Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers

YÜKSEL SOYKAN

Zonguldak Bülent Ecevit University, Department of Mathematics,

Art and Science Faculty, 67100, Zonguldak, Turkey

e-mail: yuksel_soykan@hotmail.com

Abstract. In this paper, we define Tribonacci and Tribonacci-Lucas matrix sequences and investigate their properties.

2010 Mathematics Subject Classification. 11B39, 11B83.

Keywords. Tribonacci numbers, Tribonacci matrix sequence, Tribonacci-Lucas matrix sequence.

1. Introduction and Preliminaries

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers.

The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. For example, the ratio of two consecutive Fibonacci numbers converges to the Golden section (ratio), $\alpha_F = \frac{1+\sqrt{5}}{2}$; which appears in modern research, particularly physics of the high energy particles or theoretical physics. Another example, the ratio of two consecutive Padovan numbers converges to the Plastic ratio, $\alpha_P = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$, which have many applications to such as architecture, see [9]. One last example, the ratio of two consecutive Tribonacci numbers converges to the Tribonacci ratio, $\alpha_T = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}$. For a short introduction to these three constants, see [10].

On the other hand, the matrix sequences have taken so much interest for different type of numbers. For matrix sequences of generalized Horadam type numbers, see for example [4], [5], [7], [15], [16], [17], [19], [22], and for matrix sequences of generalized Tribonacci type numbers, see for instance [2], [20], [21].

In this paper, the matrix sequences of Tribonacci and Tribonacci-Lucas numbers will be defined for the first time in the literature. Then, by giving the generating functions, the Binet formulas, and summation formulas over these new matrix sequences, we will obtain some fundamental properties on Tribonacci and Tribonacci-Lucas numbers. Also, we will present the relationship between these matrix sequences.

First, we give some background about Tribonacci and Tribonacci-Lucas numbers. Tribonacci sequence $\{T_n\}_{n\geq 0}$ (sequence A000073 in [13]) and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ (sequence A001644 in [13]) are defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{1.1}$$

and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 1, K_2 = 3 \tag{1.2}$$

respectively. Tribonacci concept was introduced by M. Feinberg [6] in 1963. Basic properties of it is given in [1], [11], [12], [18] and Binet formula for the nth number is given in [14].

The sequences $\{T_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}$$

and

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

By writting $T_{n-1} = T_{n-2} + T_{n-3} + T_{n-4}$ and eliminating T_{n-2} and T_{n-3} between this recurrence relation and recurrence relation (1.1), a useful alternative recurrence relation is obtained for $n \geq 4$:

$$T_n = 2T_{n-1} - T_{n-4}, \qquad T_0 = 0, T_1 = T_2 = 1, T_3 = 2. \tag{1.3}$$

Extension of the definition of T_n to negative subscripts can be proved by writing the recurrence relation (1.3) as

$$T_{-n} = 2T_{-n+3} - T_{-n+4}.$$

Note that $T_{-n} = T_{n-1}^2 - T_{n-2}T_n$, (see [3]).

We can give some relations between $\{T_n\}$ and $\{K_n\}$ as

$$K_n = 3T_{n+1} - 2T_n - T_{n-1} \tag{1.4}$$

and

$$K_n = T_n + 2T_{n-1} + 3T_{n-2} \tag{1.5}$$

and also

$$K_n = 4T_{n+1} - T_n - T_{n+2}. \tag{1.6}$$

Note that the last three identities hold for all integers n.

The first few Tribonacci numbers and Tribonacci Lucas numbers with positive subscript are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	...
T_{-n}	0	0	1	−1	0	2	−3	1	4	−8	5	7	−20	...

The first few Tribonacci numbers and Tribonacci Lucas numbers with negative subscript are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	...
K_{-n}	3	−1	−1	5	−5	−1	11	−15	3	23	−41	21	43	...

It is well known that for all integers n, usual Tribonacci and Tribonacci-Lucas numbers can be expressed using Binet's formulas

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.7}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n \tag{1.8}$$

respectively, where α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3),$$

is a primitive cube root of unity. Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The generating functions for the Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} T_n x^n = \frac{x}{1 - x - x^2 - x^3} \quad \text{and} \quad \sum_{n=0}^{\infty} K_n x^n = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}. \tag{1.9}$$

Note that the Binet form of a sequence satisfying (1.1) and (1.2) for non-negative integers is valid for all integers n. This result of Howard and Saidak [8] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1.1. *[8]Let $\{w_n\}$ be a sequence such that*

$$\{w_n\} = a_1 w_{n-1} + a_2 w_{n-2} + ... + a_k w_{n-k}$$

for all integers n, with arbitrary initial conditions $w_0, w_1, ..., w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$\begin{aligned} f(x) &= x^k - a_1 x^{k-1} - a_2 x^{k-2} - ... - a_{k-1} x - a_k \\ &= (x-\alpha_1)^{d_1}(x-\alpha_2)^{d_2}...(x-\alpha_h)^{d_h} \end{aligned} \tag{1.10}$$

with $d_1 + d_2 + ... + d_h = k$, and $\alpha_1, \alpha_2, ..., \alpha_k$ distinct. Then

(a): *For all n,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n \tag{1.11}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + ... + A_{r_m}^{(m)} n^{r_m - 1} = \sum_{u=0}^{r_m - 1} A_{u+1}^{(m)} n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (1.11) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0) \neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (1.10) are distinct, as they are in our examples, then

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + ... + A_k(\alpha_k)^n.$$

(b): *The Binet form for $\{w_n\}$ is valid for all integers n.*

2. The Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers

In this section we define Tribonacci and Tribonacci-Lucas matrix sequences and investgate their properties.

Definition 2.1. *For any integer $n \geq 0$, the Tribonacci matrix ($\mathcal{T}_n$) and Tribonacci-Lucas matrix ($\mathcal{K}_n$) are defined by*

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3}, \tag{2.1}$$

$$\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + \mathcal{K}_{n-3}, \tag{2.2}$$

respectively, with initial conditions

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{T}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ -1 & 4 & -1 \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The sequences $\{\mathcal{T}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{T}_{-n} = -\mathcal{T}_{-(n-1)} - \mathcal{T}_{-(n-2)} + \mathcal{T}_{-(n-3)}$$

and

$$\mathcal{K}_{-n} = -\mathcal{K}_{-(n-1)} - \mathcal{K}_{-(n-2)} + \mathcal{K}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.1) and (2.2) hold for all integers n.

The following theorem gives the nth general terms of the Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 2.2. *For any integer $n \geq 0$, we have the following formulas of the matrix sequences:*

$$\mathcal{T}_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \tag{2.3}$$

$$\mathcal{K}_n = \begin{pmatrix} K_{n+1} & K_n + K_{n-1} & K_n \\ K_n & K_{n-1} + K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-2} + K_{n-3} & K_{n-2} \end{pmatrix}. \tag{2.4}$$

Proof. We prove (2.3) by strong mathematical induction on n. (2.4) can be proved similarly.

If $n = 0$ then, since $T_1 = 1, T_2 = 1, T_0 = T_{-1} = 0$, $T_{-2} = 1$, $T_{-3} = -1$, we have

$$\mathcal{T}_0 = \begin{pmatrix} T_1 & T_0 + T_{-1} & T_0 \\ T_0 & T_{-1} + T_{-2} & T_{-1} \\ T_{-1} & T_{-2} + T_{-3} & T_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is true and

$$\mathcal{T}_1 = \begin{pmatrix} T_2 & T_1 + T_0 & T_1 \\ T_1 & T_0 + T_{-1} & T_0 \\ T_0 & T_{-1} + T_{-2} & T_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, we have

$$\begin{aligned}
\mathcal{T}_{k+1} &= \mathcal{T}_k + \mathcal{T}_{k-1} + \mathcal{T}_{k-2} \\
&= \begin{pmatrix} T_{k+1} & T_k + T_{k-1} & T_k \\ T_k & T_{k-1} + T_{k-2} & T_{k-1} \\ T_{k-1} & T_{k-2} + T_{k-3} & T_{k-2} \end{pmatrix} + \begin{pmatrix} T_k & T_{k-1} + T_{k-2} & T_{k-1} \\ T_{k-1} & T_{k-2} + T_{k-3} & T_{k-2} \\ T_{k-2} & T_{k-3} + T_{k-4} & T_{k-3} \end{pmatrix} \\
&\quad + \begin{pmatrix} T_{k-1} & T_{k-2} + T_{k-3} & T_{k-2} \\ T_{k-2} & T_{k-3} + T_{k-4} & T_{k-3} \\ T_{k-3} & T_{k-4} + T_{k-5} & T_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} T_k + T_{k-1} + T_{k+1} & T_k + T_{k-1} + T_{k-1} + T_{k-2} + T_{k-2} + T_{k-3} & T_k + T_{k-1} + T_{k-2} \\ T_k + T_{k-1} + T_{k-2} & T_{k-1} + T_{k-2} + T_{k-2} + T_{k-3} + T_{k-3} + T_{k-4} & T_{k-1} + T_{k-2} + T_{k-3} \\ T_{k-1} + T_{k-2} + T_{k-3} & T_{k-2} + T_{k-3} + T_{k-3} + T_{k-4} + T_{k-4} + T_{k-5} & T_{k-2} + T_{k-3} + T_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} T_{k+2} & T_k + T_{k+1} & T_{k+1} \\ T_{k+1} & T_k + T_{k-1} & T_k \\ T_k & T_{k-1} + T_{k-2} & T_{k-1} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on n, this proves (2.3).

We now give the Binet formulas for the Tribonacci and Tribonacci-Lucas matrix sequences.

THEOREM 2.3. *For every integer n, the Binet formulas of the Tribonacci and Tribonacci-Lucas matrix sequences are given by*

$$\mathcal{T}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \tag{2.5}$$

$$\mathcal{K}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n. \tag{2.6}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha\mathcal{T}_2 + \alpha(\alpha-1)\mathcal{T}_1 + \mathcal{T}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{T}_2 + \beta(\beta-1)\mathcal{T}_1 + \mathcal{T}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_1 = \frac{\gamma\mathcal{T}_2 + \gamma(\gamma-1)\mathcal{T}_1 + \mathcal{T}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)} \\
A_2 &= \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + \mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_2 = \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + \mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + \mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.
\end{aligned}$$

Proof. We prove the theorem only for $n \geq 0$ because of Theorem 1.1. We prove (2.5). By the assumption, the characteristic equation of (2.1) is $x^3 - x^2 - x - 1 = 0$ and the roots of it are α, β and γ. So it's general solution is given by

$$\mathcal{T}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n.$$

Using initial condition which is given in Definition 2.1, and also applying lineer algebra operations, we obtain the matrices A_1, B_1, C_1 as desired. This gives the formula for $\mathcal{T}_n$.

Similarly we have the formula (2.6).

The well known Binet formulas for Tribonacci and Tribonacci-Lucas numbers are given in (1.7) and (1.8) respectively. But, we will obtain these functions in terms of Tribonacci and Tribonacci-Lucas matrix sequences as a consequence of Theorems 2.2 and 2.3. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

COROLLARY 2.4. *For every integers n, the Binet's formulas for Tribonacci and Tribonacci-Lucas numbers are given as*

$$\begin{aligned} T_n &= \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\beta)(\gamma-\alpha)}, \\ K_n &= \alpha^n + \beta^n + \gamma^n. \end{aligned}$$

Proof. From Theorem 2.3, we have

$$\begin{aligned} \mathcal{T}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n \\ &= \frac{\alpha\mathcal{T}_2 + \alpha(\alpha-1)\mathcal{T}_1 + \mathcal{T}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{T}_2 + \beta(\beta-1)\mathcal{T}_1 + \mathcal{T}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\ &\quad + \frac{\gamma\mathcal{T}_2 + \gamma(\gamma-1)\mathcal{T}_1 + \mathcal{T}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} \alpha^3 & \alpha(\alpha+1) & \alpha^2 \\ \alpha^2 & \alpha+1 & \alpha \\ \alpha & \alpha(\alpha-1) & 1 \end{pmatrix} + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} \beta^3 & \beta(\beta+1) & \beta^2 \\ \beta^2 & \beta+1 & \beta \\ \beta & \beta(\beta-1) & 1 \end{pmatrix} \\ &\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} \gamma^3 & \gamma(\gamma+1) & \gamma^2 \\ \gamma^2 & \gamma+1 & \gamma \\ \gamma & \gamma(\gamma-1) & 1 \end{pmatrix} \end{aligned}$$

By Theorem 2.2, we know that

$$\mathcal{T}_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned} T_n &= \frac{\alpha^{n-1}\alpha^2}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}\beta^2}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}\gamma^2}{(\gamma-\beta)(\gamma-\alpha)} \\ &= \frac{\alpha^{n+1}}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n+1}}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n+1}}{(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

From Therem 2.3, we obtain

$$
\begin{aligned}
\mathcal{K}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n \\
&= \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + \mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + \mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\
&\quad + \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + \mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} 3\alpha^2+4\alpha+1 & 4\alpha^2+2 & \alpha^2+2\alpha+3 \\ \alpha^2+2\alpha+3 & 2\alpha^2+2\alpha-2 & 3\alpha^2-2\alpha-1 \\ 3\alpha^2-2\alpha-1 & -2\alpha^2+4\alpha+4 & -\alpha^2+4\alpha-1 \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} 3\beta^2+4\beta+1 & 4\beta^2+2 & \beta^2+2\beta+3 \\ \beta^2+2\beta+3 & 2\beta^2+2\beta-2 & 3\beta^2-2\beta-1 \\ 3\beta^2-2\beta-1 & -2\beta^2+4\beta+4 & -\beta^2+4\beta-1 \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} 3\gamma^2+4\gamma+1 & 4\gamma^2+2 & \gamma^2+2\gamma+3 \\ \gamma^2+2\gamma+3 & 2\gamma^2+2\gamma-2 & 3\gamma^2-2\gamma-1 \\ 3\gamma^2-2\gamma-1 & -2\gamma^2+4\gamma+4 & -\gamma^2+4\gamma-1 \end{pmatrix}.
\end{aligned}
$$

By Theorem 2.2, we know that

$$
\mathcal{K}_n = \begin{pmatrix} K_{n+1} & K_n + K_{n-1} & K_n \\ K_n & K_{n-1}+K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-2}+K_{n-3} & K_{n-2} \end{pmatrix}.
$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above last two equations, then we obtain

$$
K_n = \frac{\alpha^{n-1}(\alpha^2+2\alpha+3)}{(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{n-1}(\beta^2+2\beta+3)}{(\beta-\gamma)(\beta-\alpha)} + \frac{\gamma^{n-1}(\gamma^2+2\gamma+3)}{(\gamma-\beta)(\gamma-\alpha)}.
$$

Using the relations, $\alpha+\beta+\gamma = 1$, $\alpha\beta\gamma = 1$ and considering α, β and γ are the roots the equation $x^3 - x^2 - x - 1 = 0$, we obtain

$$
\begin{aligned}
\frac{\alpha^2+2\alpha+3}{(\alpha-\gamma)(\alpha-\beta)} &= \frac{\alpha^2+2\alpha+3}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma} = \frac{\alpha}{\alpha}\frac{\alpha^2+2\alpha+3}{\alpha^2+\alpha(-\beta-\gamma)+\beta\gamma} \\
&= \frac{\alpha}{\alpha}\frac{(\alpha^2+2\alpha+3)}{\alpha^2+\alpha(-\beta-\gamma)+\beta\gamma} = \frac{(\alpha^2+2\alpha+3)\alpha}{\alpha^3+\alpha^2(\alpha-1)+1} \\
&= \frac{(\alpha^2+2\alpha+3)\alpha}{2\alpha^3-\alpha^2+1} = \frac{(\alpha^2+2\alpha+3)\alpha}{2(\alpha^2+\alpha+1)-\alpha^2+1} \\
&= \frac{(\alpha^2+2\alpha+3)\alpha}{(\alpha^2+2\alpha+3)} = \alpha, \\
\frac{\beta^2+2\beta+3}{(\beta-\gamma)(\beta-\alpha)} &= \frac{\beta^2+2\beta+3}{\beta^2-\alpha\beta+\alpha\gamma-\beta\gamma} = \beta, \\
\frac{\gamma^2+2\gamma+3}{(\gamma-\beta)(\gamma-\alpha)} &= \frac{\gamma^2+2\gamma+3}{\gamma^2+\alpha\beta-\alpha\gamma-\beta\gamma} = \gamma.
\end{aligned}
$$

So finally we conclude that

$$K_n = \alpha^n + \beta^n + \gamma^n$$

as required.

Now, we present summation formulas for Tribonacci and Tribonacci-Lucas matrix sequences.

THEOREM 2.5. *For $m > j \geq 0$, we have*

$$\sum_{i=0}^{n-1} \mathcal{T}_{mi+j} = \frac{\mathcal{T}_{mn+m+j} + \mathcal{T}_{mn-m+j} + (1-K_m)\mathcal{T}_{mn+j}}{K_m - K_{-m}} - \frac{\mathcal{T}_{m+j} + \mathcal{T}_{j-m} + (1-K_m)\mathcal{T}_j}{K_m - K_{-m}} \tag{2.7}$$

and

$$\sum_{i=0}^{n-1} \mathcal{K}_{mi+j} = \frac{\mathcal{K}_{mn+m+j} + \mathcal{K}_{mn-m+j} + (1-K_m)\mathcal{K}_{mn+j}}{K_m - K_{-m}} - \frac{\mathcal{K}_{m+j} + \mathcal{K}_{j-m} + (1-K_m)\mathcal{K}_j}{K_m - K_{-m}}. \tag{2.8}$$

Proof. *Note that*

$$\begin{aligned} \sum_{i=0}^{n-1} \mathcal{T}_{mi+j} &= \sum_{i=0}^{n-1} (A_1\alpha^{mi+j} + B_1\beta^{mi+j} + C_1\gamma^{mi+j}) \\ &= A_1\alpha^j \left(\frac{\alpha^{mn}-1}{\alpha^m - 1}\right) + B_1\beta^j \left(\frac{\beta^{mn}-1}{\beta^m - 1}\right) + C_1\gamma^j \left(\frac{\gamma^{mn}-1}{\gamma^m - 1}\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{n-1} \mathcal{K}_{mi+j} &= \sum_{i=0}^{n-1} (A_2\alpha^{mi+j} + B_2\beta^{mi+j} + C_2\gamma^{mi+j}) \\ &= A_2\alpha^j \left(\frac{\alpha^{mn}-1}{\alpha^m - 1}\right) + B_2\beta^j \left(\frac{\beta^{mn}-1}{\beta^m - 1}\right) + C_2\gamma^j \left(\frac{\gamma^{mn}-1}{\gamma^m - 1}\right). \end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (2.7) and (2.8) as required.

As in Corollary 2.4, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

COROLLARY 2.6. *For $m > j \geq 0$, we have*

$$\sum_{i=0}^{n-1} T_{mi+j} = \frac{T_{mn+m+j} + T_{mn-m+j} + (1-K_m)T_{mn+j}}{K_m - K_{-m}} - \frac{T_{m+j} + T_{j-m} + (1-K_m)T_j}{K_m - K_{-m}} \tag{2.9}$$

and

$$\sum_{i=0}^{n-1} K_{mi+j} = \frac{K_{mn+m+j} + K_{mn-m+j} + (1-K_m)K_{mn+j}}{K_m - K_{-m}} - \frac{K_{m+j} + K_{j-m} + (1-K_m)K_j}{K_m - K_{-m}} \tag{2.10}$$

Note that using the above Corollary we obtain the following well known formulas (taking $m = 1, j = 0$):

$$\sum_{i=0}^{n-1} T_i = \frac{T_{n+2} - T_n - 1}{2} \quad \text{and} \quad \sum_{i=0}^{n-1} K_i = \frac{K_{n+2} - K_n}{2}.$$

We now give generating functions of $\mathcal{T}$ and $\mathcal{K}$.

Theorem 2.7. *The generating function for the Tribonacci and Tribonacci-Lucas matrix sequences are given as*

$$\sum_{n=0}^{\infty}\mathcal{T}_n x^n = \frac{1}{1-x-x^2-x^3}\begin{pmatrix} 1 & x+x^2 & x \\ x & 1-x & x^2 \\ x^2 & x-x^2 & 1-x-x^2 \end{pmatrix}$$

and

$$\sum_{n=0}^{\infty}\mathcal{K}_n x^n = \frac{1}{1-x-x^2-x^3}\begin{pmatrix} 1+2x+3x^2 & 2+2x-2x^2 & 3-2x-x^2 \\ 3-2x-x^2 & -2+4x+4x^2 & -1+4x-x^2 \\ -1+4x-x^2 & 4-6x & -1+5x^2 \end{pmatrix}$$

respectively.

Proof. We prove the Tribonacci case. Suppose that $g(x) = \sum_{n=0}^{\infty}\mathcal{T}_n x^n$ is the generating function for the sequence $\{\mathcal{T}_n\}_{n\geq 0}$. Then, using Definition 2.1, we obtain

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty}\mathcal{T}_n x^n = \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \sum_{n=3}^{\infty}\mathcal{T}_n x^n \\ &= \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \sum_{n=3}^{\infty}(\mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3})x^n \\ &= \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \sum_{n=3}^{\infty}\mathcal{T}_{n-1}x^n + \sum_{n=3}^{\infty}\mathcal{T}_{n-2}x^n + \sum_{n=3}^{\infty}\mathcal{T}_{n-3}x^n \\ &= \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 - \mathcal{T}_0 x - \mathcal{T}_1 x^2 - \mathcal{T}_0 x^2 + x\sum_{n=0}^{\infty}\mathcal{T}_n x^n + x^2\sum_{n=0}^{\infty}\mathcal{T}_n x^n + x^3\sum_{n=0}^{\infty}\mathcal{T}_n x^n \\ &= \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 - \mathcal{T}_0 x - \mathcal{T}_1 x^2 - \mathcal{T}_0 x^2 + xg(x) + x^2 g(x) + x^3 g(x). \end{aligned}$$

Rearranging above equation, we get

$$g(x) = \frac{\mathcal{T}_0 + (\mathcal{T}_1 - \mathcal{T}_0)x + (\mathcal{T}_2 - \mathcal{T}_1 - \mathcal{T}_0)x^2}{1-x-x^2-x^3}.$$

which equals the $\sum_{n=0}^{\infty}\mathcal{T}_n x^n$ in the Theorem. This completes the proof.

Tribonacci-Lucas case can be proved similarly.

The well known generating functions for Tribonacci and Tribonacci-Lucas numbers are as in (1.9). However, we will obtain these functions in terms of Tribonacci and Tribonacci-Lucas matrix sequences as a consequence of Theorem 2.7. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.7. Thus we have the following corollary.

Corollary 2.8. *The generating functions for the Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are given as*

$$\sum_{n=0}^{\infty}T_n x^n = \frac{x}{1-x-x^2-x^3} \text{ and } \sum_{n=0}^{\infty}K_n x^n = \frac{3-2x-x^2}{1-x-x^2-x^3}.$$

respectively.

3. Relation Between Tribonacci and Tribonacci-Lucas Matrix Sequences

The following theorem shows that there always exist interrelation between Tribonacci and Tribonacci-Lucas matrix sequences.

THEOREM 3.1. *For the matrix sequences $\{\mathcal{T}_n\}$ and $\{\mathcal{K}_n\}$, we have the following identities.*

- **(a):** $\mathcal{K}_n = 3\mathcal{T}_{n+1} - 2\mathcal{T}_n - \mathcal{T}_{n-1}$,
- **(b):** $\mathcal{K}_n = \mathcal{T}_n + 2\mathcal{T}_{n-1} + 3\mathcal{T}_{n-2}$,
- **(c):** $\mathcal{K}_n = 4\mathcal{T}_{n+1} - \mathcal{T}_n - \mathcal{T}_{n+2}$,
- **(d):** $\mathcal{K}_n = -\mathcal{T}_{n+2} + 4\mathcal{T}_{n+1} - \mathcal{T}_n$,
- **(e):** $\mathcal{T}_n = \frac{1}{22}(5\mathcal{K}_{n+2} - 3\mathcal{K}_{n+1} - 4\mathcal{K}_n)$

Proof. From (1.4), (1.5) and (1.6), (a), (b) and (c) follow. It is easy to show that $K_n = -T_{n+2} + 4T_{n+1} - T_n$ and $22T_n = 5K_{n+2} - 3K_{n+1} - 4K_n$ using Binet formulas of the numbers T_n and K_n, so now (d) and (e) follow.

LEMMA 3.2. *For all non-negative integers m and n, we have the following identities.*

- **(a):** $\mathcal{K}_0\mathcal{T}_n = \mathcal{T}_n\mathcal{K}_0 = \mathcal{K}_n$,
- **(b):** $\mathcal{T}_0\mathcal{K}_n = \mathcal{K}_n\mathcal{T}_0 = \mathcal{K}_n$.

Proof. Identities can be established easily. Note that to show (a) we need to use all the relations (1.4), (1.5) and (1.6).

Next Corollary gives another relation between the numbers T_n and K_n and also the matrices $\mathcal{T}_n$ and $\mathcal{K}_n$.

COROLLARY 3.3. *We have the following identities.*

- **(a):** $T_n = \frac{1}{22}(K_n + 5K_{n-1} + 2K_{n+1})$,
- **(b):** $\mathcal{T}_n = \frac{1}{22}(\mathcal{K}_n + 5\mathcal{K}_{n-1} + 2\mathcal{K}_{n+1})$.

Proof. From Lemma 3.2 (a), we know that $\mathcal{K}_0\mathcal{T}_n = \mathcal{K}_n$. To show (a), use Theorem 2.2 for the matrix $\mathcal{T}_n$ and calculate the matrix operation $\mathcal{K}_0^{-1}\mathcal{K}_n$ and then compare the 2nd row and 1st column entries with the matrices $\mathcal{T}_n$ and $\mathcal{K}_0^{-1}\mathcal{K}_n$. Now (b) follows from (a).

To prove the following Theorem we need the next Lemma.

LEMMA 3.4. *Let $A_1, B_1, C_1; A_2, B_2, C_2$ as in Theorem 2.3. Then the following relations hold:*

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,. \end{aligned}$$

Proof. Using $\alpha+\beta+\gamma=1$, $\alpha\beta+\alpha\gamma+\beta\gamma=-1$ and $\alpha\beta\gamma=1$, required equalities can be established by matrix calculations.

THEOREM 3.5. *For all non-negative integers m and n, we have the following identities.*

(a): $\mathcal{T}_m\mathcal{T}_n=\mathcal{T}_{m+n}=\mathcal{T}_n\mathcal{T}_m$,

(b): $\mathcal{T}_m\mathcal{K}_n=\mathcal{K}_n\mathcal{T}_m=\mathcal{K}_{m+n}$,

(c): $\mathcal{K}_m\mathcal{K}_n=\mathcal{K}_n\mathcal{K}_m=9\mathcal{T}_{m+n+2}-12\mathcal{T}_{m+n+1}-2\mathcal{T}_{m+n}+4\mathcal{T}_{m+n-1}+\mathcal{T}_{m+n-2}$,

(d): $\mathcal{K}_m\mathcal{K}_n=\mathcal{K}_n\mathcal{K}_m=\mathcal{T}_{m+n}+4\mathcal{T}_{m+n-1}+10\mathcal{T}_{m+n-2}+12\mathcal{T}_{m+n-3}+9\mathcal{T}_{m+n-4}$,

(e): $\mathcal{K}_m\mathcal{K}_n=\mathcal{K}_n\mathcal{K}_m=\mathcal{T}_{m+n}-8\mathcal{T}_{m+n+1}+18\mathcal{T}_{m+n+2}-8\mathcal{T}_{m+n+3}+\mathcal{T}_{m+n+4}$.

Proof.

(a): Using Lemma 3.4 we obtain

$$\begin{aligned}
\mathcal{T}_m\mathcal{T}_n &= (A_1\alpha^m+B_1\beta^m+C_1\gamma^m)(A_1\alpha^n+B_1\beta^n+C_1\gamma^n)\\
&= A_1^2\alpha^{m+n}+B_1^2\beta^{m+n}+C_1^2\gamma^{m+n}+A_1B_1\alpha^m\beta^n+B_1A_1\alpha^n\beta^m\\
&\quad +A_1C_1\alpha^m\gamma^n+C_1A_1\alpha^n\gamma^m+B_1C_1\beta^m\gamma^n+C_1B_1\beta^n\gamma^m\\
&= A_1\alpha^{m+n}+B_1\beta^{m+n}+C_1\gamma^{m+n}\\
&= \mathcal{T}_{m+n}.
\end{aligned}$$

(b): By Lemma 3.2, we have

$$\mathcal{T}_m\mathcal{K}_n=\mathcal{T}_m\mathcal{T}_n\mathcal{K}_0.$$

Now from (a) and again by Lemma 3.2 we obtain $\mathcal{T}_m\mathcal{K}_n=\mathcal{T}_{m+n}\mathcal{K}_0=\mathcal{K}_{m+n}$.

It can be shown similarly that $\mathcal{K}_n\mathcal{T}_m=\mathcal{K}_{m+n}$.

(c): Using (a) and Theorem 3.1 (a) we obtain

$$\begin{aligned}
\mathcal{K}_m\mathcal{K}_n &= (3\mathcal{T}_{m+1}-2\mathcal{T}_m-\mathcal{T}_{m-1})(3\mathcal{T}_{n+1}-2\mathcal{T}_n-\mathcal{T}_{n-1})\\
&= 2\mathcal{T}_n\mathcal{T}_{m-1}-6\mathcal{T}_n\mathcal{T}_{m+1}+2\mathcal{T}_m\mathcal{T}_{n-1}-6\mathcal{T}_m\mathcal{T}_{n+1}\\
&\quad +4\mathcal{T}_m\mathcal{T}_n+\mathcal{T}_{m-1}\mathcal{T}_{n-1}-3\mathcal{T}_{m-1}\mathcal{T}_{n+1}-3\mathcal{T}_{m+1}\mathcal{T}_{n-1}+9\mathcal{T}_{m+1}\mathcal{T}_{n+1}\\
&= 2\mathcal{T}_{m+n-1}-6\mathcal{T}_{m+n+1}+2\mathcal{T}_{m+n-1}-6\mathcal{T}_{m+n+1}+4\mathcal{T}_{m+n}+\mathcal{T}_{m+n-2}-3\mathcal{T}_{m+n}\\
&\quad -3\mathcal{T}_{m+n}+9\mathcal{T}_{m+n+2}\\
&= 9\mathcal{T}_{m+n+2}-12\mathcal{T}_{m+n+1}-2\mathcal{T}_{m+n}+4\mathcal{T}_{m+n-1}+\mathcal{T}_{m+n-2}
\end{aligned}$$

It can be shown similarly that $\mathcal{K}_n\mathcal{K}_m=9\mathcal{T}_{m+n+2}-12\mathcal{T}_{m+n+1}-2\mathcal{T}_{m+n}+4\mathcal{T}_{m+n-1}+\mathcal{T}_{m+n-2}$.

The remaining of identities can be proved by considering again (a) and Theorem 3.1.

Comparing matrix entries and using Teorem 2.2 we have next result.

Corollary 3.6. *For Tribonacci and Tribonacci-Lucas numbers, we have the following identities:*

(a): $T_{m+n} = T_m T_{n+1} + T_n (T_{m-1} + T_{m-2}) + T_{m-1} T_{n-1}$

(b): $K_{m+n} = T_m K_{n+1} + K_n (T_{m-1} + T_{m-2}) + K_{n-1} T_{m-1}$

(c): $K_m K_{n+1} + K_n (K_{m-1} + K_{m-2}) + K_{m-1} K_{n-1} = 9T_{m+n+2} - 12T_{m+n+1} - 2T_{m+n} + 4T_{m+n-1} + T_{m+n-2}$

(d): $K_m K_{n+1} + K_n (K_{m-1} + K_{m-2}) + K_{m-1} K_{n-1} = T_{m+n} + 4T_{m+n-1} + 10T_{m+n-2} + 12T_{m+n-3} + 9T_{m+n-4}$

(e): $K_m K_{n+1} + K_n (K_{m-1} + K_{m-2}) + K_{m-1} K_{n-1} = T_{m+n} - 8T_{m+n+1} + 18T_{m+n+2} - 8T_{m+n+3} + T_{m+n+4}$

Proof.

(a): From Theorem 3.5 we know that $\mathcal{T}_m \mathcal{T}_n = \mathcal{T}_{m+n}$. Using Theorem 2.2, we can write this result as

$$\begin{pmatrix} T_{m+1} & T_m + T_{m-1} & T_m \\ T_m & T_{m-1} + T_{m-2} & T_{m-1} \\ T_{m-1} & T_{m-2} + T_{m-3} & T_{m-2} \end{pmatrix} \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} T_{m+n+1} & T_{m+n} + T_{m+n-1} & T_{m+n} \\ T_{m+n} & T_{m+n-1} + T_{m+n-2} & T_{m+n-1} \\ T_{m+n-1} & T_{m+n-2} + T_{m+n-3} & T_{m+n-2} \end{pmatrix}.$$

Now, by multiplying the left-side matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identity in (a).

The remaining of identities can be proved by considering again Theorems 3.5 and 2.2.

The next two theorems provide us the convenience to obtain the powers of Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 3.7. *For non-negatif integers m, n and r with $n \geq r$, the following identities hold:*

(a): $\mathcal{T}_n^m = \mathcal{T}_{mn}$,

(b): $\mathcal{T}_{n+1}^m = \mathcal{T}_1^m \mathcal{T}_{mn}$,

(c): $\mathcal{T}_{n-r} \mathcal{T}_{n+r} = \mathcal{T}_n^2 = \mathcal{T}_2^n$.

Proof.

(a): We can write $\mathcal{T}_n^m$ as

$$\mathcal{T}_n^m = \mathcal{T}_n \mathcal{T}_n ... \mathcal{T}_n \ (m \text{ times}).$$

Using Theorem 3.5 (a) iteratively, we obtain the required result:

$$
\begin{aligned}
\mathcal{T}_n^m &= \underbrace{\mathcal{T}_n\mathcal{T}_n...\mathcal{T}_n}_{m \text{ times}} \\
&= \mathcal{T}_{2n}\underbrace{\mathcal{T}_n\mathcal{T}_n...\mathcal{T}_n}_{m-1 \text{ times}} \\
&= \mathcal{T}_{3n}\underbrace{\mathcal{T}_n\mathcal{T}_n...\mathcal{T}_n}_{m-2 \text{ times}} \\
&\vdots \\
&= \mathcal{T}_{(m-1)n}\mathcal{T}_n \\
&= \mathcal{T}_{mn}.
\end{aligned}
$$

(b): As a similar approach in (a) we have

$$\mathcal{T}_{n+1}^m = \mathcal{T}_{n+1}.\mathcal{T}_{n+1}...\mathcal{T}_{n+1} = \mathcal{T}_{m(n+1)} = \mathcal{T}_m\mathcal{T}_{mn} = \mathcal{T}_1\mathcal{T}_{m-1}\mathcal{T}_{mn}.$$

Using Theorem 3.5 (a), we can write iteratively $\mathcal{T}_m = \mathcal{T}_1\mathcal{T}_{m-1}$, $\mathcal{T}_{m-1} = \mathcal{T}_1\mathcal{T}_{m-2}$, ..., $\mathcal{T}_2 = \mathcal{T}_1\mathcal{T}_1$. Now it follows that

$$\mathcal{T}_{n+1}^m = \underbrace{\mathcal{T}_1\mathcal{T}_1...\mathcal{T}_1}_{m \text{ times}}\mathcal{T}_{mn} = \mathcal{T}_1^m\mathcal{T}_{mn}.$$

(c): Theorem 3.5 (a) gives

$$\mathcal{T}_{n-r}\mathcal{T}_{n+r} = \mathcal{T}_{2n} = \mathcal{T}_n\mathcal{T}_n = \mathcal{T}_n^2$$

and also

$$\mathcal{T}_{n-r}\mathcal{T}_{n+r} = \mathcal{T}_{2n} = \underbrace{\mathcal{T}_2\mathcal{T}_2...\mathcal{T}_2}_{n \text{ times}} = \mathcal{T}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{K}_n$.

THEOREM 3.8. *For non-negatif integers m, n and r with $n \geq r$, the following identities hold:*

(a): $\mathcal{K}_{n-r}\mathcal{K}_{n+r} = \mathcal{K}_n^2$,

(b): $\mathcal{K}_n^m = \mathcal{K}_0^m\mathcal{T}_{mn}$.

Proof.

(a): We use Binet's formula of Tribonacci-Lucas matrix sequence which is given in Theorem 2.2. So

$$
\begin{aligned}
& \mathcal{K}_{n-r}\mathcal{K}_{n+r} - \mathcal{K}_n^2 \\
= & (A_2\alpha^{n-r} + B_2\beta^{n-r} + C_2\gamma^{n-r})(A_2\alpha^{n+r} + B_2\beta^{n+r} + C_2\gamma^{n+r}) \\
& -(A_2\alpha^n + B_2\beta^n + C_2\gamma^n)^2 \\
= & A_2B_2\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + A_2C_2\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 \\
& +B_2C_2\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= & 0
\end{aligned}
$$

since $A_2B_2 = A_2C_2 = C_2B_2$ (see Lemma 3.4). Now we get the result as required.

(b): By Theorem 3.7, we have

$$\mathcal{K}_0^m \mathcal{T}_{mn} = \underbrace{\mathcal{K}_0\mathcal{K}_0...\mathcal{K}_0}_{m \text{ times}}\underbrace{\mathcal{T}_n\mathcal{T}_n...\mathcal{T}_n}_{m \text{ times}}.$$

When we apply Lemma 3.2 (a) iteratively, it follows that

$$\begin{aligned} \mathcal{K}_0^m \mathcal{T}_{mn} &= (\mathcal{K}_0\mathcal{T}_n)(\mathcal{K}_0\mathcal{T}_n)...(\mathcal{K}_0\mathcal{T}_n) \\ &= \mathcal{K}_n\mathcal{K}_n...\mathcal{K}_n = \mathcal{K}_n^m. \end{aligned}$$

This completes the proof.

References

[1] Bruce, I., A modified Tribonacci sequence, The Fibonacci Quarterly, 22 : 3, pp. 244–246, 1984.

[2] Cerda-Morales, G., On the Third-Order Jabosthal and Third-Order Jabosthal-Lucas Sequences and Their Matrix Representations, arXiv:1806.03709v1 [math.CO], 2018.

[3] Choi, E., Modular tribonacci Numbers by Matrix Method, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. Volume 20, Number 3 (August 2013), pages 207–221, 2013.

[4] Civciv, H., and Turkmen, R., On the (s; t)-Fibonacci and Fibonacci matrix sequences, Ars Combin. 87, 161-173, 2008.

[5] Civciv, H., and Turkmen, R., Notes on the (s; t)-Lucas and Lucas matrix sequences, Ars Combin. 89, 271-285, 2008.

[6] Feinberg, M., Fibonacci–Tribonacci, The Fibonacci Quarterly, 1 : 3 (1963) pp. 71–74, 1963.

[7] Gulec, H.H., and Taskara, N., On the (s; t)-Pell and (s; t)-Pell-Lucas sequences and their matrix representations, Appl. Math. Lett. 25, 1554-1559, 2012.

[8] Howard, F.T., Saidak, F., Congress Numer. 200 (2010), 225-237, 2010.

[9] Marohnić, L., Strmečki, T., Plastic Number: Construction and Applications, Advanced Research in Scientific Areas 2012, 1523-1528, 2012.

[10] Piezas, T., A Tale of Four Constants, https://sites.google.com/site/tpiezas/0012.

[11] Scott, A., Delaney, T., Hoggatt Jr., V., The Tribonacci sequence, The Fibonacci Quarterly, 15:3, pp. 193–200, 1977.

[12] Shannon, A., Tribonacci numbers and Pascal's pyramid, The Fibonacci Quarterly, 15:3, pp. 268-275, 1977.

[13] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/

[14] Spickerman, W., Binet's formula for the Tribonacci sequence, The Fibonacci Quarterly, 20, pp.118–120, 1981.

[15] Uslu, K., and Uygun, S., On the (s,t) Jacobsthal and (s,t) Jacobsthal-Lucas Matrix Sequences, Ars Combin. 108, 13-22, 2013.

[16] Uygun, Ş., and Uslu, K., (s,t)-Generalized Jacobsthal Matrix Sequences, Springer Proceedings in Mathematics&Statistics, Computational Analysis, Amat, Ankara, May 2015, 325-336.

[17] Uygun, Ş., Some Sum Formulas of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Matrix Sequences, Applied Mathematics, 7, 61-69, 2016.

[18] Yalavigi, C. C., Properties of Tribonacci numbers, The Fibonacci Quarterly, 10 : 3, pp. 231–246, 1972.

[19] Yazlik, Y., and Taskara, N., Uslu, K. and Yilmaz, N. The generalized (s; t)-sequence and its matrix sequence, Am. Inst. Phys. (AIP) Conf. Proc. 1389, 381-384, 2012.

[20] Yilmaz, N., and Taskara, N., Matrix Sequences in Terms of Padovan and Perrin Numbers, Journal of Applied Mathematics, Volume 2013, Article ID 941673, 7 pages, 2013.

[21] Yilmaz, N., Taskara, N., On the Negatively Subscripted Padovan and Perrin Matrix Sequences, Communications in Mathematics and Applications, Vol. 5, No. 2, 59-72, 2014.

[22] Wani, A.A., Badshah, V.H., and Rathore, G.B.S., Generalized Fibonacci and k-Pell Matrix Sequences, Punjab University Journal of Mathematics (ISSN 1016-2526) Vol. 50(1) (2018) pp. 68-79.