Graphical zonotopes with the same face vector

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Abstract

We are interested in constructing zonotopes which are combinatorially nonequivalent but have the same face vector. In this paper we introduce a quadrilateral flip operation on graphs. We show that, if one graph is obtained from another graph by a flip, then the face vectors of the graphical zonotopes of these two graphs are the same. In this way, we can easily construct a class of combinatorially nonequivalent graphical zonotopes which share the same face vector. It is known that all triangulations of the n-gon are connected through the flip operation. Thus their graphical zonotopes have the same face vector. We will compute this vector and the total number of faces.

1 Introduction

A zonotope is the Minkowski sum of several line segments. Given a set of vectors $V = \{v_1, \dots, v_m\}$ in $\mathbb{R}^n$, we define a zonotope

$$Z(V) = [0, v_1] + \cdots + [0, v_m].$$

The combinatorial structure of faces of $Z(V)$ is totally determined by the oriented matroid $\mathcal{M}(V)$ of V: the face poset of $Z(V)$ is anti-isomorphic with the poset of covectors of $\mathcal{M}(V)$ [3]. In this paper we are interested in constructing and understanding zonotopes which are combinatorially different but have the same face vector.

Let $G = (V(G), E(G))$ be a simple (no loops or multiple edges) connected graph on vertex set $\{1, 2, \dots, n\}$. The graphical zonotope Z_G of G is defined by

$$Z_G = \sum_{ij \in E(G)} [e_i, e_j],$$

where $e_1, \dots, e_n$ are the coordinate vectors in $\mathbb{R}^n$. Many properties of graph G are encoded by its graphical zonotope. For example, the volume of Z_G equals the number of spanning

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trees of G and the number of lattice points in Z_G equals to the number of forests in G [11, Proposition 2.4]. The number of vertices of Z_G is equal to the number of acyclic orientations of G. Graphical zonotope of the complete graph is just the permutahedron and all graphical zonotopes are in the class of generalized permutahedron [11]. Grujić [8] showed a relation between the face vector of Z_G and the q-analog of the chromatic symmetric function of G. Study of face vectors of graphical zonotopes can also be found in Postnikov et.al. [10].

For each edge ij of G, we denote by $\overrightarrow{ij}$ the orientation of ij from i to j. A partial orientation X of G is an orientation of a subgraph H of G. We can regard X as the directed graph whose underlying graph is H. If all the edges of G are oriented, then we say X is a full orientation. The *genus* $g(X)$ of X is defined to be the genus $g(H)$ of H, which is the edge number of H minus the vertex number of H and then plus the number of connected components of H. Let X^0 denote the subgraph of G with edge set $E(G)\setminus E(H)$. For two partial orientations X, Y of G, we define $X \leq Y$ if X is a sub-digraph of Y. Two partial orientations X and Y are *orthogonal* if either no edge is simultaneously oriented by X and Y or there exist at least two edges such that one is oriented by X and Y in a same direction and the other is oriented by X and Y in reverse directions.

For a graph G, we let $r(G)$ be the edge number of a maximal spanning forest of G. In the following, we introduce the oriented matroid of graph G in terms of partial orientations.

A partial orientation X of G is a *vector of* G if every arc (oriented edge) of X is contained in a directed circuit. Let $\mathcal{V}(G)$ denote the set of vectors of graph G. Then the poset $(\mathcal{V}(G), \leq)$ is graded and the rank $rank(X)$ of each vector X is given by $g(X)$.

A partial orientation X of G is a *covector of* G if it is orthogonal with every directed circuit of G. Let $\mathcal{L}(G)$ denote the set of covectors of graph G. Then the poset $(\mathcal{L}(G), \leq)$ is graded and the rank $rank(X)$ of each covector X is given by $r(G) - r(X^0)$ (see [3, Corollary 4.1.15]).

Given a planar graph G, let G^* be the dual planar graph of G. Then $(\mathcal{V}(G^*), \leq)$ is isomorphic with $(\mathcal{L}(G), \leq)$.

Proposition 1.1. *The face poset of Z_G is anti-isomorphic with the poset $(\mathcal{L}(G), \leq)$.*

This paper is organized as follows. In Section 2 we introduce the quadrilateral flip operation on graphs and prove that graphical zonotopes of two flip-equivalent graphs have the same face vector. In Section 3 we compute the face vector and total face number of graphical zonotopes of triangulations of the n-gon.

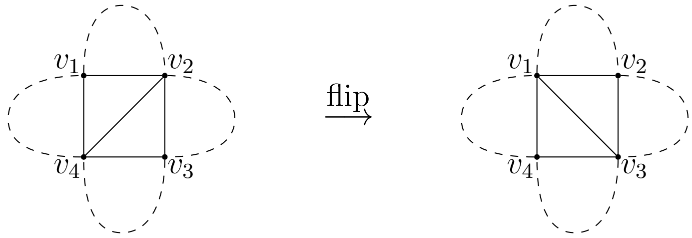
2 Flip on graphs

Flipping has long been important in the study of triangulations. Given a triangulation of a point configuration in the plane, a flip means replacing two triangles by two different triangles that cover the same quadrilateral. The flip graph is a graph defined on triangulations such that two triangulations are connected if they are related by a flip. Please refer to [4] for a survey of the flip graph of triangulations of planar point set. A generalization of flip operation to higher dimension is called bistellar flip [6].

We define the following quadrilateral flip operation on graphs.

Definition 2.1 (quadrilateral flip). *Let G be a simple connected graph. Let v_1, v_2, v_3, v_4 be four vertices of G such that the subgraph H induced by these four vertices is a 4-cycle with edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1$ and a diagonal edge v_2v_4. Suppose further that we can divide $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ into four disjoint sets V_1, V_2, V_3, V_4 such that, for every edge uv of G which is not v_2v_4, u and v belong to $V_i \cup \{v_i, v_{i+1}\}$ (v_5 is v_1) for some $i \in \{1, 2, 3, 4\}$. Then let G' be the new graph obtained from G by deleting edge v_2v_4 and then adding edge v_1v_3. Such an operation from G to G' is called a* quadrilateral flip*, or simply a* flip.

See figure below for illustration of the flip operation. We say two graphs are flip-equivalent if one can be transformed to another through a sequence of flip operations.



Theorem 2.1. *Graphical zonotopes of two flip-equivalent graphs have the same face vector.*

Proof. Let G' be a graph obtained from a graph G by a flip that removes edge v_2v_4 and adds edge v_1v_3. It is obvious that $r(G) = r(G')$. For each $i = 1, 2, 3, 4$, let G_i be the induced subgraph of G with vertex set $V_i \cup \{v_i, v_{i+1}\}$. Let H and H' be the induced subgraphs of G and G' with vertex set $\{v_1, v_2, v_3, v_4\}$, respectively. In what follows, we will explicitly construct a bijection from covectors X of G to covectors Y of G' such that $rank(X) = rank(Y)$. Note that Y is a covector of G' if its restrictions on $G_1, \ldots, G_4$ and H' are covectors of $G_1, \ldots, G_4$ and H', respectively. Let X' and Y' be the restriction of X and Y on H and H', respectively.

Case 1. No edge in H is oriented by X. Let Y equals X on edges not in H' and let Y do not orient any edge in H'. It is obvious that $rank(X) = rank(Y)$.

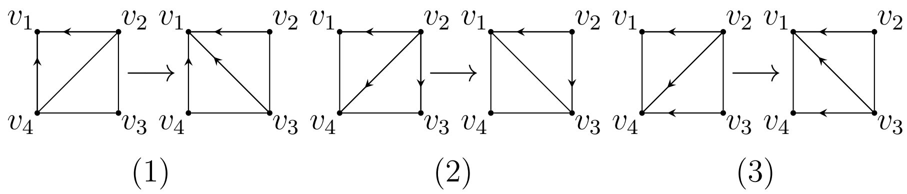


Figure 1: Case 2.

Case 2. X orients two edges in the 4-cycle of H. This case has three subcases: (1), two edges incident with v_1 or v_3 are oriented; (2), two edges incident with v_2 or v_4 are oriented, then v_2v_4 is also oriented; (3), two non-adjacent edges are oriented, then v_2v_4 is also oriented. We have shown one example for each of the three cases in Fig. 1. For each case, we let Y equals X on all edges of G' other than v_1v_3. Then the orientation situation of Y on edge v_1v_3 is determined. For example, in figure (1), since $\overrightarrow{v_2v_1}, \overrightarrow{v_4v_1} \in Y$, we must have $\overrightarrow{v_3v_1} \in Y$ to make Y a covector. It is easy to see that Y is indeed a covector of G'. Moreover, we have $rank(X) = rank(Y)$. Indeed, for example, in case (1), X^0 can be obtained from Y^0 by adding one more edge between two vertices in a same connected component of Y^0. So $r(X^0) = r(Y^0)$, and thus $rank(X) = rank(Y)$.

Case 3. X orients three edges in the 4-cycle of H. See Fig. 2 for an example. In this case u_2u_4 is also oriented. If not, then there is a triangle with only one edge oriented, contradicting the fact that X is a covector. We let Y equals X on edges of G' other than u_1u_3. Note that there is also a triangle of H' containing edge u_1u_3 and exactly one of the two boundary edges is oriented. So let Y orient edge u_1u_3 in the only allowed direction. Now $X^0 = Y^0$, so $rank(X) = rank(Y)$.

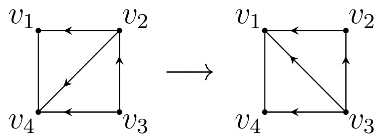
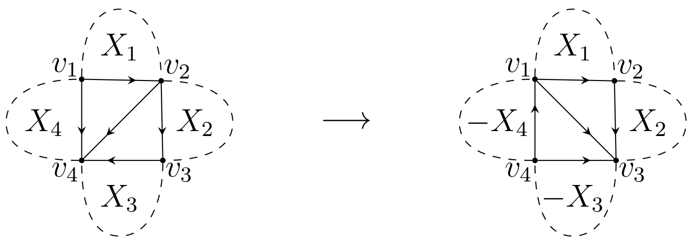


Figure 2: Case 3.

Case 4. X orients four edges in the 4-cycle of H. In this case we construct Y in the following way. We first let Y' be isomorphic with X' under the map $v_i \to v_{i+1}$. So if forgetting about labels, then Y' looks just like the rotation of X' by 90 degree. For each $i = 1, 2, 3, 4$, let X_i and Y_i be the restriction of X and Y on G_i respectively. Then for each $i = 1, 2, 3, 4$, if v_iv_{i+1} is oriented in the same direction for X' and Y', set $Y_i = X_i$; otherwise, let Y_i be the reverse of X_i, denoted as $-X_i$. See figure below for illustration. Though in example below v_2v_4 is oriented in X, we also have cases that v_2v_4 is not oriented.



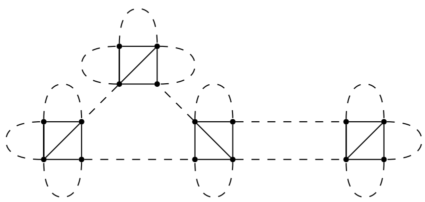
It is easy to see that Y is a covector of G'. Now we will show that $rank(X) = rank(Y)$. Note that for each $i \in \{1, 2, 3, 4\}$, there is no path from v_i to v_{i+1} in X_i^0. If not, together with edge v_iv_{i+1} we have a cycle with only one edge oriented, contradicting the fact that X is a covector. So we see that $r(X^0) = r(Y^0) = r(X_1^0) + \cdots + r(X_4^0)$ when v_2v_4 is oriented and $r(X^0) = r(Y^0) = r(X_1^0) + \cdots + r(X_4^0) + 1$ when v_2v_4 is not oriented. So $rank(X) = rank(Y)$.

At last, through above cases we have defined a map from covectors X of G to covectors Y of G'. We can see that this map is injective. As G and G' are flip-equivalent, we can also define a similar map from G' to G. Thus our map is a bijection. We have already shown that this map keeps rank. So the theorem is proved. □

Remark 2.1. *In our definition of flip on graphs, the extra condition on $V_1, \ldots, V_4$ is essential. For example, the following two graphs are skeleton graphs of two triangulations of 5 points on the plane that are related by a flip. But these two graphs are not flip-equivalent in our definition. Their graphical zonotopes do have different face vectors. For the left graph it is $(72, 150, 102, 24)$ and for the right graph it is $(78, 168, 116, 26)$. The computation is done by Polymake [7].*

Remark 2.2. *Whitney [14] characterized when two graphs have isomorphic matroids. The matroids of two flip-equivalent graphs are usually not isomorphic. Thus their graphical zonotopes are usually not combinatorially equivalent.*

Example 2.1. *The following kind of graphs admit several flip operations. These graphs are in tree shapes; namely, we can divide such a graph into blocks and then the adjacency relation between these blocks is a tree.*



3 Triangulations of the n-gon

The n-gon is a 2-dimensional polytope with n edges. In this section we consider special graphs that are skeleton graphs of triangulations of the n-gon. From definition, it is clear that if two triangulations of the n-gon are related by a flip, then their skeleton graphs are flip-equivalent. It is well known that the flip graph of triangulations of the n-gon can be realized as the skeleton graph of the $n-2$ dimensional associahedron (also called the Stasheff polytope) [9]. The diameter of this flip graph is $2n-8$ when $n > 11$ [12, 13]. See Fig. 3 for an example of flip graph of triangulations of the 6-gon.

Proposition 3.1. *The flip graph of triangulations of the n-gon is connected.*

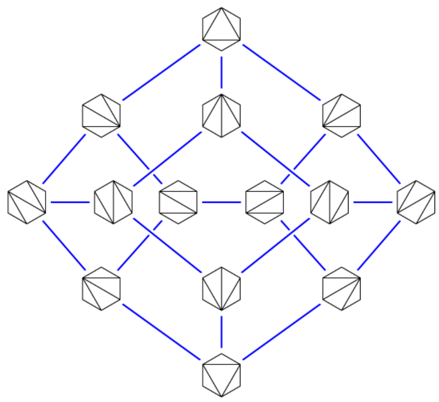


Figure 3: Flip graph of triangulations of the 6-gon [1].

In the following, we do not distinguish a triangulation of the n-gon and its skeleton graph. Combining Theorem 2.1 and Proposition 3.1, we immediately see that graphical zonotopes of all triangulations of the n-gon have a same face vector. We will prove the following results in this section.

Theorem 3.1. *The graphical zonotope of every triangulation of the n-gon has the face vector $f = (f_0, f_1, \ldots, f_{n-1})$ where for $i = 0, \ldots, n-1$:*

$$f_{n-1-i} = \sum_{m=0}^{\min\{i,n-i\}} 2^m 3^{i-m} \binom{i-1}{m-1} \binom{n}{m+i}.$$

Theorem 3.2. *The total number of faces of the graphical zonotope of a triangulation of the $(n+2)$-gon is $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which is also equal to* $3 \sum_{\frac{n}{2} \le m \le n} \binom{m}{n-m} 5^{2m-n} (-2)^{n-m} - 2 \sum_{\frac{n-1}{2} \le m \le n-1} \binom{m}{n-m-1} 5^{2m-n+1} (-2)^{n-m-1}$.

Let T_n be the triangulation of the n-gon on vertex set $\{1, 2, \ldots, n\}$ such that all its interior edges are incident with vertex 1. Note that T_n is a planar graph. Its dual planar graph T_n^* can be constructed from the *binary caterpillar tree* C_n by gluing all leaves into one vertex, where C_n is a tree such that all its interior vertices have degree three and are on a path. We assume the leaves of C_n are labelled by $1, 2, \ldots, n$ from left to right. See Fig. 4 for an example of T_5, T_5^* and C_5.

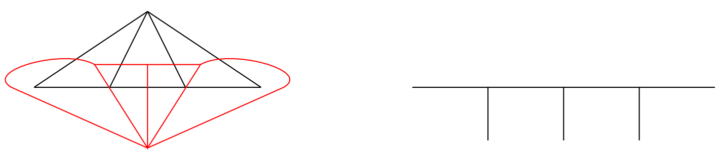


Figure 4: T_5, T_5^* and C_5.

As T_n^* is obtained from C_n by gluing its leaves, we can naturally identify partial orientations of T_n^* with partial orientations of C_n. So we let $\mathcal{V}(C_n)$ be the set of partial orientations of C_n that correspond to vectors of T_n^*, and also call elements in $\mathcal{V}(C_n)$ *vectors* of C_n. Then we have the following isomorphism

$$(\mathcal{L}(T_n), \leq) \cong (\mathcal{V}(T_n^*), \leq) \cong (\mathcal{V}(C_n), \leq).$$

The following observation is obvious.

Proposition 3.2. *A partial orientation X of C_n is a vector if and only if in X every arc is on a directed path between two leaves of C_n.*

Now we investigate the rank function of poset $(\mathcal{V}(C_n), \leq)$. For each partial orientation X of C_n, let Γ_X be the digraph on leaves of C_n such that $\overrightarrow{ij} \in \Gamma_X$ if i can reach j through a directed path in X. Let $b_0(\Gamma_X)$ be the number of weakly connected components of Γ_X.

Lemma 3.1. *The rank of an element X in $(\mathcal{V}(C_n), \leq)$ is $n - b_0(\Gamma_X)$.*

Proof. Let X' be the covector of T_n^* that corresponds to covector X of C_n. Let H be the underlying graph of X'. Then we know that $rank(X) = rank(X') = g(H)$.

Suppose that Γ_X have k isolated vertices and m weakly connected components with vertex sizes $c_1, \dots, c_m$ which are at least 2. Then the underlying graph of X is a disjoint union of m subtrees $T_1, \dots, T_m$ of C_n. For each tree T_i, let H_i be obtained from T_i by gluing its leaves, then $g(H_i) = c_i - 1$. Observe that H can be obtained from $H_1, \dots, H_m$ by taking one vertex from each graph and then gluing these m vertices into one vertex. So $g(H) = g(H_1) + \cdots + g(H_m) = c_1 + \cdots + c_m - m = n - k - m = n - b_0(\Gamma_X)$. □

Lemma 3.2. *The number of full vectors of C_n is $2 \cdot 3^{n-2}$ when $n \geq 2$.*

Proof. The subgraph obtained from C_n by deleting leaves n and $n-1$ is isomorphic with C_{n-1}. The restriction of every full vector of C_n on C_{n-1} gives rise to a full vector of C_{n-1}. On the other hand, every full vector of C_{n-1} can be extended to 3 different full vectors of C_n. Also note that C_2 has 2 full vectors. So the number of full vectors of C_n is $2 \cdot 3^{n-2}$. □

Proof of Theorem 3.1. In the following, we will compute the vector f basing on Z_{T_n}. Let i be an integer in $\{0, 1, \dots, n-1\}$. According to Proposition 1.1, f_{n-1-i} is the number of covectors of T_n of rank i, which is also the number of vectors of C_n of rank i. Then by Lemma 3.1, we conclude that f_{n-1-i} is the number of vectors X of C_n such that $b_0(\Gamma_X) = n - i$.

For each integer $m \leq n - i$, we enumerate the number of vectors X of C_n such that $b_0(\Gamma_X) = n - i$ and exactly m connected components of Γ_X are not isolated vertices. Let L be the set of isolated vertices of Γ_X. Then $|L| = n - i - m$. Let T be the subtree of C_n such that leaves of T are leaves of C_n that are not in L, and let T' be obtained from T by contracting all degree 2 vertices. Then T' is isomorphic with C_{i+m}. Note that if v is an interior vertex of C_n that is adjacent with a leaf in L, let uv and wv be another two edges incident with v, then either $uv, wv \in X^0$ or $\overrightarrow{uv}, \overrightarrow{vw} \in X$ or $\overrightarrow{wv}, \overrightarrow{vu} \in X$. So we let X' be the vector of T' such that each edge ab is oriented from a to b whenever a can reach b in X. Now $b_0(\Gamma_{X'}) = m$ and $\Gamma_{X'}$ has no isolated vertices. So X'^0 must be

$m-1$ interior edges of T' such that no two of them are adjacent. Let $T'_1, \dots, T'_m$ be the connected components of the underlying graph of X'. We assume their leaf numbers are $c_1, \dots, c_m$ respectively. For each $i = 1, \dots, m$, let X'_i be the restriction of X' on T'_i, then X'_i is a full vector of T'_i.

From above analysis we see that X is totally determined by data L, X'^0 and $X'_1, \dots, X'_m$. We have $\binom{n}{m+i}$ choice for L. To choose X'^0 from T' is equivalent with choosing $m-1$ non-adjacent numbers from $\{1, 2, \dots, i+m-3\}$, and the number of choices to this classical combinatorial problem is $\binom{(i+m-3)-(m-1)+1}{m-1} = \binom{i-1}{m-1}$. After L and X'^0 are fixed, then $T'_1, \dots, T'_m$ are also fixed. Their leaf numbers satisfy $c_1 + \cdots + c_m = m + i$. By Lemma 3.2, the number of choices of X'_i is $2 \cdot 3^{c_i-2}$ for each $i = 1, \dots, m$. So the total number of choice of $X'_1, \dots, X'_m$ is $\prod_{i=1,\dots,m} 2 \cdot 3^{c_i-2} = 2^m 3^{c_1+\cdots+c_m-2m} = 2^m 3^{i-m}$.

To sum up, the number of vectors X of C_n such that $b_0(\Gamma_X) = n - i$ and exactly m connected components of Γ_X are not isolated vertices is

$$2^m 3^{i-m} \binom{n}{m+i} \binom{i-1}{m-1}.$$

Thus $f_{n-i-1} = \sum_{0 \le m \le \min\{i, n-i\}} 2^m 3^{i-m} \binom{n}{m+i} \binom{i-1}{m-1}$. □

To enumerate the total number of faces of the graphical zonotope of a triangulation of the n-gon, we find it more convenient to do recursion than applying the formula in Theorem 3.1.

Proof of Theorem 3.2. Let $\mathcal{V}_n^0, \mathcal{V}_n^1, \mathcal{V}_n^{-1}$ be the sets of vectors of C_n such that the pendent edge of leaf n is not oriented, oriented to n, oriented from n respectively. Let $\mathcal{V}_n$ be the set of all vectors of C_n.

Let v be the interior vertex of C_n which is adjacent with leaves n and $n-1$. Let u be another vertex of C_n which is adjacent with v. Let T be the subgraph of C_n obtained by deleting leaves n and $n-1$. Note that T is isomorphic with C_{n-1}. Every vector X of C_n defines a vector X' of T which is the restriction of X on T.

For each vector $X \in \mathcal{V}_n^0$, if $vn-1, vu \in X^0$ then $X' \in \mathcal{V}_{n-1}^0$; if $\overrightarrow{uv}, \overrightarrow{vn-1} \in X$ then $X' \in \mathcal{V}_{n-1}^+$; if $\overrightarrow{vu}, \overrightarrow{n-1v} \in X$ then $X' \in \mathcal{V}_{n-1}^-$. So we see that

$$|\mathcal{V}_n^0| = |\mathcal{V}_{n-1}^0| + |\mathcal{V}_{n-1}^+| + |\mathcal{V}_{n-1}^-| = |\mathcal{V}_{n-1}|. \tag{1}$$

Similar analysis for $\mathcal{V}_n^+$ and $\mathcal{V}_n^-$ shows that

$$|\mathcal{V}_n^+| = |\mathcal{V}_{n-1}^0| + 3|\mathcal{V}_{n-1}^+| + |\mathcal{V}_{n-1}^-| \tag{2}$$

and

$$|\mathcal{V}_n^-| = |\mathcal{V}_{n-1}^0| + |\mathcal{V}_{n-1}^+| + 3|\mathcal{V}_{n-1}^-|. \tag{3}$$

Denote $\mathcal{V}_n^* = \mathcal{V}_n^+ \cup \mathcal{V}_n^-$. Then we have

$$\begin{pmatrix} |\mathcal{V}_n^0| \\ |\mathcal{V}_n^*| \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} |\mathcal{V}_{n-1}^0| \\ |\mathcal{V}_{n-1}^*| \end{pmatrix}.$$

Note that $(|\mathcal{V}_2^0|, |\mathcal{V}_2^*|) = (1, 2)$, so we have

$$\begin{pmatrix} |\mathcal{V}_{n+2}^0| \\ |\mathcal{V}_{n+2}^*| \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So $|\mathcal{V}_{n+2}| = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

If we take the summation of Eqs.(1), (2) and (3), we will also obtain the recursive formula

$$|\mathcal{V}_n| = 5|\mathcal{V}_{n-1}| - 2|\mathcal{V}_{n-2}|.$$

By method of generating function, we then get

$$|\mathcal{V}_{n+2}| = 3 \sum_{\frac{n}{2} \le m \le n} \binom{m}{n-m} 5^{2m-n}(-2)^{n-m} - 2 \sum_{\frac{n-1}{2} \le m \le n-1} \binom{m}{n-m-1} 5^{2m-n+1}(-2)^{n-m-1}.$$

□

Remark 3.1. *The sequence of total face number $|\mathcal{V}_n|$ is the sequence A052984 in OEIS [2]. They are also the Kekulé numbers for certain benzenoids [5, see p. 78].*

n \ i	0	1	2	3	4	5	6	7	Total
2	2	1							3
3	6	6	1						13
4	18	28	12	1					59
5	54	114	80	20	1				269
6	162	432	422	180	30	1			1227
7	486	1566	1962	1190	350	42	1		5597
8	1458	5508	8424	6640	2828	616	56	1	25531

Figure 5: The face vectors of graphical zonotopes of triangulations of n-gon with $n = 2, \ldots, 8$.

References

[1] *https://11011110.github.io/blog/2006/10/13/another-gratuitously-nonplanar-drawing.html.*

[2] *https://oeis.org/A052984.*

[3] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[4] Prosenjit Bose and Ferran Hurtado. Flips in planar graphs. *Comput. Geom.*, 42(1):60–80, 2009.

[5] Sven J Cyvin and Ivan Gutman. *Kekulé structures in benzenoid hydrocarbons*, volume 46 of *Lecture Notes in Chemistry*. Springer-Verlag Berlin Heidelberg, 1988.

[6] Jesus De Loera, Joerg Rambau, and Francisco Santos. *Triangulations: Structures for Algorithms and Applications*, volume 25 of *Algorithms and Computation in Mathematics*. Springer-Verlag Berlin Heidelberg, 2010.

[7] Ewgenij Gawrilow and Michael Joswig. `polymake`: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

[8] Vladimir Grujić. Counting faces of graphical zonotopes. *ARS MATHEMATICA CONTEMPORANEA*, 13(1):227–234, 2017.

[9] Carl W. Lee. The associahedron and triangulations of the n-gon. *European J. Combin.*, 10(6):551–560, 1989.

[10] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[11] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

[12] Lionel Pournin. The diameter of associahedra. *Adv. Math.*, 259:13–42, 2014.

[13] Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. *J. Amer. Math. Soc.*, 1(3):647–681, 1988.

[14] Hassler Whitney. 2-Isomorphic Graphs. *Amer. J. Math.*, 55(1-4):245–254, 1933.