(m) &= \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{23m^2}{140} - O(m),
\end{aligned}
$$

$$\lambda_6(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{27m^2}{140} - O(m),$$
$$\lambda_5(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{7m^2}{20} - O(m).$$

For $i = 3, 4$, not all i-wise crossings are at the grid vertices. There are 21 types of such crossings in total; see Fig. 9. Types 1 through 3 are 4-wise crossings and types 4 through 21 are 3-wise crossings. The bundles intersecting at each of these 21 types of vertices are listed in Table 4. For $j = 1, 2, \ldots, 21$, let w_j denote the weighted area containing all the crossings of type j; where the weight is the number of crossings per grid cell. To complete the estimates of $\lambda_i(m)$ for $i = 3, 4$, we calculate w_j for all j from the bundles intersecting at crossings of type j. The values are collected in Table 5. Observe that for two parallel strips Γ_i and Γ_j, the area of their intersection is $\text{area}(\Gamma_i \cap \Gamma_j) = \text{area}(P(2i-1, 2i, 2j-1, 2j))$; recall that $P(2i-1, 2i, 2j-1, 2j)$ denotes the parallelogram made by the two pairs of parallel lines ℓ_{2i-1}, ℓ_{2i} and ℓ_{2j-1}, ℓ_{2j}, respectively.

j	Bundles intersecting at type j vertices
1	$\mathcal{L}_6, \mathcal{L}_{12}, \mathcal{L}_3, \mathcal{L}_9$
2	$\mathcal{L}_2, \mathcal{L}_8, \mathcal{L}_{11}, \mathcal{L}_5$
3	$\mathcal{L}_{10}, \mathcal{L}_4, \mathcal{L}_1, \mathcal{L}_7$
4	$\mathcal{L}_2, \mathcal{L}_7, \mathcal{L}_9$
5	$\mathcal{L}_6, \mathcal{L}_{11}, \mathcal{L}_1$
6	$\mathcal{L}_{10}, \mathcal{L}_3, \mathcal{L}_5$
7	$\mathcal{L}_{12}, \mathcal{L}_5, \mathcal{L}_7$

j	Bundles intersecting at type j vertices
8	$\mathcal{L}_4, \mathcal{L}_{11}, \mathcal{L}_9$
9	$\mathcal{L}_8, \mathcal{L}_1, \mathcal{L}_3$
10	$\mathcal{L}_1, \mathcal{L}_5, \mathcal{L}_9$
11	$\mathcal{L}_{11}, \mathcal{L}_3, \mathcal{L}_7$
12	$\mathcal{L}_{12}, \mathcal{L}_3, \mathcal{L}_9$
13	$\mathcal{L}_4, \mathcal{L}_1, \mathcal{L}_7$
14	$\mathcal{L}_8, \mathcal{L}_{11}, \mathcal{L}_5$

j	Bundles intersecting at type j vertices
15	$\mathcal{L}_4, \mathcal{L}_8, \mathcal{L}_{12}$
16	$\mathcal{L}_6, \mathcal{L}_{12}, \mathcal{L}_3$
17	$\mathcal{L}_6, \mathcal{L}_{12}, \mathcal{L}_9$
18	$\mathcal{L}_2, \mathcal{L}_8, \mathcal{L}_{11}$
19	$\mathcal{L}_2, \mathcal{L}_8, \mathcal{L}_5$
20	$\mathcal{L}_{10}, \mathcal{L}_4, \mathcal{L}_1$
21	$\mathcal{L}_{10}, \mathcal{L}_4, \mathcal{L}_7$

Table 4: Bundles intersecting at type j vertices for $j = 1, 2, \ldots, 21$.

- For $\lambda_4(m)$, all the 4-wise crossings that are not at grid vertices are at centers of grid cells; we have

$$w_1 = \text{area}(\Gamma_6 \cap \Gamma_{12} \cap \Gamma_3 \cap \Gamma_9) = \text{area}(\Gamma_3 \cap \Gamma_9) = \text{area}(P(5, 6, 17, 18)) = \frac{\sqrt{3}}{4}.$$

Observe that types 2 and 3 are 120° and 240° rotations of type 1, respectively. So by symmetry, $w_1 = w_2 = w_3$.

Consequently, we have

$$\lambda_4(n) = \frac{a_4 + \sum_{j=1}^{3} w_j}{\text{area}(\sigma_0)} - O(m) = \left(\frac{1}{2} + \frac{3}{8}\right) m^2 - O(m) = \frac{7m^2}{8} - O(m).$$

Lastly, we estimate $\lambda_3(m)$. Besides 3-wise crossings at grid vertices in H (whose number is proportional to a_3), there are 18 types of 3-wise crossings i.e., types 4 through 21, on the boundary or in the interior of the cells in H.

- For types $4, 5$ and 6, there are 2 crossings per grid cell; and

$$w_4 = 2 \cdot \text{area}(\Gamma_2 \cap \Gamma_7 \cap \Gamma_9) = 2 \cdot (\text{area}(P(3, 4, 17, 18)) - \text{area}(1, 13, 17) - \text{area}(4, 14, 18))$$
$$= 2 \cdot \left(\frac{2}{\sqrt{3}} - \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}}\right) = \sqrt{3}.$$

  Observe that types 5 and 6 are 120° and 240° rotations of type 4, respectively. Therefore by symmetry, $w_4 = w_5 = w_6$.

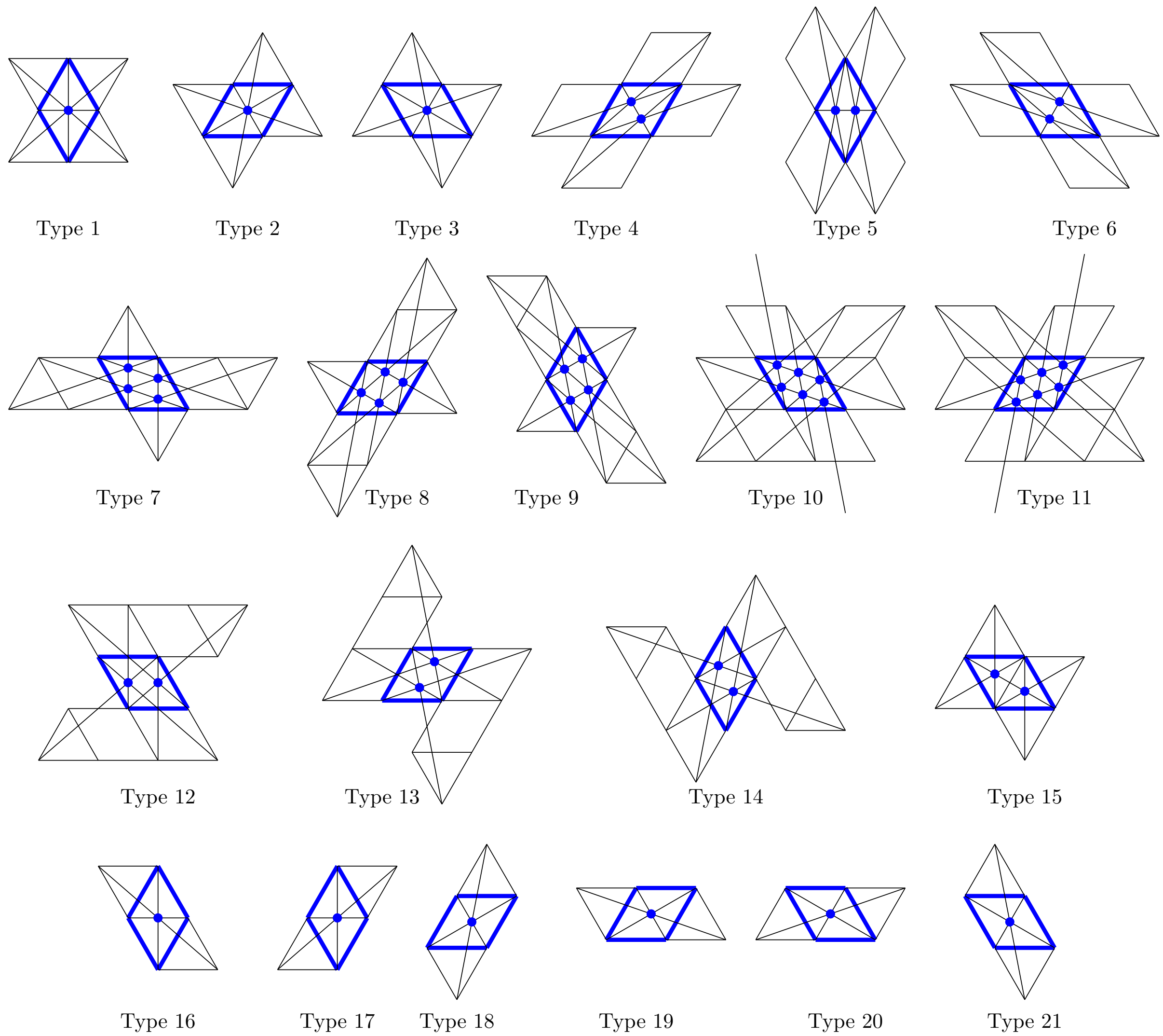


Figure 9: Other types of incidences of 3, 4 lines; 4-wise crossings: types 1 through 3; 3-wise crossings: types 4 through 21. To list the coordinates of the crossing points (shown as blue dots), we set the leftmost vertex of the grid cell (shown in blue lines) at $(0, 0)$.
For types 1 through 3 the crossings are at the center of the parallelogram.
For types 4 through 6, the crossings are on the short diagonal at $\frac{1}{3}$rd and $\frac{2}{3}$rd of the short diagonal.
For type 7, the crossings are at $(\frac{1}{2}, \frac{-\sqrt{3}}{10})$, $(\frac{1}{2}, \frac{-3\sqrt{3}}{10})$, $(1, \frac{-\sqrt{3}}{5})$, $(1, \frac{-2\sqrt{3}}{5})$.
For type 8, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{5})$, $(\frac{7}{10}, \frac{\sqrt{3}}{10})$, $(\frac{4}{5}, \frac{2\sqrt{3}}{5})$, $(\frac{11}{10}, \frac{3\sqrt{3}}{10})$.
For type 9, the crossings are at $(\frac{3}{10}, \frac{\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{-\sqrt{3}}{5})$, $(\frac{3}{5}, \frac{\sqrt{3}}{5})$, $(\frac{7}{10}, \frac{-\sqrt{3}}{10})$.
For type 10, the crossings are at $(\frac{5}{14}, \frac{-\sqrt{3}}{14})$, $(\frac{5}{7}, \frac{-\sqrt{3}}{7})$, $(\frac{15}{14}, \frac{-3\sqrt{3}}{14})$, $(\frac{3}{7}, \frac{-2\sqrt{3}}{7})$, $(\frac{11}{14}, \frac{-5\sqrt{3}}{14})$, $(\frac{8}{7}, \frac{-3\sqrt{3}}{7})$.
For type 11, the crossings are at $(\frac{8}{7}, \frac{3\sqrt{3}}{7})$, $(\frac{11}{14}, \frac{5\sqrt{3}}{14})$, $(\frac{3}{7}, \frac{2\sqrt{3}}{7})$, $(\frac{15}{14}, \frac{3\sqrt{3}}{14})$, $(\frac{5}{7}, \frac{\sqrt{3}}{7})$, $(\frac{5}{14}, \frac{\sqrt{3}}{14})$.
For type 12, the crossings are at $(\frac{1}{2}, \frac{-\sqrt{3}}{4})$ and $(1, \frac{-\sqrt{3}}{4})$.
For type 13, the crossings are at $(\frac{5}{8}, \frac{\sqrt{3}}{8})$ and $(\frac{7}{8}, \frac{3\sqrt{3}}{8})$.
For type 14, the crossings are at $(\frac{3}{8}, \frac{\sqrt{3}}{8})$ and $(\frac{5}{8}, \frac{-\sqrt{3}}{8})$.
For type 15, the crossings are on the long diagonal at $\frac{1}{3}$rd and $\frac{2}{3}$rd of the long diagonal.
For types 16 through 21 the crossings are at the center of the parallelogram.
The relative positions of all these crossings are shown in Fig. 10.

- For types $7, 8$ and 9, there are 4 crossings per grid cell, and

$$\begin{aligned} w_7 &= 4 \cdot \operatorname{area}(\Gamma_{12} \cap \Gamma_5 \cap \Gamma_7) = 4 \cdot (\operatorname{area}(P(9,10,13,14)) - \operatorname{area}(10,13,23) - \operatorname{area}(9,14,24)) \\ &= 4 \cdot \left(\frac{2\sqrt{3}}{5} - \frac{\sqrt{3}}{20} - \frac{\sqrt{3}}{20} \right) = \frac{6\sqrt{3}}{5}. \end{aligned}$$

Observe that types 8 and 9 are $120°$ and $240°$ rotations of type 7, respectively. So by symmetry, $w_7 = w_8 = w_9$.

- For types $10, 11$, there are 6 crossings per grid cell, and

$$\begin{aligned} w_{10} &= 6 \cdot \operatorname{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_9) = 6 \cdot (\operatorname{area}(P(1,2,17,18)) - \operatorname{area}(1,9,17) - \operatorname{area}(2,10,18)) \\ &= 6 \cdot \left(\frac{2\sqrt{3}}{7} - \frac{\sqrt{3}}{28} - \frac{\sqrt{3}}{28} \right) = \frac{9\sqrt{3}}{7}. \end{aligned}$$

Observe that type 11 is the reflection in a vertical line of type 10. Therefore by symmetry, we have $w_{10} = w_{11}$.

- For types $12, 13$ and 14, there are 2 crossings per grid cell, and

$$w_{12} = 2 \cdot \operatorname{area}(\Gamma_{12} \cap \Gamma_3 \cap \Gamma_9) = 2 \cdot \operatorname{area}(\Gamma_3 \cap \Gamma_9) = 2 \cdot \operatorname{area}(P(5,6,17,18)) = \frac{\sqrt{3}}{2}.$$

Observe that types 13 and 14 are $120°$ and $240°$ rotations of type 12, respectively. So by symmetry, we have $w_{12} = w_{13} = w_{14}$.

- For type 15, there are 2 crossings per grid cell and we have

$$\begin{aligned} w_{15} &= 2 \cdot \operatorname{area}(\Gamma_4 \cap \Gamma_8 \cap \Gamma_{12}) = 2 \cdot (\operatorname{area}(P(15,16,23,24)) - \operatorname{area}(7,15,24) - \operatorname{area}(8,16,23)) \\ &= 2 \cdot \left(\frac{2}{\sqrt{3}} - \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} \right) = \sqrt{3}. \end{aligned}$$

- For types 16 through 21, the crossings are at centers of grid cells; we have

$$\begin{aligned} w_{16} &= \operatorname{area}(\Gamma_6 \cap \Gamma_{12} \cap \Gamma_3 - \Gamma_9) = \operatorname{area}(\Gamma_{12} \cap \Gamma_3) - \operatorname{area}(\Gamma_{12} \cap \Gamma_3 \cap \Gamma_9) \\ &= \operatorname{area}(P(5,6,23,24)) - \operatorname{area}(P(5,6,17,18)) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}. \end{aligned}$$

Observe that type 17 is the reflection in a vertical line of type 16, types 18 and 20 are $120°$ and $240°$ rotations of type 16, respectively; and types 19 and 21 are $120°$ and $240°$ rotations of type 17, respectively. So by symmetry, we have $w_{16} = w_{17} = w_{18} = w_{19} = w_{20} = w_{21}$.

Consequently, we have

$$\lambda_3(n) = \frac{a_3 + \sum_{j=4}^{21} w_j}{\operatorname{area}(\sigma_0)} - O(m) = \left(\frac{3}{2} + \frac{3}{2} + \frac{9}{5} + \frac{9}{7} + \frac{3}{4} + \frac{1}{2} + \frac{3}{4} \right) m^2 - O(m) = \frac{283}{35} m^2 - O(m).$$

j	1	2	3	4	5	6	7	8	9	10	11
w_j	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\frac{6\sqrt{3}}{5}$	$\frac{6\sqrt{3}}{5}$	$\frac{6\sqrt{3}}{5}$	$\frac{9\sqrt{3}}{7}$	$\frac{9\sqrt{3}}{7}$

j	12	13	14	15	16	17	18	19	20	21
w_j	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$

Table 5: Values of w_j for $j = 1, \dots, 21$.

The values of $\lambda_i(m)$, for $i = 3, \dots, 12$, are summarized in Table 6; for convenience the linear terms are omitted. Since $m = n/12$, λ_i can be also viewed as a function of n.

i	3	4	5	6	7	8	9	10	11	12
$\lambda_i(m)$	$\frac{283m^2}{35}$	$\frac{7m^2}{8}$	$\frac{7m^2}{20}$	$\frac{27m^2}{140}$	$\frac{23m^2}{140}$	$\frac{19m^2}{280}$	$\frac{9m^2}{140}$	$\frac{13m^2}{280}$	$\frac{m^2}{70}$	$\frac{m^2}{10}$
$\lambda_i(n)$	$\frac{283n^2}{35\cdot 144}$	$\frac{7n^2}{8\cdot 144}$	$\frac{7n^2}{20\cdot 144}$	$\frac{27n^2}{140\cdot 144}$	$\frac{23m^2}{280\cdot 144}$	$\frac{19n^2}{280\cdot 144}$	$\frac{9n^2}{140\cdot 144}$	$\frac{13n^2}{280\cdot 144}$	$\frac{n^2}{70\cdot 144}$	$\frac{n^2}{10\cdot 144}$

Table 6: The values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, \dots, 12$.

Now we can derive the multiplicative factor in Equation (3) as follows:

$$F(n) \geq \prod_{i=3}^{12} B_i^{\lambda_i(n)} \geq 2^{\frac{283n^2}{35\cdot 144}} \cdot 8^{\frac{7n^2}{8\cdot 144}} \cdot 62^{\frac{7n^2}{20\cdot 144}} \cdot 908^{\frac{27n^2}{140\cdot 144}} \cdot 24698^{\frac{23n^2}{140\cdot 144}} \cdot 1232944^{\frac{19n^2}{280\cdot 144}}$$

$$\cdot\, 112018190^{\frac{9n^2}{140\cdot 144}} \cdot 18410581880^{\frac{13n^2}{280\cdot 144}} \cdot 5449192389984^{\frac{n^2}{70\cdot 144}} \cdot 2894710651370536^{\frac{n^2}{10\cdot 144}} \cdot 2^{-O(n)}.$$

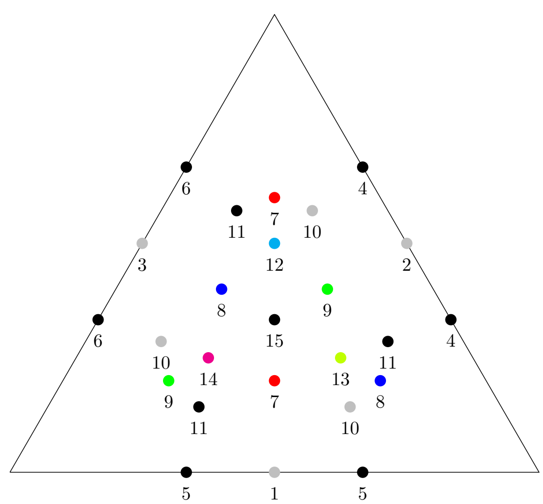


Figure 10: In the 12-gon in the middle of H, all the triangular grid cells contain 3-crossings and 4 crossings of all types 1 through 15. In other cells of the construction only some of these types appear.

We prove by induction on n that $T(n) \geq 2^{cn^2 - O(n \log n)}$ for a suitable constant $c > 0$. It suffices to choose c (using the values of B_i for $i = 3, \dots, 12$ in Table 1) so that

$$\frac{1}{144}\Big(\frac{283}{35} + \frac{7}{8}\log 8 + \frac{7}{20}\log 62 + \frac{27}{140}\log 908 + \frac{23}{140}\log 24698$$

$$+\frac{19}{280}\log 1232944+\frac{9}{140}\log 112018190+\frac{13}{280}\log 18410581880$$
$$+\frac{1}{70}\log 5449192389984+\frac{1}{10}\log 2894710651370536\Big)\geq\frac{11c}{12}.$$

The above inequality holds if we set

$$c=\frac{1}{132}\Big(\frac{283}{35}+\frac{7}{8}\log 8+\frac{7}{20}\log 62+\frac{27}{140}\log 908+\frac{23}{140}\log 24698$$
$$+\frac{19}{280}\log 1232944+\frac{9}{140}\log 112018190+\frac{13}{280}\log 18410581880 \qquad (4)$$
$$+\frac{1}{70}\log 5449192389984+\frac{1}{10}\log 2894710651370536\Big)>0.2083,$$

and the lower bound in Theorem 1 follows. □

4 Conclusion

We offered several recursive constructions derived from arrangements with lines of 3, 4, 6, 8, and 12 different slopes; in two different styles (rectangular and hexagonal). The hexagonal construction with 12 slopes yields the lower bound $b_n \geq 0.2083\, n^2$ for large n. We have no doubt that increasing the number of slopes will further increase the lower bound, and the proof complexity at the same time. The questions of how far can one go and whether there are other more efficient variants remain. We conclude with the following questions.

1. What lower bounds on B_n can be deduced from line arrangements with a higher number of slopes? In particular, hexagonal and rectangular constructions with 16 slopes seem to be the most promising candidates. Note however that the value B_{16} is currently unknown.
2. What lower bounds on B_n can be obtained from rhombic tilings of a centrally symmetric octagon? Or from those of a centrally symmetric k-gon for some other even $k \geq 10$? No closed formulas for the number of such tilings seem to be available at the time of the present writing. However, suitable estimates could perhaps be deduced from previous results; see, e.g., [1, 2, 3, 11].

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A Rectangular construction with 8 slopes

We describe and analyze a rectangular construction with lines of 8 slopes. Observe Fig. 11. Consider 8 bundles of parallel lines whose slopes are $0, \infty, \pm 1, \pm 1/2, \pm 2$. The axes of all parallel strips are all incident to the center of U. This construction yields the lower bound $b_n \geq 0.2\, n^2$ for large n.

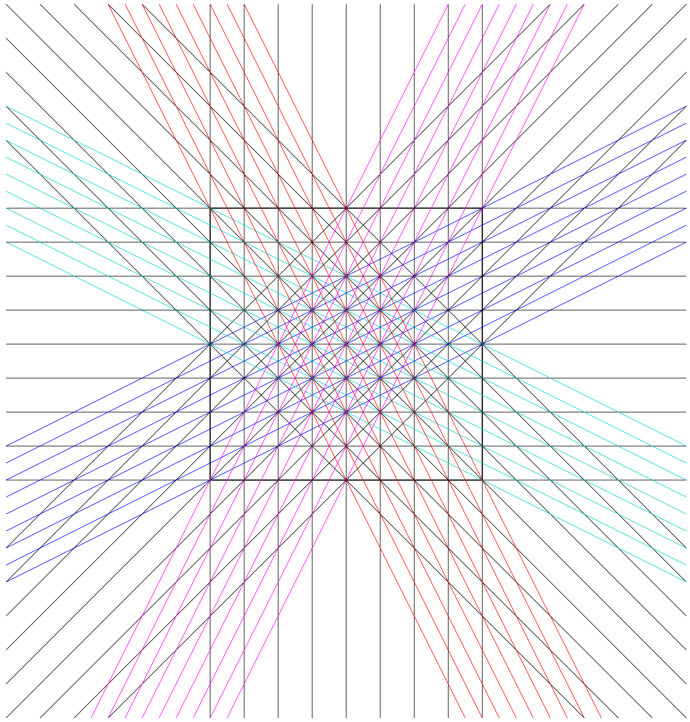


Figure 11: Construction with 8 slopes.

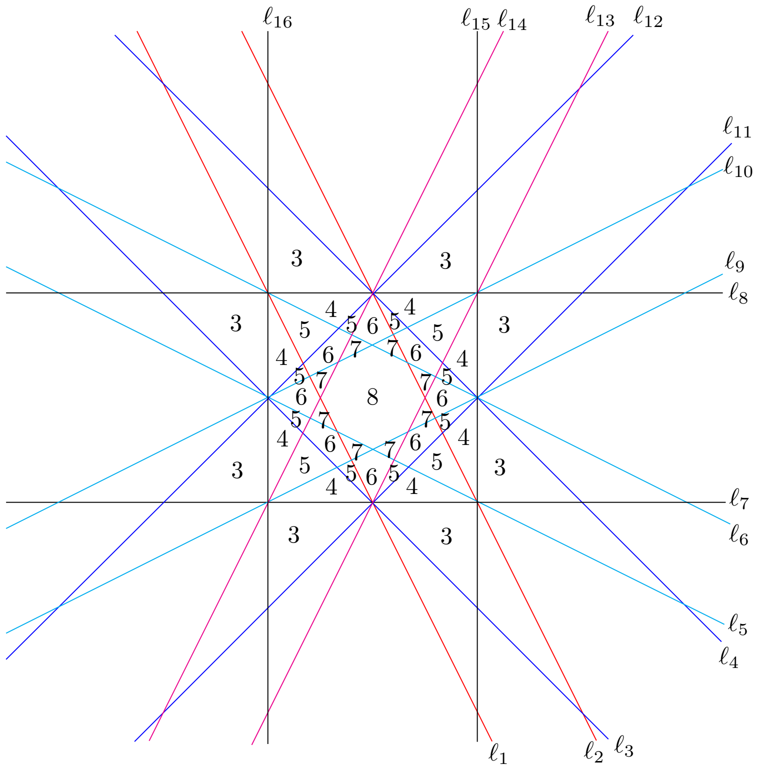
Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_8$ be the partition of $\mathcal{L}$ into eight bundles. The m lines in $\mathcal{L}_i$ are contained in the parallel strip Γ_i bounded by the two lines ℓ_{2i-1} and ℓ_{2i}, for $i = 1, \ldots, 8$. The equation of line ℓ_i is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, $i = 1, \ldots, 16$ given in Fig 12 (right). Observe that $U = \Gamma_4 \cap \Gamma_8$.

We refer to lines in $\mathcal{L}_4 \cup \mathcal{L}_8$ (i.e., axis-aligned lines) as the *primary* lines, and to rest of the lines as *secondary* lines. We refer to the intersection points of the primary lines as *grid vertices*. The slopes of the primary lines are in $\{0, \infty\}$. The slopes of the secondary lines are in $\{\pm 1/2, \pm 1, \pm 2\}$. The final construction is obtained by a small clockwise rotation, so that there are no vertical lines. Observe that

- the distance between consecutive lines in $\mathcal{L}_4$ or $\mathcal{L}_8$ is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in $\mathcal{L}_2$ or $\mathcal{L}_6$ is $\frac{1}{m\sqrt{2}}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_5$ or $\mathcal{L}_7$ is $\frac{1}{m\sqrt{5}}\left(1 - O\left(\frac{1}{m}\right)\right)$.

Let $\sigma_0 = \sigma_0(m)$ denote the basic parallelogram (here, square) determined by axis-aligned lines (i.e., lines in $\mathcal{L}_4 \cup \mathcal{L}_8$); the side length of σ_0 is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$. We refer to these basic parallelograms as *grid cell*. Let U' be the smaller square made by $\ell_5, \ell_6, \ell_{13}, \ell_{14}$, i.e., $U' = \Gamma_3 \cap \Gamma_7$; the similarity ratio $\rho(U', U)$ is equal to $\frac{1}{\sqrt{5}}$. We have

$$\begin{aligned}
\text{area}(U) &= 1, \\
\text{area}(U') &= \frac{\text{area}(U)}{5} = \frac{1}{5}, \\
\text{area}(\sigma_0) &= \frac{1}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right).
\end{aligned}$$



i	α_i	β_i	γ_i
1	2	1	-1
2	2	1	-2
3	1	1	-0.5
4	1	1	-1.5
5	1	2	-1
6	1	2	-2
7	0	1	0
8	0	1	-1
9	1	-2	0
10	1	-2	1
11	1	-1	-0.5
12	1	-1	0.5
13	2	-1	-1
14	2	-1	0
15	1	0	-1
16	1	0	0

Figure 12: Left: The eight parallel strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by axis-aligned lines. Right: Coefficients of the lines $\ell_i\ :\ i = 1, \ldots, 16$.

Let a_i, for $i = 3, \ldots, 8$, denote the area of the region covered by exactly i of the 8 strips. Recall that area(i, j, k) denotes the area of the triangle made by ℓ_i, ℓ_j and ℓ_k. Observe that

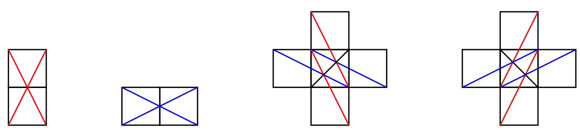
$$\begin{aligned}
a_3 &= 8 \cdot \text{area}(3,7,15) = 1,\\
a_4 &= 8 \cdot \text{area}(5,7,11) = \frac{1}{3},\\
a_5 &= 4\,(2 \cdot \text{area}(5,11,13) + \text{area}(2,5,11)) = \frac{7}{30},\\
a_6 &= 4\,(\text{area}(6,11,13) - 2 \cdot \text{area}(2,11,9) - \text{area}(2,9,13)) = \frac{1}{5},\\
a_7 &= 8 \cdot \text{area}(5,9,13) = \frac{1}{15},\\
a_8 &= \text{area}(U') - 4 \cdot \text{area}(5,9,13) = \frac{1}{5} - \frac{1}{30} = \frac{1}{6}.
\end{aligned}$$

Observe that $a_4 + a_5 + a_6 + a_7 + a_8 = \text{area}(U) = 1$. Recall that $\lambda_i(m)$ denote the number i-wise crossings where each bundle consists of m lines. Also that $\lambda_i(m)$ is proportional to a_i, for $i = 4, 5, 6, 7, 8$. Indeed, $\lambda_i(m)$ is equal to the number of grid points that lie in a region covered by i parallel strips, which is roughly equal to the ratio $\frac{a_i}{\text{area}(\sigma_0)}$, for $i = 4, 5, 6, 7, 8$. More precisely, taking also the boundary effect of the relevant regions into account, we obtain

$$\lambda_4(m) = \frac{a_4}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{3} - O(m),$$

$$\lambda_5(m) = \frac{a_5}{\text{area}(\sigma_0)} - O(m) = \frac{7m^2}{30} - O(m),$$
$$\lambda_6(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{5} - O(m),$$
$$\lambda_7(m) = \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{15} - O(m),$$
$$\lambda_8(m) = \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{6} - O(m).$$

For estimating $\lambda_3(m)$, in addition to considering 3-wise crossings in the exterior of U, we also observe 3-wise crossings on the boundaries or in the interior of the small grid squares contained in some regions of U. Specifically, we distinguish exactly four types of 3-wise crossings, as illustrated in Fig. 13 (left). For $j = 1, 2, 3, 4$, let w_j denote the weighted area containing all crossings of type j; where the weight is the number of 3-wise crossings per grid cell. To complete the estimate of $\lambda_3(m)$, we calculate w_j for all j, from the bundles intersecting at crossings of type j; listed in Fig. 13 (right).



j	Bundles intersecting at vertices of type j
1	$\mathcal{L}_4, \mathcal{L}_1, \mathcal{L}_7$
2	$\mathcal{L}_8, \mathcal{L}_3, \mathcal{L}_5$
3	$\mathcal{L}_6, \mathcal{L}_1, \mathcal{L}_3$
4	$\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_7$

Figure 13: Left: Other types of 3-wise crossings. Right: Intersecting bundles for these crossings.

Recall that for two parallel strips Γ_i and Γ_j, the area of their intersection is $\text{area}(\Gamma_i \cap \Gamma_j) = \text{area}(P(2i-1, 2i, 2j-1, 2j))$; where $P(2i-1, 2i, 2j-1, 2j)$ denotes the parallelogram made by the two pairs of parallel lines ℓ_{2i-1}, ℓ_{2i} and ℓ_{2j-1}, ℓ_{2j}, respectively. For types 1 and 2, there are 1 crossing per grid cell and for types 3 and 4, there are 2 crossings per grid cell. Therefore we have,

- $w_1 = \text{area}(\Gamma_4 \cap \Gamma_1 \cap \Gamma_7) = \text{area}(\Gamma_1 \cap \Gamma_7) = \text{area}(P(1, 2, 13, 14)) = 1/4,$
- $w_2 = \text{area}(\Gamma_8 \cap \Gamma_3 \cap \Gamma_5) = \text{area}(\Gamma_3 \cap \Gamma_5) = \text{area}(P(5, 6, 9, 10)) = 1/4,$
- $w_3 = 2 \cdot \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_6) = 2 \cdot (\text{area}(P(1, 2, 5, 6)) - 2 \cdot \text{area}(2, 5, 11)) = 2 \cdot (1/3 - 1/12) = 1/2,$
- $w_4 = 2 \cdot \text{area}(\Gamma_2 \cap \Gamma_5 \cap \Gamma_7) = 1/2.$

It follows that

$$\lambda_3(m) = \frac{a_3 + \sum_{j=1}^{4} w_j}{\text{area}(\sigma_0)} - O(m) = \left(1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\right) m^2 - O(m) = \frac{5m^2}{2} - O(m).$$

The values of $\lambda_i(m)$, for $i = 3, 4, \ldots, 8$, are summarized in Table 7; for convenience the linear terms are omitted. Since $m = n/8$, λ_i can be also viewed as a function of n.

Now we can derive the multiplicative factor in Equation (3) as follows:

$$F(n) \geq \prod_{i=3}^{8} B_i^{\lambda_i(n)} \geq 2^{\frac{5n^2}{2\cdot 64}} \cdot 8^{\frac{n^2}{3\cdot 64}} \cdot 62^{\frac{7n^2}{30\cdot 64}} \cdot 908^{\frac{n^2}{5\cdot 64}} \cdot 24698^{\frac{n^2}{15\cdot 64}} \cdot 1232944^{\frac{n^2}{6\cdot 64}} \cdot 2^{-O(n)}.$$

i	3	4	5	6	7	8
$\lambda_i(m)$	$\frac{5m^2}{2}$	$\frac{m^2}{3}$	$\frac{7m^2}{30}$	$\frac{m^2}{5}$	$\frac{m^2}{15}$	$\frac{m^2}{6}$
$\lambda_i(n)$	$\frac{5n^2}{2\cdot 64}$	$\frac{n^2}{3\cdot 64}$	$\frac{7n^2}{30\cdot 64}$	$\frac{n^2}{5\cdot 64}$	$\frac{n^2}{15\cdot 64}$	$\frac{n^2}{6\cdot 64}$

Table 7: The values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, 4, \ldots, 8$.

We prove by induction on n that $T(n) \geq 2^{cn^2 - O(n \log n)}$ for a suitable constant $c > 0$. It suffices to choose c (using the values of B_i for $i = 3, \ldots, 8$ in Table 1) so that

$$\frac{1}{64}\left(\frac{5}{2} + \frac{1}{3}\log 8 + \frac{7}{30}\log 62 + \frac{1}{5}\log 908 + \frac{1}{15}\log 24698 + \frac{1}{6}\log 1232944\right) \geq \frac{7c}{8}.$$

The above inequality holds if we set

$$c = \frac{1}{56}\left(\frac{5}{2} + 1 + \frac{7}{30}\log 62 + \frac{1}{5}\log 908 + \frac{1}{15}\log 24698 + \frac{1}{6}\log 1232944\right) > \frac{1}{5}, \tag{5}$$

and this yields the lower bound $B_n \geq 2^{cn^2 - O(n \log n)}$, for some constant $c > 0.2$. In particular, we have $B_n \geq 2^{0.2n^2}$ for large n.

Interestingly enough, evaluating the weighted sum of logarithms in the LHS of (5) with finite precision does not quite give the RHS of (5), due to the approximations introduced (it only yields $0.1999947\ldots$). We can however obtain (5) if we revert to calculations over the integers; as shown below.

After simplifying the denominator by 5 and multiplying by 6, inequality (5) is equivalent to the following:

$$\begin{aligned}
\frac{1}{56}\left(105 + 7\log 62 + 6\log 908 + 2\log 24698 + 5\log 1232944\right) &> 6,\\
7\log 62 + 6\log 908 + 2\log 24698 + 5\log 1232944 &> 231,\\
2^7 \cdot 31^7 \cdot 2^{12} \cdot 227^6 \cdot (2^2 \cdot 53^2 \cdot 233^2) \cdot (2^{20} \cdot 263^5 \cdot 293^5) &> 2^{231},\\
2^{43} \cdot 31^7 \cdot 227^6 \cdot 53^2 \cdot 233^2 \cdot 263^5 \cdot 293^5 &> 2^{231},\\
31^7 \cdot 227^6 \cdot 53^2 \cdot 233^2 \cdot 263^5 \cdot 293^5 &> 2^{188}.
\end{aligned}$$

To justify the last line above, observe that

$$\begin{aligned}
31^7 \cdot 227^6 &> 2^{81},\\
53^2 \cdot 233^2 &> 2^{27},\\
263^5 \cdot 293^5 &> 2^{81},
\end{aligned}$$

and so by taking the product we get

$$31^7 \cdot 227^6 \cdot 53^2 \cdot 233^2 \cdot 263^5 \cdot 293^5 > 2^{81+27+81} = 2^{189} > 2^{188},$$

as required.

B Rectangular construction with 12 slopes

We next describe and analyze a rectangular construction with lines of 12 slopes. Consider 12 bundles of parallel lines whose slopes are $0, \infty, \pm 1/3, \pm 1/2, \pm 1, \pm 2, \pm 3$. The axes of all parallel strips are all incident to the center of $U = [0,1]^2$. Refer to Fig. 14. This construction yields the lower bound $b_n \geq 0.2053\, n^2$ for large n.

i	α_i	β_i	γ_i
1	3	1	-1.5
2	3	1	-2.5
3	2	1	-1
4	2	1	-2
5	1	1	-0.5
6	1	1	-1.5
7	1	2	-1
8	1	2	-2

i	α_i	β_i	γ_i
9	1	3	-1.5
10	1	3	-2.5
11	0	1	0
12	0	1	-1
13	-1	3	-0.5
14	-1	3	-1.5
15	1	-2	0
16	1	-2	1

i	α_i	β_i	γ_i
17	1	-1	-0.5
18	1	-1	0.5
19	2	-1	-1
20	2	-1	0
21	3	-1	-1.5
22	3	-1	-0.5
23	1	0	-1
24	1	0	0

Table 8: Coefficients of the 24 lines.

Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{12}$ be the partition of $\mathcal{L}$ into twelve bundles. The m lines in $\mathcal{L}_i$ are contained in the parallel strip Γ_i bounded by the two lines ℓ_{2i-1} and ℓ_{2i}, for $i = 1, \ldots, 12$. The equation of line ℓ_i is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, $i = 1, \ldots, 24$ given in Table 8. Observe that $U = \Gamma_6 \cap \Gamma_{12}$.

We refer to lines in $\mathcal{L}_6 \cup \mathcal{L}_{12}$ (i.e., axis-aligned lines) as the *primary* lines, and to rest of the lines as the *secondary* lines. We refer to the intersection points of the primary lines as *grid vertices*. The slopes of the primary lines are in $\{0, \infty\}$, and the slopes of the secondary lines are in $\{\pm 1/3, \pm 1/2, \pm 1, \pm 2, \pm 3\}$. The final construction is obtained by a small clockwise rotation, so that there are no vertical lines. Observe that

- the distance between consecutive lines in $\mathcal{L}_6$ or $\mathcal{L}_{12}$ is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in $\mathcal{L}_3$ or $\mathcal{L}_9$ is $\frac{1}{m\sqrt{2}}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_8$, or $\mathcal{L}_{10}$ is $\frac{1}{m\sqrt{5}}\left(1 - O\left(\frac{1}{m}\right)\right)$;
- the distance between consecutive lines in $\mathcal{L}_1$, $\mathcal{L}_5$, $\mathcal{L}_7$, or $\mathcal{L}_{11}$ is $\frac{1}{m\sqrt{10}}\left(1 - O\left(\frac{1}{m}\right)\right)$.

Let $\sigma_0 = \sigma_0(m)$ denote the basic parallelogram (here, square) determined by axis-aligned lines (i.e., lines in $\mathcal{L}_6 \cup \mathcal{L}_{12}$); the side length of σ_0 is $\frac{1}{m}\left(1 - O\left(\frac{1}{m}\right)\right)$. We refer to these basic parallelograms as *grid cell*. Let $U_1 = \Gamma_1 \cap \Gamma_7$, be the square made by $\ell_1, \ell_2, \ell_{13}, \ell_{14}$, and $U_2 = \Gamma_2 \cap \Gamma_8$, be the smaller square made by $\ell_3, \ell_4, \ell_{15}, \ell_{16}$. Note that $\rho(U_1, U) = \frac{1}{\sqrt{10}}$ and $\rho(U_2, U) = \frac{1}{\sqrt{5}}$. We have

$$\begin{aligned}
\text{area}(U) &= 1, \\
\text{area}(U_1) &= \frac{\text{area}(U)}{10} = \frac{1}{10}, \\
\text{area}(U_2) &= \frac{\text{area}(U)}{5} = \frac{1}{5}, \\
\text{area}(\sigma_0) &= \frac{1}{m^2}\left(1 - O\left(\frac{1}{m}\right)\right).
\end{aligned}$$

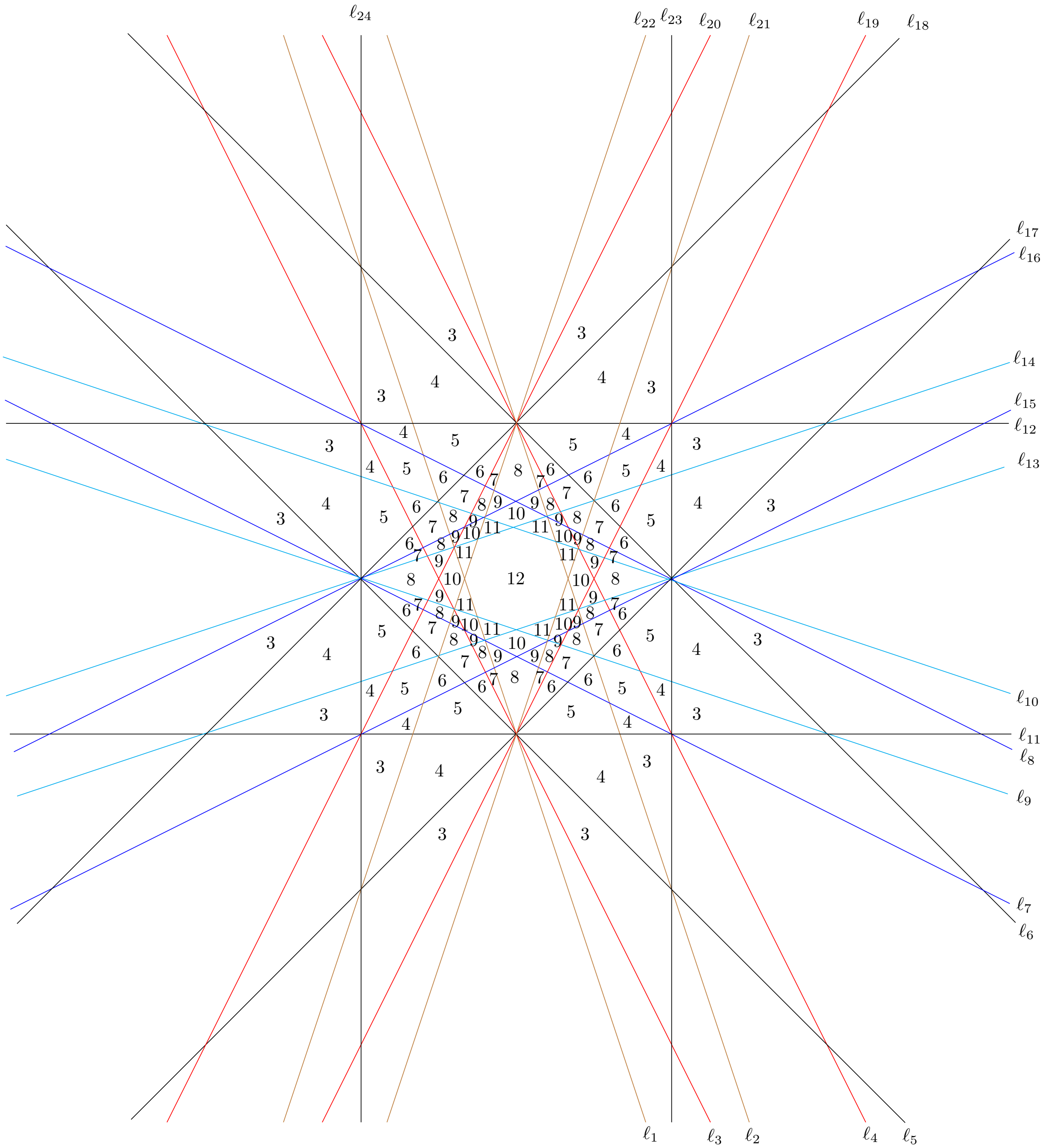


Figure 14: Construction with 12 slopes. The twelve parallel strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by axis-aligned lines.

Let a_i, for $i = 3, \ldots, 12$, denote the area of the region covered by exactly i of the 12 strips. Recall that $\mathrm{area}(i, j, k)$ denotes the area of the triangle made by ℓ_i, ℓ_j and ℓ_k. Observe that

$$a_3 = 8 \cdot (\mathrm{area}(2, 11, 13) + \mathrm{area}(3, 5, 23)) = 8 \cdot \left(\frac{1}{8} + \frac{1}{24}\right) = \frac{4}{3},$$

$$a_4 = 8 \cdot (\mathrm{area}(2, 5, 11) + \mathrm{area}(2, 7, 11)) = 8 \cdot \left(\frac{1}{12} + \frac{1}{120}\right) = \frac{11}{15},$$

$$
\begin{aligned}
a_5 &= 4\cdot(\text{area}(11,17,23) - 2\cdot\text{area}(2,7,11) - 2\cdot\text{area}(2,7,17)) = 4\left(\frac{1}{8} - \frac{2}{120} - \frac{2}{120}\right) = \frac{11}{30},\\
a_6 &= 4\cdot(2\cdot\text{area}(7,17,19) + 2\cdot\text{area}(2,7,17)) = 4\cdot\left(\frac{2}{120} + \frac{2}{120}\right) = \frac{2}{15},\\
a_7 &= 4\cdot(2\cdot\text{area}(7,19,21) + 2\cdot(\text{area}(9,17,19) - \text{area}(7,17,19)))\\
&= 4\cdot\left(\frac{1}{140} + 2\cdot\left(\frac{1}{56} - \frac{1}{120}\right)\right) = \frac{11}{105},\\
a_8 &= 8\cdot(\text{area}(9,19,21) - \text{area}(7,19,21)) + 4\cdot(\text{area}(2,9,15) - \text{area}(9,15,19))\\
&\quad + 8\cdot\text{area}(7,21,25) = \frac{13}{105},\\
a_9 &= 8\cdot(\text{area}(7,15,21) + \text{area}(9,15,19)) = 8\cdot\left(\frac{1}{280} + \frac{1}{840}\right) = \frac{4}{105},\\
a_{10} &= 4\cdot((\text{area}(7,13,15) - \text{area}(9,13,15)) + (\text{area}(13,19,21) - \text{area}(15,19,21)))\\
&= 4\cdot\left(\left(\frac{1}{40} - \frac{1}{60}\right) + \left(\frac{1}{80} - \frac{1}{120}\right)\right) = 4\cdot\left(\frac{1}{120} + \frac{1}{240}\right) = \frac{1}{20},\\
a_{11} &= 8\cdot\text{area}(2,13,21) = \frac{8}{240} = \frac{1}{30},\\
a_{12} &= \text{area}(U_1) - 4\cdot\text{area}(9,13,21) = \frac{1}{10} - \frac{4}{240} = \frac{1}{12}.
\end{aligned}
$$

Observe that the region whose area is $\sum_{i=4}^{12} a_i$ consists of U and 8 triangles outside U. Therefore,

$$\sum_{i=4}^{12} a_i = \text{area}(U) + 8\cdot\text{area}(2,5,11) = 1 + 2/3 = 5/3.$$

Recall that $\lambda_i(m)$ denote the number i-wise crossings where each bundle consists of m lines. Observe that $\lambda_i(m)$ is proportional to a_i, for $i = 7, 8, \ldots, 12$. Indeed, $\lambda_i(m)$ is equal to the number of intersection points of axis-aligned lines (i.e., lines in $\mathcal{L}_6 \cup \mathcal{L}_{12}$) that lie in a region covered by i parallel strips, which is roughly equal to the ratio $\frac{a_i}{\text{area}(\sigma_0)}$, for $i = 7, 8, \ldots, 12$. More precisely, taking also the boundary effect of the relevant regions into account, we obtain

$$
\begin{aligned}
\lambda_7(m) &= \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{11m^2}{105} - O(m),\\
\lambda_8(m) &= \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{13m^2}{105} - O(m),\\
\lambda_9(m) &= \frac{a_9}{\text{area}(\sigma_0)} - O(m) = \frac{9m^2}{105} - O(m),\\
\lambda_{10}(m) &= \frac{a_{10}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{20} - O(m),\\
\lambda_{11}(m) &= \frac{a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{30} - O(m),\\
\lambda_{12}(m) &= \frac{a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{12} - O(m).
\end{aligned}
$$

For $i = 3, 4, 5, 6$, not all the i-wise crossings are at grid vertices. There are 29 types of such crossings in total; see Fig. 15. The bundles intersecting at each of these 29 types of vertices are listed in Table 9. For $j = 1, 2, \ldots, 29$, let w_j denote the weighted area containing all crossings of type j; where the weight is the number of crossings per grid cell. To complete the estimates of $\lambda_i(m)$ for $i = 3, 4, 5, 6$, we calculate w_j for all j from the bundles intersecting at type j crossings. The values are listed in Table 10.

j	Bundles intersecting at type j vertices	j	Bundles intersecting at type j vertices	j	Bundles intersecting at type j vertices
1	$\mathcal{L}_2, \mathcal{L}_6, \mathcal{L}_{10}$	11 & 12	$\mathcal{L}_1, \mathcal{L}_6, \mathcal{L}_{11}$	21	$\mathcal{L}_3, \mathcal{L}_7, \mathcal{L}_9, \mathcal{L}_{11}$
2	$\mathcal{L}_4, \mathcal{L}_8, \mathcal{L}_{12}$	13	$\mathcal{L}_2, \mathcal{L}_8, \mathcal{L}_{11}$	22	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_9$
3	$\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_9$	14	$\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}_{10}$	23	$\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}_7, \mathcal{L}_{10}$
4	$\mathcal{L}_3, \mathcal{L}_8, \mathcal{L}_{10}$	15	$\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_8$	24	$\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_8, \mathcal{L}_{11}$
5	$\mathcal{L}_3, \mathcal{L}_7, \mathcal{L}_9$	16	$\mathcal{L}_4, \mathcal{L}_7, \mathcal{L}_{10}$	25	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_7, \mathcal{L}_9, \mathcal{L}_{11}$
6	$\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_9$	17	$\mathcal{L}_1, \mathcal{L}_5, \mathcal{L}_9$	26	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_9, \mathcal{L}_{11}$
7	$\mathcal{L}_3, \mathcal{L}_9, \mathcal{L}_{11}$	18	$\mathcal{L}_3, \mathcal{L}_7, \mathcal{L}_{11}$	27	$\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9, \mathcal{L}_{11}$
8	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_9$	19	$\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9$	28	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9$
9 & 10	$\mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_{12}$	20	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_9, \mathcal{L}_{11}$	29	$\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9, \mathcal{L}_{11}$

Table 9: Bundles intersecting at type j vertices for $j = 1, 2, \ldots, 29$.

For $\lambda_6(m)$, all the 6-wise crossings that are not at grid vertices, are at the centers of cells; we have

$$w_{29} = \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 \cap \Gamma_{11}) = \text{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_{11}) = a_{12}.$$

It follows that

$$\lambda_6(m) = \frac{a_6 + w_{29}}{\text{area}(\sigma_0)} - O(m) = \frac{a_6 + a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{2m^2}{15} + \frac{m^2}{12} - O(m) = \frac{13m^2}{60} - O(m).$$

Similarly for $\lambda_5(m)$, all the 5-wise crossings that are not at grid vertices, i.e., types 25 through 28, are in the interiors of cells contained in eight small triangles inside U. For example,

$$w_{28} = \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_{11}) = \text{area}(1, 14, 22) + \text{area}(2, 13, 21) = 1/120.$$

Observe that sum of the areas of these eight small triangles equals to a_{11}. It follows that

$$\lambda_5(m) = \frac{a_5 + \sum_{j=25}^{28} w_j}{\text{area}(\sigma_0)} - O(m) = \frac{a_5 + a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{11m^2}{30} + \frac{m^2}{30} - O(m) = \frac{2m^2}{5} - O(m).$$

To estimate $\lambda_4(m)$, note that besides 4-wise crossings at grid vertices, there are 6 types of 4-wise crossings i.e., types 19 through 24, in the interiors of grid cells:

- For types 19 and 20, there is one crossing per grid cell; and

$$\begin{aligned} w_{19} &= \text{area}(\Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_1 - \Gamma_{11}) \\ &= (\text{area}(2, 10, 13) - \text{area}(2, 10, 21)) + (\text{area}(9, 14, 22) - \text{area}(1, 14, 22)) = 1/15, \end{aligned}$$

  Observe that type 20 is a $90°$ rotation of type 19. Therefore by symmetry, $w_{19} = w_{20}$.

- For types 21 and 22, there is one crossing per grid cell; and

$$\begin{aligned} w_{21} &= \text{area}(\Gamma_3 \cap \Gamma_7 \cap \Gamma_9 \cap \Gamma_{11} - \Gamma_1 - \Gamma_5) \\ &= (\text{area}(2, 14, 21) - \text{area}(2, 10, 21)) + (\text{area}(1, 13, 22) - \text{area}(1, 9, 22)) = 1/40. \end{aligned}$$

  Observe that type 22 is the reflection in a vertical line of type 21. Therefore by symmetry, $w_{21} = w_{22}$.

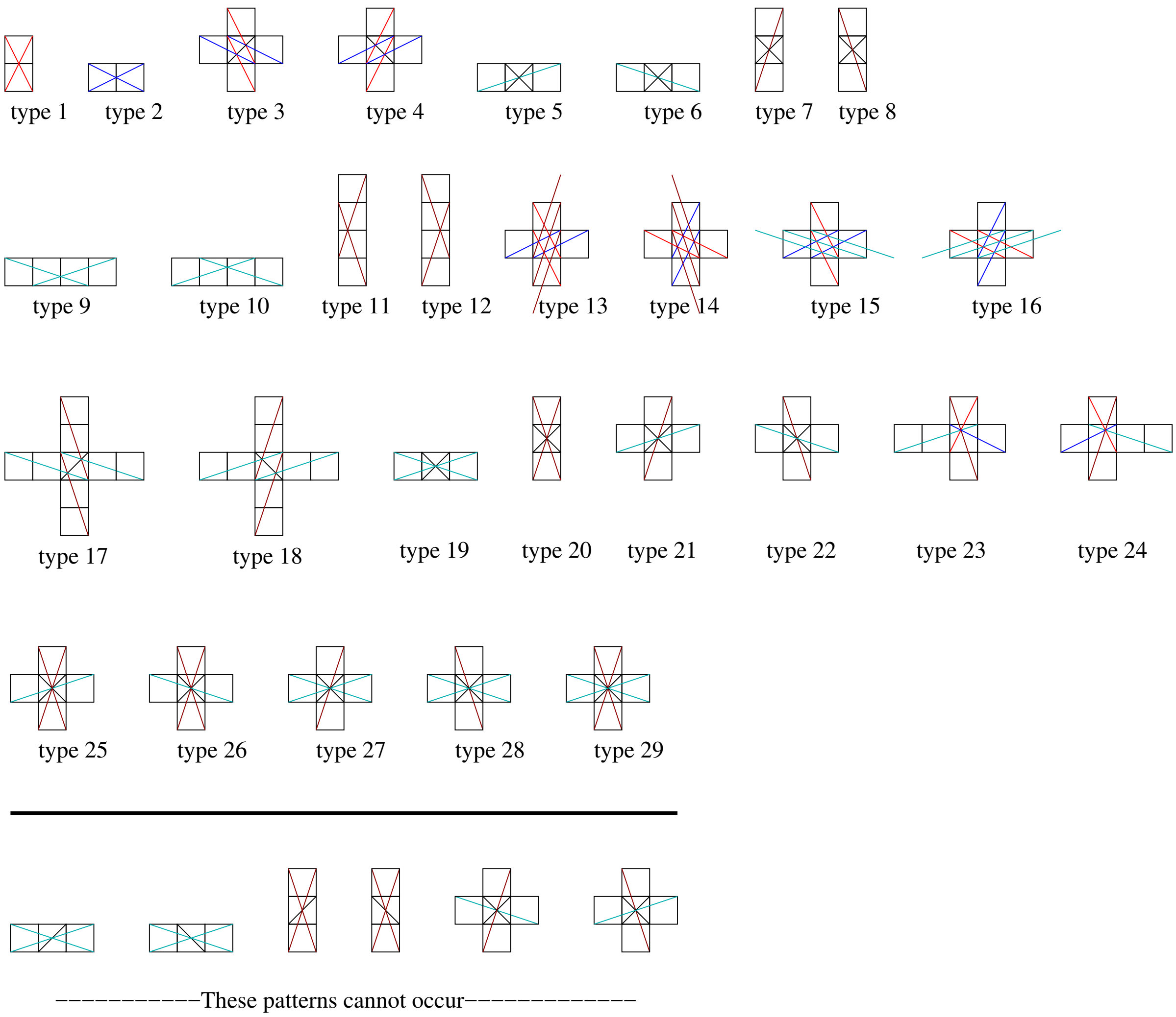


Figure 15: Other types of incidences of 3, 4, 5, and 6 lines; 3-wise crossings: types 1 through 18; 4-wise crossings: types 19 through 24; 5-wise crossings: types 25 through 28; 6-wise crossings: type 29.
For some types, the crossings are in the middle of a cell. To list the coordinates of crossing points, we rescale the grid cells to the unit square $[0,1]^2$.
For types 1 and 2, the crossings are at the midpoint of the horizontal and the vertical grid edges respectively.
For type 3, the crossings are at $(1/3,1/3)$ and $(2/3,2/3)$.
For type 4, the crossings are at $(1/3,2/3)$ and $(2/3,1/3)$.
For types 9 and 10, the crossings are on vertical grid edges at height $1/3$ and $2/3$ from the horizontal line below, respectively.
For types 11 and 12, the crossings are on horizontal grid edges at distance $1/3$ and $2/3$ from the vertical line on the left, respectively.
For type 13, the crossings are at $(1/5,3/5)$ and $(3/5,4/5)$ and $(4/5,2/5)$ and $(2/5,1/5)$.
For type 14, the crossings are at $(1/5,2/5)$ and $(2/5,4/5)$ and $(4/5,3/5)$ and $(3/5,1/5)$.
For type 15, the crossings are at $(1/5,3/5)$ and $(3/5,4/5)$ and $(4/5,2/5)$ and $(2/5,1/5)$.
For type 16, the crossings are at $(1/5,2/5)$ and $(2/5,4/5)$ and $(4/5,3/5)$ and $(3/5,1/5)$.
For type 17, the crossings are at $(1/4,1/4)$ and $(3/4,3/4)$.
For type 18, the crossings are at $(1/4,3/4)$ and $(3/4,1/4)$.
For type 23, the crossings are at $(1/5,2/5)$ and $(2/5,4/5)$ and $(4/5,3/5)$ and $(3/5,1/5)$.
For type 24, the crossings are at $(1/5,3/5)$ and $(3/5,4/5)$ and $(4/5,2/5)$ and $(2/5,1/5)$.
For the other types, the crossings are at $(1/2,1/2)$.

- For types 23 and 24, there are 4 crossings per grid cell; and

$$w_{23} = 4 \cdot \text{area}(\Gamma_1 \cap \Gamma_4 \cap \Gamma_7 \cap \Gamma_{10}) = 4 \cdot \text{area}(\Gamma_1 \cap \Gamma_7) = 4 \cdot \text{area}(U_1) = 2/5.$$

  Observe that type 24 is the reflection in a vertical line of type 23. Therefore by symmetry, $w_{23} = w_{24}$.

Consequently, we have

$$\lambda_4(m) = \frac{a_4 + \sum_{j=19}^{24} w_j}{\text{area}(\sigma_0)} - O(m) = \Big(\frac{11}{15} + \frac{2}{15} + \frac{1}{20} + \frac{4}{5}\Big)m^2 - O(m) = \frac{103m^2}{60} - O(m).$$

Lastly, we estimate $\lambda_3(m)$. Besides 3-wise crossings at grid vertices, there are 18 types of 3-wise crossings i.e., types 1 through 18, in the interior of grid cells:

- For types 1 and 2, there are 1 crossing per grid cell; and

$$w_1 = \text{area}(\Gamma_2 \cap \Gamma_6 \cap \Gamma_{10}) = \text{area}(\Gamma_2 \cap \Gamma_{10}) = \text{area}(P(3, 4, 19, 20)) = 1/4.$$

  Observe that type 2 is a 90° rotation of type 1. Therefore by symmetry, $w_1 = w_2$.

- For types 3 and 4, there are 2 crossings per grid cell; and

$$w_3 = 2 \cdot (\text{area}(\Gamma_2 \cap \Gamma_4 \cap \Gamma_9)) = 2 \cdot (\text{area}(P(3, 4, 7, 8)) - \text{area}(3, 8, 18) - \text{area}(4, 7, 17)) = 1/2.$$

  Observe that type 4 is the reflection in a vertical line of type 3. Therefore by symmetry, $w_3 = w_4$.

- For types $5, 6, 7, 8$, there is one crossing per grid cell; and

$$w_5 = \text{area}(\Gamma_3 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_1 - \Gamma_5 - \Gamma_{11}) = \text{area}(5, 9, 22) + \text{area}(6, 10, 21)) = 1/20.$$

  Observe that type 6 is the reflection in a vertical line of type 5, and types 7 and 8 are 90° rotations of types 6 and 5, respectively. Therefore by symmetry, $w_5 = w_6 = w_7 = w_8$.

- For types $9, 10, 11, 12$, there is one crossing on the boundary of each grid cell; and

$$w_9 = \text{area}(\Gamma_5 \cap \Gamma_7 \cap \Gamma_{12}) = \text{area}(\Gamma_5 \cap \Gamma_7) = \text{area}(P(9, 10, 13, 14)) = 1/6.$$

  Observe that type 10 is the the reflection in a horizontal line of type 9, and types 11 and 12 are 90° rotations of types 9 and 10, respectively. Therefore by symmetry, $w_9 = w_{10} = w_{11} = w_{12}$.

- For types $13, 14, 15, 16$, there are 4 crossings per grid cell; and

$$w_{13} = 4 \cdot (\text{area}(\Gamma_2 \cap \Gamma_8 \cap \Gamma_{11} - \Gamma_5)) = 4 \cdot (\text{area}(3, 9, 13) + \text{area}(4, 10, 16)) = 1/5.$$

  Observe that type 14 is the reflection in a vertical line of type 13, and types 15 and 16 are 90° rotations of types 13 and 14, respectively. Therefore by symmetry, $w_{13} = w_{14} = w_{15} = w_{16}$.

- For type 17 and 18, there are 2 crossings per grid cell; and

$$w_{17} = 2 \cdot (\text{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_9)) = 2 \cdot (\text{area}(\Gamma_1 \cap \Gamma_5)) = 2 \cdot \text{area}(P(1, 2, 9, 10)) = 1/4.$$

  Observe that type 18 is the reflection in a vertical line of type 17. Therefore by symmetry, $w_{17} = w_{18}$.

j	w_j
1	1/4
2	1/4
3	1/2
4	1/2
5	1/20
6	1/20
7	1/20

j	w_j
8	1/20
9 & 10	1/3
11 & 12	1/3
13	1/5
14	1/5
15	1/5
16	1/5

j	w_j
17	1/4
18	1/4
19	1/15
20	1/15
21	1/40
22	1/40
23	2/5

j	w_j
24	2/5
25	1/120
26	1/120
27	1/120
28	1/120
29	1/12

Table 10: Values of w_j for $j = 1, \dots, 29$.

Consequently, we have

$$\begin{aligned}\lambda_3(m) &= \frac{a_3 + \sum_{j=1}^{18} w_j}{\operatorname{area}(\sigma_0)} - O(m)\\ &= \Big(\frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{10} + \frac{1}{10} + \frac{1}{3} + \frac{1}{3} + \frac{4}{5} + \frac{1}{2}\Big)m^2 - O(m) = 5m^2 - O(m).\end{aligned}$$

The values of $\lambda_i(m)$, for $i = 3, 4, \dots, 12$, are summarized in Table 11; for convenience the linear terms are omitted. Since $m = n/12$, λ_i can be also viewed as a function of n.

i	3	4	5	6	7	8	9	10	11	12
$\lambda_i(m)$	$5m^2$	$\frac{103m^2}{60}$	$\frac{2m^2}{5}$	$\frac{13m^2}{60}$	$\frac{11m^2}{105}$	$\frac{13m^2}{105}$	$\frac{9m^2}{105}$	$\frac{m^2}{20}$	$\frac{m^2}{30}$	$\frac{m^2}{12}$
$\lambda_i(n)$	$\frac{5n^2}{144}$	$\frac{103n^2}{60\cdot 144}$	$\frac{2n^2}{5\cdot 144}$	$\frac{13n^2}{60\cdot 144}$	$\frac{11n^2}{105\cdot 144}$	$\frac{13n^2}{105\cdot 144}$	$\frac{9n^2}{105\cdot 144}$	$\frac{n^2}{20\cdot 144}$	$\frac{n^2}{30\cdot 144}$	$\frac{n^2}{12\cdot 144}$

Table 11: The values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, 4, \dots, 12$.

Now we can derive the multiplicative factor in Equation (3) as follows:

$$\begin{aligned}F(n) \geq \prod_{i=3}^{12} B_i^{\lambda_i(n)} &\geq 2^{\frac{5n^2}{144}} \cdot 8^{\frac{103n^2}{60\cdot 144}} \cdot 62^{\frac{2n^2}{5\cdot 144}} \cdot 908^{\frac{13n^2}{60\cdot 144}} \cdot 24698^{\frac{11n^2}{105\cdot 144}} \cdot 1232944^{\frac{13n^2}{105\cdot 144}}\\ &\cdot 112018190^{\frac{9n^2}{105\cdot 144}} \cdot 18410581880^{\frac{n^2}{20\cdot 144}} \cdot 5449192389984^{\frac{n^2}{30\cdot 144}} \cdot 2894710651370536^{\frac{n^2}{12\cdot 144}} \cdot 2^{-O(n)}.\end{aligned}$$

We prove by induction on n that $T(n) \geq 2^{cn^2 - O(n \log n)}$ for a suitable constant $c > 0$. It suffices to choose c (using the values of B_i for $i = 3, \dots, 12$ in Table 1) so that

$$\begin{aligned}&\frac{1}{144}\Big(5 + \frac{103}{60}\log 8 + \frac{2}{5}\log 62 + \frac{13}{60}\log 908 + \frac{11}{105}\log 24698 + \frac{13}{105}\log 1232944 + \frac{9}{105}\log 112018190\\ &\quad + \frac{1}{20}\log 18410581880 + \frac{1}{30}\log 5449192389984 + \frac{1}{12}\log 2894710651370536\Big) \geq \frac{11c}{12}.\end{aligned}$$

The above inequality holds if we set

$$\begin{aligned}c = \frac{1}{132}\Big(&5 + \frac{103}{60}\log 8 + \frac{2}{5}\log 62 + \frac{13}{60}\log 908 + \frac{11}{105}\log 24698\\ &+ \frac{13}{105}\log 1232944 + \frac{9}{105}\log 112018190 + \frac{1}{20}\log 18410581880\\ &+ \frac{1}{30}\log 5449192389984 + \frac{1}{12}\log 2894710651370536\Big) > 0.2053.\end{aligned} \tag{6}$$