Permutation Invariant Gaussian Matrix Models

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ABSTRACT

Permutation invariant Gaussian matrix models were recently developed for applications in computational linguistics. A 5-parameter family of models was solved. In this paper, we use a representation theoretic approach to solve the general 13-parameter Gaussian model, which can be viewed as a zero-dimensional quantum field theory. We express the two linear and eleven quadratic terms in the action in terms of representation theoretic parameters. These parameters are coefficients of simple quadratic expressions in terms of appropriate linear combinations of the matrix variables transforming in specific irreducible representations of the symmetric group S_D where D is the size of the matrices. They allow the identification of constraints which ensure a convergent Gaussian measure and well-defined expectation values for polynomial functions of the random matrix at all orders. A graph-theoretic interpretation is known to allow the enumeration of permutation invariants of matrices at linear, quadratic and higher orders. We express the expectation values of all the quadratic graph-basis invariants and a selection of cubic and quartic invariants in terms of the representation theoretic parameters of the model.

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Contents

1   Introduction

In the context of distributional semantics [1, 2], the meaning of words is represented by vectors which are constructed from the co-occurrences of a word of interest with a set of context words. In tensorial compositional distributional semantics [3, 4, 5, 6, 7], different types of words, depending on their grammatical role, are associated with vectors, matrices or higher rank tensors. In [8, 9] we initiated a study of the statistics of these tensors in the framework of matrix/tensor models. We focused on matrices associated with adjectives or verbs, constructed by a linear regression method, from the vectors for nouns and for adjective-noun composites or verb-noun composites.

We developed a 5-parameter Gaussian model,

$$\mathcal{Z}(\Lambda, a, b, J^0, J^S) \quad = \int dM e^{-\frac{\Lambda}{2}\sum_{i=1}^{D} M_{ii}^2 - \frac{1}{4}(a+b)\sum_{i<j}(M_{ij}^2 + M_{ji}^2)}$$

$$e^{-\frac{1}{2}(a-b)\sum_{i<j} M_{ij}M_{ji} + J^0 \sum_i M_{ii} + J^S \sum_{i<j}(M_{ij} + M_{ji})} \quad (1.1)$$

The parameters J^S, J^0, a, b, Λ are coefficients of five linearly independent linear and quadratic functions of the D^2 random matrix variables $M_{i,j}$ which are permutation invariant, i.e. obey the equation

$$f(M_{i,j}) = f(M_{\sigma(i),\sigma(j)}) \quad (1.2)$$

for $\sigma \in S_D$, the symmetric group of all permutations of D distinct objects. This S_D invariance implements the notion that the meaning represented by the word-matrices is independent of the ordering of the D context words. General observables of the model are polynomials $f(M)$ obeying the condition (1.2). At quadratic order there are 11 linearly independent polynomials, which are listed in Appendix B of [8]. A three dimensional subspace of quadratic invariants was used in the model above. The most general Gaussian matrix model compatible with S_D symmetry considers all the eleven parameters and allow coefficients for each of them. What makes the 5-parameter model relatively easy to handle is that the diagonal variables M_{ii} are each decoupled from each other and from the off-diagonal elements, and there are $D(D-1)/2$ pairs of off-diagonal elements. For each $i < j$, M_{ij} and M_{ji} mix with each other so the solution of the model requires an inversion of a 2×2 matrix.

Expectation values of $f(M)$ are computed as

$$\langle f(M) \rangle \equiv \frac{1}{\mathcal{Z}} \int dM f(M) \text{EXP} \quad (1.3)$$

where EXP is the product of exponentials in (1.1).

Representation theory of S_D offers the techniques to solve the general permutation invariant Gaussian model. The D^2 matrix elements M_{ij} transform as the tensor product

$V_D \otimes V_D$ of two copies of the natural representation V_D. We first decompose $V_D \otimes V_D$ into irreducible representations of the diagonal S_D.

$$V_D \otimes V_D = 2V_0 \oplus 3V_H \oplus V_2 \oplus V_3 \qquad (1.4)$$

The trivial (one-dimensional) representation V_0 occurs with multiplicity 2. The $(D-1)$-dimensional irreducible representation (irrep) V_H occurs with multiplicity 3. V_2 is an irrep of dimension $\frac{(D-1)(D-3)}{2}$ which occurs with multiplicity 1. Likewise, V_3 of dimension $\frac{(D-1)(D-2)}{2}$ occurs with multiplicity 1. As a result of these multiplicities, the 11 parameters can be decomposed as

$$11 = 1 + 1 + 3 + 6 \qquad (1.5)$$

3 is the size of a symmetric 2×2 matrix. 6 is the size of a symmetric 3×3 matrix. More precisely the parameters form

$$\mathcal{M} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{M}_2^+ \times \mathcal{M}_3^+ \qquad (1.6)$$

where $\mathbb{R}^+$ is the set of real numbers greater or equal to zero, $\mathcal{M}_r^+$ is the space of positive semi-definite matrices of size r. Calculating the correlators of this Gaussian model amounts to inverting a symmetric 2×2 matrix, inverting a symmetric 3×3 matrix, and applying Wick contraction rules, as in quantum field theory, for calculating correlators. There is a Graph basis for permutation invariant functions of M. This is explained in Appendix B of [8] which gives examples of graph basis invariants and representation theoretic counting formulae which make contact with the sequence A052171 - directed multi-graphs with loops on any number of nodes - of the Online Encyclopaedia of Integer Sequences (OEIS) [10].

In this paper we show how all the linear and quadratic moments of the graph-basis invariants are expressed in terms of the representation theoretic parameters of (1.6). We also show how some cubic and quartic graph basis invariants are expressed in terms of these parameters. These results are analytic expressions valid for all D.

The paper is organised as follows. Section 2 introduces the relevant facts from the representation theory of S_D we need in a fairly self-contained way, which can be read with little prior familiarity of rep theory, but only knowledge of linear algebra. This is used to define the 13-parameter family of Gaussian models (equations (2.62) ,(2.63), (2.64)). Section 3 calculates the expectation values of linear and quadratic graph-basis invariants in the Gaussian model. Sections 4 and 5 describe calculations of expectation values of a selection of cubic and quartic graph-basis invariants in the model.

2 General permutation invariant Gaussian Matrix models

We solved a permutation invariant Gaussian Matrix model with 2 linear and 3 quadratic parameters [8]. The linear parameters are coefficients of linear permutation invariant

functions of M and the quadratic parameters (denoted Λ, a, b) are coefficients of quadratic functions. We explained the existence of a $2+11$ parameter family of models, based on the fact that there are 11 linearly independent quadratic permutation invariant functions of a matrix. The general $2+11$-parameter family of models can be solved by using techniques from the representation theory of S_D. Useful background references on representation theory are [11, 12, 13]. We begin by collecting the relevant facts which will allow a useful parametrisation of the quadratic invariants of a matrix M_{ij}. An important step is to form linear combinations of the M_{ij} labelled by irreducible representations of S_D. The results of this step are in (2.42)-(2.45). The quadratic terms in the action of the Gaussian model are close to diagonal in these S-variables.

The matrix elements M_{ij}, where i, j run over $\{1, 2, \cdots, D\}$ span a vector space of dimension D^2. It is isomorphic to the tensor product $V_D \otimes V_D$, where V_D is a D-dimensional space. Consider V_D as a span of D basis vectors $\{e_1, e_2, \cdots, e_D\}$. This is a representation of S_D. For every permutation $\sigma \in S_D$, there is a linear operator $\rho_{V_D}(\sigma)$ defined by

$$\rho_{V_D}(\sigma)e_i = e_{\sigma^{-1}(i)} \tag{2.1}$$

on the basis vectors and extended by linearity. With this definition, ρ_{V_D} is a homomorphism from S_n to linear operators acting on V_D.

$$\rho_{V_D}(\sigma_1)\rho_{V_D}(\sigma_2) = \rho_{V_D}(\sigma_1\sigma_2) \tag{2.2}$$

We can take the basis vectors to be orthonormal.

$$(e_i, e_j) = \delta_{ij} \tag{2.3}$$

We can form the following linear combinations

$$
\begin{aligned}
E_0 &= \frac{1}{\sqrt{D}}(e_1 + e_2 + \cdots + e_D) \\
E_1 &= \frac{1}{\sqrt{2}}(e_1 - e_2) \\
E_2 &= \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3) \\
&\vdots \\
E_a &= \frac{1}{\sqrt{a(a+1)}}(e_1 + e_2 + \cdots + e_a - ae_{a+1}) \\
&\vdots \\
E_{D-1} &= \frac{1}{\sqrt{D(D+1)}}(e_1 + e_2 + \cdots + e_{D-1} - (D-1)e_D)
\end{aligned}
\tag{2.4}
$$

E_0 is invariant under the action of S_D

$$\rho_{V_D}(\sigma)E_0 = E_0 \tag{2.5}$$

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The one-dimensional vector space spanned by E_0 is an S_D invariant vector subspace of V_D. We can call this vector space V_0. The vector space spanned by E_a, where $1 \leq a \leq (D-1)$, which we call V_H, is also an S_D-invariant subspace.

$$\rho_{V_D}(\sigma)E_a \in V_H \tag{2.6}$$

We have some matrices $D_{ab}^H(\sigma)$ such that

$$\rho_{V_D}(\sigma)E_a = \sum_b D_{ba}^H(\sigma)E_b \tag{2.7}$$

These matrices are obtained by using the action on the e_i and the change of basis coefficients. The vectors E_A for $0 \leq A \leq D$ are orthonormal.

$$(E_A, E_B) = \delta_{A,B} \tag{2.8}$$

All the above facts are summarised by saying that the natural representation V_D of S_D decomposes as an orthogonal direct sum of irreducible representations of S_D as

$$V_D = V_0 \oplus V_H \tag{2.9}$$

By reading off the coefficients in the expansion of the E_0, E_a in V_H, we can define the coefficients

$$\begin{aligned} C_{0,i} &= (E_0, e_i) \\ C_{a,i} &= (E_a, e_i) \end{aligned} \tag{2.10}$$

They are

$$\begin{aligned} C_{0,i} &= \frac{1}{\sqrt{D}} \\ C_{a,i} &= \mathcal{N}_a \left(-(a+1) + \sum_{j=1}^{a} \delta_{ji} \right) \\ \mathcal{N}_a &= \frac{1}{\sqrt{a(a+1)}} \end{aligned} \tag{2.11}$$

The orthonormality means that

$$\sum_i C_{0,i}C_{0,i} = 1$$
$$\sum_i C_{a,i}C_{b,i} = \delta_{a,b}$$
$$\sum_i C_{0,i}C_{a,i} = 0$$

$$(2.12)$$

The last equation implies that

$$\sum_i C_{a,i} = 0 \tag{2.13}$$

From

$$\sum_{A=0}^{D-1} C_{A,i} C_{A,j} = C_{0,i} C_{0,i} + \sum_{a=1}^{D-1} C_{a,i} C_{a,j} = \delta_{i,j} \tag{2.14}$$

we deduce

$$\sum_a C_{a,i} C_{a,j} = (\delta_{ij} - \frac{1}{D}) \equiv F(i,j) \tag{2.15}$$

As we will see, this function $F(i,j)$ will play an important role in calculations of correlators in the Gaussian model. It is the projector in V_D for the subspace V_H, obeying

$$\sum_j F(i,j) F(j,k) = F(i,k)$$
$$\sum_i F(i,i) = (D-1) \tag{2.16}$$

Now we will use these coefficient $C_{A,i}$ to build linear combinations of the matrix elements $M_{i,j}$ which have well-defined transformation properties under $S_D \times S_D$. Define

$$
\begin{aligned}
S^{00} &= \sum_{i,j} C_{0,i} C_{0,j} M_{ij} = \frac{1}{D} \sum_{i,j=1}^{D} M_{ij} \\
S_a^{0H} &= \sum_{i,j} C_{0,i} C_{a,j} M_{ij} = \frac{1}{\sqrt{D}} \sum_{i,j} C_{a,j} M_{ij} \\
S_a^{H0} &= \sum_{i,j} C_{a,i} C_{0,j} M_{ij} = \frac{1}{\sqrt{D}} \sum_{i,j} C_{a,i} M_{ij} \\
S_{ab}^{HH} &= \sum_{i,j} C_{a,i} C_{b,j} M_{i,j}
\end{aligned}
\tag{2.17}
$$

The a, b indices range over $1 \cdots (D-1)$. These variables are irreducible under $S_D \times S_D$, transforming as $V_0 \otimes V_0, V_0 \otimes V_H, V_H \otimes V_0, V_H \otimes V_H$. Under the diagonal S_D, the first three transform as V_0, V_H, V_H while S_{ab}^{HH} form a reducible representation.

Conversely, we can write these M variables in terms of the S variables, using the orthogonality properties of the $C_{0,i}, C_{a,i}$.

$$M_{ij} = C_{0,i} C_{0,j} S^{00} + \sum_{a=1}^{D-1} C_{0,i} C_{a,j} S_a^{0H} + \sum_{a=1}^{D-1} C_{a,i} C_{0,j} S_a^{H0} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} S_{ab}^{HH}$$

$$= \frac{1}{D}S^{00} + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1} C_{a,j}S_a^{0H} + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1} C_{a,i}S_a^{H0} + \sum_{a,b=1}^{D-1} C_{a,i}C_{b,j}S_{ab}^{HH} \quad (2.18)$$

The next step is to consider quadratic products of these S-variables, and identify the products which are invariant. In order to do this we need to understand the transformation properties of the above S variables in terms of the diagonal action of S_D. It is easy to see that S^{00} is invariant. S_a^{0H} and S_a^{H0} both have a single a index running over $\{1, 2, \cdots, (D-1)\}$, and they transform in the same way as V_H. The vector space spanned by S_{ab}^{HH} form a space of dimension $(D-1)^2$ which is

$$V_H \otimes V_H \quad (2.19)$$

Permutations act on this as

$$\sigma(S_{ab}^{HH}) = \sum_{a_1, b_1} D_{a_1 a}^H(\sigma) \, D_{b_1 b}^H(\sigma) \, S_{a_1, b_1}^{HH} \quad (2.20)$$

2.1 Useful facts from representation theory

- The representation space $V_H \otimes V_H$ can be decomposed into irreducible representations (irreps) of the diagonal S_D action as

$$V_H \otimes V_H = V_0 \oplus V_H \oplus V_2 \oplus V_3 \quad (2.21)$$

In Young diagram notation for irreps of S_D

$$\begin{aligned}
V_0 &\to [n] \\
V_H &\to [n-1, 1] \\
V_2 &\to [n-2, 2] \\
V_3 &\to [n-2, 1, 1]
\end{aligned} \quad (2.22)$$

- These irreps are known to have dimensions $1, (D-1), \frac{D(D-3)}{2}, \frac{(D-1)(D-2)}{2}$. They add up to $(D-1)^2$ which is the dimension of $V_H^{\otimes 2}$.

- The vector $\sum_{a=1}^{D-1} e_a \otimes e_a$ is invariant under diagonal action of the σ. The action of σ on V_H is given by

$$D_{ab}^H(\sigma) = (E_a, \sigma E_b) = \sum_i C_{a,i}C_{b,\sigma(i)} \quad (2.23)$$

These can be verified to satisfy the homomorphism property

$$D_{ab}^H(\sigma)D_{bc}^H(\tau) = D_{ac}^H(\sigma\tau) \quad (2.24)$$

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We also have $D_{ab}^H(\sigma^{-1}) = D_{ba}^H(\sigma)$. Using these properties, we can show that $e_a \otimes e_a$ is invariant under the diagonal action. The vector

$$\frac{1}{D-1}\sum_a e_a \otimes e_a = \frac{1}{(D-1)}\sum_a e_a \otimes e_a \tag{2.25}$$

- The vector in V_0 coming from S_{ab} is simply

$$\sum_{i,j} C_{0,i}C_{0,j}M_{ij} = \frac{1}{D}\sum_{i,j} M_{ij} \equiv S^{HH\to 0} \tag{2.26}$$

The vector in V_H inside (2.21) is some linear combination

$$\sum_{b,c} C_{b,\ c\ ;\ a}^{H,H\to H} S_{bc}^{HH} \equiv S_a^{H,H\to H} \tag{2.27}$$

The coefficients $C_{b,\ c\ ;\ a}^{H,H\to H}$ are some representation theoretic numbers (called Clebsch-Gordan coefficients) which satisfy the orthonormality condition

$$\sum_{b,c} C_{b,\ c\ ;\ a}^{H,H\to H} C_{b,\ c\ ;\ d}^{H,H\to H} = \delta_{a,d} \tag{2.28}$$

As shown in Appendix B, these Clebsch-Gordan coefficients are proportional to $C_{a,b,c} \equiv \sum_i C_{a,i}C_{b,i}C_{c,i}$

$$C_{a,\ b\ ;\ c}^{HH\to H} = \sqrt{\frac{D}{(D-2)}}C_{a,b,c} \tag{2.29}$$

It is a useful fact that the Clebsch-Gordan coefficients for $V_H \otimes V_H \to V_H$ can be usefully written in terms of the $C_{a,i}$ describing V_H as a subspace of the natural representation. This has recently played a role in the explicit description of a ring structure on primary fields of free scalar conformal field theory [14]. It would be interesting to explore the more general construction of explicit Clebsch-Gordan coefficients and projectors in the representation theory of S_D in terms of the $C_{a,i}$.

- Similarly for V_2, V_3 we have corresponding vectors and clebsch-Gordan coefficients

$$\sum_{b,c} C_{b,\ c\ ;\ a}^{H,H\to V_2} S_{b,c} \equiv S_a^{HH\to V_2} \tag{2.30}$$

where a ranges from 1 to $Dim(V_2) = \frac{D(D-3)}{2}$. We have the orthogonality property

$$\sum_{b,c} C_{b,\ c\ ;\ a_1}^{H,H\to V_2} C_{b,\ c\ ;\ a_2}^{H,H\to V_2} = \delta_{a_1,a_2} \tag{2.31}$$

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And for V_3

$$\sum_{b,c} C^{H,H\to V_3}_{b,\ c\ ;\ a} S^{HH}_{bc} \equiv S^{HH\to V_3}_a$$

$$\sum_{b,c} C^{H,H\to V_3}_{b,\ c\ ;\ a_1} C^{H,H\to V_3}_{b,\ c\ ;\ a_2} = \delta_{a_1,a_2} \tag{2.32}$$

Here the a, a_1, a_2 runs over 1 to $\frac{(D-1)(D-2)}{2}$.

- The projector for the subspace of $V_H \otimes V_H$ transforming as V_H under the diagonal S_D is

$$
\begin{aligned}
(P^{H,H\to H})_{a,b;c,d} &= \sum_e C^{H,H\to H}_{a,\ b\ ;\ e} C^{H,H\to H}_{c,\ d\ ;\ e} \\
&= \frac{D}{(D-2)} \sum_e C_{a,b,e} C_{c,d,e} \tag{2.33}
\end{aligned}
$$

- The projector $P^{H,H\to V_0}$ for V_0 in $V_H \otimes V_H$ is

$$(P_{H,H\to V_0})_{a,b;c,d} = \frac{1}{(D-1)} \delta_{a,b} \delta_{c,d} \tag{2.34}$$

$V_{[n-2,1,1]} = V_3$ is just the anti-symmetric of $V_H \otimes V_H$. It is the orthogonal complement to $V_H \oplus V_0$ inside the symmetric subspace of $V_H \otimes V_H$ which is invariant under the swop of the two factors (often denoted $\mathrm{Sym}^2(V_H)$)

$$
\begin{aligned}
(P^{H,H\to V_2}) &= (1 - P^{H,H\to H} - P^{H,H\to V_0}) \frac{(1+s)}{2} \\
(s)_{a,b;c,d} &= \delta_{a,d} \delta_{b,c} \tag{2.35}
\end{aligned}
$$

The quadratic invariant corresponding to V_2 is

$$S^{HH}_{ab} (P^{H,H\to V_2})_{a,b;c,d} S^{HH}_{cd} \tag{2.36}$$

The quadratic invariant corresponding to V_3 is similar. We just have to calculate

$$P^{H,H\to V_3} = \frac{1}{2}(1-s) \tag{2.37}$$

- The inner product

$$\langle M_{ij}, M_{kl} \rangle = \delta_{ik} \delta_{jl} \tag{2.38}$$

is invariant under the action $\sigma(M_{i,j}) = M_{\sigma^{-1}(i),\sigma^{-1}(j)}$.

$$\langle \sigma(M_{ij}), \sigma(M_{kl}) \rangle = \langle M_{ij}, M_{kl} \rangle \tag{2.39}$$

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- The following is an important fact about invariants. Every irreducible representation of S_D, let us denote it by V_R has the property that

$$\mathrm{Sym}^2(V_R) \tag{2.40}$$

contains the trivial irrep once. This invariant is formed by taking the sum over an orthonormal basis $\sum_A e_A^V \otimes e_A^V$.

$$
\begin{aligned}
D^{V \otimes V}(\sigma) \sum_A e_A^V \otimes e_A^V &= \sum_A D^V(\sigma) e_A^V \otimes D^V(\sigma) e_A^V \\
&= \sum_B D_{BA}^V(\sigma) D_{CA}^V(\sigma) e_B^V \otimes e_C^V \\
&= \sum_B D_{BA}^V(\sigma) D_{AC}^V(\sigma^{-1}) e_B^V \otimes e_C^V \\
&= \sum_A e_A^V \otimes e_A^V
\end{aligned}
\tag{2.41}
$$

- To summarize the matrix variables M_{ij} can be linearly transformed to the following variables, organised according to representations of the diagonal S_D.

$$
\begin{array}{ll}
\text{Trivial rep:} & S^{00}, S^{HH \to 0} \\
\text{Hook rep:} & S_a^{0,H}, S_a^{H,0}, S_a^{H,H \to H} \\
\text{The rep } V_2: & S_a^{HH \to V_2} \\
\text{The rep } V_3: & S_a^{HH \to V_3}
\end{array}
\tag{2.42}
$$

- For convenience, we will also use simpler names

$$
\begin{aligned}
S^{V_0;1} &= S^{00} \\
S^{V_0;2} &= S^{H,H \to 0}
\end{aligned}
\tag{2.43}
$$

where we introduced labels $1, 2$ to distinguish two occurrences of the trivial irrep V_0 in the space spanned by the M_{ij}. We will also use

$$
\begin{aligned}
S_a^{H;1} &= S_a^{0,H \to H} \\
S_a^{H;2} &= S_a^{H,0 \to H} \\
S_a^{H;3} &= S_a^{H,H \to H}
\end{aligned}
\tag{2.44}
$$

where we introduced labels $1, 2, 3$ to distinguish the three occurrences of V_H the space spanned by M_{ij}. For the multiplicity-free cases, we introduce

$$
\begin{aligned}
S_a^{V_2} &= S_a^{HH \to V_2} \\
S_a^{V_3} &= S_a^{HH \to V_3}
\end{aligned}
\tag{2.45}
$$

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The M_{ij} variables can be written as linear combinations of the S variables. Rep-basis expansion of M_{ij}

$$
\begin{aligned}
M_{ij} &= C_{0,i}C_{0,j}S^{00} + C_{a,i}C_{b,j}S^{HH}_{ab} + C_{0,i}C_{a,j}S^{0H}_a + C_{a,i}C_{0,j}S^{H0}_a \\
&= \frac{1}{D}S^{00} + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,j}S^{0H}_a + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,i}S^{H0}_a + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}S^{HH}_{ab} \\
&= \frac{1}{D}S^{00} + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,j}S^{0H}_a + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,i}S^{H0}_a \\
&\quad + \sum_{a,b}C_{a,i}C_{b,j}\sum_{V\in\{V_0,V_H,V_2,V_3\}}C^{HH\to V}_{a,\ b\ ;\ c}S^{HH\to V}_c \\
&= \frac{1}{D}S^{00} + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,j}S^{0H}_a + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,i}S^{H0}_a + \frac{1}{\sqrt{D-1}}\sum_a C_{a,i}C_{a,j}S^{HH\to V_0} \\
&\quad + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to H}_{a,\ b\ ;\ c}S^{HH\to H}_c + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to V_2}_{a,\ b\ ;\ c}S^{HH\to V_2}_c \\
&\quad + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to V_3}_{a,\ b\ ;\ c}S^{HH\to V_3}_c
\end{aligned}
\tag{2.46}
$$

In going from first to second line, we have used the fact that the transition from the natural representation to the trivial is given by simple constant coefficients. In the third line, we have used the Clebsch-Gordan coefficients for $V_H \otimes V_H \to V$, obeying the orthogonality

$$
\sum_{a,b}C^{H,H\to V}_{abc}C^{H,H\to V}_{abc'} = \delta_{cc'}
\tag{2.47}
$$

For $V = V_0$, which is one dimensional, we just have

$$
C^{HH\to V_0}_{ab} = \frac{\delta_{ab}}{\sqrt{D-1}}
\tag{2.48}
$$

It is now useful to collect together the terms corresponding to each irrep V_0, V_H, V_2, V_3

$$
\begin{aligned}
M_{ij} &= \left(\frac{1}{D}S^{00} + \frac{1}{\sqrt{D-1}}\sum_a C_{a,i}C_{a,j}S^{HH\to 0}\right) \\
&\quad + \left(\frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,j}S^{0H}_a + \frac{1}{\sqrt{D}}\sum_{a=1}^{D-1}C_{a,i}S^{H0}_a + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to H}_{a,\ b\ ;\ c}S^{HH\to H}_c\right) \\
&\quad + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to V_2}_{a,\ b\ ;\ c}S^{HH\to V_2}_c + \sum_{a,b=1}^{D-1}C_{a,i}C_{b,j}C^{HH\to V_3}_{a,\ b\ ;\ c}S^{HH\to V_3}_c
\end{aligned}
\tag{2.49}
$$

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Using the notation of (2.43), (2.44), (2.45) , we write this as

$$
\begin{aligned}
M_{ij} = {} & \left(\frac{1}{D} S^{V_0;1} + \frac{1}{\sqrt{D-1}} \sum_a C_{a,i} C_{a,j} S^{V_0;2} \right) \\
& + \left(\frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,j} S_a^{H;1} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S_a^{H;2} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} C_{a,\ b\ ;\ c}^{HH \to H} S_c^{H;3} \right) \\
& + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} C_{a,\ b\ ;\ c}^{HH \to V_2} S_c^{HH \to V_2} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} C_{a,\ b\ ;\ c}^{HH \to V_3} S_c^{HH \to V_3}
\end{aligned}
\tag{2.50}
$$

- The discussion so far has included explicit bases for V_H inside V_D which are easy to write down. A key object in the above discussion is the projector $F(i,j)$ defined in (2.15). For the irreps V_2, V_3 which appear in $V_H \otimes V_H$, we will not need to write down explicit bases. Although Clebsch-Gordan coefficients for $H, H \to V_2$ and $H, H \to V_3$ appear in some of the above formulae, we will only need some of their orthogonality properties rather than their explicit forms. The projectors for V_2, V_3 in $V_D \otimes V_D$ can be written in terms of the $F(i,j)$, and it is these projectors which play a role in the correlators we will be calculating.

2.2 Representation theoretic description of quadratic invariants

With the above background of facts from representation theory at hand, we can give a useful description of quadratic invariants. Quadratic invariant functions of M_{ij} form the invariant subspace of $\mathrm{Sym}^2(V_D \otimes V_D)$ since M_{ij} transform as $V_D \otimes V_D$.

$$
\begin{aligned}
(V_D \otimes V_D) &= (V_0 \oplus V_H) \otimes (V_0 \oplus V_H) \\
&= (V_{00} \oplus V_{0,H}^H \oplus V_{H,0}^H \oplus V_{H,H}^0 \oplus V_{H,H}^H \oplus V_{H,H}^{V_2} \oplus V_{H,H}^{V_3})
\end{aligned}
\tag{2.51}
$$

So there are two copies of V_0, namely $V_{0,0}^0, V_{H,H}^0$. $\mathrm{Sym}^2(V_{0,0}^0 \oplus V_{H,H}^0)$ contains three invariants.

$$
\begin{aligned}
(S^{00})^2 &= (S^{V_0;1})^2 \\
(S^{00} S^{HH \to 0}) &= S^{V_0;2} S^{V_0;1} = S^{V_0;1} S^{V_0;2} \\
(S^{HH \to 0})^2 &= (S^{V_0;2})^2
\end{aligned}
\tag{2.52}
$$

These are all easy to write in terms of the original matrix variables (using the formulae for S-variables in terms of M). The general invariant quadratic function of the $S^{V_0;\alpha}$ variables is

$$
\sum_{\alpha,\beta=1}^{2} (\Lambda_{V_0})_{\alpha\beta} S^{V_0;\alpha} S^{V_0;\beta}
\tag{2.53}
$$

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Λ_{V_0} is a 2×2 symmetric matrix. As we will see later, in defining the Gaussian model, this matrix will be restricted to be positive semi-definite.

There are three copies of V_H, namely $V_{0,H}^H, V_{H,0}^H, V_{H,H}^H$. These lead to 6 invariants.

$$
\begin{aligned}
\sum_a S_a^{0,H\to H} S_a^{0,H\to H} &= \sum_a S_a^{H;1} S_a^{H;1} \\
\sum_a S_a^{H,0\to H} S_a^{H,0\to H} &= \sum_a S_a^{H;2} S_a^{H;2} \\
\sum_a S_a^{H,H\to H} S_a^{H,H\to H} &= \sum_a S_a^{H;3} S_a^{H;3} \\
\sum_a S_a^{0,H\to H} S_a^{H,0\to H} &= \sum_a S_a^{H;1} S_a^{H;2} &= \sum_a S_a^{H;2} S_a^{H;1} \\
\sum_a S_a^{0,H\to H} S_a^{H,H\to H} &= \sum_a S_a^{H;1} S_a^{H;3} &= \sum_a S_a^{H;3} S_a^{H;1} \\
\sum_a S^{H,0\to H_a} S_a^{H,H\to H} &= \sum_a S_a^{H;2} S_a^{H;3} &= \sum_a S_a^{H;3} S_a^{H;2}
\end{aligned}
\tag{2.54}
$$

The sum over a runs over $(D-1) = \text{Dim}(V_H)$ elements of a basis for V_H. Thus the general quadratic invariants arising from the H representation among the M_{ij} are

$$
\sum_{\alpha,\beta=1}^{3} (\Lambda_H)_{\alpha\beta} \sum_a S_a^{H;\alpha} S_a^{H;\beta}
\tag{2.55}
$$

We introduced parameters $(\Lambda_H)_{\alpha\beta}$ forming a symmetric 3×3 matrix. When we define the general Gaussian measure, we will see that this matrix will be required to be a positive definite matrix.

The quadratic invariants constructed from the V_2, V_3 variables are

$$
\begin{aligned}
(\Lambda_{V_2}) \sum_{a=1}^{\text{Dim}V_2} S_a^{V_2} S_a^{V_2} \\
(\Lambda_{V_3}) \sum_{a=1}^{\text{Dim}V_3} S_a^{V_3} S_a^{V_3}
\end{aligned}
\tag{2.56}
$$

When we define the general Gaussian measure, we will take the parameters $\Lambda_{V_2}, \Lambda_{V_3}$ to obey $\Lambda_{V_2}, \Lambda_{V_3} \geq 0$.

2.3 Definition of the Gaussian models

The measure dM for integration over the matrix variables M_{ij} is taken to be the Euclidean measure on $\mathbb{R}^{D^2}$ parametrised by the D^2 variables

$$
dM \equiv \prod_i dM_{ii} \prod_{i\neq j} dM_{ij}
\tag{2.57}
$$

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Since the variables $S_a^{V;\alpha}$ defined in (2.43), (2.44), (2.45) are given by an orthogonal change of basis, we can show that

$$dM = dS^{V_0;1}dS^{V_0;2} \prod_{a=1}^{\mathrm{Dim}V_H} dS_a^{H;1}dS_a^{H;2}dS_a^{H;3} \prod_{a=1}^{\mathrm{Dim}V_2} dS_a^{V_2} \prod_{a=1}^{\mathrm{Dim}V_3} dS_a^{V_3} \tag{2.58}$$

Indeed writing M_A for the matrix variables, where A runs over the D^2 pairs (i,j) and S_B for S_D-covariant variables, where B runs over all the factors in 2.58, we have

$$dM = \prod_A dM_A = |\det J| \prod_B dS_B \tag{2.59}$$

with

$$J_{AB} = \frac{\partial M_A}{\partial S_B} \tag{2.60}$$

Now the S_B variables are obtained from M_A by an orthogonal basis change, and symmetric group properties also allow the matrix to be chosen to be real. This implies that the matrix is orthogonal

$$JJ^T = 1 \tag{2.61}$$

Hence $\det J$ has magnitude 1, and we have the claimed identity (2.58).

The model is defined by integration. The partition function is

$$\mathcal{Z}(\mu_1, \mu_2; \Lambda_{V_0}, \Lambda_{V_H}, \Lambda_{V_2}, \Lambda_{V_3}) = \int dM e^{-\mathcal{S}} \tag{2.62}$$

where the action is a combination of linear and quadratic functions.

$$\mathcal{S} = -\sum_{\alpha=1}^{2} \mu_\alpha^{V_0} S^{V_0;\alpha} + \frac{1}{2} \sum_{\alpha,\beta=1}^{2} S^{V_0;\alpha}(\Lambda_{V_0})_{\alpha\beta} S^{V_0;\beta} + \frac{1}{2} \sum_{a=1}^{D-1} \sum_{\alpha,\beta=1}^{3} S_a^{V_H;\alpha}(\Lambda_H)_{\alpha\beta} S_a^{V_H;\beta}$$
$$+ \frac{1}{2}\Lambda_{V_2} \sum_{a=1}^{(D-1)(D-2)/2} S_a^{V_2} S_a^{V_2} + \frac{1}{2}\Lambda_{V_3} \sum_{a=1}^{D(D-3)/2} S_a^{V_3} S_a^{V_3} \tag{2.63}$$

The expectation values of permutation invariant polynomials $f(M)$ are defined by

$$\langle f(M) \rangle = \frac{1}{\mathcal{Z}} \int dM e^{-\mathcal{S}} f(M) \tag{2.64}$$

These expectation values can be computed using standard techniques from quantum field theory, specialised to matrix fields in zero space-time dimensions (See Appendix A) for some explanations). Textbook discussions of these techqniques are given, for example

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in [23],[24]. For linear functions, the non-vanishing expectation values are those of the invariant variables, which transform as V_0 under the S_D action

$$\langle S^{V_0;\alpha} \rangle = \sum_\beta (\Lambda^{-1})_{\alpha\beta} \, \mu_\beta \equiv \widetilde{\mu}_\alpha \tag{2.65}$$

We have defined variables $\widetilde{\mu}_1, \widetilde{\mu}_2$ for convenience. The variables transforming according to V_H, V_2, V_3 have vanishing expectation values

$$\begin{aligned} \langle S_a^{H;\alpha} \rangle &= 0 \\ \langle S_a^{V_2} \rangle &= 0 \\ \langle S_a^{V_3} \rangle &= 0 \end{aligned} \tag{2.66}$$

The quadratic expectation values are

$$\langle S^{V_i;\alpha} S^{V_j;\beta} \rangle = \langle S^{V_i;\alpha} S^{V_j;\beta} \rangle_{\text{conn}} + \langle S^{V_i;\alpha} \rangle \, \langle S^{V_j;\beta} \rangle \tag{2.67}$$

where

$$\langle S_a^{V_i;\alpha} S_b^{V_j;\beta} \rangle_{\text{conn}} = \delta(V_i, V_j)(\Lambda_{V_i}^{-1})_{\alpha\beta}\delta_{ab} \tag{2.68}$$

The V_i, V_j can be V_0, V_H, V_2, V_3. The delta function means that these expectation values vanish unless the two irreps V_i, V_j are equal. While δ_{ab} is the identity in the state space for each V_i. The fact that the mixing matrix in the multiplicity indices α, β is the inverse of the coupling matrix Λ_V is a special (zero-dimensional) case of a standard result in quantum field theory, where the propagator is the inverse of the operator defining the quadratic terms in the action.

The requirement of an S_D invariant Gaussian measure has led us to define variables $S^{V,\alpha}$, transforming in irreducible representations of S_D. The action is simple in terms of these variables. This is reflected in the fact that the above one and two-point functions are simple in terms of the parameters of the model.

When $\Lambda_{V_2} > 0, \Lambda_{V_3} > 0$ and Λ_H, Λ_{V_0} are positive-definite real symmetric matrices (i.e real symmetric matrices with positive eigenvalues), then the partition function $\mathcal{Z}$ is well defined as well as the numerators in the definition of $\langle f(M) \rangle$. We can relax these conditions, allowing $\Lambda_{V_2}, \Lambda_{V_3} \geq 0$ and Λ_H, Λ_{V_0} positive semi-definite, by appropriately restricting the $f(M)$ we consider. For example, if $\Lambda_{V_2} = 0$, we consider functions $f(M)$ which do not depend on S^{V_2}, which ensures that the ratios defining $\langle f(M) \rangle$ are well-defined.

Thus the complete set of constraints on the representation theoretic parameters are

$$\begin{aligned} \text{Det}(\Lambda_{V_0}) &\geq 0 \\ \text{Det}(\Lambda_{V_H}) &\geq 0 \\ \Lambda_{V_2} &\geq 0 \end{aligned}$$

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$$\Lambda_{V_3} \geq 0 \tag{2.69}$$

More explicitly

$$(\Lambda_{V_0})_{11}(\Lambda_{V_0})_{22} - ((\Lambda_{V_0})_{12})^2 \geq 0$$
$$(\Lambda_H)_{11}((\Lambda_H)_{22}(\Lambda_H)_{33} - ((\Lambda_H)_{23})^2) - (\Lambda_H)_{12}((\Lambda_H)_{12}(\Lambda_H)_{33} - (\Lambda_H)_{23}(\Lambda_H)_{13})$$
$$+(\Lambda_H)_{13}((\Lambda_H)_{12}(\Lambda_H)_{23} - (\Lambda_H)_{13}(\Lambda_H)_{22}) \geq 0$$
$$\Lambda_{V_2} \geq 0$$
$$\Lambda_{V_3} \geq 0 \tag{2.70}$$

With these linear and quadratic expectation values of representation theoretic matrix variables S available, the expectation value of a general polynomial function of M_{ij} can be expressed in terms of finite sums of products involving these linear and quadratic expectation values. This is an application of Wick's theorem in the context of QFT. We will explain this for the integrals at hand in Appendix A and describe the consequences of Wick's theorem explicitly for expectation values of functions up to quartic in the matrix variables. We will be particularly interested in the expectation values of polynomial functions of the M_{ij} which are invariant under S_D action and can be parametrised by graphs. While the mixing between between the S variables in the quadratic action is simple, there are non-trivial couplings between the D^2 variables M_{ij} if we expand the action in terms of the M variables. This will lead to non-trivial expressions for the expectation values of the graph-basis polynomials.

These expectation values were computed for the 5-parameter Gaussian model in [8]. They were referred to as theoretical expectation values $\langle f(M) \rangle$, which were compared with experimental expectation values $\langle f(M) \rangle_{EXPT}$. These experimental expectation values were calculated by considering a list of words labelled by an index A ranging from 1 to N, and their corresponding matrices M^A,

$$\langle f(M) \rangle_{EXPT} = \frac{1}{N} \sum_{A=1}^{N} f(M^A) \tag{2.71}$$

We will now proceed to explicitly apply Wick's theorem to calculate the expectation values of permutation invariant functions labelled by graphs for the case of quadratic functions (2-edge graphs), cubic (3-edge graphs) and quartic functions (4-edge graphs). We will leave the comparison of the results of this 13-parameter Gaussian model to linguistic data for the future.

3 Graph basis invariants in terms of rep theory parameters

In the graph theoretic description of S_D invariants constructed from M_{ij}, nodes in the graph correspond to indices, M corresponds to directed edges. At linear order we have

the one-node invariant $\sum_i M_{ii}$ and the two-node invariant $\sum_{i,j} M_{ij}$. At quadratic order in M we have up to three nodes. In this section, we calculate the expectation values of the linear and quadratic invariants in the Gaussian model defined in Section 2. The quadratic expectation values show non-trivial mixing between the different M_{ij}, unlike the 5-parameter model, where the M_{ii} are decoupled from the off-diagonal elements and from each other. In that simple model, M_{ij} only mixes with M_{ji}. Here the mixings are more non-trivial, but controlled by S_D representation theory.

The quantity $F(i,j)$ defined in (2.15)

$$F(i,j) = \sum_a C_{a,i} C_{a,j} = \left(\delta_{ij} - \frac{1}{D} \right) \tag{3.1}$$

will play an important role in the following. Its meaning is that it is the projector for the hook representation in the natural representation. Deriving expressions for expectation values of permutation invariant polynomial functions of the matrix variable M amounts to doing appropriate sums of products of F factors, with the arguments of these F factors being related to each other according to the nature of the polynomial under consideration. In terms of the variables

$$\widetilde{\mu}_\alpha = \sum_{b=1}^2 (\Lambda_{V_0}^{-1})_{\alpha\beta} \mu_\beta$$

defined in section 2, repeated here for convenience,

$$\langle S^{V_0;1} \rangle = \widetilde{\mu}_1$$
$$\langle S^{V_0;2} \rangle = \widetilde{\mu}_2 \tag{3.2}$$

Using the expansion (2.50) of the matrix variables in terms of the rep-thepretic S variables and the 1-point functions of these in (2.65) and (2.66), we have the 1-point function for the matrix variables

$$\langle M_{ij} \rangle = \frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}} F(i,j) \tag{3.3}$$

Using (A.4) (A.5) along with the expansion of M in terms of S variables, and the two-point functions of these, we have

$$\langle M_{ij} M_{kl} \rangle = \langle M_{ij} M_{kl} \rangle_{\text{conn}} + \langle M_{ij} \rangle \langle M_{kl} \rangle \tag{3.4}$$

where

$$\langle M_{ij} M_{kl} \rangle_{\text{conn}} = \frac{1}{D^2} \langle S^{V_0;1} S^{V_0;1} \rangle_{\text{conn}} + \frac{1}{D-1} \sum_{a_1,a_2} C_{a_1,i} C_{a_1,j} C_{a_2,k} C_{a_2,l} \langle S^{V_0;2} S^{V_0;2} \rangle_{\text{conn}}$$

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$$+\frac{1}{D\sqrt{D-1}}\sum_a \langle S^{V_0;1}S^{V_0;2}\rangle_{\mathrm{conn}}C_{a,k}C_{a,l}+\frac{1}{D\sqrt{D-1}}\sum_a \langle S^{V_0;2}S^{V_0;1}\rangle_{\mathrm{conn}}C_{a,i}C_{a,j}$$

$$+\frac{1}{D}\sum_{a_1,a_2=1}^{D-1}C_{a_1,j}C_{a_2,l}\langle S^{H;1}_{a_1}S^{H;1}_{a_2}\rangle_{\mathrm{conn}}+\frac{1}{D}\sum_{a_1,a_2=1}^{D-1}C_{a_1,i}C_{a_2,k}\langle S^{H;2}_{a_1}S^{H;2}_{a_2}\rangle_{\mathrm{conn}}$$

$$+\sum_{a_1,b_1,a_2,b_2=1}^{D-1}C_{a_1,i}C_{b_1,j}C_{a_2,k}C_{b_2,l}C^{HH\to H}_{a_1,b_1;\ c_1}C^{HH\to H}_{a_2,b_2;\ c_2}\langle S^{H;3}_{c_1}S^{H;3}_{c_2}\rangle_{\mathrm{conn}}$$

$$+\frac{1}{D}\sum_{a_1,a_2=1}^{D-1}C_{a_1,j}C_{a_2,k}\langle S^{H;1}_{a_1}S^{H;2}_{a_2}\rangle_{\mathrm{conn}}+\frac{1}{D}\sum_{a_1,a_2=1}^{D-1}C_{a_1,i}C_{a_2,l}\langle S^{H;2}_{a_1}S^{H;1}_{a_2}\rangle_{\mathrm{conn}}$$

$$+\frac{1}{\sqrt{D}}\sum_{a_1=1}^{D-1}\sum_{a_2,b_2=1}^{D-1}C_{a_1,j}C_{a_2,k}C_{b_2,l}C^{HH\to H}_{a_2,\ b_2\ ;\ c_2}\langle S^{H;1}_{a_1}S^{H;3}_{c_2}\rangle_{\mathrm{conn}}$$

$$+\frac{1}{\sqrt{D}}\sum_{a_1,b_1=1}^{D-1}\sum_{a_2=1}^{D-1}C_{a_1,i}C_{b_1,j}C^{HH\to H}_{a_1,b_1;\ c_1}C_{a_2,l}\langle S^{H;3}_{c_1}S^{H;1}_{a_2}\rangle_{\mathrm{conn}}$$

$$+\frac{1}{\sqrt{D}}\sum_{a_1=1}^{D-1}\sum_{a_2,b_2=1}^{D-1}C_{a_1,i}C_{a_2,k}C_{b_2,l}C^{HH\to H}_{a_2,b_2;\ c_2}\langle S^{H;2}_{a_1}S^{H;3}_{c_2}\rangle_{\mathrm{conn}}$$

$$+\frac{1}{\sqrt{D}}\sum_{a_1,b_1=1}^{D-1}\sum_{a_2=1}^{D-1}C_{a_1,i}C_{b_1,j}C^{HH\to H}_{a_1,b_1;\ c_1}C_{a_2,k}\langle S^{H;3}_{c_1}S^{H;2}_{a_2}\rangle_{\mathrm{conn}}$$

$$+\sum_{a_1,b_1=1}^{D-1}\sum_{a_2,b_2=1}^{D-1}C_{a_1,i}C_{b_1,j}C^{HH\to V_2}_{a_1,b_1;\ c_1}C_{a_2,k}C_{b_2,l}C^{HH\to V_2}_{a_2,b_2;\ c_2}\langle S^{V_2}_{c_1}S^{V_2}_{c_2}\rangle_{\mathrm{conn}}$$

$$+\sum_{a_1,b_1=1}^{D-1}\sum_{a_2,b_2=1}^{D-1}C_{a_1,i}C_{b_1,j}C^{HH\to V_3}_{a_1,\ b_1\ ;\ c_1}C_{a_2,k}C_{b_2,l}C^{HH\to V_3}_{a_2,b_2;\ c_2}\langle S^{V_3}_{c_1}S^{V_3}_{c_2}\rangle_{\mathrm{conn}}\qquad(3.5)$$

All the terms can be expressed in terms of the F-function defined in (3.1)

$$\langle M_{ij}M_{kl}\rangle_{\mathrm{conn}}=\frac{1}{D^2}(\Lambda_{V_0}^{-1})_{11}+\frac{(\Lambda_{V_0}^{-1})_{22}}{(D-1)}F(i,j)F(k,l)+\frac{(\Lambda_{V_0}^{-1})_{12}}{D\sqrt{D-1}}\left(F(k,l)+F(i,j)\right)$$

$$+\frac{(\Lambda_H^{-1})_{11}}{D}F(j,l)+\frac{(\Lambda_H^{-1})_{22}}{D}F(i,k)+\frac{D(\Lambda_H^{-1})_{33}}{(D-2)}\sum_{p,q}F(i,p)F(j,p)F(k,q)F(l,q)F(p,q)$$

$$+\frac{(\Lambda_H^{-1})_{12}}{D}\left(F(j,k)+F(i,l)\right)+\frac{(\Lambda_H^{-1})_{13}}{\sqrt{D-2}}\left(\sum_p F(j,p)F(k,p)F(l,p)+F(i,p)F(j,p)F(l,p)\right)$$

$$+\frac{(\Lambda_H^{-1})_{23}}{\sqrt{D-2}}\left(\sum_p F(i,p)F(k,p)F(l,p)+F(i,p)F(j,p)F(k,p)\right)$$

$$+(\Lambda_{V_2}^{-1})\left(\frac{1}{2}F(i,k)F(j,l)+\frac{1}{2}F(i,l)F(j,k)-\frac{D}{D-2}\sum_{p,q}F(i,p)F(j,p)F(k,q)F(l,q)F(p,q)\right.$$

$$-\frac{1}{(D-1)}F(i,j)F(k,l)\Big)$$

$$+\frac{(\Lambda_{V_3}^{-1})}{2}\left(F(i,k)F(j,l)-F(i,l)F(j,k)\right) \tag{3.6}$$

We will refer to the terms depending on Λ_{V_0} as V_0-channel contributions to the 2-point functions, those on Λ_H as V_H-channel (or H-channel) contributions, those on Λ_{V_2} as V_2-channel and those on Λ_{V_3} as V_3-channel contributions. It will be convenient to denote these different channel contributions as $\langle M_{ij}M_{kl}\rangle_V$ where $V \in \{V_0, V_H, V_2, V_3\}$, so that we have

$$\langle M_{ij}M_{kl}\rangle_{\mathrm{conn}} = \langle M_{ij}M_{kl}\rangle_{\mathrm{conn}}^{V_0} + \langle M_{ij}M_{kl}\rangle_{\mathrm{conn}}^{V_H} + \langle M_{ij}M_{kl}\rangle_{\mathrm{conn}}^{V_2} + \langle M_{ij}M_{kl}\rangle_{\mathrm{conn}}^{V_3} \tag{3.7}$$

In arriving at the expressions for the last two terms in (3.6), we used the fact that these terms in 3.5 can be expressed as

$$\Lambda_{V_2}^{-1}\ (e_i \otimes e_j, P^{V_D \otimes V_D \to V_2}(e_k \otimes e_l)) + \Lambda_{V_3}^{-1}\ (e_i \otimes e_j, P^{V_D \otimes V_D \to V_3}(e_k \otimes e_l)) \tag{3.8}$$

Here $P^{V_D \otimes V_D \to V_2}$ is the projector from the tensor product of natural reps. to V_2, the irrep of S_D associated with Young diagram $[n-2,2]$. Similarly for $V_3 = [n-2,1,1]$. Now it is useful to observe that V_3 is just the anti-symmetric part of $V_H \otimes V_H$. The symmetric part decomposes as $V_0 \oplus V_H \oplus V_2$. The projector for V_0 is

$$(P^{HH \to V_0})_{a_1 b_1; a_2 b_2} = \frac{1}{D-1}\delta_{a_1 a_2}\delta_{b_1 b_2} \tag{3.9}$$

For V_H it is

$$(P^{HH \to H})_{a_1 b_1; a_2 b_2} = \sum_c C^{H,H \to H}_{a_1, b_1;\ c} C^{H,H \to H}_{a_2, b_2;\ c} = \frac{D}{D-2}\sum_c C_{a_1, b_1, c} C_{a_2, b_2, c} \tag{3.10}$$

The factor $\frac{D}{D-2}$ is explained in Appendix C.

3.1 Calculation of $\sum_{i,j}\langle M_{ij}M_{ij}\rangle$

Following (3.7) the expectation value $\langle M_{ij}M_{ij}\rangle$ can be written as a sum over V-channel contributions, where V ranges over the four irreps.

3.1.1 Contributions from V_2, V_3

From (3.8) we find

$$\sum_{i,j}\langle M_{ij}M_{ij}\rangle_{\mathrm{conn}}^{V_2} = (\Lambda_{V_2})^{-1}\mathrm{tr}_{V_D \otimes V_D}(P^{V_D \otimes V_D \to V_2})$$

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$$= (\text{Dim } V_2)(\Lambda_{V_2})^{-1} = \frac{D(D-3)}{2}(\Lambda_{V_2})^{-1} \tag{3.11}$$

The projector has eigenvalue 1 on the subspace transforming in the irrep V_2 and zero elsewhere, hence the $(\text{Dim } V_2)$. Similarly

$$\sum_{i,j} \langle M_{ij} M_{ij} \rangle_{\text{conn}}^{V_3} = (\text{Dim } V_3)(\Lambda_{V_3})^{-1} = \frac{(D-1)(D-2)}{2}(\Lambda_{V_3})^{-1} \tag{3.12}$$

Since V_2, V_3 appear inside the $V_H \otimes V_H$ subspace of $V_{\text{nat}} \otimes V_{\text{nat}}$, we can also write the trace in $V_H \otimes V_H$ and express this in terms of irreducible characters

$$\begin{aligned} tr_{V_H \otimes V_H} P_{V_2} \\ = \frac{(\text{Dim } V_2)}{D!} \sum_{\sigma \in S_D} \chi_{V_2}(\sigma) \chi_H(\sigma) \chi_H(\sigma) \\ = (\text{Dim } V_2) \end{aligned} \tag{3.13}$$

In the last line we have used the fact that the Kroncker coefficients for $V_H \otimes V_H \to V_2$ is 1.

3.1.2 Contribution from V_H channel

The $(\Lambda_H^{-1})_{11}$ contribution is

$$(\text{Dim } V_H)(\Lambda_H^{-1})_{11} = (D-1)(\Lambda_H^{-1})_{11} \tag{3.14}$$

which can be obtained easily from (3.5) or (3.6). Similarly, the $(\Lambda_H^{-1})_{12}$ contribution is zero. And the $(\Lambda_H^{-1})_{22}$ contribution is

$$(\text{Dim } V_H)(\Lambda_H^{-1})_{22} = (D-1)(\Lambda_H^{-1})_{22} \tag{3.15}$$

From either (3.5) or (3.6) we easily conclude that the $(\Lambda_H)^{-1})_{13}, (\Lambda_H)^{-1})_{23}$ contributions vanish. Starting from (3.5), we make repeated use of (2.13).

The $(\Lambda_H^{-1})_{33}$ contribution is

$$(\Lambda_H^{-1})_{33} tr_{V_H \otimes V_H}(P^{HH \to H}) = (\Lambda_H^{-1})_{33}(\text{Dim } V_H) = (\Lambda_H^{-1})_{33}(D-1) \tag{3.16}$$

We used the second equation in (2.12).

3.1.3 Contribution from V_0 channel

This is

$$\begin{aligned} \sum_{i,j} \langle M_{ij} M_{ij} \rangle_{\text{conn}}^{V_0} &= (\Lambda_{V_0}^{-1})_{11} + \frac{1}{\text{Dim } V_H}(\Lambda_{V_0}^{-1})_{22}\text{Dim } V_H \\ &= (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} \end{aligned} \tag{3.17}$$

3.1.4 Summing all channels

$$\sum_{i,j}\langle M_{ij}M_{ij}\rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} + (D-1)(\Lambda_H^{-1})_{22} + (D-1)(\Lambda_H^{-1})_{33} + (D-1)(\Lambda_H^{-1})_{11}$$

$$+\frac{D(D-3)}{2}(\Lambda_{V_2})^{-1} + \frac{(D-1)(D-2)}{2}(\Lambda_{V_3})^{-1} \tag{3.18}$$

The disconnected piece is

$$\sum_{i,j}\langle M_{ij}\rangle \langle M_{ij}\rangle = \sum_{i,j}\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(i,j)\right)^2$$

$$= \widetilde{\mu}_1^2 + \widetilde{\mu}_2^2 \tag{3.19}$$

and using (3.4)

$$\sum_{i,j}\langle M_{ij}M_{ij}\rangle = \widetilde{\mu}_1^2 + \widetilde{\mu}_2^2 + \sum_{i,j}\langle M_{ij}M_{ij}\rangle_{\text{conn}}$$

$$= \widetilde{\mu}_1^2 + \widetilde{\mu}_2^2 + (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} + (D-1)(\Lambda_H^{-1})_{22} + (D-1)(\Lambda_H^{-1})_{33} + (D-1)(\Lambda_H^{-1})_{11}$$

$$+\frac{D(D-3)}{2}(\Lambda_{V_2})^{-1} + \frac{(D-1)(D-2)}{2}(\Lambda_{V_3})^{-1} \tag{3.20}$$

3.2 Calculation of $\sum_{i,j}\langle M_{ij}M_{ji}\rangle$

As in (3.7) the expectation value $\langle M_{ij}M_{ij}\rangle$ can be written as a sum over V-channel contributions, where V ranges over the four irreps.

3.2.1 Contribution from multiplicity 1 channels V_2, V_3

The $(\Lambda_{V_2}^{-1})$ contribution is

$$(\Lambda_{V_2}^{-1})\text{tr}_{V_H\otimes V_H}(P^{V_2}\tau)$$

$$= (\Lambda_{V_2}^{-1})d_{V_2} = (\Lambda_{V_2}^{-1})\frac{D(D-3)}{2} \tag{3.21}$$

τ is the swop which acts on the two factors of V_H. We have used the fact that V_2 appears in the symmetric part of $V_H \otimes V_H$.

The $(\Lambda_{V_3}^{-1})$ contribution is

$$-(\Lambda_{V_3}^{-1})\frac{(D-1)(D-2)}{2} \tag{3.22}$$

We use the fact that V_3 is the antisymmetric part of $V_H \otimes V_H$.

3.2.2 Contribution from V_H channel

The $(\Lambda_H^{-1})_{11}$ contribution From (3.5), we have

$$\frac{1}{D}\sum_{a_1,a_2}\sum_{i,j}C_{a_1,j}C_{a_2,i}(\Lambda_H^{-1})_{11}\delta_{a_1a_2} = \frac{1}{D}(\Lambda_H^{-1})_{11}\sum_a C_{a,i}C_{a,j} = 0 \tag{3.23}$$

using (2.13). Similarly the $(\Lambda_H^{-1})_{22}$ contribution is zero.

The $(\Lambda_H^{-1})_{12}$ contribution is

$$\frac{2(D-1)}{D}\sum_{a,j}C_{a,j}C_{a,j}(\Lambda_H^{-1})_{12} = 2\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} \tag{3.24}$$

The $(\Lambda_H^{-1})_{33}$ contribution is

$$\begin{aligned}
&\sum_{i,j}\sum_{a,b,c,d,e}(\Lambda_H^{-1})_{33}C_{a,i}C_{b,j}C_{c,j}C_{d,i}C^{HH\to H}_{a,\ b\ ;\ e}C^{HH\to H}_{c,\ d\ ;\ e}\\
&= (\Lambda_H^{-1})_{33}\sum_{a,b,c,d,e}C^{HH\to H}_{a,\ b\ ;\ e}C^{HH\to H}_{b,\ a\ ;\ e}\\
&= (\Lambda_H^{-1})_{33}\,\mathrm{tr}_{H\otimes H}P^{H\otimes H\to H}s = (\Lambda_H^{-1})_{33}\,\mathrm{tr}_{H\otimes H}P^{H\otimes H\to H}(\frac{1+s}{2}-\frac{1-\tau}{2})\\
&= (\Lambda_H^{-1})_{33}d_H = (D-1)(\Lambda_H^{-1})_{33}
\end{aligned} \tag{3.25}$$

In the penultimate line, we have introduced the swop s which exchanges the two factors in $H \otimes H$. We know that H appears in the symmetric part of $H \otimes H$, so swop leaves it invariant.

Uses of the equation (2.13) shows that the $(\Lambda_H^{-1})_{13}, (\Lambda_H^{-1})_{23}$ dependent terms vanish. Collecting all the V_H-channel contributions, we have

$$\sum_{i,j}\langle M_{ij}M_{ji}\rangle^{V_H}_{\mathrm{conn}} = 2\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + (D-1)(\Lambda_H^{-1})_{33} \tag{3.26}$$

3.2.3 Contribution from V_0 channel

The first term from (3.5) is

$$(\Lambda_{V_0}^{-1})_{11} \tag{3.27}$$

The second term is

$$\frac{(\Lambda_{V_0}^{-1})_{22}}{d_H}\sum_{i,j}\sum_{a_1,a_2}C_{a_1,i}C_{a_1,j}C_{a_2,j}C_{a_2,i} = \frac{(\Lambda_{V_0}^{-1})_{22}}{D-1}\sum_{a_1,a_2}\delta_{a_1a_2}\delta_{a_1a_2} = (\Lambda_{V_0}^{-1})_{22} \tag{3.28}$$

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The third term is

$$\frac{1}{D\sqrt{D-1}}\sum_{i,j}\sum_a C_{a,j}C_{a,i}(\Lambda_{V_0}^{-1})_{12} \tag{3.29}$$

which vanishes using (2.13). The last term vanishes for the same reason.

So collecting the V_0-channel contributions to $\sum_{i,j}\langle M_{ij}M_{ji}\rangle_{\text{conn}}$, we have

$$\sum_{i,j}\langle M_{ij}M_{ji}\rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} \tag{3.30}$$

3.2.4 summing all channels

$$\sum_{i,j}\langle M_{ij}M_{ji}\rangle_{\text{conn}} = (\Lambda_{V_2}^{-1})\frac{(D)(D-3)}{2} - (\Lambda_{V_3}^{-1})\frac{(D-1)(D-2)}{2}$$
$$+ \; 2\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + (D-1)(\Lambda_H^{-1})_{33} + \; (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} \tag{3.31}$$

Since $F(i,j) = F(j,i)$, the disconnected piece is the same in (3.19)

$$\sum_{i,j}\langle M_{ij}\rangle\langle M_{ji}\rangle = \sum_{i,j}\langle M_{ij}\rangle\langle M_{ij}\rangle$$
$$= \widetilde{\mu}_1^2 + \widetilde{\mu}_2^2 \tag{3.32}$$

and $\sum_{i,j}\langle M_{ij}M_{ij}\rangle$ in terms of the μ, Λ parameters of the Gaussian model is the sum of the expressions in (3.31) and (3.32).

3.3 Calculation of $\sum_{i,j}\langle M_{ii}M_{ij}\rangle$

Important observation here is the the sum of j projects to V_0. This means that when we expand M_{ii} and M_{ij} into S variables as in the first line of (2.46), we only need to keep the term S^{H0} or S^{00} from the expansion of M_{ij}.

3.3.1 Contribution from V_2, V_3 channels

From the above observation, and since V_2, V_3 appear only in $\langle S^{HH}S^{HH}\rangle$, we immediately see that

$$\sum_{i,j}\langle M_{ii}M_{ij}\rangle_{\text{conn}}^{V_2} = \sum_{i,j}\langle M_{ii}M_{ij}\rangle_{\text{conn}}^{V_3} = 0 \tag{3.33}$$

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3.3.2   contribution from V_H channel

From the above observation, the only non-zero contributions in the V_H channel come from $\langle S^{HH \to H} S^{H0} \rangle$, $\langle S^{H0} S^{H0} \rangle$ and $\langle S^{0H} S^{H0} \rangle$. These are

$$\frac{1}{D} \sum_{a_1, a_2, i} C_{a_1, i} C_{a_2, i} (\Lambda_H^{-1})_{12} \delta_{a_1 a_2} = \frac{(D-1)}{D} (\Lambda_H^{-1})_{12} \qquad (3.34)$$

$$\frac{1}{D} \sum_{a_1, a_2, i} C_{a_1, i} C_{a_2, i} (\Lambda_H^{-1})_{22} \delta_{a_1, a_2} = \frac{(D-1)}{D} (\Lambda_H^{-1})_{22} \qquad (3.35)$$

$$\frac{1}{\sqrt{D}} \sum_{a_1, b_1, a_2} C_{a_1, i} C_{b_1, i} C_{a_1, b_1;\ c_1}^{H, H \to H} C_{a_2, i} \delta_{c_1 a_2} (\Lambda_H^{-1})_{32}$$
$$= \frac{1}{\sqrt{D-2}} \sum_{a_1, b_1, a_2} C_{a_1, i} C_{b_1, i} C_{a_1, b_1, c_1} C_{a_2, i} \delta_{c_1 a_2} (\Lambda_H^{-1})_{32}$$
$$= \frac{1}{D} (D-1) \sqrt{(D-2)} (\Lambda_H^{-1})_{23} \qquad (3.36)$$

Note that Λ_H is a symmetric 3×3 matrix and $\Lambda_H^{-1})_{23} = (\Lambda_H^{-1})_{32}$. These add up to

$$\langle M_{ii} M_{ij} \rangle_{\text{conn}}^{V_H} = \frac{(D-1)}{D} (\Lambda_H^{-1})_{12} + \frac{(D-1)}{D} (\Lambda_H^{-1})_{22} + \frac{1}{D} (D-1) \sqrt{(D-2)} (\Lambda_H^{-1})_{23} \qquad (3.37)$$

3.3.3   contribution from V_0 channel

The non-zero contributions come from $\langle S^{00} S^{00} \rangle$ and $\langle S^{HH \to 0} S^{00} \rangle$. They are

$$\sum_{i,j} \langle M_{ii} M_{ij} \rangle_{\text{conn}}^{V_0} = \frac{1}{D^2} \sum_{i,j} (\Lambda_{V_0}^{-1})_{11} + \frac{1}{D \sqrt{d_H}} \sum_{i,j} \sum_a (\Lambda_{V_0}^{-1})_{12} C_{a,i} C_{a,i}$$
$$= (\Lambda_{V_0}^{-1})_{11} + \frac{1}{\sqrt{D-1}} \sum_{i,a} (\Lambda_{V_0}^{-1}) C_{a,i} C_{a,i}$$
$$= (\Lambda_{V_0}^{-1})_{11} + \sqrt{d_H} (\Lambda_{V_0}^{-1})_{12} = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)} (\Lambda_{V_0}^{-1})_{12} \qquad (3.38)$$

3.3.4   summing all channels

$$\sum_{i,j} \langle M_{ii} M_{ij} \rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)} (\Lambda_{V_0}^{-1})_{12}$$
$$+ \frac{(D-1)}{D} (\Lambda_H^{-1})_{12} + \frac{(D-1)}{D} (\Lambda_H^{-1})_{22} + \frac{1}{D} (D-1) \sqrt{(D-2)} (\Lambda_H^{-1})_{23}$$

$$(3.39)$$

The disconnected piece is

$$
\begin{aligned}
\sum_{i,j}\langle M_{ii}\rangle\langle M_{ij}\rangle &= \left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(i,i)\right)\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(i,j)\right) \\
&= \widetilde{\mu}_1^2 + \frac{\widetilde{\mu}_1\widetilde{\mu}_2}{\sqrt{D-1}}\sum_{i,j}(1 - \frac{1}{D}) \\
&= \widetilde{\mu}_1^2 + \widetilde{\mu}_1\widetilde{\mu}_2 D\sqrt{(D-1)}
\end{aligned}
\tag{3.40}
$$

so we have

$$
\sum_{i,j}\langle M_{ii}M_{ij}\rangle = \sum_{i,j}\langle M_{ii}M_{ij}\rangle_{\text{conn}} + \widetilde{\mu}_1^2 + \widetilde{\mu}_1\widetilde{\mu}_2 D\sqrt{D-1}
\tag{3.41}
$$

with the first term given by (3.39)

3.4 Calculation of $\sum_{i,j}\langle M_{ii}M_{ji}\rangle$

We can write down the answer from inspection of (3.39)

$$
\begin{aligned}
\sum_{i,j}\langle M_{ii}M_{ji}\rangle_{\text{conn}} &= (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)}(\Lambda_{V_0}^{-1})_{12} \\
&+ \frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + \frac{(D-1)}{D}(\Lambda_H^{-1})_{11} + \frac{1}{D}(D-1)\sqrt{(D-2)}(\Lambda_H^{-1})_{13}
\end{aligned}
\tag{3.42}
$$

The reasoning is as follows. The sum over j projects to V_0. This means that the only non-zero contributions are, from the V_0 channel,

$$
\begin{aligned}
&\langle S^{00}S^{00}\rangle_{\text{conn}} \\
&\langle S^{HH\to 0}S^{00}\rangle_{\text{conn}}
\end{aligned}
\tag{3.43}
$$

and from the V_H channel

$$
\begin{aligned}
&\langle S^{0H}S^{0H}\rangle_{\text{conn}} \\
&\langle S^{H0}S^{0H}\rangle_{\text{conn}} \\
&\langle S^{HH\to H}S^{0H}\rangle_{\text{conn}}
\end{aligned}
\tag{3.44}
$$

This identifies the contributing entries of $\Lambda_{V_0}^{-1}, \Lambda_H^{-1}$ using the indexing in (2.43) and (2.44). given the similarity between the expectation value in section 3.3, we have contributions of the same form, up to taking care of the right indices on $\Lambda_{V_0}^{-1}, \Lambda_H^{-1}$.

Given the symmetry of $F(i,j)$ under exchange of i,j, the disconnected piece is the same as above

$$
\sum_{i,j}\langle M_{ii}M_{ji}\rangle = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)}(\Lambda_{V_0}^{-1})_{12}
$$

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$$+\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + \frac{(D-1)}{D}(\Lambda_H^{-1})_{11} + \frac{1}{D}(D-1)\sqrt{(D-2)}(\Lambda_H^{-1})_{13}$$
$$+\widetilde{\mu}_1^2 + \widetilde{\mu}_1\widetilde{\mu}_2 D\sqrt{(D-1)} \tag{3.45}$$

3.5 Calculation of $\sum_{i,j,k}\langle M_{ij}M_{ik}\rangle$

The sums over j,k project to V_0. The non-vanishing contributions are $\langle S^{00}S^{00}\rangle_{\text{conn}}$ and $\langle S^{H0}S^{H0}\rangle_{\text{conn}}$. They add up to

$$\sum_{i,j,k}\langle M_{ij}M_{ik}\rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{22} \tag{3.46}$$

The disconnected part is

$$\sum_{i,j,k}\langle M_{ij}\rangle\,\langle M_{ik}\rangle = \widetilde{\mu}_1^2 D \tag{3.47}$$

leading to

$$\sum_{i,j,k}\langle M_{ij}M_{ik}\rangle = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{22} + \widetilde{\mu}_1^2 D \tag{3.48}$$

3.6 Calculation of $\sum_{i,j,k}\langle M_{ij}M_{kj}\rangle$

Now we are projecting to V_0 on the first index of both M's. This means that the contributing terms are $\langle S^{00}S^{00}\rangle$ and $\langle S^{0H}S^{0H}\rangle$.

Repeat the same steps as above in (3.46) to get

$$\sum_{i,j,k}\langle M_{ij}M_{kj}\rangle_{con} = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{11} \tag{3.49}$$

The only difference is that we are picking up the $(1,1)$ matrix element of (Λ_H^{-1}) instead of the $(2,2)$ element, since we defined $S^{0H} = S^{\{V_0;1\}}$ and $S^{H0} = S^{\{V_0;2\}}$.

Adding the disconnected piece, which is the same as (3.47), we have

$$\sum_{i,j,k}\langle M_{ij}M_{kj}\rangle = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{11} + \widetilde{\mu}_1^2 D \tag{3.50}$$

3.7 $\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle$

We are now projecting to V_0 on first index of one of the matrices and second index of the other. Hence the contributing terms are $\langle S^{00}S^{00}\rangle$ and $\langle S^{0H}S^{H0}\rangle$. The result is

$$\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{12} \tag{3.51}$$

and

$$\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_H^{-1})_{12} + \widetilde{\mu}_1^2 D \qquad (3.52)$$

3.8 $\sum_{i,j,k,l}\langle M_{ij}M_{kl}\rangle$

Here we project to V_0 on all four indices, so

$$\sum_{i,j,k,l}\langle M_{ij}M_{kl}\rangle_{\mathrm{conn}} = D^2\langle S^{00}S^{00}\rangle = D^2(\Lambda_{V_0}^{-1})_{11} \qquad (3.53)$$

Adding the disconnected piece we have

$$\sum_{i,j,k,l}\langle M_{ij}M_{kl}\rangle = D^2(\Lambda_{V_0}^{-1})_{11} + \widetilde{\mu}_1^2 D^2 \qquad (3.54)$$

3.9 $\sum_i\langle M_{ii}^2\rangle$

3.9.1   The V_0 channel

$$\sum_i\langle M_{ii}^2\rangle_{\mathrm{conn}}^{V_0}$$
$$= \frac{1}{D^2}\sum_i(\Lambda_{V_0}^{-1})_{11} + \frac{1}{(D-1)}\sum_i F(i,i)^2(\Lambda_{V_0}^{-1})_{22} + \frac{2}{D\sqrt{D-1}}\sum_i F(i,i)(\Lambda_{V_0}^{-1})_{12}$$
$$= \frac{1}{D}(\Lambda_{V_0}^{-1})_{11} + \frac{(D-1)}{D}(\Lambda_{V_0}^{-1})_{22} + 2\frac{\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12} \qquad (3.55)$$

3.9.2   The V_H channel

It is convenient to use (3.6) to arrive at

$$\sum_i\langle M_{ii}^2\rangle_{\mathrm{conn}}^{V_H} = \frac{D-1}{D}(\Lambda_H^{-1})_{11} + (\Lambda_H^{-1})_{22}\frac{(D-1)}{D} + D^{-1}(D-1)(D-2)(\Lambda_H^{-1})_{33}$$
$$+2\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + 2\frac{(\Lambda_H^{-1})_{13}}{D}(D-1)\sqrt{(D-2)} + 2\frac{(\Lambda_H^{-1})_{23}}{D}(D-1)\sqrt{(D-2)} \qquad (3.56)$$

Useful equations in arriving at the above are the sums

$$\sum_{i,p,q}F(i,p)F(i,p)F(i,q)F(i,q)F(p,q) = \frac{(D-2)^2(D-1)}{D^2}$$
$$\sum_{i,p}(F(i,p))^3 = \frac{(D-1)(D-2)}{D} \qquad (3.57)$$

which can be obtained by hand or with the help of Mathematica. In the latter case, it is occasionally easier to evaluate for a range of integer D and fit to a form $Polynomial(D)/D^{some\ power}$.

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3.9.3 the V_2, V_3 channels

Now calculate the $HH \to V_2$ and $HH \to V_3$ channel.

$$\sum_i \langle M_{ii}^2 \rangle_{\text{conn}}^{V_2} = \sum_i \sum_{a,b,c,d} C_{a,i} C_{b,i} C_{c,i} C_{d,i} C_{ab;e}^{HH;V_2} C_{cd;f}^{HH;V_2} \langle S_e^{HH \to V_2} S_f^{HH \to V_2} \rangle_{\text{conn}}$$

$$= (\Lambda_{V_2}^{-1}) \sum_{i,j} \sum_{a,b,c,d,e} C_{a,i} C_{b,i} \delta_{ij} C_{c,j} C_{d,j} C_{a,\ b\ ;\ e}^{HH;V_2} C_{c,\ d\ ;\ e}^{HH;V_2}$$

$$= (\Lambda_{V_2}^{-1}) \sum_{a,b,c,d,e} C_{a,i} C_{b,i} \left(C_{e,i} C_{e,j} + \frac{1}{D} \right) C_{c,j} C_{d,j} C_{a,\ b\ ;\ e}^{HH;V_2} C_{c,\ d\ ;\ e}^{HH;V_2}$$

$$= \frac{D-2}{D} (\Lambda_{V_2}^{-1}) tr_{H \otimes H}(P_H P_{V_2}) + \frac{1}{D} (\Lambda_{V_2}^{-1}) \sum_{a,b} C_{a,\ a\ ;\ e}^{HH \to V_2} C_{b,\ b\ ;\ e}^{HH \to V_2}$$

$$= \frac{D-2}{D} (\Lambda_{V_2}^{-1}) tr_{H \otimes H}(P^{H,H \to H} P^{H,H \to V_2}) + \frac{1}{D} (\Lambda_{V_2}^{-1}) \sum_{a,b} C_{a,\ a\ ;\ e}^{HH \to V_2} C_{b,\ b\ ;\ e}^{HH \to V_2}$$

$$= 0 + \frac{(\Lambda_{V_2}^{-1})}{D-2} \sum_{a,b,i,j} C_{a,i} C_{a,i} C_{e,i} C_{b,j} C_{b,j} C_{e,j}$$

$$= \frac{(\Lambda_{V_2}^{-1})}{D-2} \sum_{i,j} F(i,j) F(i,i) F(j,j) = (\Lambda_{V_2}^{-1}) \frac{(D-1)^2}{D^2(D-2)} \sum_{i,j} F(i,j) = 0 \qquad (3.58)$$

We used the fact that the projectors for H, V_2 are orthogonal.

Similarly, the contribution from V_3 is zero. Another to arrive at the same answer is to recognise that V_3 is the antisymmetric part, so

$$P_{ab;cd}^{H,H \to V_3} = \frac{1}{2} \left(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} \right) \qquad (3.59)$$

3.9.4 summing the channels

$$\sum_i \langle M_{ii}^2 \rangle_{\text{conn}} = \frac{1}{D} (\Lambda_{V_0}^{-1})_{11} + \frac{(D-1)}{D} (\Lambda_{V_0}^{-1})_{22} + 2 \frac{\sqrt{D-1}}{D} (\Lambda_{V_0}^{-1})_{12}$$

$$+ \frac{D-1}{D} (\Lambda_H^{-1})_{11} + (\Lambda_H^{-1})_{22} \frac{(D-1)}{D} + D^{-1}(D-1)(D-2)(\Lambda_H^{-1})_{33}$$

$$+ 2 \frac{(D-1)}{D} (\Lambda_H^{-1})_{12} + 2 \frac{(\Lambda_H^{-1})_{13}}{D} (D-1) \sqrt{(D-2)} + 2 \frac{(\Lambda_H^{-1})_{23}}{D} (D-1) \sqrt{(D-2)} \qquad (3.60)$$

The disconnected part is

$$\sum_i \langle M_{ii} \rangle \langle M_{ii} \rangle = \sum_i \left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}} F(i,i) \right)^2$$

$$= \frac{\widetilde{\mu}_1^2}{D} + 2 \widetilde{\mu}_1 \widetilde{\mu}_2 \frac{\sqrt{D-1}}{D} + \widetilde{\mu}_2^2 \frac{(D-1)}{D} \qquad (3.61)$$

3.10 $\quad \sum_{i,j} \langle M_{ii} M_{jj} \rangle$

Since $\sum_i M_{ii}$ and $\sum_j M_{jj}$ are S_D invariant, we only have contributions from the V_0 channel. Use the first four terms of (3.5) to get

$$\sum_{i,j} \langle M_{ii} M_{jj} \rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + d_H(\Lambda_{V_0}^{-1})_{22} + 2\frac{\sqrt{d_H}}{D}(\Lambda_0^{-1})_{12} \tag{3.62}$$

Using

$$\sum_i \left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}} F(i,i) \right) = \widetilde{\mu}_1 + \widetilde{\mu}_2 \sqrt{D-1} \tag{3.63}$$

The disconnected part is

$$\sum_{i,j} \langle M_{ii} \rangle \langle M_{jj} \rangle = \widetilde{\mu}_1^2 + 2\widetilde{\mu}_1\widetilde{\mu}_2\sqrt{D-1} + \widetilde{\mu}_2^2(D-1) \tag{3.64}$$

so that

$$\sum_{i,j} \langle M_{ii} M_{jj} \rangle = \sum_{i,j} \langle M_{ii} M_{jj} \rangle_{\text{conn}} + \widetilde{\mu}_1^2 + 2\widetilde{\mu}_1\widetilde{\mu}_2\sqrt{D-1} + \widetilde{\mu}_2^2(D-1) \tag{3.65}$$

3.11 $\quad \sum_{i,j,k} \langle M_{ii} M_{jk} \rangle$

Here we get contributions from $\langle S^{HH \to 0} S^{00} \rangle_{\text{conn}}$ and $\langle S^{00} S^{00} \rangle_{\text{conn}}$. Adding these up from (3.5)

$$\sum_{i,j,k} \langle M_{ii} M_{jk} \rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + \frac{\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12} \tag{3.66}$$

The disconnected part is

$$\sum_{i,j,k} \langle M_{ii} \rangle \langle M_{jk} \rangle = (\widetilde{\mu}_1 + \widetilde{\mu}_2\sqrt{D-1})\, D\widetilde{\mu}_1 \tag{3.67}$$

hence

$$\sum_{i,j,k} \langle M_{ii} M_{jk} \rangle = D(\Lambda_{V_0}^{-1})_{11} + \frac{\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12} + (\widetilde{\mu}_1 + \widetilde{\mu}_2\sqrt{D-1})\, D\widetilde{\mu}_1 \tag{3.68}$$

3.12 Summary of results for quadratic expectation values in a large D limit

It is interesting to collect the results for the connected quadratic expectation values and consider the large D limit. Let us assume that all the $\Lambda_{V_0}, \Lambda_H, \Lambda_{V_2}, \Lambda_{V_3}$ scale in the same way as $D \to \infty$ and consider the sums normalized by the appropriate factor of D

$$\frac{1}{D^2}\sum_{i,j}\langle M_{ij}M_{ij}\rangle_{\text{conn}} = \frac{1}{2}\left((\Lambda_{V_2})^{-1} + (\Lambda_{V_3})^{-1}\right)$$

$$\frac{1}{D^2}\sum_{i,j}\langle M_{ij}M_{ji}\rangle_{\text{conn}} = \frac{1}{2}\left((\Lambda_{V_2})^{-1} - (\Lambda_{V_3})^{-1}\right)$$

$$\frac{1}{D^2}\sum_{i,j}\langle M_{ii}M_{ij}\rangle_{\text{conn}} = \frac{1}{D^{3/2}}(\Lambda_{V_0}^{-1})_{12} + \frac{1}{D^{3/2}}(\Lambda_H^{-1})_{23}$$

$$\frac{1}{D^2}\sum_{i,j}\langle M_{ii}M_{ji}\rangle_{\text{conn}} = \frac{1}{D^{3/2}}(\Lambda_{V_0}^{-1})_{12} + \frac{1}{D^{3/2}}(\Lambda_H^{-1})_{13}$$

$$\frac{1}{D^3}\sum_{i,j,k}\langle M_{ij}M_{ik}\rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{22}$$

$$\frac{1}{D^3}\sum_{i,j,k}\langle M_{ij}M_{kj}\rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{11}$$

$$\frac{1}{D^3}\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{12}$$

$$\frac{1}{D^4}\sum_{i,j,k,l}\langle M_{ij}M_{kl}\rangle_{\text{conn}} = \frac{1}{D^2}(\Lambda_{V_0}^{-1})_{11}$$

$$\frac{1}{D}\sum_{i}\langle M_{ii}^2\rangle_{\text{conn}} = (\Lambda_H^{-1})_{33}$$

$$\frac{1}{D^2}\sum_{i,j}\langle M_{ii}M_{jj}\rangle_{\text{conn}} = \frac{1}{D}(\Lambda_{V_0}^{-1})_{11} + \frac{1}{D}(\Lambda_{V_0}^{-1})_{22}$$

$$\frac{1}{D^3}\sum_{i,j,k}\langle M_{ii}M_{jk}\rangle_{\text{conn}} = \frac{1}{D^2}(\Lambda_{V_0}^{-1})_{11}$$

The dominant expectation values in this limit are the first, second and eighth. These are the quadratic expressions which enter the simplified 5-parameter model considered in [8] (see equation (1.1)). It will be interesting to systematically explore the different large D scalings of the parameters in real world data, e.g. the computational linguistics setting of [8] or in any other situation where permutation invariant matrix Gaussian matrix distributions can be argued to be appropriate.

4   A selection of cubic expectation values

In this section we use Wick's theorem from Appendix A to express expectation values of cubic functions of Matrix variables in terms of linear and quadratic expectation values. The permutation invariance condition requires sums of indices over the range $\{1, \cdots, D\}$. This leads to non-trivial sums of products of the natural-to-hook projector $F(i, j)$. The invariants at cubic order are 52 in number (Appendix B of [8]) and correspond to graphs with up to 6 nodes.

4.1   1-node case $\sum_i \langle M_{ii}^3 \rangle$

Using A.6, we have

$$\sum_i \langle M_{ii}^3 \rangle = 3 \sum_i \langle M_{ii}^2 \rangle_{\text{conn}} \langle M_{ii} \rangle + \langle M_{ii} \rangle^3 \tag{4.1}$$

Specialising (3.3)

$$\langle M_{ii} \rangle = \frac{1}{D}\widetilde{\mu}_1 + \frac{\sqrt{(D-1)}}{D}\widetilde{\mu}_2 \tag{4.2}$$

Since this is independent of i, we can use (3.60) to get

$$\begin{aligned}
\sum_i \langle M_{ii}^3 \rangle = & \left(\frac{1}{D}\widetilde{\mu}_1 + \frac{\sqrt{(D-1)}}{D})\widetilde{\mu}_2 \right) \times \\
& \left(\frac{1}{D}(\Lambda_{V_0}^{-1})_{11} + \frac{(D-1)}{D}(\Lambda_{V_0}^{-1})_{22} + 2\frac{\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12} \right. \\
& + \frac{D-1}{D}(\Lambda_H^{-1})_{11} + (\Lambda_H^{-1})_{22}\frac{(D-1)}{D} + D^{-1}(D-1)(D-2)(\Lambda_H^{-1})_{33} \\
& \left. + 2\frac{(D-1)}{D}(\Lambda_H^{-1})_{12} + 2\frac{(\Lambda_H^{-1})_{13}}{D}(D-1)\sqrt{(D-2)} + 2\frac{(\Lambda_H^{-1})_{23}}{D}(D-1)\sqrt{(D-2)} \right) \\
& + \frac{1}{D^2} \left(\widetilde{\mu}_1 + \sqrt{(D-1)}\,\widetilde{\mu}_2 \right)^3
\end{aligned} \tag{4.3}$$

4.2   A 2-node case $\sum_{i,j} \langle M_{ij}^3 \rangle$

Using A.6 we have

$$\langle M_{ij}^3 \rangle = \sum_{i,j} 3\langle M_{ij}^2 \rangle_{\text{conn}} \langle M_{ij} \rangle + \sum_{i,j} \langle M_{ij} \rangle^3 \tag{4.4}$$

Calculating this requires doing a few sums, which can be done by hand or with Mathematica (the function KoneckerDelta is handy).

$$\sum_{i,j} F(i,j) = 0$$
$$\sum_{i,j} (F(i,j))^2 = (D-1)$$
$$\sum_{i,j} (F(i,j))^3 = D^{-1}(D-1)(D-2) \tag{4.5}$$

Using (3.3), we find for the second term in (4.4)

$$\sum_{i,j} \langle M_{ij}\rangle^3 = \frac{\widetilde{\mu}_1^3}{D} + \frac{3}{D}\,\widetilde{\mu}_1\widetilde{\mu}_2^2 + \frac{(D-2)}{D\sqrt{D-1}}\widetilde{\mu}_2^3 \tag{4.6}$$

For the first term on the RHS of (4.4)

$$\sum_{i,j} \langle M_{ij}\rangle_{\text{conn}} \langle M_{ij}\rangle = \frac{3\widetilde{\mu}_1}{D}\sum_{i,j}\langle M_{ij}^2\rangle_{\text{conn}} + \frac{3\widetilde{\mu}_2}{\sqrt{D-1}}\sum_{i,j}\langle M_{ij}^2\rangle_{\text{conn}}\,F(i,j) \tag{4.7}$$

The first term in (4.7) can be expressed as a function of the parameters of the Gaussian model using (3.18). The second term is calculated by specialising the fundamental quadratic moments (3.6) and doing the resulting sums over the F-factors. Consider the V_0 contributions to the second term above. The term proportional to $(\Lambda_{V_0}^{-1})_{11}$ vanishes due to the first of (4.5). The (22) contribution, using the third of (4.5) is

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}} \times \frac{D-2}{D}(\Lambda_{V_0}^{-1})_{22} = 3\widetilde{\mu}_2(\Lambda_{V_0}^{-1})_{22}\frac{(D-2)}{D\sqrt{D-1}} \tag{4.8}$$

The (12) contributions, using the second of (4.5) is

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}} \times \frac{2\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12} = \frac{6\widetilde{\mu}_2(\Lambda_{V_0}^{-1})_{12}}{D} \tag{4.9}$$

Now consider the V_H contribution to the second term in (4.7). The $(\Lambda_H^{-1})_{11}$ term is

$$\frac{1}{D}(\Lambda_H^{-1})_{11}\sum_{i,j}F(i,j)F(j,j) = 0 \tag{4.10}$$

The $(\Lambda_H^{-1})_{22}$ contribution is similarly zero. The $(\Lambda_H^{-1})_{33}$ contribution is

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}} \times \frac{D(\Lambda_H^{-1})_{33}}{(D-2)} \times \sum_{i,j,p,q}F(i,p)F(i,q)F(j,p)F(j,q)F(i,j)F(p,q)$$

$$= 3\widetilde{\mu}_2 (\Lambda_H^{-1})_{33} \sqrt{D-1} \frac{(D-3)}{D} \tag{4.11}$$

The sum of products of six F's is readily done with Mathematica to give $D^{-2}(D-1)(D-2)(D-3)$.

Contributions from the $(1,2)$ matrix element of symmetric matrix Λ_H give

$$\frac{6\widetilde{\mu}_2 \sqrt{(D-1)}}{D} (\Lambda_H^{-1})_{12} \tag{4.12}$$

This uses the second of (4.5). From $(1,3)$ and $(3,1)$ we have

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}} \times \frac{\Lambda_H^{-1}}{\sqrt{D-2}} \times 2 \sum_{i,j,p} F(i,p)(F(j,p))^2 F(i,j)$$

$$= \frac{3\widetilde{\mu}_2}{\sqrt{D-1}} \times \frac{\Lambda_H^{-1}}{\sqrt{D-2}} \times 2D^{-1}(D-1)(D-2)$$

$$= \frac{6\widetilde{\mu}_2}{D} (\Lambda_H^{-1})_{13} \sqrt{(D-1)(D-2)} \tag{4.13}$$

From $(2,3)$ and $(3,2)$, we have

$$\frac{6\widetilde{\mu}_2}{D} \sqrt{(D-1)(D-2)} (\Lambda_H^{-1})_{23} \tag{4.14}$$

Now consider the contribution from V_2. It is convenient to use (3.5)

$$(\Lambda_{V_2}^{-1}) \sum_{i,j} \sum_{a_1,b_1,c,a_2,b_2,d} C_{a_1 i} C_{b_1 j} C_{a_1 b_1 c}^{H,H \to V_2} C_{a_2 i} C_{b_2 j} C_{a_2,b_2,c}^{H,H \to V_2} C_{di} C_{dj}$$

$$= (\Lambda_{V_2}^{-1}) \sum_{i,j} \sum_{a_1,b_1,c,a_2,b_2,d} C_{a_1 i} C_{a_2 i} C_{di} C_{b_1 j} C_{b_2 j} C_{dj} (P^{H,H \to V_2})_{a_1 b_1 ; a_2 b_2}$$

$$= (\Lambda_{V_2}^{-1}) \sum_{a_1,b_1,c,a_2,b_2,d} C_{a_1 a_2 d} C_{b_1 b_2 d} (P^{H,H \to V_2})_{a_1 b_1 ; a_2 b_2}$$

$$= (\Lambda_{V_2}^{-1}) \frac{D-2}{D} \sum_{a_1,a_2,b_1,b_2} (P^{H,H \to H})_{a_1,a_2 ; b_1,b_2} (P^{H,H \to V_2})_{a_1 b_1 ; a_2 b_2} \tag{4.15}$$

In the last line, we have used the second equation in C.10 which gives the relation between the invariant C_{abc} in $V_{\mathrm{nat}}^{\otimes 3}$ and the normalized Clebsch-Gordan Gordan coefficients $C_{abc}^{HH \to H}$, and the formula for the projector in terms of the Clebsch-Gordan coefficients.

The symmetric part of $V_H \otimes V_H$, i.e the subspace invariant under the swop of the two factors, decomposes into irreducible representations of the diagonal S_D action as $V_0 \oplus V_H \oplus V_2$. This means that

$$P^{H,H \to V_2} = (1 - P^{H,H \to H} - P^{H,H \to V_0}) \frac{(1+\tau)}{2}$$

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$$P^{H,H\to H}\frac{(1+\tau)}{2} = P^{H,H\to H} \tag{4.16}$$

This means that

$$(P^{H,H\to V_2})_{a_1b_1;a_2b_2} = \frac{1}{2}(\delta_{a_1a_2}\delta_{b_1b_2} + \delta_{a_1b_2}\delta_{a_2b_1}) - P^{H,H\to H}_{a_1,b_1;a_2,b_2} - P^{H,H\to V_0}_{a_1,b_1;a_2,b_2} \tag{4.17}$$

Useful fact following from (2.15) and (2.13) is

$$\sum_a C_{aab} = 0 \tag{4.18}$$

When the expression (4.17) is substituted in (4.15) the first term on the RHS of (4.17) does not contribute because of (4.18). The second term gives

$$(\Lambda_{V_2}^{-1})\frac{D-2}{2D} \sum_{a_1,a_2,b_1,b_2} (P^{H,H\to H})_{a_1,a_2;a_2,a_1} = (\Lambda_{V_2}^{-1})\frac{(D-1)(D-2)}{2D} \tag{4.19}$$

The third term gives

$$-(\Lambda_{V_2}^{-1})\frac{D-2}{D} \sum_{a_1,a_2,b_1,b_2} (P^{H,H\to H})_{a_1,a_2;b_1,b_2}(P^{H,H\to H})_{a_1b_1;a_2b_2}$$

$$= -(\Lambda_{V_2}^{-1})\frac{D}{(D-2)} \sum_{a_1,a_2,b_1,b_2,c_1,c_2} C_{a_1,a_2,c_1}C_{b_1,b_2,c_1}C_{a_1,b_1,c_2}C_{a_2,b_2,c_2}$$

$$= -(\Lambda_{V_2}^{-1})\frac{D}{(D-2)} \sum_{a_1,a_2,b_1,b_2,c_1,c_2} \sum_{i,j,p,q} F(i,p)F(i,q)F(i,j)F(j,p)F(j,q)F(p,q)$$

$$= -(\Lambda_{V_2}^{-1})\frac{D}{(D-2)}\frac{(D-1)(D-2)(D-3)}{D^2}$$

$$= -(\Lambda_{V_2}^{-1})\frac{(D-1)(D-3)}{D} \tag{4.20}$$

The fourth term gives

$$-(\Lambda_{V_2}^{-1})\frac{D-2}{D} \sum_{a_1,a_2,b_1,b_2} (P^{H,H\to H})_{a_1,a_2;b_1,b_2}(P^{H,H\to V_0})_{a_1,b_1;a_2,b_2}$$

$$-(\Lambda_{V_2}^{-1})\frac{D-2}{D}\frac{1}{(D-1)} \sum_{a_1,a_2,b_1,b_2} (P^{H,H\to H})_{a_1,a_2;b_1,b_2}\delta_{a_1b_1}\delta_{a_2b_2}$$

$$= -(\Lambda_{V_2}^{-1})\frac{D-2}{D} \tag{4.21}$$

Collecting terms from (4.19), (4.20) and (4.21) we get

$$(\Lambda_{V_2}^{-1})\frac{(D^2-3D+3)}{2D} \tag{4.22}$$

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Multiplying the factor $\frac{3\widetilde{\mu}_2}{\sqrt{D-1}}$ from (4.7) to get a contribution to $\sum_{i,j}\langle M_{ij}^3\rangle$, we get

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}}\,(\Lambda_{V_2}^{-1})\frac{(D^2-3D+3)}{2D}=\widetilde{\mu}_2\Lambda_{V_2}^{-1}\frac{3(D^2-3D+3)}{2D\sqrt{D-1}} \tag{4.23}$$

The contribution from V_3 is

$$\frac{3\widetilde{\mu}_2}{\sqrt{D-1}}(\Lambda_{V_3}^{-1})\frac{D-2}{D}\sum_{a_1,a_2,b_1,b_2}(P^{H,H\to H})_{a_1,a_2;b_1,b_2}(P^{H,H\to V_3})_{a_1b_1;a_2b_2} \tag{4.24}$$

Now use the fact that

$$(P^{H,H\to V_3})_{a_1b_1;a_2b_2}=\frac{1}{2}(\delta_{a_1a_2}\delta_{b_1b_2}-\delta_{a_1b_2}\delta_{a_2b_1}) \tag{4.25}$$

along with

$$\sum_{a,b}(P^{H,H\to H})_{aa;bb}=0$$
$$\sum_{a_1,a_2}(P^{H,H\to H})_{a_1a_2;a_2a_1}=\sum_{a_1,a_2}(P^{H,H\to H})_{a_1a_2;a_1a_2}=(D-1) \tag{4.26}$$

to find

$$-(\Lambda_{V_3}^{-1})\frac{3\widetilde{\mu}_2}{\sqrt{D-1}}\frac{(D-2)}{D}(D-1)=-3\,\widetilde{\mu}_2\,(\Lambda_{V_3}^{-1})\frac{(D-2)\sqrt{D-1}}{D} \tag{4.27}$$

Collecting all the contributions we have

$$\sum_{i,j}\langle M_{ij}^3\rangle=\frac{\widetilde{\mu}_1^3}{D}+\frac{3}{D}\,\widetilde{\mu}_1\widetilde{\mu}_2^2+\frac{(D-2)}{D\sqrt{D-1}}\widetilde{\mu}_2^3$$
$$+\frac{3\widetilde{\mu}_1}{D}\,\left((\Lambda_{V_0}^{-1})_{11}+(\Lambda_{V_0}^{-1})_{22}+(D-1)(\Lambda_H^{-1})_{22}+(D-1)(\Lambda_H^{-1})_{33}+(D-1)(\Lambda_H^{-1})_{11}\right.$$
$$\left.+\frac{D(D-3)}{2}(\Lambda_{V_2})^{-1}+\frac{(D-1)(D-2)}{2}(\Lambda_{V_3})^{-1}\right)+3\widetilde{\mu}_2(\Lambda_{V_0}^{-1})_{22}\frac{D-2}{D\sqrt{D-1}}+\frac{6\widetilde{\mu}_2(\Lambda_{V_0}^{-1})_{12}}{D}$$
$$+3\widetilde{\mu}_2(\Lambda_H^{-1})_{33}\sqrt{D-1}\frac{(D-3)}{D}+6\widetilde{\mu}_2(\Lambda_H^{-1})_{12}\frac{\sqrt{(D-1)}}{D}+6\widetilde{\mu}_2(\Lambda_H^{-1})_{13}\frac{\sqrt{(D-1)(D-2)}}{D}$$
$$+6\widetilde{\mu}_2(\Lambda_H^{-1})_{13}\frac{\sqrt{(D-1)(D-2)}}{D}+\widetilde{\mu}_2\Lambda_{V_2}^{-1}\frac{3(D^2-3D+3)}{2D\sqrt{D-1}}-3\,\widetilde{\mu}_2\,(\Lambda_{V_3}^{-1})\frac{(D-2)\sqrt{D-1}}{D}$$
$$\tag{4.28}$$

4.3 A 3-node case $\sum_{i,j,k}\langle M_{ij}M_{jk}M_{ki}\rangle$

Using Wick's theorem (A.6)

$$\sum_{i,j,k}\langle M_{ij}M_{jk}M_{ki}\rangle = \sum_{i,j,k}\langle M_{ij}\rangle\langle M_{jk}\rangle\langle M_{ki}\rangle + \langle M_{ij}M_{jk}\rangle\langle M_{ki}\rangle + \langle M_{ij}M_{ki}\rangle\langle M_{jk}\rangle + \langle M_{jk}M_{ki}\rangle\langle M_{ij}\rangle$$

$$(4.29)$$

The first term is

$$\sum_{i,j,k}\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(i,j)\right)\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(j,k)\right)\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(k,i)\right) \quad (4.30)$$

Using

$$\sum_j F(i,j)F(j,k) = F(i,k) \tag{4.31}$$

along with the first and second of (4.5) we can show that

$$\sum_{i,j,k}\langle M_{ij}\rangle\langle M_{jk}\rangle\langle M_{ki}\rangle = \widetilde{\mu}_1^3 + \frac{\widetilde{\mu}_2^3}{\sqrt{D-1}} \tag{4.32}$$

Consider the remaining three terms. Focus on the first of these.

$$\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle_{\text{conn}}\,\langle M_{ki}\rangle$$

$$= \sum_{i,j,k}\langle M_{ij}M_{jk}\rangle_{\text{conn}}\left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(k,i)\right)$$

$$= \frac{\widetilde{\mu}_1}{D}\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle_{\text{conn}} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}\sum_{i,j,k}\langle M_{ij}M_{jk}\rangle F(k,i) \tag{4.33}$$

We already know the first term from (3.51). So let us consider the second. Easy calculation using (3.5) (or equivalently using (3.6)) shows that the contribution from the V_0 channel is

$$\frac{\widetilde{\mu}_2}{\sqrt{D-1}}(\Lambda_{V_0}^{-1})_{22} \tag{4.34}$$

From the V_H channel, the contributions are

$$\widetilde{\mu}_2\sqrt{D-1}(\Lambda_H^{-1})_{33} + \frac{\widetilde{\mu}_2}{D}\sqrt{D-1}(\Lambda_H^{-1})_{12} \tag{4.35}$$

From the V_2 channel, we get

$$\widetilde{\mu}_2 \Lambda_{V_2}^{-1} \frac{D(D-3)}{2\sqrt{D-1}} - \widetilde{\mu}_2 (\Lambda_{V_3}^{-1}) \frac{D - 2\sqrt{D-1}}{2} \tag{4.36}$$

Collecting terms

$$\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle_{\text{conn}} \langle M_{ki} \rangle = \widetilde{\mu}_1 \left((\Lambda_{V_0}^{-1})_{11} + (D-1)(\Lambda_H^{-1})_{12} \right)$$

$$+ \widetilde{\mu}_2 \sqrt{D-1}(\Lambda_0^{-1})_{22} + \widetilde{\mu}_2 \sqrt{D-1}(\Lambda_H^{-1})_{33} + \frac{\widetilde{\mu}_2}{D} \sqrt{D-1}(\Lambda_H^{-1})_{12}$$

$$+ \widetilde{\mu}_2 \Lambda_{V_2}^{-1} \frac{D(D-3)}{2\sqrt{D-1}} - \widetilde{\mu}_2 (\Lambda_{V_3}^{-1}) \frac{(D-2)\sqrt{D-1}}{2} \tag{4.37}$$

By relabelling indices, it is easy to see that

$$\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle_{\text{conn}} \langle M_{ki} \rangle = \sum_{i,j,k} \langle M_{ij} M_{ki} \rangle_{\text{conn}} \langle M_{jk} \rangle = \sum_{i,j,k} \langle M_{jk} M_{ki} \rangle_{\text{conn}} \langle M_{ij} \rangle \tag{4.38}$$

Hence, we have

$$\sum_{i,j,k} \langle M_{ij} M_{jk} M_{ki} \rangle = \widetilde{\mu}_1^3 + \frac{\widetilde{\mu}_2^3}{\sqrt{D-1}} + 3\widetilde{\mu}_1 \left((\Lambda_{V_0}^{-1})_{11} + (D-1)(\Lambda_H^{-1})_{12} \right)$$

$$+ 3\widetilde{\mu}_2 \sqrt{D-1}(\Lambda_{V_0}^{-1})_{22} + 3\widetilde{\mu}_2 \sqrt{D-1}(\Lambda_H^{-1})_{33} + 3\frac{\widetilde{\mu}_2}{D} \sqrt{D-1}(\Lambda_H^{-1})_{12}$$

$$+ 3\widetilde{\mu}_2 \Lambda_2^{-1} \frac{D(D-3)}{2\sqrt{D-1}} - 3\widetilde{\mu}_2 (\Lambda_3^{-1}) \frac{(D-2)\sqrt{D-1}}{2} \tag{4.39}$$

4.4   a 6-node case $\sum_{i_1,\cdots,i_6} \langle M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \rangle$

The sums over $i_1, \cdots, i_6$ project to the V_0 representations. As a result, using (A.6), along with (2.50), we have

$$\sum_{i_1,\cdots,i_6} \langle M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \rangle = \frac{3}{D^3} \sum_{i_1,\cdots,i_6} \langle S^{V_0;1} S^{V_0;1} \rangle \langle S^{V_0;1} \rangle + \frac{1}{D^3} \sum_{i_1,\cdots,i_6} (\langle S^{V_0;1} \rangle)^3$$

$$= 3\widetilde{\mu}_1 D^3 (\Lambda_{V_0}^{-1})_{11} + \widetilde{\mu}_1^3 D^3 \tag{4.40}$$

5   A selection of quartic expectation values

The methods we have used to calculate the cubic expectation values, which were explained in detail above, extend straightforwardly to quartic expectation values. The first step is to use Wick's theorem A.7. Then we use the formulae for quadratic and linear expectation values from Sections 2 and 3. In order to arrive at the final result as a function of $D, \widetilde{\mu}_1, \widetilde{\mu}_2, \Lambda_{V_0}, \Lambda_H, \Lambda_{V_2}, \Lambda_{V_3}$ we have to do certain sums over products of the natural-to-Hook projector $F(i,j)$. We will give some formulae below to illustrate these steps for the quartic case, without producing detailed formulae as in previous sections.

5.1 $\sum_{i,j}\langle M_{ij}^4 \rangle$

$$\sum_{i,j}\langle M_{ij}^4 \rangle = \sum_{i,j}\langle M_{ij}\rangle^4 + 6\sum_{i,j}\langle M_{ij}^2\rangle_{\text{conn}}\langle M_{ij}\rangle^2 + 3\sum_{i,j}\langle M_{ij}^2\rangle_{\text{conn}}\langle M_{ij}^2\rangle_{\text{conn}} \tag{5.1}$$

The quadratic average is

$$
\begin{aligned}
\langle M_{i,j}^2\rangle_{\text{conn}} &= \frac{(\Lambda_{V_0}^{-1})_{11}}{D^2} + \frac{(\Lambda_{V_0}^{-1})_{22}}{(D-1)^2}F(i,j)^2 + \frac{2(\Lambda_{V_0}^{-1})_{12}}{D\sqrt{D-1}}F(i,j) \\
&+ \frac{(\Lambda_H^{-1})_{11}}{D}F(j,j) + \frac{(\Lambda_H^{-1})_{22}}{D}F(i,i) + \frac{(\Lambda_H^{-1})_{33}}{D-2}\sum_{p,q}F(i,p)F(j,q))F(i,q)F(j,q)F(p,q) \\
&+ \frac{2(\Lambda_H^{-1})_{12}}{D}F(i,j) + \frac{2(\Lambda_H^{-1})_{13}}{D-2}\sum_{p}F(i,p)F(j,p)F(j,p) \\
&+ \frac{2(\Lambda_H^{-1})_{23}}{\sqrt{D-2}}\sum_{p}F(i,p)F(j,p)F(j,p) \\
&+ (\Lambda_{V_2})^{-1}\left(\frac{1}{2}F(i,i)F(j,j) + \frac{1}{2}F(i,j)F(i,j) - \frac{D}{D-2}\sum_{p,q}F(i,p)F(i,q)F(j,p)F(j,q)F(p,q)\right) \\
&+ \frac{(\Lambda_{V_3}^{-1})}{2}\left(\frac{1}{2}F(i,i)F(j,j) - F(i,j)F(i,j)\right)
\end{aligned}
\tag{5.2}
$$

Using this and

$$\langle M_{ij}\rangle = \left(\frac{\widetilde{\mu}_1}{D} + \frac{\widetilde{\mu}_2}{\sqrt{D-1}}F(i,j)\right) \tag{5.3}$$

we can work out the formula for $\sum_{i,j}\langle M_{ij}^4 \rangle$ as a function of the 13 Gaussian model parameters. Mathematica would be handy in doing the sums over products of $F(i,j)$ which arise.

5.2 $\sum_{i,j,k,p,q}\langle M_{ij}M_{jk}M_{kp}M_{pq}\rangle$

From A.7, we have

$$
\begin{aligned}
\sum_{i,j,k,p,q}\langle M_{ij}M_{jk}M_{kp}M_{pq}\rangle = &\sum_{i,j,k,p,q}\langle M_{ij}\rangle\langle M_{jk}\rangle\langle M_{kp}\rangle\langle M_{pq}\rangle \\
&+ \sum\langle M_{ij}M_{jk}\rangle_{\text{conn}}\langle M_{kp}\rangle\langle M_{pq}\rangle + \sum\langle M_{ij}M_{kp}\rangle_{\text{conn}}\langle M_{jk}\rangle\langle M_{pq}\rangle + \sum\langle M_{ij}M_{pq}\rangle_{\text{conn}}\langle M_{jk}\rangle\langle M_{kp}\rangle \\
&+ \sum\langle M_{jk}M_{kp}\rangle_{\text{conn}}\langle M_{ij}\rangle\langle M_{pq}\rangle + \sum\langle M_{jk}M_{pq}\rangle_{\text{conn}}\langle M_{ij}\rangle\langle M_{kp}\rangle + \sum\langle M_{kp}M_{pq}\rangle_{\text{conn}}\langle M_{ij}\rangle\langle M_{jk}\rangle \\
&+ \sum\langle M_{ij}M_{jk}\rangle_{\text{conn}}\langle M_{kp}M_{pq}\rangle_{\text{conn}} + \sum\langle M_{ij}M_{kp}\rangle_{\text{conn}}\langle M_{jk}M_{pq}\rangle_{\text{conn}} + \sum\langle M_{ij}M_{pq}\rangle_{\text{conn}}\langle M_{jk}M_{kp}\rangle_{\text{conn}}
\end{aligned}
\tag{5.4}
$$

All the summands on the RHS can be evaluated using 3.5 or 3.6 in terms of $F(i,j)$. The sums can be done with the help of Mathematica to obtain expressions in terms of $D, \widetilde{\mu}_1, \widetilde{\mu}_2, \Lambda_V$.

6   Summary and Outlook

We have used the representation theory of symmetric groups S_D in order to define a 13-parameter permutation invariant Gaussian matrix model, to compute the expectation values of all the graph-basis permutation invariant quadratic functions of the random matrix, and a selection of cubic and quartic invariants. In [8] analogous computations with a 5-parameter model were compared with matrix data constructed from a corpus of the English language. A natural direction is to extend that discussion of the English language, or indeed other languages, to the present 13-parameter model. Combining the experimental methods employed in [8] with machine learning methods such those used in [17], in the investigation of the 13-parameter model, would also be interesting to explore.

As a theoretical extension of the present work, it will be useful to generalise the representation theoretic parametrisation of the Gaussian models to perturbations of the Gaussian model, where we add cubic and quartic terms to the Gaussian action. Identifying parameter spaces of these deformations which allow well-defined convergent partition functions and expectation values will be useful for eventual comparison to data. If we ignore the convergence constraints, the general perturbed model at cubic and quartic order has 348 parameters, since there are 52 cubic invariants and 296 quartic invariants (Appendix A of [8]). As in the Gaussian case, we can expect that representation theory methods will be useful in handling this more general problem. Further techniques involving partition algebras underlying the representation theory of tensor products of the natural representation will likely play a role (see e.g. [18] for recent work in these directions).

It is worth noting that permutation invariant random matrix distributions have been approached from a different perspective, based on non-commutative probability theory [19, 20, 21]. The approach of the present paper and [8] is based on the connection between statistical physics and zero dimensional quantum field theory (QFT). It would seem that the approach of the present paper can complement the theory developed in these papers [19, 20, 21] by producing integral representations (Gaussians or perturbed Gaussians) of random matrix distributions having finite expectation values for permutation invariant polynomial functions of matrices. The results on the central limit theorem from the above references would be very interesting to interpret from the present QFT perspective.

The computation of expectation values in Gaussian matrix models admits generalization to higher tensors. Indeed the motivating framework in computational linguistics disucssed in [8] involves matrices as well as higher tensors in a natural way. Generalizations of the present work on representation theoretic parametrisation of Gaussian models

and computation of graph-basis observables to the tensor case is an interesting avenue for future research.

In this paper, we have focused on the explicit computation of permutation invariant correlators for general D. Some simplifications at large D were discussed in section 3.12. For traditional matrix models having $U(D)$ (or $SO(D)/Sp(D)$ symmetries), there is a rich geometry of two dimensional surfaces and maps in the large D expansions which allows these expansions of matrix quantum field theories to have deep connections to string theory [26, 27]. It will be interesting to explore the possibility of two dimensional geometrical interpretations of the large D expansion in permutation invariant matrix models.

Acknowledgements

A Multi-dimensional Gaussian Integrals and Wick's theorem

Consider the multi-variable integral with a Gaussian integrand

$$\mathcal{Z} = \int dx \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i s_i x_i \right) = \sqrt{\frac{(2\pi)^N}{\det A}} \exp\left(\frac{1}{2} s_i (A^{-1})_{ij} s_j \right) \quad (A.1)$$

$x \in \mathbb{R}^N$. $A \in \mathbb{C}^{N \times N}$ is a real symmetric positive definite matrix. $s \in \mathbb{R}^N$ is an arbitrary complex vector (see for example [15], [16], Appendix A euqations (8) and (9) of [24]). One can also consider A more generally to be complex with positive definite real part, but to keep a probabilistic interpretation we keep A real symmetric. Expectation values of functions $f(x)$ are defined by

$$\langle f(x) \rangle = \frac{1}{\mathcal{Z}} \int dx \ f(x) \ \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i s_i x_i \right) \quad (A.2)$$

These expectation values can be calculated by taking derivatives with respect to s_i on both sides of (A.1). For the x variables

$$\langle x_i \rangle = \sum_j (A^{-1})_{ij} \, x_j = \sum_j x_j \, (A^{-1})_{ji} \tag{A.3}$$

Application of this equation, along with the formula for dM in terms of the representation theoretic S-variables (2.58) leads to (2.65),(2.66). For expectation values of quadratic monomials

$$\langle x_i x_j \rangle = (A^{-1})_{ij} + \langle x_i \rangle \langle x_j \rangle \tag{A.4}$$

We define the connected part as

$$\langle x_i x_j \rangle_{\mathrm{conn}} \equiv \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = (A^{-1})_{ij} \tag{A.5}$$

The expressions (2.67) and (2.3) follow from these.

For cubic expressions

$$\langle x_i x_j x_k \rangle = \langle x_i x_j \rangle_{\mathrm{conn}} \langle x_k \rangle + \langle x_i x_k \rangle_{\mathrm{conn}} \langle x_j \rangle + \langle x_j x_k \rangle_{\mathrm{conn}} \langle x_i \rangle + \langle x_i \rangle \langle x_j \rangle \langle x_k \rangle \tag{A.6}$$

For quartic expressions

$$\begin{aligned}
\langle x_i x_j x_k x_l \rangle \quad &= \langle x_i x_j \rangle_{\mathrm{conn}} \langle x_k x_l \rangle_{\mathrm{conn}} + \langle x_i x_k \rangle_{\mathrm{conn}} \langle x_j x_l \rangle_{\mathrm{conn}} + \langle x_i x_l \rangle_{\mathrm{conn}} \langle x_j x_k \rangle_{\mathrm{conn}} \\
&\quad + \langle x_i x_j \rangle_{\mathrm{conn}} \langle x_k \rangle \langle x_l \rangle + \langle x_i x_k \rangle_{\mathrm{conn}} \langle x_j \rangle \langle x_l \rangle + \langle x_i x_l \rangle_{\mathrm{conn}} \langle x_j \rangle \langle x_k \rangle + \langle x_i \rangle \langle x_j \rangle \langle x_k \rangle \langle x_l \rangle
\end{aligned} \tag{A.7}$$

These illustrate a general fact (known as Wick's theorem in the quantum field theory context and Isserlis' theorem in probablity theory [25]) about Gaussian expectation values. Higher order expectation values can be expressed in terms of linear and quadratic expectation values. When applied to permutation invariant matrix models, we still have non-trivial sums left to do, after Wick's theorem has been applied. This is illustrated in the calculations of section 4 and section 5.

B   Rep theory of V_H and its tensor products

Some basics of rep theory of V_H can be presented in a self-contained way, assuming only knowledge of linear algebra and index notation.

Alternatively, we can observe that the matrices in V_H are the same as Young's orthogonal basis. If we just follow the self-contained route, we define

$$\begin{aligned}
D^H_{ab}(\sigma) \quad &= (E_a, \sigma E_b) \\
&= \sum_i \sum_j C_{a,i} C_{b,j} (e_i, e_j)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j} C_{a,i} C_{b,j} (e_i, e_{\sigma^{-1}(j)}) \\
&= \sum_{i,j} C_{a,i} C_{b,j} \delta_{i,\sigma^{-1}(j)} \\
&= \sum_{i} C_{a,i} C_{b,\sigma(i)}
\end{aligned} \tag{B.1}$$

To get the orthogonality property :

$$\begin{aligned}
D_{ab}^H(\sigma^{-1}) &= \sum_{i} C_{a,i} C_{b,\sigma^{-1}(i)} \\
&= \sum_{i} C_{a,\sigma(i)} C_{b,i} \\
&= D_{ba}^H(\sigma)
\end{aligned} \tag{B.2}$$

The homomorphism property

$$\begin{aligned}
D_{ab}^H(\sigma) D_{bc}^H(\tau) &= \sum_{i} C_{a,i} C_{b,\sigma(i)} \sum_{j} C_{b,j} C_{c,\tau(j)} \\
&= \sum_{i,j} C_{a,i} C_{b,j} C_{b,\sigma(i)} C_{c,\tau(j)} \\
&= \sum_{i,j} C_{a,i} C_{b,j} C_{b,\sigma(i)} C_{c,\tau(j)} \\
&= \sum_{i,j} C_{a,i} \left(\delta_{j,\sigma(i)} - \frac{1}{D} \right) C_{c,\tau(j)} \\
&= \sum_{i,j} C_{a,\sigma^{-1}(j)} C_{c,\tau(j)} - \frac{1}{D} \sum_{i,j} C_{a,i} C_{c,\tau(j)} \\
&= \sum_{j} C_{a,j} C_{c,\tau(\sigma(j))} \\
&= \sum_{j} C_{a,j} C_{c,\sigma\tau(j)} \\
&= D_{ac}^H(\sigma\tau)
\end{aligned} \tag{B.3}$$

We used

$$\begin{aligned}
\sum_{a} C_{a,i} C_{a,j} &= \left(\delta_{ij} - \frac{1}{D} \right) \\
\sum_{a} C_{a,i} &= 0
\end{aligned} \tag{B.4}$$

Using the definition (B.1), we prove

$$\sum_{b} D_{ba}^H(\sigma) C_{b,i} = C_{a,\sigma(i)} \tag{B.5}$$

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Indeed

$$\sum_b D_{ba}^H(\sigma) C_{b,i} = \sum_j C_{b,j} C_{a,\sigma(j)} C_{b,i}$$

$$= \sum_j C_{a,\sigma(j)} \left(\delta_{i,j} - \frac{1}{D}\right)$$

$$= C_{a,\sigma(i)} \tag{B.6}$$

It is useful to define $C_{\sigma(a),i} = \sum_b D_{ba}^H(\sigma) C_{bi}$ so the above can be expressed as an **equivariance** property

$$C_{\sigma(a),i} = C_{a,\sigma(i)} \tag{B.7}$$

which is an equivariance condition for the map $V_H \to V_{nat}$ given by the coefficients $C_{a,i}$. This map intertwines the S_n action on the V_H and V_{nat}. Now define $C_{a,b,c}$

$$C_{a,b,c} = \sum_i C_{a,i} C_{b,i} C_{c,i} \tag{B.8}$$

We show that this is an **invariant tensor**.

$$C_{\sigma(a),\sigma(b),\sigma(c)} = C_{a,b,c} \tag{B.9}$$

Indeed

$$C_{\sigma(a),\sigma(b),\sigma(c)} = \sum_{i=1}^n C_{\sigma(a),i} C_{\sigma(b),i} C_{\sigma(c),i}$$

$$= \sum_i C_{a,\sigma(i)} C_{b,\sigma(i)} C_{c,\sigma(i)}$$

$$= \sum_i C_{a,i} C_{b,i} C_{c,i}$$

$$= C_{a,b,c} \tag{B.10}$$

WE used the equivariance of the C's, then the relabelled the sum $i \to \sigma(i)$.

Using vectors $\{e_a\}$ spanning V_H, we write a basis for the tensor product $V_H \otimes V_H$.

$$e_a \otimes e_b \tag{B.11}$$

There is a subspace of $V_H \otimes V_H$, which transforms as the irrep V_H. This is constructed using the invariant 3-index tensor $C_{a,b,c}$. Define the linear combinations

$$E_a = \sum_{a,b,c} C_{a,b,c} e_b \otimes e_c \tag{B.12}$$

Act on this by the diagonal action of $\sigma \in S_D$.

$$
\begin{aligned}
\sigma E_a &= \sum_{a,b,c} C_{a,b,c}(\sigma e_b) \otimes (\sigma e_c) \\
&= \sum_{a,b,c} \sum_{b',c'} C_{a,b,c} D_{b'b}^H(\sigma) D_{c'c}^H(\sigma)(e_{b'} \otimes e_{c'}) \\
&= \sum_{a,b,c} \sum_{d,b',c'} C_{d,b,c} D_{ad}^H(\sigma^{-1}\sigma) D_{b'b}^H(\sigma) D_{c'c}^H(\sigma)(e_{b'} \otimes e_{c'}) \\
&= \sum_{a,b,c} \sum_{a',d,b',c'} C_{d,b,c} D_{aa'}^H(\sigma^{-1}) D_{a'd}^H(\sigma) D_{b'b}^H(\sigma) D_{c'c}^H(\sigma)(e_{b'} \otimes e_{c'}) \\
&= \sum_{a,a',b',c'} C_{\sigma^{-1}(a'),\sigma^{-1}(b'),\sigma^{-1}(c')} D_{a',a}^H(\sigma)(e_{b'} \otimes e_{c'}) \\
&= \sum_{a'} D_{a',a}^H(\sigma) \sum_{a,b,c} C_{a',b',c'}(e_{b'} \otimes e_{c'}) \\
&= \sum_{a'} D_{a'a}^H(\sigma) E_{a'} \qquad (B.13)
\end{aligned}
$$

Useful orthogonality property

$$
\begin{aligned}
(E_{a_1}, E_{a_2}) &= \sum_{b,c} C_{a_1,b,c} C_{a_2,b,c} \\
&= \sum_{i,j} \sum_{b,c} C_{a_1,i} C_{b,i} C_{c,i} C_{a_2,j} C_{b,j} C_{c,j} \\
&= \sum_{i,j} C_{a_1,i} C_{a_2,j} \left(\delta_{i,j} - \frac{1}{D} \right) \left(\delta_{i,j} - \frac{1}{D} \right) \\
&= (1 - \frac{2}{D}) \sum_i C_{a_1,i} C_{a_2,i} \\
&= \frac{(D-2)}{D} \delta_{a_1,a_2} \qquad (B.14)
\end{aligned}
$$

C   Clebsch-Gordan coefficients and normalizations

$$
\sum_{a,b} C_{a,b,c}^{H,H \to V} C_{a,b,c'}^{H,H \to V} = \delta_{cc'} \qquad (C.1)
$$

This means that

$$
\sum_{a,b,c} C_{a,b,c}^{H,H \to V} C_{a,b,c}^{H,H \to V} = \text{Dim } V \qquad (C.2)
$$

If instead we consider the invariant state in $H \otimes H \otimes V$, normalized to one, then

$$
\sum_{a,b,c} \sum_{a',b',c'} (C_{a,b,c}^{H,H,V} e_a \otimes e_b \otimes e_c, C_{a',b',c'}^{H,H,V} e_{a'} \otimes e_{b'} \otimes e_{c'}) = \sum_{a,b,c} (C_{a,b,c}^{H,H,V})^2 = 1 \qquad (C.3)
$$

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The equivariance property of the map $C^{H,H\to V}$ is

$$D^{H\otimes H}(\sigma\otimes\sigma)\sum_{a,b}C^{H,H\to V}_{a,b,c}e_a\otimes e_b \quad = \sum_{b',c'}D^H_{a'a}(\sigma)D^H_{b'b}(\sigma)C^{H,H\to V}_{a,b,c}e_{a'}\otimes e_{b'}$$

$$= \sum_{c'}D^H_{c'c}(\sigma)\sum_{a',b'}C^{H,H\to V}_{a',b',c'}e_{a'}\otimes e_{b'} \qquad (C.4)$$

which means

$$\sum_{b',c'}D^H_{a'a}(\sigma)D^H_{b'b}(\sigma)C^{H,H\to V}_{a,b,c} = D^H_{c'c}(\sigma)C^{H,H\to V}_{a,b,c'} \qquad (C.5)$$

Multiplying on both sides by $D^H_{ce}(\sigma^{-1})$ and summing over c, we have, after using $D^H_{ab}(\sigma^{-1}) = D^H_{ba}(\sigma)$ and relabelling indices

$$\sum_{b',c'}D^H_{a'a}(\sigma)D^H_{b'b}(\sigma)D^H_{c'c}(\sigma)C^{H,H\to V}_{a,b,c} = C^{H,H\to V}_{a,b,c} \qquad (C.6)$$

This means that we can identify

$$C^{H,H\to V}_{a,b,c} = \sqrt{\text{Dim } V}\ C^{H,H,V}_{a,b,c} \qquad (C.7)$$

We also know that

$$C_{a,b,c} = \sum_i C_{a,i}C_{b,i}C_{c,i} \qquad (C.8)$$

has the invariance property of $C^{H,H,H}_{a,b,c}$. Since there is a unique invariant state in $V_H\otimes V_H\otimes V_H$, C_{abc} must be proportional to $C^{H,H,H}_{a,b,c}$. We calculate

$$\sum_{a,b,c}C_{a,b,c}C_{a,b,c} = \sum_{i,j}\sum_{a,b,c}C_{a,i}C_{b,i}C_{c,i}C_{a,j}C_{b,j}C_{c,j}$$

$$= \sum_{i,j}(\delta_{ij}-\frac{1}{D})(\delta_{ij}-\frac{1}{D})(\delta_{ij}-\frac{1}{D})$$

$$= 1\times D - \frac{3}{D}\times D + \frac{3}{D^2}\times D - \frac{1}{D^3}\times D^2$$

$$= D - 3 + 2D^{-1} = D^{-1}(D-1)(D-2) \qquad (C.9)$$

We can therefore identify

$$C^{H,H,H}_{a,b,c} \quad = \sqrt{\frac{D}{(D-1)(D-2)}}\ C_{abc}$$

$$C^{H,H\to H}_{a,b,c} \quad = \sqrt{\frac{D}{(D-2)}}\ C_{a,b,c} \qquad (C.10)$$

The second equation is also consistent, as expected, with (B.14). The projector for V_H in $V_H\otimes V_H$ is given in terms of the normalized Clebsch-Gordan coefficients $C^{H,H\to H}_{a,b,c}$ as

$$P^{H,H\to H}_{ab;cd} = \sum_d C^{H,H\to H}_{a,b,e}C^{H,H\to H}_{c,d,e} \qquad (C.11)$$

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