On the growth of the Möbius function of permutations*

Vít Jelínek † Ida Kantor ‡ Jan Kynčl §
Martin Tancer ¶

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Abstract

We study the values of the Möbius function μ of intervals in the containment poset of permutations. We construct a sequence of permutations π_n of size $2n-2$ for which $\mu(1, \pi_n)$ is given by a polynomial in n of degree 7. This construction provides the fastest known growth of $|\mu(1, \pi)|$ in terms of $|\pi|$, improving a previous quadratic bound by Smith.

Our approach is based on a formula expressing the Möbius function of an arbitrary permutation interval $[\alpha, \beta]$ in terms of the number of embeddings of the elements of the interval into β.

keywords: Möbius function, permutation poset, permutation embedding.

1 Introduction

The Möbius function of a poset is a classical parameter with applications in combinatorics, number theory and topology. From the combinatorial per-

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†Computer Science Institute, Charles University, Faculty of Mathematics and Physics, Malostranské náměstí 25, Prague, Czech Republic, jelinek@iuuk.mff.cuni.cz

‡Computer Science Institute, Charles University, Faculty of Mathematics and Physics, Malostranské náměstí 25, Prague, Czech Republic, ida@iuuk.mff.cuni.cz

§Department of Applied Mathematics and Institute for Theoretical Computer Science, Charles University, Faculty of Mathematics and Physics, Malostranské náměstí 25, Prague, Czech Republic, kyncl@kam.mff.cuni.cz

¶Department of Applied Mathematics, Charles University, Faculty of Mathematics and Physics, Malostranské náměstí 25, Prague, Czech Republic, tancer@kam.mff.cuni.cz

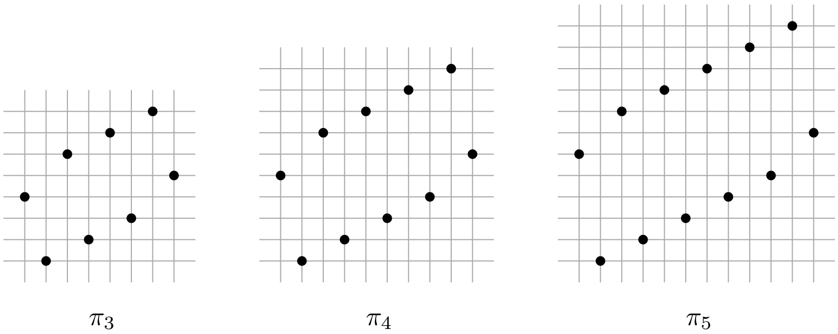


Figure 1: Permutations π_3, π_4 and π_5.

spective, an important problem is to study the Möbius function of containment posets of basic combinatorial structures, such as words [1, 2, 10], integer compositions [9, 12], integer partitions [21], or set partitions [7, 8].

Wilf [20] was the first to propose the study of the Möbius function of the containment poset of permutations, and it quickly became clear that in its full generality this is a challenging topic. This is not too surprising, considering that it is computationally hard even to determine whether two permutations are comparable in the permutation poset [3], and moreover, the poset of permutations is also hard to tackle by topological tools: for instance, most of its intervals are not shellable [11].

Thus, the known formulas for the Möbius function of the permutation poset are restricted to permutations of specific structure, such as layered permutations [12], 132-avoiding permutations [18], separable permutations [6], or permutations with a fixed number of descents [14, 15].

In this paper, we study the growth of the value $\max\{|\mu(1,\pi)|; |\pi| = n\}$ as a function of n. Here $\mu(1,\pi)$ is the Möbius function of the interval $[1,\pi]$, where 1 is the unique permutation of size one; see Section 2 for precise definitions. We give a construction showing that the rate of growth of this value is $\Omega(n^7)$, improving a previous result by Smith [14], who obtained a quadratic lower bound.

Specifically, for $n \geq 1$, we define the permutation $\pi_n \in \mathcal{S}_{2n+2}$ by

$$\pi_n = n+1, 1, n+3, 2, n+4, 3, n+5, \ldots, n, 2n+2, n+2.$$

See Figure 1. Our main result is the following formula for $\mu(1,\pi_n)$.

Theorem 1.1. *For every $n \geq 2$, we have*

$$\mu(1,\pi_n) = -\binom{n+2}{7} - \binom{n+1}{7} + 2\binom{n+2}{5} - \binom{n+2}{3} - \binom{n}{2} - 2n.$$

The Möbius function is closely related to the topological properties of the underlying poset. In particular, the Möbius function is equal to the reduced Euler characteristic of the order complex of the poset, making it a homotopy invariant of the order complex. For an overview of the topological aspects of posets, the interested reader may consult the survey by Wachs [19]. Several previous results on the Möbius function of posets are based on topological tools [12, 16]. In this paper, however, our approach is purely combinatorial and requires no topological background.

Our main tool is a formula relating the Möbius function of the permutation poset to the number of embeddings between pairs of permutations. We believe this formula (Proposition 2.7 and the closely related Corollary 2.8) may find further applications in the study of the Möbius function. In fact, several such applications already emerged from our joint work with Brignall and Marchant [4], which is being prepared for publication in parallel with this paper.

2 Definitions and preliminaries

Permutations and their diagrams. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A *permutation* of size n is a bijection π of $[n]$ onto itself. We represent such a permutation π as the sequence of values $\pi(1), \pi(2), \ldots, \pi(n)$. If there is no risk of ambiguity, we omit the commas and write, for example, 312 for the permutation π with $\pi(1) = 3$, $\pi(2) = 1$ and $\pi(3) = 2$.

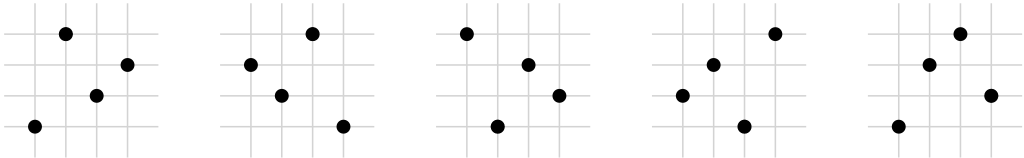
The *diagram* of a permutation π is the set of points $\{(i, \pi(i));\ i \in [n]\}$ in the plane; in other words, it is the graph of π as a function. Let $\mathcal{S}_n$ be the set of permutations of size n, and let $\mathcal{S} = \bigcup_{n\geq 1} \mathcal{S}_n$ be the set of all finite permutations.

Embeddings. A sequence of real numbers $a_1, a_2, \ldots, a_n$ is *order-isomorphic* to a sequence $b_1, b_2, \ldots, b_n$ if for every $i, j \in [n]$ we have $a_i < a_j \Leftrightarrow b_i < b_j$.

An *embedding* of a permutation $\sigma \in \mathcal{S}_k$ into a permutation $\pi \in \mathcal{S}_n$ is a function $f \colon [k] \to [n]$ such that $f(1) < f(2) < \cdots < f(k)$, and the sequence $\pi(f(1)), \pi(f(2)), \ldots, \pi(f(k))$ is order-isomorphic to $\sigma(1), \sigma(2), \ldots, \sigma(k)$. The *image* of an embedding f is the set $\mathrm{Img}(f) = \{f(i);\ i \in [k]\}$. Observe that for a given π, the set $\mathrm{Img}(f)$ determines both f and σ uniquely.

If there is an embedding of σ into π, we say that π *contains* σ, and write $\sigma \leq \pi$, otherwise we say that π *avoids* σ. The containment relation $\leq$ is a partial order on $\mathcal{S}$. We will call the pair $(\mathcal{S}, \leq)$ the *permutation poset.*

We let $\mathcal{E}(\sigma, \pi)$ denote the set of embeddings of σ into π, and we let $\mathrm{E}(\sigma, \pi)$ denote the cardinality of $\mathcal{E}(\sigma, \pi)$.



$\sigma = 1423$ $\quad \sigma^r = 3241$ $\quad \sigma^c = 4132$ $\quad \sigma^{rc} = 2314$ $\quad \sigma^{-1} = 1342$

Figure 2: The main permutation symmetries.

Permutation symmetries. For a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$, its *reverse* π^r is the permutation $\pi(n)\pi(n-1)\dots\pi(1)$, its *complement* π^c is the permutation $n+1-\pi(1), n+1-\pi(2), \dots, n+1-\pi(n)$, its *reverse-complement* π^{rc} is the permutation $(\pi^r)^c$, and its *inverse* is the permutation $\pi^{-1} \in \mathcal{S}_n$ with the property $\pi^{-1}(\pi(i)) = i$ for every $i \in [n]$. Observe that these operations correspond to reflections or rotations of the diagram of π; see Figure 2.

Note that these operations are poset automorphisms of $(\mathcal{S}, \leq)$, that is, $\sigma \leq \pi$ implies $\sigma^r \leq \pi^r$, $\sigma^c \leq \pi^c$, $\sigma^{rc} \leq \pi^{rc}$, and $\sigma^{-1} \leq \pi^{-1}$.

The Möbius function. For a poset $(P, \leq)$, we let $[x, y]$ denote the closed interval $\{z \in P;\ x \leq z \leq y\}$, and $[x, y)$ the half-open interval $\{z \in P;\ x \leq z < y\}$. A poset $(P, \leq)$ is *locally finite* if each of its intervals is finite. Given a locally finite poset $(P, \leq)$, we define its *Möbius function* $\mu\colon P \times P \to \mathbb{Z}$ by the recurrences

$$\mu(x,y) = \begin{cases} 0 \text{ if } x \not\leq y \\ 1 \text{ if } x = y \\ -\sum_{z \in [x,y)} \mu(x,z) \text{ if } x < y. \end{cases}$$

A *chain* from $x \in P$ to $y \in P$ is a set $C = \{x_0, x_1, \dots, x_k\} \subseteq P$ such that $x_0 = x$, $x_k = y$, and $x_{i-1} < x_i$ for every $i \in [k]$. The *length* of a chain C, denoted by $\ell(C)$, is defined as $|C| - 1$. We let $\mathfrak{C}(x, y)$ denote the set of all chains from x to y.

For an arbitrary set $\mathfrak{C}$ of chains, we define the *weight of* $\mathfrak{C}$, denoted by $w(\mathfrak{C})$, as $\sum_{C \in \mathfrak{C}} (-1)^{\ell(C)}$.

We will need a classical identity known as Philip Hall's Theorem, which expresses the Möbius function of an interval as the reduced Euler characteristic of the corresponding order complex. For details, see, for example, Stanley [17, Proposition 3.8.5] or Wachs [19, Proposition 1.2.6].

Fact 2.1 (Philip Hall's Theorem)**.** *If $(P, \leq)$ is a locally finite poset with elements x and y, then $\mu(x, y) = w(\mathfrak{C}(x, y))$.*

The following symmetry property of the Möbius function is a direct consequence of Philip Hall's Theorem.

Corollary 2.2. *Let $(P, \leq)$ be a locally finite poset with Möbius function μ. Let $\leq^*$ be the partial order on P defined by $x \leq^* y \Leftrightarrow y \leq x$. The Möbius function μ^* of the poset $(P, \leq^*)$ then satisfies $\mu^*(x, y) = \mu(y, x)$.*

We will often use the following easy identities, where the first one follows from the definition of μ, and the second one from Corollary 2.2.

Fact 2.3. *For a locally finite poset P and a pair of elements $x, y \in P$ with $x < y$, we have $\sum_{z \in [x,y]} \mu(x, z) = 0$ and $\sum_{z \in [x,y]} \mu(z, y) = 0$.*

We will also use the Möbius inversion formula, which is a basic property of the Möbius function. The following form of the formula can be deduced, for example, from Proposition 3.7.2 in Stanley's book [17].

Fact 2.4 (Möbius inversion formula)**.** *Let P be a locally finite poset with maximum element y, let μ be the Möbius function of P, and let $f\colon P \to \mathbb{R}$ be a function. If a function $g\colon P \to \mathbb{R}$ is defined by*

$$g(x) = \sum_{z \in [x,y]} f(z),$$

then for every $x \in P$, we have

$$f(x) = \sum_{z \in [x,y]} \mu(x, z) g(z).$$

A lemma on decreasing patterns. From now on, we only deal with the poset $(\mathcal{S}, \leq)$ of permutations ordered by the containment relation, and μ refers to the Möbius function of this poset.

Lemma 2.5. *Let δ_k be the decreasing permutation of size k; that is, $\delta_k = k, (k-1), \ldots, 1$. For any permutation π other than 1 or 12, we have*

$$\sum_{k=1}^{|\pi|} \mu(\delta_k, \pi) = 0. \tag{1}$$

Proof. Consider the set of chains $\mathfrak{C} = \bigcup_{k=1}^{|\pi|} \mathfrak{C}(\delta_k, \pi)$. In view of Fact 2.1, equation (1) is equivalent to $w(\mathfrak{C}) = 0$.

Define two sets of chains $\mathfrak{C}_1$ and $\mathfrak{C}_2$ as follows:

$\mathfrak{C}_1 = \{C \in \mathfrak{C};\ C \text{ contains a decreasing permutation of size at least } 2\}$, and
$\mathfrak{C}_2 = \mathfrak{C} \setminus \mathfrak{C}_1$.

Clearly, $w(\mathfrak{C}) = w(\mathfrak{C}_1) + w(\mathfrak{C}_2)$. We will show that both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ have zero weight.

To see that $w(\mathfrak{C}_1) = 0$, consider a parity-exchanging involution Φ_1 on $\mathfrak{C}_1$ defined as follows: if $C \in \mathfrak{C}_1$ contains the permutation 1, define $\Phi_1(C) = C \setminus \{1\}$, otherwise define $\Phi_1(C) = C \cup \{1\}$. We see that Φ_1 is an involution on $\mathfrak{C}_1$ that maps chains of odd length to chains of even length and vice versa. Therefore $w(\mathfrak{C}_1) = 0$.

To deal with $\mathfrak{C}_2$, consider the mapping Φ_2 that maps a chain $C \in \mathfrak{C}_2$ to $C \setminus \{12\}$ if C contains 12, and it maps C to $C \cup \{12\}$ otherwise. This is again easily seen to be a parity-exchanging involution on $\mathfrak{C}_2$, showing that $w(\mathfrak{C}_2) = 0$. □

In our applications, we will use Lemma 2.5 in the situation when π avoids 321. In such cases, the sum on the left-hand side of (1) has at most two nonzero summands, and the identity can be rephrased as follows.

Corollary 2.6. *Any 321-avoiding permutation π other than 1 or 12 satisfies*

$$\mu(1, \pi) = -\mu(21, \pi).$$

We remark that a slightly more restricted case of Corollary 2.6 has already been proven by Smith [15, Lemma 3.6], by a topological argument.

Möbius function via embeddings. The core of our argument is the following general formula expressing the Möbius function in terms of another function f. It can be seen as a version of the Möbius inversion formula. It can be generalized in a straightforward way to an arbitrary locally finite poset, although for our purposes, we only state it in the permutation setting.

Proposition 2.7. *Let σ and π be arbitrary permutations, and let $f\colon [\sigma, \pi] \to \mathbb{R}$ be a function satisfying $f(\pi) = 1$. We then have*

$$\mu(\sigma, \pi) = f(\sigma) - \sum_{\lambda \in [\sigma, \pi)} \mu(\sigma, \lambda) \sum_{\tau \in [\lambda, \pi]} f(\tau). \tag{2}$$

Proof. Fix σ, π and f. For $\lambda \in [\sigma, \pi]$, define $g(\lambda) = \sum_{\tau \in [\lambda, \pi]} f(\tau)$. Using Fact 2.4 for the poset $P = [\sigma, \pi]$, we obtain

$$f(\sigma) = \sum_{\lambda \in [\sigma, \pi]} \mu(\sigma, \lambda) g(\lambda). \tag{3}$$

Substituting the definition of $g(\lambda)$ into the identity (3) and using the assumption $f(\pi) = 1$, we get

$$\begin{aligned} f(\sigma) &= \sum_{\lambda \in [\sigma,\pi]} \mu(\sigma,\lambda) \sum_{\tau \in [\lambda,\pi]} f(\tau) \\ &= \mu(\sigma,\pi) + \sum_{\lambda \in [\sigma,\pi)} \mu(\sigma,\lambda) \sum_{\tau \in [\lambda,\pi]} f(\tau), \end{aligned}$$

from which the proposition follows. □

In our applications of Proposition 2.7, we shall always use the function f defined as $f(\tau) = (-1)^{|\tau|-|\pi|}\mathrm{E}(\tau,\pi)$, where π is assumed to be fixed. We state this special case of Proposition 2.7 as a corollary.

Corollary 2.8. *For any two permutations σ and π, we have*

$$\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}\mathrm{E}(\sigma,\pi) - \sum_{\lambda \in [\sigma,\pi)} \mu(\sigma,\lambda) \sum_{\tau \in [\lambda,\pi]} (-1)^{|\pi|-|\tau|}\mathrm{E}(\tau,\pi).$$

We remark that the formula of Corollary 2.8 has a similar structure to another summation formula for the Möbius function derived previously by Smith [16, Theorem 19].

Descents and inverse descents. The *inverse descent* of a permutation $\pi = \pi(1)\pi(2)\ldots\pi(m)$ is a pair of indices $i, j \in [m]$ such that $i < j$ and $\pi(i) = \pi(j) + 1$. Let $\mathrm{ides}(\pi)$ be the number of inverse descents of π. For example, 315264 has two inverse descents, corresponding to $(i,j) = (1,4)$ and $(i,j) = (3,6)$. Observe that if σ is contained in π, then $\mathrm{ides}(\sigma) \leq \mathrm{ides}(\pi)$.

The inverse descent statistic is closely related to the more familiar descent statistic, where a *descent* in a permutation π is a pair of indices i, j such that $\pi(i) > \pi(j)$ and $j = i+1$. The number of descents of π is denoted by $\mathrm{des}(\pi)$. Note that $\mathrm{des}(\pi) = \mathrm{ides}(\pi^{-1})$.

Suppose that π has only one inverse descent, occurring at positions $i < j$ with $\pi(i) = \pi(j) + 1$. We say that an element $\pi(k)$ is a *top element* if $\pi(k) \geq \pi(i)$, and it is a *bottom element* if $\pi(k) \leq \pi(j)$. We also say that k is a *top position* of π if $\pi(k)$ is a top element, and bottom positions are defined analogously. Note that each element of π is either a top element or a bottom element, and that the top elements, as well as the bottom elements, form an increasing subsequence of π.

By replacing each top element of π by the symbol '$\mathtt{t}$' and each bottom element by the symbol '$\mathtt{b}$', we encode a permutation π with $\mathrm{ides}(\pi) = 1$ into a word $w(\pi)$ over the alphabet $\{\mathtt{b}, \mathtt{t}\}$. For example, for $\pi = 31245$ we get

$w(\pi) = \texttt{tbbtt}$. Note that $w(\pi)$ determines π uniquely. On the other hand, some words over the alphabet $\{\texttt{b}, \texttt{t}\}$ do not correspond to any permutation π with $\mathrm{ides}(\pi) = 1$; for example, the word $\texttt{bbtt}$, and in general, every word where all symbols '$\texttt{b}$' appear before all symbols '$\texttt{t}$'.

This encoding of permutations into words was introduced by Smith [15], who also generalized it to permutations with k inverse descents[1], by encoding them into words over an alphabet of size $k+1$. The key feature of Smith's encoding is that if σ and π have the same number of inverse descents, then $\sigma \leq \pi$ if and only if $w(\sigma)$ is a subword of $w(\pi)$, that is, the word $w(\sigma)$ forms a (not necessarily consecutive) subsequence of $w(\pi)$. In other words, if $\mathrm{ides}(\sigma) = \mathrm{ides}(\pi)$ then the interval $[\sigma, \pi]$ is isomorphic, as a poset, to the interval $[w(\sigma), w(\pi)]$ in the subword order.

To express the Möbius function in the subword order, Björner [1, 2] has introduced the notion of normal embeddings among words. This notion was adapted by Smith [15] to the permutation setting, to express the Möbius function of permutations with a fixed number of descents. We will present Smith's definition of normal embeddings below. Let us remark that other authors have used different notions of normal embeddings, suitable for other special types of permutations [4, 6, 12, 16].

Let $\pi = \pi(1)\pi(2)\dots\pi(n)$ be a permutation. An *adjacency* in π is a maximal consecutive sequence $\pi(i)\pi(i+1)\dots\pi(i+k)$ satisfying $\pi(i+j) = \pi(i)+j$ for each $j = 1, \dots, k$; in other words, it is a maximal sequence of consecutive increasing values at consecutive positions in π. An embedding f of a permutation σ into π is *normal* if for each adjacency $\pi(i)\pi(i+1)\dots\pi(i+k)$ of π, the positions $i+1, \dots, i+k$ all belong to $\mathrm{Img}(f)$. Let $\mathcal{NE}(\sigma, \pi)$ be the set of normal embeddings of σ into π, and let $\mathrm{NE}(\sigma, \pi)$ be the cardinality of $\mathcal{NE}(\sigma, \pi)$.

As an example, consider the permutations $\sigma = 123$ and $\pi = 165234$. There are four embeddings of σ into π, with images $\{1,4,5\}$, $\{1,4,6\}$, $\{1,5,6\}$ and $\{4,5,6\}$. The permutation π has one adjacency of length more than 1, namely the sequence 234 at positions 4, 5 and 6. Thus, an embedding f into π is normal if $\mathrm{Img}(f)$ contains both 5 and 6. In particular, $\mathrm{NE}(\sigma, \pi) = 2$.

Note that if all the adjacencies in π have length 1, then every embedding of a permutation σ into π is normal.

Using Björner's formula for the Möbius function of the subword order [1, 2], Smith [15] obtained the following result.

[1] Actually, Smith works with descents rather than inverse descents, but since the inverse operation is a poset automorphism of $(\mathcal{S}, \leq)$, this change does not affect the relevant results.

π_3

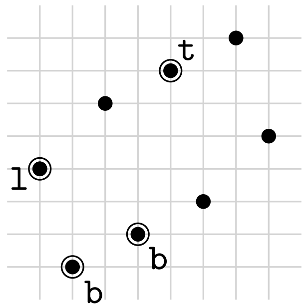


Figure 3: The permutation π_3. The circled elements correspond to the embedding $f =$ `lb-bt---` of the permutation 3124.

Fact 2.9 (Smith [15, Proposition 3.3])**.** *If* σ *and* π *satisfy* $\mathrm{des}(\sigma) = \mathrm{des}(\pi)$*, then* $\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|}\mathrm{NE}(\sigma, \pi)$*.*

Observing that $\mathrm{NE}(\sigma^{-1}, \pi^{-1}) = \mathrm{NE}(\sigma, \pi)$, and recalling that $\mu(\sigma^{-1}, \pi^{-1}) = \mu(\sigma, \pi)$ and $\mathrm{ides}(\pi^{-1}) = \mathrm{des}(\pi)$, we can rephrase Fact 2.9 as follows.

Corollary 2.10. *If* σ *and* π *satisfy* $\mathrm{ides}(\sigma) = \mathrm{ides}(\pi)$*, then*

$$\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|}\mathrm{NE}(\sigma, \pi).$$

Let $\pi \in \mathcal{S}_{2n}$ be the permutation with one inverse descent and encoding $w(\pi) = \texttt{tbtbtb}\ldots\texttt{tb}$. By Corollary 2.10 and Corollary 2.6, $\mu(1, \pi) = -\mu(21, \pi) = -\mathrm{NE}(21, \pi) = -\binom{n+1}{2}$. This example, pointed out by Smith [14], gave the largest previously known growth of $|\mu(1, \pi)|$ in terms of $|\pi|$. We note that Brignall and Marchant [5] have recently found another, substantially different example of a family of permutations π for which they conjecture that $|\mu(1, \pi)|$ is quadratic in $|\pi|$.

3 Proof of Theorem 1.1

In this section, we present a proof of Theorem 1.1. We will assume throughout that n is a fixed integer greater than 1.

Recall that we defined the permutation $\pi_n \in \mathcal{S}_{2n+2}$ by

$$\pi_n = n+1, 1, n+3, 2, n+4, 3, n+5, \ldots, n, 2n+2, n+2.$$

See Figures 1 and 3. Note that by transposing the two values $n+1$ and $n+2$ in π_n, we would obtain the permutation $\pi \in \mathcal{S}_{2n+2}$ with $w(\pi) = \texttt{tbtbt}\ldots\texttt{tb}$, considered by Smith.

We will refer to the leftmost element of π_n, that is, the value $n+1$, as the *left element*, the rightmost one as the *right element*, the elements $1, 2, \ldots, n$ as the *bottom elements*, and $n+3, n+4, \ldots, 2n+2$ as the *top elements*. If $\pi_n(i)$ is a bottom element, we say that i is a *bottom position* of π_n, and similarly for left, right and top positions.

We use the letters $\mathtt{t}, \mathtt{b}, \mathtt{l}, \mathtt{r}$ to represent the top, bottom, left and right elements, respectively. By replacing each element of π_n by the corresponding letter, we obtain *the encoding* of π_n; for example, the encoding of π_3 is `lbtbtbtr`.

Note that any subword of the encoding of π_n uniquely determines the subpermutation formed by the corresponding elements of π_n. For example, the subsequence `lbbt` corresponds to the subpermutation 3124. However, a permutation $\sigma \leq \pi_n$ may correspond to several different words: for example, 41253 corresponds to either `lbbtb` or `tbbtb` or `tbbtr`. In particular, an interval $[\sigma, \pi_n]$ will not in general be isomorphic to any interval in the subword order, and we cannot use the results of Björner and Smith directly to obtain a formula for $\mu(\sigma, \pi_n)$. On the positive side, if $[\sigma, \pi_n]$ is not isomorphic to an interval in the subword order, then $|\mu(\sigma, \pi_n)|$ can be much larger than $\mathrm{NE}(\sigma, \pi_n)$, as witnessed by Theorem 1.1.

In the rest of this section, we only consider embeddings of permutations into π_n, unless we explicitly specify otherwise. For our convenience, we will often represent an embedding f into π_n by a word, using the letters $\mathtt{b}, \mathtt{t}, \mathtt{l}, \mathtt{r}$ for the bottom, top, left and right positions in $\mathrm{Img}(f)$, and the symbol `-` for positions not in $\mathrm{Img}(f)$. For example, $f =$ `lb-bt---` is an embedding of 3124 into $\pi_3 = 41627385$ using the first, second, fourth and fifth elements (in other words, $\mathrm{Img}(f) = \{1, 2, 4, 5\}$; see Figure 3). We call this the *hyphen-letter encoding* of an embedding.

3.1 Outline of the proof of Theorem 1.1

Since π_n avoids 321, we know that $\mu(1, \pi_n) = -\mu(21, \pi_n)$ by Corollary 2.6. We therefore focus on finding the value of $\mu(21, \pi_n)$. By Corollary 2.8, $\mu(21, \pi_n)$ equals

$$(-1)^{|\pi_n|-|21|}\mathrm{E}(21, \pi_n) - \sum_{\lambda \in [21, \pi_n)} \mu(21, \lambda) \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\pi_n|-|\tau|}\mathrm{E}(\tau, \pi_n),$$

which, since π_n has even size, simplifies to

$$\mu(21, \pi_n) = \mathrm{E}(21, \pi_n) - \sum_{\lambda \in [21, \pi_n)} \mu(21, \lambda) \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\tau|}\mathrm{E}(\tau, \pi_n). \tag{4}$$

We say that a permutation $\lambda \in [21, \pi_n)$ is *vanishing* if the expression $\mu(21, \lambda) \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\tau|} \mathrm{E}(\tau, \pi_n)$, which is the outer summand on the right-hand side of (4), is equal to zero. To simplify equation (4), we will first establish several sufficient conditions for λ to be vanishing; in particular, it will turn out that there are only polynomially many non-vanishing λ, and we can describe their structure explicitly.

Our next concern will be to express, for a fixed non-vanishing $\lambda \in [21, \pi_n)$, the value

$$S_\lambda = \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\tau|} \mathrm{E}(\tau, \pi_n),$$

that is, the value of the inner sum in equation (4).

Recall that $\mathcal{E}(\tau, \pi_n)$ is the set of embeddings of τ into π_n. We further let

$$\mathcal{E}_\lambda(*, \pi_n) = \bigcup_{\tau \in [\lambda, \pi_n]} \mathcal{E}(\tau, \pi_n).$$

For an embedding g, we let $|g|$ denote the size of the permutation being embedded; equivalently, $|g| = |\operatorname{Img}(g)|$. With this notation, S_λ can be written as follows:

$$S_\lambda = \sum_{g \in \mathcal{E}_\lambda(*, \pi_n)} (-1)^{|g|}.$$

Call an embedding g *odd* if $|g|$ is odd, and even otherwise. To find a formula for S_λ, we will find a partial matching on $\mathcal{E}_\lambda(*, \pi_n)$ between odd and even embeddings, thereby cancelling out their contribution to S_λ. The unmatched embeddings will have a very specific structure, and we will be able to count them exactly.

Let g be an embedding of a permutation τ into π_n, and let $i \in [2n+2]$ be an index corresponding to a position in π_n. The *i-switch* of the embedding g, denoted by $\Delta_i(g)$, is the embedding uniquely determined by the following properties:

$$\begin{aligned} &\operatorname{Img}(\Delta_i(g)) = \operatorname{Img}(g) \cup \{i\} \text{ if } i \notin \operatorname{Img}(g), \text{ and} \\ &\operatorname{Img}(\Delta_i(g)) = \operatorname{Img}(g) \setminus \{i\} \text{ if } i \in \operatorname{Img}(g). \end{aligned}$$

Note that for any $i \in [2n+2]$, the i-switch is an involution on the set of embeddings into π_n, that is, $\Delta_i(\Delta_i(g)) = g$ for any g. Note also that Δ_i is parity-exchanging, that is, it maps even embeddings to odd ones and vice versa. Switches will be our main tool to obtain cancellations between even and odd embeddings contributing to S_λ for a fixed λ. The idea of using switches to get parity-exchanging involutions on a set of embeddings is quite common in the literature, and can be traced back at least to Björner's work on the subword order [2].

3.2 Vanishing lambdas

We will now identify sufficient conditions for λ to be vanishing, that is, for $\mu(21, \lambda)S_\lambda$ to be equal to zero.

Lemma 3.1. *Any permutation $\lambda \in [21, \pi_n)$ with $\mathrm{ides}(\lambda) = 2$ is vanishing.*

Proof. Fix $\lambda \in [21, \pi_n)$ with $\mathrm{ides}(\lambda) = 2$. Since $\mathrm{ides}(\pi_n) = 2$, it follows that any $\tau \in [\lambda, \pi_n]$ has $\mathrm{ides}(\tau) = 2$. In particular, for any such τ we have $(-1)^{|\tau|}\mathrm{NE}(\tau, \pi_n) = \mu(\tau, \pi_n)$ by Corollary 2.10. Since π_n has no adjacency of length more than 1, all the embeddings into π_n are normal, and in particular $\mathrm{NE}(\tau, \pi_n) = \mathrm{E}(\tau, \pi_n)$. Therefore,

$$\begin{aligned} S_\lambda &= \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\tau|}\mathrm{E}(\tau, \pi_n) \\ &= \sum_{\tau \in [\lambda, \pi_n]} \mu(\tau, \pi_n) \\ &= 0 && \text{by Fact 2.3,} \end{aligned}$$

showing that λ is vanishing. □

Note that any λ containing 21 has at least one inverse descent. Lemma 3.1 therefore implies that any non-vanishing λ in $[21, \pi_n)$ satisfies $\mathrm{ides}(\lambda) = 1$. From now on, we focus on permutations λ with one inverse descent.

We say that a permutation $\lambda \in [21, \pi_n)$ *omits* a position $i \in [2n + 2]$ if there is no embedding f of λ into π_n such that $i \in \mathrm{Img}(f)$.

Lemma 3.2. *If a permutation $\lambda \in [21, \pi_n)$ omits a position $i \in [2n + 2]$, then λ is vanishing.*

Proof. We easily see that the i-switch Δ_i is a parity-exchanging involution on the set $\mathcal{E}_\lambda(*, \pi_n)$, and therefore $S_\lambda = 0$. □

Corollary 3.3. *Let λ be a permutation of size m with one inverse descent. Then λ is vanishing whenever it satisfies at least one of the following conditions.*

a) The two leftmost elements $\lambda(1)$ and $\lambda(2)$ are both top elements.

b) The leftmost element $\lambda(1)$ is a bottom element, and λ has at least one other bottom element $\lambda(i)$ with $1 < i < m$.

c) The two rightmost elements $\lambda(m - 1)$ and $\lambda(m)$ are both bottom elements.

d) *The rightmost element $\lambda(m)$ is a top element, and λ has at least one other top element $\lambda(i)$ with $1 < i < m$.*

e) *The permutation λ has only one top element and at least three bottom elements.*

f) *The permutation λ has only one bottom element and at least three top elements.*

Proof. If case a) occurs, then λ omits the position $i = 2$, and if b) occurs, λ omits $i = 1$. The cases c) and d) are symmetric to the cases a) and b) via the reverse-complement symmetry (note that $\pi_n^{rc} = \pi_n$, and consequently if λ is vanishing then so is λ^{rc}). If case e) occurs, then at least one of b) and c) occurs as well. Case f) is again symmetric to e). $\square$

We say that a permutation λ of size m with one inverse descent is a *cup* if $\lambda(1)$ and $\lambda(m)$ are top elements and the remaining elements are bottom elements; in other words, λ is the permutation $(m-1), 1, 2, \ldots, (m-2), m$. We say that λ is a *cap* if $\lambda(1)$ and $\lambda(m)$ are bottom elements and the remaining elements are top elements, that is, $\lambda = 1, 3, 4, \ldots, m, 2$.

Suppose that $\lambda \le \tau \le \pi_n$, f is an embedding of λ and g is an embedding of τ. We say that f is *compatible* with g and also that g is *compatible* with f if $\mathrm{Img}(f)$ is a subset of $\mathrm{Img}(g)$.

Lemma 3.4. *If λ is a cup or a cap of size $m \ge 3$, then λ is vanishing.*

Proof. Let λ be a cup. Recall that we are considering embeddings into the permutation π_n with $n > 1$, that is, π_n has size $2n + 2 \ge 6$. We say that an embedding f of λ into π_n is *broad* if $f(m) = 2n + 2$, and f is *narrow* otherwise. Observe that any broad embedding must satisfy $f(1) = 1$. We partition the set $\mathcal{E}_\lambda(*, \pi_n)$ into two subsets A and B, where A is the set of those embeddings $g \in \mathcal{E}_\lambda(*, \pi_n)$ that are compatible with at least one narrow embedding of λ, while B contains those embeddings $g \in \mathcal{E}_\lambda(*, \pi_n)$ that are only compatible with broad embeddings of λ. Note that Δ_{2n+2} is an involution on the set A, showing that A does not contribute to S_λ.

We now deal with the set B. Consider first the situation when $m \ge 4$. We claim that in such case Δ_3 is an involution on the set B. To see this, observe first that for any $g \in B$ we have $\Delta_3(g) \in \mathcal{E}_\lambda(*, \pi_n)$, since any broad embedding of λ compatible with g is also compatible with $\Delta_3(g)$. It remains to show that there is no narrow embedding of λ compatible with $\Delta_3(g)$. Indeed, if f were such a narrow embedding, we would necessarily have $f(1) = 3$, and therefore $f(m) < 2n + 2$, since $\pi_n(f(1))$ must be smaller than $\pi_n(f(m))$. But then, if we redefine the value of $f(1)$ from 3 to 1, we obtain a narrow embedding of λ compatible with g, which is impossible since g is in B. Thus, B does not contribute to S_λ either, and λ is vanishing.

Now suppose λ has size 3, that is, $\lambda = 213$. Consider an embedding $g \in B$. Note that both 1 an $2n+2$ are in $\mathrm{Img}(g)$, and in the hyphen-letter encoding of g, all occurrences of the symbol `b` must be to the right of any occurrence of `t`, otherwise g would be compatible with a narrow embedding of λ. Let $B' \subseteq B$ be the set of those embeddings $g \in B$ whose hyphen-letter encoding has only one symbol `b` and this symbol appears at position $2n$, and let B'' be the set $B \setminus B'$. We note that Δ_{2n} is an involution on B'', while Δ_3 is an involution on B' (notice that here we require that $n \neq 1$). We conclude that any cup λ is vanishing.

A cap is the reverse-complement of a cup, and therefore caps are vanishing as well, by symmetry. □

So far we have identified several cases when λ is vanishing because S_λ is zero. We now focus on the situations when $\mu(21, \lambda) = 0$, which also implies that λ is vanishing. For a permutation λ with $\mathrm{ides}(\lambda) = 1$, we define a *top repetition* to be a pair $(\lambda(i), \lambda(i+1))$ of two consecutive top elements in λ, and similarly a *bottom repetition* to be a pair of two consecutive bottom elements.

By Corollary 2.10, $|\mu(21, \lambda)| = \mathrm{NE}(21, \lambda)$ whenever $\mathrm{ides}(\lambda) = 1$. Observing that a normal embedding of 21 into λ must contain the right element of any repetition in its image, we reach the following conclusion.

Observation 3.5. *Let λ be a permutation with $\mathrm{ides}(\lambda) = 1$. If λ has at least two top repetitions, or at least two bottom repetitions, or a top repetition appearing to the right of a bottom repetition, then 21 has no normal embedding into λ and consequently $\mu(21, \lambda) = 0$. In particular, such λ is vanishing.*

3.3 Proper lambdas

We will say that a permutation $\lambda \in [21, \pi_n)$ of size m is *proper* if it satisfies the following three conditions:

1) $\mathrm{ides}(\lambda) = 1$,

2) $\lambda(1)$ and $\lambda(m-1)$ are top elements, while $\lambda(2)$ and $\lambda(m)$ are bottom elements (for $m = 2$ the elements of each pair coincide and $\lambda = 21$),

3) λ has at most one top repetition and at most one bottom repetition; moreover, if it has both a top repetition and a bottom repetition, then the top repetition is to the left of the bottom repetition.

Condition 1) and Corollary 2.10 imply the following identity.

Corollary 3.6. *For every proper permutation λ, we have*

$$\mu(21, \lambda) = (-1)^{|\lambda|}\mathrm{NE}(21, \lambda).$$

Let $\mathcal{P}_n \subseteq [21, \pi_n)$ be the set of proper permutations, and let $\mathcal{P}_{n,m}$ be the set of proper permutations of size m. By Lemma 3.1, Corollary 3.3, Lemma 3.4 and Observation 3.5, any non-vanishing permutation $\lambda \in [21, \pi_n)$ is proper. In particular, we may simplify identity (4) as follows:

$$\mu(21, \pi_n) = \mathrm{E}(21, \pi_n) - \sum_{\lambda \in \mathcal{P}_n} \mu(21, \lambda) S_\lambda. \tag{5}$$

Note that 21 is the smallest proper permutation, and that there are no proper permutations of size 3. For future reference, we state several easy facts about embeddings of proper permutations.

Lemma 3.7. *If λ is a proper permutation, then its reverse-complement λ^{rc} is proper as well. Moreover, we have $\mu(21, \lambda) = \mu(21, \lambda^{rc})$ and $S_\lambda = S_{\lambda^{rc}}$.*

Proof. The fact that λ^{rc} is proper follows directly from the definition of proper permutation. The identity $21^{rc} = 21$ and the fact that the reverse-complement operation is an automorphism of the permutation poset imply that the intervals $[21, \lambda]$ and $[21, \lambda^{rc}]$ are isomorphic as posets, and hence $\mu(21, \lambda) = \mu(21, \lambda^{rc})$. It remains to prove that $S_\lambda = S_{\lambda^{rc}}$. Recall that $\pi_n^{rc} = \pi_n$. Thus, for any permutation τ we have $\mathrm{E}(\tau, \pi_n) = \mathrm{E}(\tau^{rc}, \pi_n)$. Moreover, τ belongs to $[\lambda, \pi_n]$ if and only if τ^{rc} belongs to $[\lambda^{rc}, \pi_n]$. Together, this gives

$$S_\lambda = \sum_{\tau \in [\lambda, \pi_n]} (-1)^{|\tau|}\mathrm{E}(\tau, \pi_n) = \sum_{\tau^{rc} \in [\lambda^{rc}, \pi_n]} (-1)^{|\tau^{rc}|}\mathrm{E}(\tau^{rc}, \pi_n) = S_{\lambda^{rc}},$$

as claimed. $\square$

Lemma 3.8. *Let λ be a proper permutation of size m, and let $f\colon [m] \to [2n+2]$ be a function. Then f is an embedding of λ into π_n if and only if it satisfies the following three conditions:*

1) *The function f is strictly increasing, that is, $f(i) < f(i+1)$ for every $i \in [m-1]$.*

2) *If $i \in [m]$ is a top position of λ, then $f(i)$ is a left or top position of π_n, and if i is a bottom position of λ then $f(i)$ is a bottom or right position of π_n.*

3) *At most one of the two values* 1 *and* $2n+2$ *is in* $\mathrm{Img}(f)$*.*

Proof. Suppose f is an embedding of λ. Then f is strictly increasing by definition. Moreover, each top element of λ has a smaller bottom element to its right, and therefore it has to be mapped to an element of π_n that has a smaller element of π_n to its right, and similarly for the bottom elements of λ. Finally, to see that $\mathrm{Img}(f)$ cannot contain both 1 and $2n+2$, recall that for any $\lambda \in \mathcal{P}_{n,m}$ we have $\lambda(1) > \lambda(m)$. Therefore, every embedding of λ satisfies the three properties of the lemma. Conversely, it is easy to observe that any function satisfying the three properties is an embedding of λ. □

As a direct consequence of Lemma 3.8, we get the following result.

Corollary 3.9. *Let f and f' be two embeddings of a proper permutation λ into π_n. Define a function f^* by $f^*(i) = \max\{f(i), f'(i)\}$. Then f^* is also an embedding of λ.*

We remark that Corollary 3.9 does not generalize to improper permutations. Consider for instance $\lambda = 3124$, which is a cup and therefore not proper. Take the two embeddings $f =$ `l--b-b-r`, with $\mathrm{Img}(f) = \{1, 4, 6, 8\}$, and $f' =$ `--tb-bt-`, with $\mathrm{Img}(f') = \{3, 4, 6, 7\}$. Their pointwise maximum $f^* =$ `--tb-b-r` is not an embedding of $\lambda = 3124$ but of 4123.

3.4 Determining S_λ for a proper λ

Fix a proper permutation λ. Our goal now is to determine the value $S_\lambda = \sum_{g\in\mathcal{E}_\lambda(*,\pi_n)}(-1)^{|g|}$. To this end, we will describe cancellations between odd and even embeddings in $\mathcal{E}_\lambda(*, \pi_n)$, so that the value of S_λ can be determined by a small and well-structured subset of uncancelled embeddings.

Let $<_L$ denote the *lexicographic order* on the set $\mathcal{E}(\lambda, \pi_n)$, which is a total order defined as follows. Let f and f' be two embeddings of λ into π_n, and let i be the smallest index for which $f(i) \neq f'(i)$. If $f'(i) < f(i)$, then put $f' <_L f$. If $g \in \mathcal{E}_\lambda(*, \pi_n)$ is an embedding, then $<_L$ can be restricted to a total order on the set of embeddings of λ that are compatible with g. The maximum element in this ordered set is called the *rightmost embedding of λ compatible with g*, or just the *rightmost embedding of λ in g*. Corollary 3.9 implies the following fact.

Corollary 3.10. *The rightmost embedding of λ in g is the pointwise maximum of all the embeddings of λ compatible with g.*

The notion of rightmost embedding will serve us to establish cancellations between odd and even embedding in $\mathcal{E}_\lambda(*, \pi_n)$. We remark that a similar approach has been used by Björner [2] in the subword poset (who uses the term *final embedding* for rightmost embedding) as well as by Sagan and Vatter [12].

We now show that rightmost embeddings can be constructed by a natural greedy right-to-left procedure. (Alternatively, they could also be characterized as 'locally rightmost', in the sense that no element can be shifted to the right alone.) Let λ be a proper permutation of size m, and let $g \in \mathcal{E}_\lambda(*, \pi_n)$ be an embedding. We say that an embedding f of λ is *greedy in g* if f is constructed by the following rules:

- $f(m)$ is equal to the largest (that is, rightmost) bottom or right position in $\mathrm{Img}(g)$,
- for each top position $i \in [m-1]$ of λ, assuming $f(i+1)$ has already been defined, $f(i)$ is equal to the largest left or top position $j \in \mathrm{Img}(g)$ such that $j < f(i+1)$, and similarly,
- for each bottom position $i \in [m-1]$ of λ, assuming $f(i+1)$ has already been defined, $f(i)$ is equal to the largest bottom position $j \in \mathrm{Img}(g)$ such that $j < f(i+1)$.

We say that an embedding f of λ is *almost greedy in g* if $\mathrm{Img}(g)$ contains the rightmost position $2n+2$, and f is greedy in the embedding g^- defined by $\mathrm{Img}(g^-) = \mathrm{Img}(g) \setminus \{2n+2\}$.

Lemma 3.11. *For any proper permutation λ and any $g \in \mathcal{E}_\lambda(*, \pi_n)$, the rightmost embedding of λ in g is greedy or almost greedy in g. Moreover, if the rightmost embedding is almost greedy, then every embedding f' of λ into g satisfies $1 \in \mathrm{Img}(f')$ and therefore $2n+2 \notin \mathrm{Img}(f')$, and there is no greedy embedding of λ in g.*

Proof. Let m be the size of λ, and let f be the rightmost embedding of λ in g. Suppose that f is not greedy in g, and let i be the largest index for which $f(i)$ differs from the value prescribed by the definition of the greedy embedding.

First consider the case $i < m$. Since $f(i)$ differs from its greedy value, there must be a position $j \in \mathrm{Img}(g)$ such that $f(i) < j < f(i+1)$, and either both j and $f(i)$ are bottom positions, or j is a top position and $f(i)$ is a top or left position. In any case, we can define a new embedding f^+ by

$$f^+(x) = \begin{cases} f(x) \text{ for } x \neq i, \\ j \text{ for } x = i. \end{cases}$$

By Lemma 3.8, f^+ is an embedding of λ, and it is clearly compatible with g. However, we have $f <_L f^+$, contradicting the choice of f.

Suppose now that $i = m$, that is, the rightmost bottom or right position $j \in \mathrm{Img}(g)$ is greater than $f(m)$. By defining f^+ as in the previous paragraph, we again get contradiction, except for the case when $f(1) = 1$ and $j = 2n + 2$. In such situation f is almost greedy. Furthermore, since $f(1) = 1$, there can be no embedding f' of λ into g with $f'(1) > 1$, because the pointwise maximum of f and f' would then contradict the choice of f. Since any embedding f' of λ compatible with g satisfies $f'(1) = 1$, we see that $2n + 2 \notin \mathrm{Img}(f')$ by Lemma 3.8, and therefore f' is not greedy. $\square$

For $f \in \mathcal{E}(\lambda, \pi_n)$, let $\mathcal{E}_f(*, \pi_n)$ be the set of all the embeddings $g \in \mathcal{E}_\lambda(*, \pi_n)$ such that f is the rightmost embedding of λ in g. Let

$$S_f = \sum_{g \in \mathcal{E}_f(*,\pi_n)} (-1)^{|g|}.$$

In particular, we have

$$\mathcal{E}_\lambda(*, \pi_n) = \bigcup_{f \in \mathcal{E}(\lambda,\pi_n)} \mathcal{E}_f(*, \pi_n), \text{ and}$$
$$S_\lambda = \sum_{f \in \mathcal{E}(\lambda,\pi_n)} S_f.$$

Proper pairs. We will now show that $S_f = 0$ except when f has a specific form.

For $f \in \mathcal{E}(\lambda, \pi_n)$, a *gap* in f is an open interval $(f(i), f(i+1))$ of integers such that $f(i + 1) > f(i) + 1$. We say that the gap has *type* `tb`, or that it is a `tb`-gap, if $\lambda(i)$ is a top element and $\lambda(i + 1)$ a bottom element; types `bt`, `bb` and `tt` are defined analogously. For instance, the embedding $f =$ `lbt--b-btb--` has two gaps, namely the `tb`-gap $(f(3), f(4)) = \{4, 5\}$ and the `bb`-gap $(f(4), f(5)) = \{7\}$.

Note that a top repetition or a bottom repetition in λ will necessarily form a gap of type `tt` or `bb`, respectively, in any embedding of λ in π_n.

Lemma 3.12. *Let λ be a proper permutation and let f be an embedding of λ into π_n with $S_f \neq 0$. Then f satisfies the following three conditions:*

1) *Position 1 is in* $\mathrm{Img}(f)$.

2) *If f has a* `tb`*-gap $(f(i), f(i+1))$, then $i \geq 4$, and $\lambda(i-1)$ and $\lambda(i-2)$ are both bottom elements.*

	f	g	f'	g'
After the definition of f':	`t----b`	`t-???b`	`??????`	`tb???b`
No `t` in the gap in g':	`t----b`	`t--?-b`	`??-?-?`	`tb-?-b`
`b` on position $j+1$ in f':	`t----b`	`t--?-b`	`?b-?-?`	`tb-?-b`
`b` on position k in f':	`t----b`	`t--?-b`	`?b-?-b`	`tb-?-b`
at most one `bb`-gap in f':	`t----b`	`t----b`	`?b---b`	`tb---b`

Table 1: Evolution of the conditions on f, g, f' and g' inside a `tb` gap of f. (For sake of example, $k = j + 5$.)

3) *If f has a* `bt`*-gap $(f(i), f(i+1))$, then $i \geq 4$ and either $\lambda(i-1)$ and $\lambda(i-2)$ are both top elements, or $\lambda(i-1)$ is a bottom element and $\lambda(i-2)$ and $\lambda(i-3)$ are both top elements.*

Proof. Let m be the size of λ. To prove part 1), note that if $1 \notin \mathrm{Img}(f)$, then Δ_1 is an involution on $\mathcal{E}_f(*, \pi_n)$, and hence $S_f = 0$.

Now we prove part 2). Let $(f(i), f(i+1))$ be a `tb`-gap, and let $j = f(i)$ and $k = f(i+1)$.

We will show that if $\lambda(i-1)$ and $\lambda(i-2)$ are not both bottom elements, then Δ_{j+1} is an involution on $\mathcal{E}_f(*, \pi_n)$ and hence $S_f = 0$. To see this, suppose that Δ_{j+1} is not such an involution, that is, there is an embedding $g \in \mathcal{E}_f(*, \pi_n)$ such that $\Delta_{j+1}(g)$ is not in $\mathcal{E}_f(*, \pi_n)$. Let $g' = \Delta_{j+1}(g)$. Since $j+1$ is not in $\mathrm{Img}(f)$, f is compatible with g'. As $g' \notin \mathcal{E}_f(*, \pi_n)$, g' is compatible with an embedding of λ greater than f in the $<_L$-order. In particular, $\mathrm{Img}(g') = \mathrm{Img}(g) \cup \{j+1\}$. Let f' be the rightmost embedding of λ in g'. We have $f <_L f'$, and also $f(l) \leq f'(l)$ for every $l \in [m]$ by Corollary 3.10.

We observe that $\mathrm{Img}(g)$ contains no top position j' in the gap (j, k), otherwise f would would not be rightmost in g, since it could be modified to map i to j' instead of j. Therefore $\mathrm{Img}(g')$ contains no such top position either. Follow Table 1 for steps in this paragraph. Since f' is compatible with g' but not with g, $\mathrm{Img}(f')$ contains $j+1$. Also, $\mathrm{Img}(f')$ contains k, otherwise we could shift the rightmost `b` in f' inside the `tb`-gap to the right, contradicting the choice of f'. Since $j+1$ and k are both bottom positions of π_n, and $\mathrm{Img}(f')$ has no top position in the gap, we conclude that λ has a bottom repetition mapped to positions $j+1$ and k by f'. In particular, the bottom repetition in λ must appear to the left of the element $\lambda(i)$.

It remains to show that the bottom repetition of λ appears at positions $i-2$ and $i-1$. Suppose that the bottom repetition appears at positions i' and $i'+1$ for some $i' < i-2$. The positions $i'+2, i'+3, \ldots, m$ do not have any repetition in λ, that is, they correspond to alternating top and bottom elements, starting with a top one. Moreover, $i > i' + 2$ by assumption, therefore in fact $i \geq i' + 4$, since $i' + 2$ and i are both top positions of λ. Note that $f'(i' + 2) > k$, since $f'(i' + 1) = k$.

We define a mapping $f^+ \colon [m] \to [2n + 2]$, contradicting the choice of f, as follows:

$$f^+(x) = \begin{cases} f(x) \text{ for } x < i, \\ f'(x-2) \text{ for } x \geq i. \end{cases}$$

We easily verify that f^+ is an embedding of λ using Lemma 3.8: condition 2) follows directly from the definition of f^+, condition 1) follows from $f(i - 1) < f(i) = j < k = f'(i' + 1) < f'(i - 2)$, and condition 3) follows from $f^+(m) = f'(m - 2) < f'(m) \leq 2n + 2$. Moreover, f^+ is compatible with g since $j + 1 \notin \mathrm{Img}(f^+)$, and $f^+(i) > f(i)$, contradicting the choice of f. This proves part 2) of the lemma.

The proof of part 3) is similar. Let $(f(i), f(i + 1))$ be a `bt`-gap, and let $j = f(i)$ and $k = f(i + 1)$. We will again show that Δ_{j+1} is an involution on $\mathcal{E}_f(*, \pi_n)$, unless λ satisfies the conditions of part 3).

Again, let $g \in \mathcal{E}_f(*, \pi_n)$ be an embedding such that $\Delta_{j+1}(g)$ is not in $\mathcal{E}_f(*, \pi_n)$. Let $g' = \Delta_{j+1}(g)$ and let f' be the rightmost embedding of λ in g'. For the same reason as in part 2), we have $\mathrm{Img}(g') = \mathrm{Img}(g) \cup \{j + 1\}$, $f <_L f'$, and $f(l) \leq f'(l)$ for every $l \in [m]$. No bottom position in the `bt`-gap can belong to $\mathrm{Img}(g)$, otherwise f would not be rightmost in g. Again, for the same reason as in part 2), both $j + 1$ and k are in $\mathrm{Img}(f')$. Thus, λ contains a top repetition to the left of $\lambda(i)$, at positions i_1 and $i_1 + 1$ for some $i_1 < i$, such that $f'(i_1) = j + 1$ and $f'(i_1 + 1) = k$. Let i_2 be the largest top position of λ smaller than i. In particular, i_2 is equal to $i - 1$ or $i - 2$. We need to prove that $i_2 = i_1 + 1$, which is equivalent to the condition in part 3).

Suppose that $i_2 > i_1 + 1$. This implies $i_2 \geq i_1 + 3$, since $i_1 + 2$ is a bottom position in λ. Since λ is proper, there is no repetition among the elements $\lambda(1), \lambda(2), \ldots, \lambda(i_1)$, that is, these elements form an alternation of top and bottom elements, starting with a top one.

We define a mapping $f^+ \colon [m] \to [2n + 2]$, contradicting the choice of f, as follows:

$$f^+(x) = \begin{cases} f(x+2) \text{ for } x \leq i_1 - 2 \\ f(i_2 - 1) \text{ for } x = i_1 - 1 \\ f(i_2) \text{ for } x = i_1 \\ f'(x) \text{ for } x > i_1 \end{cases}$$

We verify that f^+ is an embedding of λ using Lemma 3.8: condition 2) follows directly from the definition of f^+ and from the 'alternating property' of λ, condition 1) follows from $f(i_1) < f(i_2 - 1)$ and $f(i_2) < f(i) = j < f'(i_1) <$

$f'(i_1+1)$, and condition 3) follows from $f^+(1) = f(3) > 1$. Moreover, f^+ is compatible with g since $j+1 \notin \mathrm{Img}(f^+)$, and $f <_L f^+$, contradicting the definition of f. □

We say that (λ, f) is a *proper pair* if λ is a proper permutation and f is an embedding of λ into π_n that satisfies the three conditions of Lemma 3.12. Let $\mathcal{PP}_n$ be the set of all proper pairs (λ, f) where f is an embedding into π_n. Combining formula (5), Corollary 3.6 and Lemma 3.12, we get

$$\mu(21, \pi_n) = \mathrm{E}(21, \pi_n) - \sum_{(\lambda, f) \in \mathcal{PP}_n} (-1)^{|\lambda|} \mathrm{NE}(21, \lambda) S_f. \tag{6}$$

Singular embeddings. Our goal is now to compute, for a proper pair (λ, f), the value $S_f = \sum_{g \in \mathcal{E}_f(*, \pi_n)} (-1)^{|g|}$. To this end, we will introduce further cancellations on the set $\mathcal{E}_f(*, \pi_n)$. Let $j \in [2n+2]$ be the smallest index not belonging to $\mathrm{Img}(f)$. For every embedding $g \in \mathcal{E}_f(*, \pi_n)$, let $g' = \Delta_j(g)$. Clearly, $g' \in \mathcal{E}_\lambda(*, \pi_n)$, since f is compatible with g'. However, g' is not necessarily in $\mathcal{E}_f(*, \pi_n)$, because g' may be compatible with another embedding f' of λ with $f <_L f'$.

Example 3.13. Let $\lambda = 3142$ and $g =$ `lbtb-----r`. The rightmost embedding of λ in g is $f =$ `lbtb------`, hence g is in $\mathcal{E}_f(*, \pi_n)$. The first position not in $\mathrm{Img}(f)$ is the fifth one and we have $g' =$ `lbtbt----r`, where the rightmost embedding of λ is `--tbt----r`. Hence $g' \notin \mathcal{E}_f(*, \pi_n)$.

We say that an embedding $g \in \mathcal{E}_f(*, \pi_n)$ is *f-regular* if $g' \in \mathcal{E}_f(*, \pi_n)$; otherwise we say that g is *f-singular*. Let $\mathcal{SE}_f(*, \pi_n)$ be the set of f-singular embeddings in $\mathcal{E}_f(*, \pi_n)$.

Observe that if g is f-regular then g' is also f-regular. Thus, the j-switch restricts to a parity-exchanging involution on the set of f-regular embeddings. This shows that the contributions of f-regular embeddings to S_f cancel out, and therefore

$$S_f = \sum_{g \in \mathcal{SE}_f(*, \pi_n)} (-1)^{|g|}. \tag{7}$$

We will now analyze f-singular embeddings in detail.

For an embedding f of a permutation λ of size m into π_n, the set $\{i \in [2n+2];\ i > f(m)\}$ is called the *tail* of f. A *segment* of f is a maximal subset of consecutive integers belonging to $\mathrm{Img}(f)$. Thus, if $1 \in \mathrm{Img}(f)$, then the set $[2n+2]$ can be partitioned into segments, gaps and the tail of f.

Lemma 3.14. *Let (λ, f) be a proper pair where $|\lambda| = m$ and f is an embedding into π_n. Let $j = \min([2n+2] \setminus \mathrm{Img}(f))$. Let g be an f-singular* *embedding and let $g' = \Delta_j(g)$. Let f' be the rightmost embedding of λ in g'. Then the following properties hold:*

(a) $\mathrm{Img}(g') = \mathrm{Img}(g) \cup \{j\}$, $j \notin \mathrm{Img}(g)$, $j \in \mathrm{Img}(f')$, *and* $f <_L f'$.

(b) *For every* $i \in [m]$ *we have* $f(i) < f'(i)$.

(c) *The embedding f is almost greedy in g, and f' is greedy in g'. In particular, there is no greedy embedding of λ in g, and* $\mathrm{Img}(f')$ *and* $\mathrm{Img}(g)$ *both contain the position* $2n + 2$.

(d) $\mathrm{Img}(g)$ *has no bottom position in the tail of f, no top position in any* `tt`*-gap or* `tb`*-gap of f, and no bottom position in any* `bb`*-gap or* `bt`*-gap of f.*

(e) *If f has at least one gap, then* $\mathrm{Img}(g)$ *has no position in the leftmost gap of f, and has at least one top position in the tail of f.*

(f) *If f has at least two gaps, and the second gap from the left is a* `bt`*-gap or a* `tb`*-gap, then* $\mathrm{Img}(g)$ *has no position in the second gap.*

(g) *If f has a* `tt`*-gap* $(f(i), f(i+1))$ *and a* `bb`*-gap* $(f(i+2), f(i+3))$*, then* $\mathrm{Img}(g)$ *contains at most one top position in the* `bb`*-gap.*

Proof. (a) These facts directly follow from g being f-singular.

(b) By Corollary 3.10, we have $f(i) \leq f'(i)$ for every i. Since the interval $[1, j-1]$ is a segment in f, we have $j = f'(i')$ for some $i' < j - 1$ that has the same parity as j. Since f' is rightmost, we have $f'(i) = i + (j - i')$ for every $i \leq i'$. Consequently, $f(i) < f'(i)$ for every $i \leq j - 1$.

Suppose that for some $i_0 \geq j$ we have $f(i_0) = f'(i_0)$. Define a mapping f^+ by

$$f^+(x) = \begin{cases} f'(x) \text{ if } j-1 \leq x < i_0 \\ f(x) \text{ otherwise.} \end{cases}$$

Clearly, f^+ satisfies all three conditions of Lemma 3.8 and thus it is an embedding of λ. Also $j \notin \mathrm{Img}(f^+)$, so f^+ is compatible with g. Finally, $f(j-1) = j-1$ and $f'(j-1) > j$ imply $f <_L f^+$; this is a contradiction with f being rightmost in g.

(c) By Lemma 3.11, we know that f is greedy or almost greedy in g, and f' is greedy or almost greedy in g'. Note that the value $j \in \mathrm{Img}(g') \setminus \mathrm{Img}(g)$ cannot be equal to either of $f(m)$ or $f'(m)$: indeed, we have either $j < m$ (in case f has a gap), or $j = m + 1$ (when f has no gap) and in the latter case j is a top position and m a bottom one. In particular, $f'(m) \in \mathrm{Img}(g)$. By part (b), we have $f(m) < f'(m)$, which implies that f is almost greedy in g, further implying that $f'(m) = 2n+2$ and f' is greedy in g'. By Lemma 3.11, this implies that there is no greedy embedding of λ in g.

(d) If $i \in \mathrm{Img}(g)$ is a bottom position in the tail of f, or a top position in a `tt`-gap or `tb`-gap of f, or a bottom position in a `bb`-gap or `bt`-gap of f, we get a contradiction with the almost-greedy property of f, since the position of λ mapped by f to the largest position of $\mathrm{Img}(f)$ to the left of i would be mapped to i or to the right of i by an almost-greedy embedding.

(e) Suppose that $\mathrm{Img}(g)$ has a position in the leftmost gap $(f(j-1), f(j))$ of f, and let k be the leftmost such position. Assume that $\lambda(j-1)$ is a top element; the other case is analogous, with the roles of bottom and top elements exchanged. Thus, j is a bottom position of π_n. By parts (a) and (d) of the current lemma, $\mathrm{Img}(g)$, and therefore also $\mathrm{Img}(g')$ and $\mathrm{Img}(f')$, have no top position in the gap $(f(j-1), f(j))$, so k is a bottom position and $k > j$.

The facts that $j \in \mathrm{Img}(f')$, f' is greedy, and j and k are consecutive bottom positions in $\mathrm{Img}(g')$, imply that $k \in \mathrm{Img}(f')$. Thus, the two positions of λ that are mapped to j and k by f' form a bottom repetition in λ. This bottom repetition forms a gap in f which is to the left of j, contradicting the definition of j. This proves that $\mathrm{Img}(g)$ has no position in the leftmost gap of f.

Now we show that $\mathrm{Img}(g)$ has a top position in the tail of f. Since λ is proper, the position $m-1$ is the rightmost top position in λ. By part (b) of the current lemma, we have $f'(m-1) > f(m-1)$. We claim that $f'(m-1)$ is in the tail of f: if not, then $(f(m-1), f(m))$ would be a `tb`-gap in f and $f'(m-1)$ would be a top position in this `tb`-gap, contradicting part (d) of the current lemma. Therefore, $f'(m-1)$ is a top position in the intersection of $\mathrm{Img}(g')$ and the tail of f. Finally, since f has at least one gap, we have $f(m-1) \geq j-1$, implying $f'(m-1) > j$, and hence $f'(m-1)$ is in $\mathrm{Img}(g)$ as well.

(f) Let $(f(i), f(i+1))$ be the second gap of f from the left. Suppose that this is a `bt`-gap; the other case is analogous, with the roles of bottom and top elements exchanged. By Lemma 3.12 we have $i \geq 4$, and since f cannot have both a `tt`-gap and a `bb`-gap to the left of $f(i)$, the leftmost gap of f is the `tt`-gap $(f(i-2), f(i-1))$. For contradiction, suppose that g contains a position k in the `bt`-gap $(f(i), f(i+1))$ of f. By part (d), we know that k is a top position.

By part (b), we have $f'(i) > f(i)$, and since $\mathrm{Img}(g')$ has no bottom position in the `bt`-gap $(f(i), f(i+1))$ of f, this implies $f'(i) > f(i+1)$. Since f' is greedy, it follows that $f'(i-1) \geq f(i+1)$, $f'(i-2) \geq k$, and

$f'(i-3) \geq f(i) > f(i-2)$. We may now define a mapping f^+ as

$$f^+(x) = \begin{cases} f'(x) \text{ if } x \geq i-3 \\ f(x+2) \text{ if } x \leq i-4. \end{cases}$$

Since $1 \notin \operatorname{Img} f'$, f^+ is an embedding of λ, clearly satisfying $f <_L f^+$. Since f^+ does not use the position j from the first gap of f, f^+ is compatible with g, which contradicts f being the rightmost in g.

(g) Suppose first that $(f(i), f(i+1))$ is a tt-gap and $(f(i+2), f(i+3))$ a bb-gap. Since λ is proper, we have $i \geq 3$. Since λ has at most one top repetition and at most one bottom repetition, Lemma 3.12 implies that the tt-gap $(f(i), f(i+1))$ is the leftmost gap of f. By part (d), $\operatorname{Img}(g)$, and consequently also $\operatorname{Img}(g')$ and $\operatorname{Img}(f')$, have no bottom position in the bb-gap $(f(i+2), f(i+3))$. By part (b), we have $f'(i+2) > f(i+2)$, and therefore $f'(i+2) \geq f(i+3)$. For contradiction, suppose that $\operatorname{Img}(g)$ contains at least two top positions $k_1 < k_2$ in the bb-gap of f. The greediness of f' implies $f'(i+1) \geq k_2$, $f'(i) \geq k_1$ and $f'(i-1) \geq f(i+2)$. We then define f^+ by

$$f^+(x) = \begin{cases} f'(x) \text{ if } x \geq i-1 \\ f(x+2) \text{ if } x \leq i-2. \end{cases}$$

By the same reasoning as in part (f), we get a contradiction with f being the rightmost in g. □

3.5 Adding up all contributions

Let π_n be fixed. Recall from (6) and (7) that

$$\mu(21, \pi_n) = \mathrm{E}(21, \pi_n) - \sum_{(\lambda, f) \in \mathcal{PP}_n} (-1)^{|\lambda|} \mathrm{NE}(21, \lambda) S_f, \tag{8}$$

where $S_f = \sum_{g \in \mathcal{SE}_f(*, \pi_n)} (-1)^{|g|}$. We will now evaluate the sum on the right-hand side of (8). We will distinguish the proper pairs (λ, f) depending on the number of repetitions of λ and the number of gaps of f. For integers $a \leq b$, we let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$. When representing the structure of an embedding by its hyphen-letter notation, we underline the individual segments for added clarity, and we use the ellipsis '…' for segments of unknown length. We will use an auxiliary symbol '*' to denote a sequence of hyphens of arbitrary length, possibly empty. In particular, -*- represents a sequence of hyphens of length at least 2, and if '*' is adjacent to a segment from both left and right, the two segments may possibly form a single segment. We say that such a potentially empty sequence of hyphens represents a *potential gap*.

Case A: λ has no repetitions. Then f has no gaps, by Lemma 3.12. In the hyphen-letter notation, we have

$$f = \underline{\texttt{lbtb}\ldots\texttt{tbtb}}\texttt{-*-}.$$

Let $\mathcal{PP}_A$ be the set of proper pairs (λ, f) of this form; similarly, $\mathcal{PP}_B, \mathcal{PP}_C$, $\mathcal{PP}_D$ and $\mathcal{PP}_E$ will be the sets of proper pairs to be considered in subsequent cases.

Fix a proper pair $(\lambda, f) \in \mathcal{PP}_A$. We claim that $\mathcal{SE}_f(*, \pi_n)$ contains exactly one embedding g_A, determined by $\mathrm{Img}(g_A) = \mathrm{Img}(f) \cup \{2n + 2\}$; that is,

$$g_A = \underline{\texttt{lbtb}\ldots\texttt{tbtb}}\texttt{-*}\underline{\texttt{r}}.$$

It is easy to see that f is the rightmost embedding in g_A. Since $|g_A|$ is odd, the contribution of g_A to S_f is $(-1)^{|g_A|} = -1$.

Now assume that $g \in \mathcal{SE}_f(*, \pi_n)$. We have $2n+2 \in \mathrm{Img}(g)$ by Lemma 3.14 (c). By Lemma 3.14 (d), $\mathrm{Img}(g)$ has no bottom position in the tail of f. We claim that $\mathrm{Img}(g)$ has no top position in the tail of f either. Indeed, if $\mathrm{Img}(g)$ contained a top position k in the tail of f, then g would be compatible with a greedy embedding f^+ of λ satisfying $\mathrm{Img}(f^+) = (\mathrm{Img}(f) \cup \{k, 2n + 2\}) \setminus \{1, 2\}$, and this would contradict Lemma 3.14 (c). Therefore, $\mathrm{Img}(g) = \mathrm{Img}(f) \cup \{2n + 2\}$.

To compute $\sum_{(\lambda,f)\in\mathcal{PP}_A} (-1)^{|\lambda|}\mathrm{NE}(21, \lambda)S_f$, we reason as follows: to every triple (i_1, i_2, i_3) with $1 \le i_1 \le i_2 \le i_3 \le n$ we associate a proper λ of size $2i_3$ with no repetitions, and a normal embedding h of 21 into λ with $\mathrm{Img}(h) = \{2i_1 - 1, 2i_2\}$. As there are $\binom{n+2}{3}$ triples (i_1, i_2, i_3) of this form, and $S_f = -1$ for all $(\lambda, f) \in \mathcal{PP}_A$, we get

$$\sum_{(\lambda,f)\in\mathcal{PP}_A} (-1)^{|\lambda|}\mathrm{NE}(21, \lambda)S_f = -\binom{n+2}{3}.$$

Case B: λ has a bottom repetition but no top repetition. Then f has a bb-gap, and possibly also a tb-gap immediately following it; that is,

$$f = \underline{\texttt{lb}\ldots\texttt{tb}}\texttt{-*}\underline{\texttt{bt}}\texttt{*}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{-*-}$$

with the second segment of length exactly 2 and the third of length at least 1, and these two are possibly combined into a single segment.

By Lemma 3.14 (c) and (e), if $g \in \mathcal{SE}_f(*, \pi_n)$, then $\mathrm{Img}(g)$ contains $2n+2$ as well as at least one top position in the tail of f. On the other hand, by Lemma 3.14 (e) and (f), $\mathrm{Img}(g)$ has no position in the gaps of f.

Conversely, we claim that if $\mathrm{Img}(g) = \mathrm{Img}(f) \cup T \cup \{2n+2\}$ where T is a nonempty set of top positions in the tail of f, then $g \in \mathcal{SE}_f(*, \pi_n)$. Clearly g is compatible with f since $\mathrm{Img}(f) \subset \mathrm{Img}(g)$. To show that $g \in \mathcal{E}_f(*, \pi_n)$, we observe that f is almost greedy in g and that there is no greedy embedding of λ in g, since every embedding of λ compatible with g must coincide with f on all top positions of λ before the repetition. It remains to show that g is singular. If j is the leftmost position in the leftmost gap of f, $g' = \Delta_j(g)$, k is the second position in the second segment of f, and l is the rightmost position of T, then the embedding f^+ of λ with image $\mathrm{Img}(f) \setminus \{1, 2, k\} \cup \{j, l, 2n+2\}$ is greedy in g' and satisfies $f <_L f^+$.

We now compute the value of S_f. We can apply the involution Δ_{2n+1} to cancel out the contribution of those embeddings $g \in \mathcal{SE}_f(*, \pi_n)$ that contain at least one top position in the tail different from $2n+1$. This leaves exactly one embedding in $\mathcal{SE}_f(*, \pi_n)$ that is not cancelled, namely,

$$g_B = \underline{\mathtt{lb}\ldots\mathtt{tb}}\mathtt{-*}\underline{\mathtt{bt}}\mathtt{*}\underline{\mathtt{btb}\ldots\mathtt{tb}}\mathtt{*}\underline{\mathtt{tr}}.$$

Since $|g_B|$ is odd, the contribution of g_B to S_f is $(-1)^{|g_B|} = -1$ and hence $S_f = -1$.

Note that a normal embedding of 21 into λ must map the second element of 21 to the second element of the bottom repetition of λ. To compute $\sum_{(\lambda,f)\in\mathcal{PP}_B}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f$, we encode the contributions to this sum as quintuples $(i_1, i_2, \ldots, i_5)$ with $1 \le i_1 \le i_2 < i_3 < i_4 \le i_5 \le n$, corresponding to the embedding f with segments $[1, 2i_2]$, $\{2i_3, 2i_3+1\}$, and $[2i_4, 2i_5]$ (the latter two segments possibly merged into a single one), and the normal embedding h of 21 into λ specified by $\mathrm{Img}(fh) = \{2i_1 - 1, 2i_3\}$, where fh is the embedding of 21 to π_n that is a composition of h and f. Since $|\lambda|$ is odd, we have

$$\sum_{(\lambda,f)\in\mathcal{PP}_B}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f = \binom{n+2}{5}.$$

Case C: λ has a top repetition and no bottom repetition. The proper permutations λ of this form are precisely the reverse-complements of the permutations considered in Case B. From this, we may deduce that the contributions of the two cases are equal. To see this, let $\mathcal{P}_B$ denote the set of all the proper permutations with a bottom repetition and no top repetition, and $\mathcal{P}_C$ the set of all the proper permutations with a top repetition and no bottom repetition. We then obtain

$$\begin{aligned}
\sum_{(\lambda,f)\in\mathcal{PP}_C}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f &= \sum_{\lambda\in\mathcal{P}_C}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_\lambda \\
&= \sum_{\lambda\in\mathcal{P}_B}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_\lambda \qquad \text{(Lemma 3.7)} \\
&= \sum_{(\lambda,f)\in\mathcal{PP}_B}(-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f \\
&= \binom{n+2}{5}.
\end{aligned}$$

Case D: λ has two repetitions, and the top repetition is not adjacent to the bottom one. For such λ we have $\mathrm{NE}(21,\lambda)=1$, and f has the form

$$f = \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{*}\underline{\texttt{tb}\ldots\texttt{tb}}\texttt{-*}\underline{\texttt{bt}}\texttt{*}\underline{\texttt{bt}\ldots\texttt{tb}}\texttt{-*-},$$

with the second and the fourth segments of length 2, either of them possibly combined with the following segment into a segment of length at least 3.

Fix $g\in\mathcal{SE}_f(*,\pi_n)$. By Lemma 3.14 (c), (e) and (f), $\mathrm{Img}(g)$ contains $2n+2$ and at least one top position in the tail of f, but it has no position in the tt-gap or the bt-gap of f. Moreover, By Lemma 3.14 (d), $\mathrm{Img}(g)$ has no bottom position in the bb-gap of f and no top position in the tb-gap of f. We conclude that $\mathrm{Img}(g)=\mathrm{Img}(f)\cup T_{\texttt{bb}}\cup B_{\texttt{tb}}\cup T\cup\{2n+2\}$ where $T_{\texttt{bb}}$ is a set of top positions in the bb-gap of f, $B_{\texttt{tb}}$ is a set of bottom positions in the tb-gap of f, and T is a nonempty set of top positions in the tail of f.

Moreover, at least one of $T_{\texttt{bb}}$ and $B_{\texttt{tb}}$ must be nonempty, otherwise the mapping f' defined as in Lemma 3.14 would have to map the top elements in the third segment of f to the same positions as f, contradicting Lemma 3.14 (b).

On the other hand, it cannot happen that $T_{\texttt{bb}}$ and $B_{\texttt{tb}}$ are both nonempty, because in this case g would admit a greedy embedding of λ, contradicting Lemma 3.14 (c); we illustrate this in the following example, where f^+ is the greedy embedding of λ into g:

$$\begin{aligned}
f &= \underline{\texttt{lbtbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{tbtb}}\texttt{---}\underline{\texttt{bt}}\texttt{--}\underline{\texttt{btb}}\texttt{----},\\
g &= \underline{\texttt{lbtbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{tbtbt}}\texttt{--}\underline{\texttt{btb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}},\\
f^+ &= \texttt{--}\underline{\texttt{tbtb}}\texttt{----}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{t}}\texttt{-}\underline{\texttt{tbt}}\texttt{--}\underline{\texttt{btb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}}.
\end{aligned}$$

We conclude that for any $g\in\mathcal{SE}_f(*,\pi_n)$, exactly one of the two sets $T_{\texttt{bb}}$ and $B_{\texttt{tb}}$ is empty.

Conversely, it is straightforward to verify that if g is an embedding whose image has the form $\mathrm{Img}(f) \cup T_{\texttt{bb}} \cup B_{\texttt{tb}} \cup T \cup \{2n+2\}$ as above, with exactly one of $T_{\texttt{bb}}$ and $B_{\texttt{tb}}$ being empty, then $g \in \mathcal{SE}_f(*, \pi_n)$; see the following example:

$$
\begin{aligned}
f &= \underline{\texttt{lbtbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{tbtb}}\texttt{---}\underline{\texttt{bt}}\texttt{--}\underline{\texttt{btb}}\texttt{----},\\
g_1 &= \underline{\texttt{lbtbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{tbtbt}}\texttt{--}\underline{\texttt{bt}}\texttt{--}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}},\\
f_1' &= \texttt{--}\underline{\texttt{tbtbtb}}\texttt{--}\underline{\texttt{t}}\texttt{---}\underline{\texttt{tbtbt}}\texttt{--}\underline{\texttt{b}}\texttt{---}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}},\\
g_2 &= \underline{\texttt{lbtbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{--}\underline{\texttt{tbtb}}\texttt{---}\underline{\texttt{btb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}},\\
f_2' &= \texttt{--}\underline{\texttt{tbtbtb}}\texttt{--}\underline{\texttt{t}}\texttt{---}\underline{\texttt{tbt}}\texttt{----}\underline{\texttt{btb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}}.
\end{aligned}
$$

Note that $B_{\texttt{tb}}$ can be nonempty only when f has a $\texttt{tb}$-gap.

As in Case B, we can apply the involution Δ_{2n+1} to cancel out the contributions of all $g \in \mathcal{SE}_f(*, \pi_n)$ except those for which $T = \{2n+1\}$. By an analogous argument, we cancel out all $g \in \mathcal{SE}_f(*, \pi_n)$ except those for which $T_{\texttt{bb}}$ is either empty or a singleton set containing the leftmost element of the $\texttt{bb}$-gap of f, and $B_{\texttt{tb}}$ is either empty or a singleton set containing the leftmost element of the $\texttt{tb}$-gap of f. After these cancellations, the contribution of $\mathcal{SE}_f(*, \pi_n)$ restricts to just two embeddings g_1 and g_2 shown in the two examples, where g_2 is only applicable if f has a $\texttt{tb}$-gap. Since both $|g_1|$ and $|g_2|$ are odd, the contribution of each of g_1 and g_2 to S_f is -1.

The embeddings of the form g_1 can be encoded by 7-tuples $1 \le i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \le i_7 \le n$ where $\mathrm{Img}(f) = [1, 2i_1+1] \cup \{2i_2+1, 2i_2+2\} \cup [2i_3+1, 2i_4] \cup \{2i_5, 2i_5+1\} \cup [2i_6, 2i_7]$. The embeddings g_2 can be encoded in the same way, only now we have the extra condition that $i_6 > i_5 + 1$. Therefore, there are $\binom{n+1}{7}$ embeddings of the form g_1 and $\binom{n}{7}$ embeddings of the form g_2, so the total contribution from Case D is

$$\sum_{(\lambda, f) \in \mathcal{PP}_D} (-1)^{|\lambda|} \mathrm{NE}(21, \lambda) S_f = -\binom{n+1}{7} - \binom{n}{7}.$$

Case E: λ has two repetitions, and they are adjacent to each other. Again, we have $\mathrm{NE}(21, \lambda) = 1$. The form of f is

$$f = \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{-*}\underline{\texttt{b}}\texttt{*}\underline{\texttt{t}}\texttt{*}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{-*-},$$

where the third and fourth segment, as well as the fourth and fifth segment, are separated by a potential gap; that is, any of these two pairs of consecutive segments may in fact be merged into a single segment.

Fix $g \in \mathcal{SE}_f(*, \pi_n)$. By Lemma 3.14 (d) and (e), we have $\mathrm{Img}(g) = \mathrm{Img}(f) \cup T_{\texttt{bb}} \cup T_{\texttt{bt}} \cup B_{\texttt{tb}} \cup T \cup \{2n+2\}$ where $T_{\texttt{bb}}$ is a set of top positions in the $\texttt{bb}$ gap of f, $T_{\texttt{bt}}$ and $B_{\texttt{tb}}$ are defined analogously, and T is a nonempty set of top positions in the tail of f. In addition, Lemma 3.14 (g) implies $|T_{\texttt{bb}}| \leq 1$.

First we assume that $B_{\texttt{tb}}$ is nonempty; this is of course only possible when f has a $\texttt{tb}$-gap. Then $T_{\texttt{bb}}$ and $T_{\texttt{bt}}$ are both empty, otherwise g would admit a greedy embedding of λ; see the following example:

$$\begin{aligned}
f &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{---}\underline{\texttt{b}}\texttt{--}\underline{\texttt{t}}\texttt{--}\underline{\texttt{btb}}\texttt{----},\\
g_1 &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{---}\underline{\texttt{bt}}\texttt{-}\underline{\texttt{tb}}\texttt{-}\underline{\texttt{btbt}}\texttt{-}\underline{\texttt{tr}},\\
f_1' &= \texttt{--}\underline{\texttt{tb}}\texttt{----}\underline{\texttt{t}}\texttt{----}\underline{\texttt{bt}}\texttt{-}\underline{\texttt{tb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}},\\
g_2 &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tbt}}\texttt{--}\underline{\texttt{b}}\texttt{--}\underline{\texttt{tb}}\texttt{-}\underline{\texttt{btbt}}\texttt{-}\underline{\texttt{tr}},\\
f_2' &= \texttt{--}\underline{\texttt{tb}}\texttt{----}\underline{\texttt{tbt}}\texttt{-----}\underline{\texttt{tb}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}}.
\end{aligned}$$

Thus $\mathrm{Img}(g) = \mathrm{Img}(f) \cup B_{\texttt{tb}} \cup T \cup \{2n+2\}$ with $B_{\texttt{tb}}$ and T both nonempty. Conversely, every g with $\mathrm{Img}(g)$ of this form belongs to $\mathcal{SE}_f(*, \pi_n)$; see the following example:

$$\begin{aligned}
f &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{---}\underline{\texttt{b}}\texttt{--}\underline{\texttt{t}}\texttt{----}\underline{\texttt{btb}}\texttt{----},\\
g &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{---}\underline{\texttt{b}}\texttt{--}\underline{\texttt{tb}}\texttt{-}\underline{\texttt{b}}\texttt{-}\underline{\texttt{btbt}}\texttt{-}\underline{\texttt{tr}},\\
f' &= \texttt{--}\underline{\texttt{tbtb}}\texttt{--}\underline{\texttt{t}}\texttt{-------}\underline{\texttt{t}}\texttt{--}\underline{\texttt{b}}\texttt{-}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}}.
\end{aligned}$$

Applying analogous cancellations as in Case B and Case D, we cancel out all g of this form except those with $T = \{2n+1\}$ and $B_{\texttt{tb}}$ a singleton set containing the leftmost position in the $\texttt{tb}$-gap of f. Thus the contribution of this type of g to $\mathcal{SE}_f(*, \pi_n)$ reduces to the following embedding g, shown along with f for clarity:

$$\begin{aligned}
f &= \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{-*}\underline{\texttt{b}}\texttt{*}\underline{\texttt{t}}\texttt{-*-}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{-*-},\\
g &= \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{-*}\underline{\texttt{b}}\texttt{*}\underline{\texttt{tb}}\texttt{-*}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{*}\underline{\texttt{tr}}.
\end{aligned}$$

Since $|g|$ is odd, the contribution of g to S_f is -1. Every g of this form can be encoded by a 6-tuple $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq n$ where $\mathrm{Img}(f) = [1, 2i_1+1] \cup \{2i_2+1, 2i_2+2, 2i_3+2, 2i_4+1\} \cup [2i_5+2, 2i_6]$. Therefore, the number of these embeddings g is $\binom{n}{6}$.

Now we assume that $B_{\texttt{tb}}$ is empty. We claim that $T_{\texttt{bb}}$ is nonempty and thus $|T_{\texttt{bb}}| = 1$: if $T_{\texttt{bb}}$ were empty, every embedding of λ compatible with g' would coincide with f on the top positions in the first two segments of f; in particular, there would be no greedy embedding of λ in g'.

We conclude that $\mathrm{Img}(g) = \mathrm{Img}(f) \cup \{k\} \cup T_{\texttt{bt}} \cup T \cup \{2n+2\}$ where k is a top position in the $\texttt{bb}$-gap of f, $T_{\texttt{bt}}$ is a possibly empty set of top positions in the $\texttt{bt}$-gap of f, and T is a nonempty set of top positions in the tail of f.

Conversely, every such g belongs to $\mathcal{SE}_f(*, \pi_n)$; see the following example:

$$\begin{aligned} f &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tb}}\texttt{---}\underline{\texttt{b}}\texttt{----}\underline{\texttt{t}}\texttt{--}\underline{\texttt{btb}}\texttt{----}, \\ g &= \underline{\texttt{lbtbt}}\texttt{---}\underline{\texttt{tbt}}\texttt{--}\underline{\texttt{bt}}\texttt{-}\underline{\texttt{t}}\texttt{-}\underline{\texttt{t}}\texttt{--}\underline{\texttt{btbt}}\texttt{-}\underline{\texttt{tr}}, \\ f' &= \texttt{--}\underline{\texttt{tbtb}}\texttt{--}\underline{\texttt{t}}\texttt{-}\underline{\texttt{t}}\texttt{--}\underline{\texttt{b}}\texttt{-------}\underline{\texttt{btb}}\texttt{--}\underline{\texttt{tr}}. \end{aligned}$$

The contributions of these embeddings sum to zero whenever f has a $\texttt{bt}$-gap, since for any top position i in the $\texttt{bt}$-gap, the i-switch Δ_i is a parity-exchanging involution on these embeddings. If f has no $\texttt{bt}$-gap, usual cancellations restrict the contributions of this type of g to the case $T = \{2n+1\}$, corresponding to the following f and g:

$$\begin{aligned} f &= \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{-*-}\underline{\texttt{bt}}\texttt{*}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{-*-}, \\ g &= \underline{\texttt{lbt}\ldots\texttt{bt}}\texttt{-*}\underline{\texttt{tb}}\texttt{*}\underline{\texttt{t}}\texttt{*}\underline{\texttt{bt}}\texttt{*}\underline{\texttt{btb}\ldots\texttt{tb}}\texttt{*}\underline{\texttt{tr}}. \end{aligned}$$

Since $|g|$ is odd, the contribution of g to S_f is -1. Every g of this form can be encoded by a 6-tuple $1 \le i_1 < i_2 < i_3 < i_4 < i_5 \le i_6 \le n$ where $\mathrm{Img}(g) = [1, 2i_1+1] \cup \{2i_2+1, 2i_2+2, 2i_3+1, 2i_4, 2i_4+1\} \cup [2i_5, 2i_6] \cup \{2n+1, 2n+2\}$. Therefore, the number of these embeddings g is $\binom{n+1}{6}$.

Adding the contributions of the two types of embeddings considered in Case E, we get

$$\sum_{(\lambda,f)\in\mathcal{PP}_E} (-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f = -\binom{n}{6} - \binom{n+1}{6}.$$

Final count. Adding the contributions of Case D and E, we get $-\binom{n+1}{7} - \binom{n}{7} - \binom{n+1}{6} - \binom{n}{6} = -\binom{n+2}{7} - \binom{n+1}{7}$. Noting that $\mathrm{E}(21,\pi_n) = \binom{n}{2} + 2n$, and summing the five preceding cases together, we get

$$\begin{aligned} \mu(1,\pi_n) = -\mu(21,\pi_n) &= -\mathrm{E}(21,\pi_n) + \sum_{(\lambda,f)\in\mathcal{PP}_n} (-1)^{|\lambda|}\mathrm{NE}(21,\lambda)S_f \\ &= -\binom{n+2}{7} - \binom{n+1}{7} + 2\binom{n+2}{5} - \binom{n+2}{3} - \binom{n}{2} - 2n, \end{aligned}$$

and Theorem 1.1 is proved.

4 Further directions and open problems

Determining the fastest possible growth of $|\mu(1,\pi)|$ as a function of $|\pi|$ is still widely open. Defining

$$f(n) = \max\{|\mu(1,\pi)|; |\pi| = n\},$$

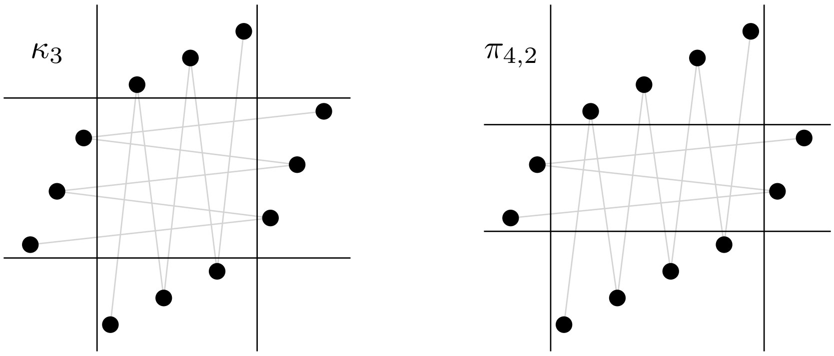


Figure 4: The permutations κ_3 (left) and $\pi_{4,2}$ (right).

Theorem 1.1 gives an asymptotic lower bound $f(n) \geq \Omega(n^7)$. We believe this is just a first step towards proving much better lower bounds on $f(n)$. The main obstacle here is our inability to compute or even estimate $|\mu(1, \pi)|$ for a general π.

Our computational experiments suggest that μ might grow exponentially fast even for permutations of seemingly simple structure. Let $\kappa_n \in \mathcal{S}_{4n}$ be a permutation defined as

$$\kappa_n = n+1, n+3, \ldots, 3n-1, 1, 3n+1, 2, 3n+2, \ldots, n, 4n, n+2, n+4, \ldots, 3n;$$

see Figure 4. Note that κ_n is a 321-free permutation that can be split into four 'quadrants', each consisting of an increasing subsequence of length n.

Conjecture 4.1. *The absolute value of $\mu(1, \kappa_n)$ is exponential in n.*

The permutation π_n is a subpermutation of κ_n for which we were able to compute $\mu(1, \pi_n)$ precisely due to a relatively simple structure of the interval $[1, \pi_n]$. For $k \leq n$, let $\pi_{n,k}$ be the subpermutation of κ_n induced by the values $n+1, n+3, \ldots, n+2k-1, 1, 3n+1, 2, 3n+2, \ldots, n, 4n, n+2, n+4, \ldots, n+2k$ in κ_n; see Figure 4. In particular, $\pi_{n,1} = \pi_n$. Our intuition and some preliminary results support the following conjecture.

Conjecture 4.2. *For every fixed $k \geq 1$, the absolute value of $\mu(1, \pi_{n,k})$ grows as $\Theta(n^{k^2+6k})$.*

From Philip Hall's Theorem (Fact 2.1), we see that $|\mu(1, \pi)|$ is bounded from above by the number of chains from 1 to π in the interval $[1, \pi]$, which is further bounded from above by the number of chains from $\emptyset$ to $[n]$ in the poset $(\mathcal{P}([n]), \subseteq)$ of all subsets of $[n]$; see A000670 in OEIS [13]. This gives the rough upper bound

$$f(n) \leq \left(\frac{1}{\log_e 2}\right)^n \cdot n! < (1.443)^n \cdot n!$$

for large n.

This bound can be further improved using Ziegler's result [22, Lemma 4.6], which states that the Möbius function of an interval $[x, y]$ in any locally finite poset is bounded from above by the number of maximal chains from x to y. Again, by counting maximal chains in $(\mathcal{P}([n]), \subseteq)$, we get the upper bound

$$f(n) \leq n!.$$

This is still far even from the exponential lower bound proposed in Conjecture 4.1.

Problem 4.3. *Is $f(n) \leq 2^{O(n)}$?*

We note without giving further details that the number of maximal chains from 1 to π can grow as $2^{\Omega(n \log n)}$ for some permutations π of size n, including our permutation π_n.

Hereditary classes. Suppose that π is restricted to a given proper down-set $\mathcal{C}$ of $(\mathcal{S}, \leq)$, that is, to a hereditary permutation class. Determining whether the values of $\mu(1, \pi)$ are bounded by a constant for $\pi \in \mathcal{C}$ is also an interesting problem. Burstein et al. [6] show that $\mu(1, \pi)$ is bounded on the class of the so-called separable permutations, while Smith [14, 15] shows that it is unbounded on permutations with at most one descent. These results suggest that the growth of $\mu(1, \pi)$ might depend the so-called simple permutations in the class, where a permutation is *simple* if it does not map a nontrivial interval of consecutive positions to an interval of consecutive values.

Problem 4.4. *On which hereditary permutation classes is $\mu(1, \pi)$ bounded? Is it bounded on every class with finitely many simple permutations? Is it unbounded on every class with infinitely many simple permutations?*

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