SYMMETRIC POWERS, INDECOMPOSABLES AND REPRESENTATION STABILITY

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ABSTRACT. Working over the prime field $\mathbb{F}_p$, the structure of the indecomposables Q^* for the action of the algebra of Steenrod reduced powers $\mathscr{A}(p)$ on the symmetric power functors S^* is studied by exploiting the theory of strict polynomial functors.

In particular, working at the prime 2, representation stability is exhibited for certain related functors, leading to a Conjectural representation stability description of quotients of Q^* arising from the polynomial filtration of symmetric powers.

1. INTRODUCTION

Symmetric powers give a rich source and far from understood source of representations working over a finite field $\mathbb{k}$: for V a finite-dimensional $\mathbb{k}$-vector space, $S^n(V)$ is a representation of the general linear group $GL(V)$ and, in general, the multiplicities of its composition factors are not known. The symmetric powers have further useful structure, since $S^*(V)$ is a bicommutative Hopf algebra, naturally in V.

Algebraic topologists have long been interested in these representations, since $S^*(V)$ is the polynomial part of the cohomology of the classifying space $BV^\sharp$ of the dual of V when $\mathbb{k}$ is the prime field $\mathbb{F}_p$. (To simplify the presentation, the prime p is taken to be two in this introduction.) In particular, the mod 2 Steenrod algebra $\mathscr{A}$ acts upon $S^*(V)$ naturally with respect to V and one can consider the representations given by the indecomposables $Q^*(V) := \mathbb{F}_2 \otimes_{\mathscr{A}} S^*(V)$.

This is of interest since Singer's algebraic transfer relates $Q^*(V)$ to $\mathrm{Ext}_{\mathscr{A}}^{\dim V}(\mathbb{F}_2, \mathbb{F}_2)$, the cohomology of the Steenrod algebra, and thus to the stable homotopy groups of spheres. More precisely, there is a map of algebras from the cohomology of the Steenrod algebra to the dual of the $GL(V)$-invariants $Q^*(V)^{GL(V)}$, with $\dim V$ corresponding to the cohomological grading and the algebra structure induced by the coproduct on symmetric powers. To apply this requires understanding the structure of $Q^*(V)$ as a $GL(V)$-module. Much effort has gone into this, frequently referred to as the Peterson hit problem (see the recent books by Walker and Wood [WW18b, WW18a]), especially for low dimensions. Whereas some generic results are known, the current research frontier is $\dim V = 5$ (see the work of Nguyen Sum [Sum15, Sum14] for example).

The viewpoint taken here is different, namely to consider Q^n as an object of $\mathscr{F}$, the category of functors between (finite-dimensional) $\mathbb{F}_2$-vector spaces. This is a finite functor of Eilenberg-MacLane polynomial degree n (see Section 2 for background) with cosocle the nth exterior power Λ^n. This has an interesting structural interpretation: the simple functor Λ^n is indexed by the

partition (1^n) and concatenation of partitions provides the identification $(1^{n+1}) = (1^n) \bullet (1)$; from this viewpoint, the cosocle corresponds to a periodic family with period $2^0 = 1$.

 A contrario, and to indicate the current lack of knowledge, beyond calculations for small n and certain special cases, little is known about the socle of Q^n.

 Hence, in order to get further general information on the functors Q^*, rather than studying $Q^*(V)$ for small V, an orthogonal approach is taken, by starting at the top. Namely, the polynomial filtration $(p_d S^n)_{d \in \mathbb{N}} \subset S^n$ induces a filtration

$$\ldots \subset Q^n[d] \subset Q^n[d+1] \subset \ldots \subset Q^n$$

for each $n \in \mathbb{N}$. The subquotient $Q^n[d]/Q^n[d-1]$ is written Q_d^n and this is zero if $d > n$. These subquotients are studied for small $n - d$, noting that the first non-trivial case, $Q_n^n \cong \Lambda^n$, corresponds to the cosocle of Q^n discussed above.

 The first step is to approximate these subquotients by using the fact that the Steenrod algebra $\mathscr{A}$ acts naturally upon $p_d S^*/p_{d-1} S^*$ so that one can form the indecomposables $\mathfrak{Q}_d^* := \mathbb{F}_2 \otimes_{\mathscr{A}} p_d S^*/p_{d-1} S^*$. There is a natural surjection:

$$\mathfrak{Q}_d^n \twoheadrightarrow Q_d^n$$

that is close to being an isomorphism. (Corollary 8.14 gives a criterion for this to be an isomorphism, whereas Corollary 9.27 gives an example where it is not.)

 A crucial additional observation is that $p_d S^n/p_{d-1} S^n$ has the structure of a *strict polynomial functor* of degree d. (The category of such strict polynomial functors $\mathscr{P}_d$ is equivalent to the category of modules over the appropriate Schur algebra and comes equipped with a forgetful functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}_d$ to Eilenberg-MacLane polynomial functors of degree d; see Section 3 for more details.) More precisely:

$$p_d S^n/p_{d-1} S^n \cong \bigoplus_{\omega \in \mathsf{Seq}_d^p(n)} \bigotimes_{i \in \mathbb{N}} \Lambda^{\omega_i},$$

where the sum is taken over sequences of natural numbers ω such that $\sum \omega_i = d$ and $\sum \omega_i p^i = n$. Moreover, the $\mathscr{A}$-action arises from $\mathscr{P}_d$, so that $\mathfrak{Q}_d^n$ can be considered as a strict polynomial functor. This allows the more rigid structure of the categories of strict polynomial functors to be exploited.

 Write $\mathscr{P}_d^{\geq c}$, $c \in \mathbb{N}$, for the full subcategory of strict polynomial functors P such that $\mathscr{O} P(\mathbb{F}_2^{c-1}) = 0$. Harman [Har15] has shown that these categories satisfy a form of *representation stability* (or, rather, periodicity), as explained here (in the current notation) in Theorem 5.10. Namely, for natural numbers $e > d, t$ such that (d, t) is stable (i.e. $d \geq 2t$) and $e \equiv d \mod 2^{\lceil \log_2 t \rceil}$, there is an equivalence of categories:

$$\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-t} \xrightarrow{\cong} \mathscr{P}_e^{\geq e-t}.$$

This is proved by using the fact that these are highest weight categories and by exploiting Ringel duality (the proof is outlined in Section 5). The periodicity equivalence acts on simple objects via:

$$\mathrm{Per}_{d,e}(L_\lambda) = L_{\lambda \bullet 1^{e-d}},$$

where λ is a partition of d, $\lambda \bullet 1^{e-d}$ is given by concatenation and L_λ denotes the simple object indexed by λ, namely L_λ is the highest weight composition factor of $\Lambda^{\lambda'}$, where λ' denotes the *conjugate partition* to λ.

 This leads directly to:

Theorem 1. *(Theorem 10.2.) For natural numbers $d \leq n$ such that $(d, n-d)$ is stable, if $e > d \in \mathbb{N}$ such that $e \equiv d \mod 2^{\lceil \log_2(n-d) \rceil}$, then $\mathfrak{Q}_d^n \in \mathsf{Ob}\, \mathscr{P}_d^{\geq d-(n-d)}$ and $\mathfrak{Q}_e^n \in \mathsf{Ob}\, \mathscr{P}_e^{\geq e-(n-d)}$*

and, under the equivalence of categories of Theorem 5.10,

$$\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-(n-d)} \xrightarrow{\cong} \mathscr{P}_e^{\geq e-(n-d)}$$
$$\mathfrak{Q}_d^n \mapsto \mathfrak{Q}_e^{n+e-d}.$$

This is a far-reaching generalization of the periodicity of Q_n^n with period one.

An important observation is the following: Q_d^n is a quotient of $\bigoplus_{\omega \in \mathsf{Seq}_d^p(n)} \bigotimes_{i \in \mathbb{N}} \Lambda^{\omega_i}$, where the ω are sequences of natural numbers (not partitions). This means that concatenation with ω is not necessarily well-defined (note that with the current conventions, concatenation acts on the conjugate partition). However, the strict stability hypothesis ensures that ω_0 is the maximal element of the sequence and thus concatenation is defined and corresponds to the operation $\omega_0 \mapsto \omega_0 + (e - d)$.

The next step is to pass to Q_d^n. Here there are two major (and unresolved) difficulties: a lack of understanding of the kernel of $\mathfrak{Q}_d^n \twoheadrightarrow Q_d^n$ in general together with the fact that one can no longer work entirely with strict polynomial functors. The analogous functor categories $\mathscr{F}_d^{\geq d-(n-d)}$ are not known to satisfy representation stability, although weaker results can be established (these are not developed here).

A first step is to work at the level of the Grothendieck groups of the relevant categories, for which representation stability can be proved. The weak representation stability result is the following:

Theorem 2. *(Theorem 5.21.) Suppose that $\Bbbk = \mathbb{F}_2$ and that (d,t) is a strictly stable pair of integers (i.e. $d > 2t$). For $d < e \in \mathbb{N}$ such that $d \equiv e \mod 2^{\lceil \log_2 t \rceil}$, the periodicity equivalence $\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-t} \to \mathscr{P}_e^{\geq e-t}$ induces a commutative diagram of abelian groups:*

$$
\begin{array}{ccc}
G_0(\mathscr{P}_d^{\geq d-t}) & \xrightarrow{G_0(\mathscr{O})} & G_0(\mathscr{F}_d^{\geq d-t}) \\
{\scriptstyle G_0(\mathrm{Per}_{d,e})} \downarrow {\scriptstyle\cong} & & {\scriptstyle\cong} \downarrow {\scriptstyle \bullet 1^{e-d}} \\
G_0(\mathscr{P}_e^{\geq e-t}) & \xrightarrow{G_0(\mathscr{O})} & G_0(\mathscr{F}_e^{\geq e-t})
\end{array}
$$

in which the vertical morphisms are isomorphisms, where $\bullet 1^{e-d}$ is induced by concatenation of partitions.

This result is proved by studying the behaviour of representation stability with respect to the tensor product and by applying the Steinberg tensor product theorem for the simple strict polynomial functors (see Section 5, in particular Proposition 5.13).

With this in hand, the following is immediate:

Corollary 3. *(Corollary 10.3.) Let $d, e, n \in \mathbb{N}$ satisfy the hypotheses of Theorem 2 and, in addition, suppose that the pair $(d, n-d)$ is strictly stable (i.e. $d > 2(n-d)$). Then, under the isomorphism $\bullet 1^{e-d}$ between Grothendieck groups:*

$$\bullet 1^{e-d} : G_0(\mathscr{F}_d^{\geq d-(n-d)}) \xrightarrow{\cong} G_0(\mathscr{F}_d^{\geq e-(n-d)})$$
$$[\mathfrak{Q}_d^n] \mapsto [\mathfrak{Q}_e^{n+e-d}].$$

However, rather than the subquotients $\mathfrak{Q}_d^n$ (and hence Q_d^n), one is more interested in studying the quotients of the form $Q^n/Q^n[d-1]$. Corollary 3 suggests:

Conjecture 1. (Conjecture 10.5.) Suppose that $d, e, n \in \mathbb{N}$ satisfy the hypotheses of Corollary 10.3, then under the isomorphism of Grothendieck groups

$$\bullet 1^{e-d} : G_0(\mathscr{F}_n^{\geq d-(n-d)}) \xrightarrow{\cong} G_0(\mathscr{F}_{n+e-d}^{\geq e-(n-d)})$$
$$[Q^n/Q^n[d-1]] \mapsto [Q^{n+e-d}/Q^{n+e-d}[e-1]].$$

Initial calculations suggest that a stronger result should be true:

Conjecture 2. (Conjecture 10.8.) Suppose that $d, e, n \in \mathbb{N}$ satisfy the hypotheses of Corollary 10.3, then the lattices of subobjects of $Q^n/Q^n[d-1]$ and $Q^{n+e-d}/Q^{n+e-d}[e-1]$ are isomorphic, compatibly with the identification of Conjecture 1.

If Conjecture 2 holds, then the structure of the graded functor $Q^*/Q^*[*-t]$, for fixed t, is determined by $Q^n/Q^n[n-t]$ for $n \leq N(t)$ (with an explicit bound), higher functors being determined by periodicity. It is an interesting open question to relate this conjectural periodicity to behaviour arising from the fact that $Q^*/Q^*[*-t]$ arises from the action of a *finite* sub-Hopf algebra of $\mathscr{A}$ on $S^*/p_{*-t}S^*$.

Conjecture 2 also suggests that the study of Q^n should be approached in two parts, namely the *stable part* $Q^n/Q^n[\lceil\frac{2n}{3}\rceil]$ and the *unstable part*, $Q^n[\lceil\frac{2n}{3}\rceil]$. To study Q^n one can first consider the stable part of $Q^{n_{\mathrm{stab}}}$, where $n_{\mathrm{stab}} := n + 2^{\tilde{\pi}(n)+1}$ for $\tilde{\pi}(n) := \lceil\log_2 n - 1\rceil$. The coproduct structure of Q^* induces a surjection:

$$(Q^{n_{\mathrm{stab}}}; \Lambda^{2^{\tilde{\pi}(n)}} \otimes \Lambda^{2^{\tilde{\pi}(n)}}) \twoheadrightarrow Q^n,$$

where $(-; \Lambda^{2^{\tilde{\pi}(n)}} \otimes \Lambda^{2^{\tilde{\pi}(n)}})$ denotes the division functor. This factors as

$$(Q^{n_{\mathrm{stab}}}/Q^{n_{\mathrm{stab}}}[\lceil\frac{2n_{\mathrm{stab}}}{3}\rceil]; \Lambda^{2^{\tilde{\pi}(n)}} \otimes \Lambda^{2^{\tilde{\pi}(n)}}) \twoheadrightarrow Q^n,$$

thus relating Q^n to the stable part of $Q^{n_{\mathrm{stab}}}$ which (conjecturally) lies in the stable zone. This establishes the interest of the division functors $(-; \Lambda^i)$ in this context.

The above conjectural picture shows how understanding the *stable* behaviour of the functors $Q^*/Q^*[*-t]$ gives explicit information on the global behaviour of Q^*. This viewpoint can be developed much further, for instance considering the coalgebra structure of Q^*.

For small n, the size of the unstable part can be limited by using the fact that $Q^n[d]$ vanishes for $d < \deg_{\min,2}^{\mathrm{ess}}(n)$, where $\deg_{\min,2}^{\mathrm{ess}}(n)$ is a function of n given by an explicit recursive formula (see Corollary 9.23); this exploits the multiplicative structure of the symmetric powers together with the fact that the modules considered are unstable modules over $\mathscr{A}(p)$ (see Section 9).

Unfortunately, asymptotically the unstable part cannot be neglected, since Proposition 9.16 shows that

$$\lim_{n\to\infty} \frac{\deg_{\min,2}^{\mathrm{ess}}(n)}{n} = 0.$$

A major open problem is how to extend these results to odd primes. Indeed, the results of Section 9 are independent of the parity of p and suggest that the above Conjectures should have analogues at odd primes. However, for this it is not possible to use connectivity as in the case $p = 2$ and the connection with the theory of highest weight categories is less direct. This stems from the fact that the exterior powers appearing above are replaced by p-truncated symmetric powers in the general case.

Organization of the paper: Background on functors and on strict polynomial functors is provided in Sections 2, 3, and Section 4 covers the relevant material on highest weight categories.

The representation stability results are provided in Section 5, including the treatment of the tensor product.

The short Section 6 reviews the natural transformations between symmetric power functors and Section 7 the polynomial filtration of symmetric powers, presented so as to be able to exploit strict polynomial functors.

The indecomposables Q are introduced in Section 8 together with their approximations $\mathfrak{Q}$. Section 9 steps back to look at general structural results that follow from the instability conditions

on the action of the Steenrod algebra. Finally, Section 10 puts everything together, notably stating the main results and the Conjectures.

1.1. General notions and notation.

Definition 1.1. For X an object of an abelian category,
  (1) X is finite if it has a finite composition series;
  (2) the socle of X, soc X, is the largest semi-simple subobject of X;
  (3) the cosocle of X, cosoc X, is the largest semi-simple quotient of X.

Notation 1.2. For objects X, Y of an abelian category, $X \cdot Y$ is shorthand for an object occurring in a short exact sequence $0 \to X \to \mathscr{E} \to Y \to 0$.

Notation 1.3. For $\{X_i | i \in \mathscr{I}\}$ a finite set of objects in an abelian category $\mathscr{C}$, let $\mathrm{Filt}(X_i | i \in \mathscr{I})$ denote the full subcategory of $\mathscr{C}$ of objects that admit a finite filtration with filtration quotients isomorphic to objects of $\{X_i | i \in \mathscr{I}\}$.

Notation 1.4. $\mathbb{F}_p$ denotes the prime field of characteristic $p > 0$.

Contents

2. Functors on $\Bbbk$-vector spaces

The purpose of this section is to review the theory of functors between $\Bbbk$-vector fields; in the applications, $\Bbbk$ will be taken to be a finite field. This material is readily available in the standard references (see [Kuh94a] and [FFSS99] and the references therein).

2.1. The category of functors.

Let $\Bbbk$ be a field, $\mathcal{V}$ be the category of $\Bbbk$-vector spaces and $\mathcal{V}^f \subset \mathcal{V}$ the full subcategory of finite-dimensional vector spaces.

Notation 2.1. Let $\mathscr{F}$ denote the category of functors from $\mathcal{V}^f$ to $\mathcal{V}$; the full subcategory of functors taking values in $\mathcal{V}^f$ is written $\mathscr{F}^{\mathrm{fd}}$. Write $D : \mathscr{F}^{\mathrm{op}} \to \mathscr{F}$ for the duality functor defined by $DF(V) := F(V^\sharp)^\sharp$, where $\sharp$ denotes $\Bbbk$-vector space duality.

The category $\mathscr{F}$ inherits an abelian structure from $\mathcal{V}$ and has tensor product $\otimes$ defined pointwise, with unit the constant functor $\Bbbk$.

Example 2.2. For $d \in \mathbb{N}$, the dth tensor power T^d is given by $V \mapsto V^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts by place permutations on T^d and the dth divided power Γ^d is given by the invariants $(T^d)^{\mathfrak{S}_d}$ and the dth symmetric power S^d by the coinvariants $(T^d)_{\mathfrak{S}_d}$. The dth exterior power is denoted by Λ^d; this is both a subfunctor and a quotient of T^d.

If $\Bbbk$ is a field of characteristic $p > 0$, the dth p-truncated symmetric power $\overline{S}^d$ is the quotient of S^d by the degree d component of the graded ideal in S^* (see Example 2.5) generated by pth powers. This identifies as $\operatorname{cosoc} S^d$; in particular $\overline{S}^d$ is simple.

These functors are all finite; T^d, Λ^d (and $\overline{S}^d$ when $\Bbbk$ has characteristic p) are self-dual under D, whereas $D\Gamma^d \cong S^d$.

2.2. Exponential functors.
Exponential functors provide a powerful calculational tool. As a general reference, the reader is referred to [FFSS99]. (Note that the exponential functors considered here are the Hopf exponential functors of *loc. cit.*.)

Consider the category $\mathcal{V}^f$ as a symmetric monoidal category with respect to $\oplus$.

Definition 2.3. For $(\mathscr{C}, \otimes, \mathbb{1})$ a symmetric monoidal category, the category of exponential functors from $\mathcal{V}^f$ to $\mathscr{C}$ is the category of strict monoidal functors from $\mathcal{V}^f$ to $\mathscr{C}$. Thus, a functor E is exponential if, for $V, W \in \mathsf{Ob}\, \mathcal{V}^f$, there is a natural isomorphism $E(V \oplus W) \cong E(V) \otimes E(W)$ and these satisfy the associativity and symmetry axioms.

The additive structure of $\mathcal{V}^f$ induces additional structure on exponential functors. For example for exponential functors with values in $\mathcal{V}$ one has:

Lemma 2.4. *For $V \in \mathsf{Ob}\, \mathcal{V}^f$ and E an exponential functor taking values in $\mathcal{V}$, the diagonal $V \to V \oplus V$ and codiagonal $V \oplus V \to V$ induce respectively a natural commutative product $\mu_E : E \otimes E \to E$ and cocommutative coproduct $\Delta_E : E \to E \otimes E$, so that E takes values naturally in bicommutative bialgebras. Moreover, multiplication by -1 on $\mathcal{V}^f$ induces a natural conjugation, so that E takes values in bicommutative Hopf algebras.*

We work here with exponential functors taking values in $\mathcal{V}^{\mathbb{N}}$, the category of $\mathbb{N}$-graded vector spaces, equipped with the usual graded tensor product and the symmetry without Koszul signs and refer to these as graded exponential.

Example 2.5. Let $\Bbbk$ be an arbitrary field.
(1) The functor S^* is graded exponential. In particular there are natural commutative products $S^i \otimes S^j \to S^n$ and cocommutative coproducts $S^n \to S^i \otimes S^j$, where $n = i + j$.
(2) If $\Bbbk$ has characteristic $p > 0$, $\overline{S}^*$ is graded exponential and the natural surjection $S^* \to \overline{S}^*$ is a morphism of graded exponential functors.

Notation 2.6. For $\Bbbk$ a field of characteristic p and $X \in \mathsf{Ob}\, \mathscr{F}^{\mathbb{N}}$ an $\mathbb{N}$-graded functor, let ΦX be the $\mathbb{N}$-graded functor given by $(\Phi X)^{pn} = X^n$ and $(\Phi X)^i = 0$ if $i \not\equiv 0 \mod p$.

Lemma 2.7. *Let E, E' be graded exponential functors.*
(1) The functor ΦE is graded exponential.
(2) If E takes values in $\mathcal{V}^f$ then the graded dual DE is graded exponential.
(3) The graded tensor product $E \otimes E'$ is graded exponential.

The following basic results explain the effectiveness of exponential functors:

Proposition 2.8. *For E a graded exponential functor and $F, G \in \mathsf{Ob}\, \mathscr{F}$, there are natural graded isomorphisms:*

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{F}}(E, F \otimes G) &\cong \operatorname{Hom}_{\mathscr{F}}(E, F) \otimes \operatorname{Hom}_{\mathscr{F}}(E, G) \\
\operatorname{Hom}_{\mathscr{F}}(F \otimes G, E) &\cong \operatorname{Hom}_{\mathscr{F}}(F, E) \otimes \operatorname{Hom}_{\mathscr{F}}(G, E).
\end{aligned}$$

Corollary 2.9. *For E a graded exponential functor, the bigraded object $\operatorname{Hom}_{\mathscr{F}}(E, E)$ has a natural bigraded, bicommutative Hopf algebra structure.*

2.3. **Division functors.**

Notation 2.10. For $F \in \mathsf{Ob}\ \mathscr{F}^{\mathrm{fd}}$, write $(-; F) : \mathscr{F} \to \mathscr{F}$ for the division functor defined as the left adjoint to the functor $F \otimes -$.

Notation 2.11.
  (1) Let $I_{\Bbbk}$ denote the functor $V \mapsto \Bbbk^{V^{\sharp}}$, the $\Bbbk$-vector space of maps from the dual of V to $\Bbbk$ and $\overline{I_{\Bbbk}}$ the reduced functor defined by the decomposition $I_{\Bbbk} \cong \Bbbk \oplus \overline{I_{\Bbbk}}$, where $\Bbbk$ is considered as a constant functor.
  (2) Let $\Delta_{\Bbbk} : \mathscr{F} \to \mathscr{F}$ denote the shift functor given by $\Delta_{\Bbbk} F(V) := F(V \oplus \Bbbk)$.
  (3) Let $\overline{\Delta_{\Bbbk}} : \mathscr{F} \to \mathscr{F}$ denote the difference functor given by $\overline{\Delta_{\Bbbk}} F(V) := \Delta_{\Bbbk} F(V)/F(V)$ for the canonical inclusion $F(V) \hookrightarrow F(V \oplus \Bbbk)$.

Lemma 2.12. [Kuh94a] *Let $\Bbbk$ be a finite field. Then*
  (1) the functors $I_{\Bbbk}$, $\Bbbk$ and $\overline{I_{\Bbbk}}$ are injective in $\mathscr{F}$;
  (2) $I_{\Bbbk}$ is locally finite but is not finite;
  (3) $I_{\Bbbk}$ is a (non-graded) exponential functor.

The Yoneda lemma gives the well-known:

Proposition 2.13. *For $\Bbbk$ a finite field, the functor $(-; I_{\Bbbk})$ is isomorphic to the shift functor $\Delta_{\Bbbk} : \mathscr{F} \to \mathscr{F}$ and $(-; \overline{I_{\Bbbk}})$ is isomorphic to the difference functor $\overline{\Delta_{\Bbbk}} : \mathscr{F} \to \mathscr{F}$.*

The following general properties hold for division functors:

Proposition 2.14. *Let $X, Y \in \mathsf{Ob}\ \mathscr{F}^{\mathrm{fd}}$ and $F, G \in \mathsf{Ob}\ \mathscr{F}$.*
  (1) The functor $(-; X) : \mathscr{F} \to \mathscr{F}$ is right exact.
  (2) The functor $(F; -) : (\mathscr{F}^{\mathrm{fd}})^{\mathrm{op}} \to \mathscr{F}$ is right exact.
  (3) There are natural isomorphisms $(-; X \otimes Y) \cong ((-; X); Y) \cong ((-; Y); X)$.
  (4) If E is a graded exponential functor taking values in $\mathcal{V}^f$, there is a natural isomorphism

$$(F \otimes G; E^n) \cong \bigoplus_{i+j=n} (F; E^i) \otimes (G; E^j).$$

Example 2.15. Let $\Bbbk$ be the prime field $\mathbb{F}_p$ of characteristic $p > 0$. There is a monomorphism (unique up to non-zero scalar multiple) $S^1 \to I_{\Bbbk}$. Hence, for $F, G \in \mathsf{Ob}\ \mathscr{F}$, there is a natural surjection $\Delta F \twoheadrightarrow (F; S^1)$. Moreover, there is a canonical isomorphism $(F \otimes G; S^1) \cong ((F; S^1) \otimes G) \oplus (F \otimes (G; S^1))$.

For $\Bbbk$ a field of characteristic $p > 0$, the division functors $(-; \overline{S}^n)$, for $n \in \mathbb{N}$ are of interest and arise in Proposition 8.5. One can reduce to considering the cases $n = p^k$, $k \in \mathbb{N}$, by the following standard result (cf. [FFSS99, Lemma 2.2]):

Lemma 2.16. *Let $\Bbbk$ be a field of characteristic $p > 0$. For $n \in \mathbb{N}$ with p-adic expansion $n = \sum_{i \in \mathbb{N}} n_i p^i$ $(0 \le n_i < p)$, $\overline{S}^n$ is a direct summand of $\bigotimes_{i \in \mathbb{N}} (\overline{S}^{p^i})^{\otimes n_i}$. Hence $(-; \overline{S}^n)$ is naturally a direct summand of the composite of the division functors $(-; \overline{S}^{p^i})^{\circ n_i}$, $i \in \mathbb{N}$.*

Lemma 2.17. *Let $\Bbbk$ be a finite field of characteristic $p > 0$. For $a, n \in \mathbb{N}$, there are natural isomorphisms $(S^n; \overline{S}^a) \cong S^{n-a}$ and $(\overline{S}^n; \overline{S}^a) \cong \overline{S}^{n-a}$.*

2.4. **Polynomial functors.** The difference functor $\overline{\Delta_{\Bbbk}} : \mathscr{F} \to \mathscr{F}$ leads to the following simple definition of polynomial functors (in the sense of Eilenberg and MacLane):

Definition 2.18. (Cf. [Kuh94a].) A functor $F \in \mathsf{Ob}\ \mathscr{F}$ is polynomial of degree $\leq d$, for $d \in \mathbb{N}$, if and only if $\overline{\Delta_{\Bbbk}}^{d+1} F = 0$. The full subcategory of functors of Eilenberg-MacLane polynomial degree at most d is denoted $\mathscr{F}_d \subset \mathscr{F}$. Let $p_d : \mathscr{F} \to \mathscr{F}_d$ (respectively $q_d : \mathscr{F} \to \mathscr{F}_d$) be the right (respectively left) adjoint to the inclusion.

Example 2.19.
  (1) The functors of Example 2.2 are polynomial of degree d.
  (2) Let $\Bbbk = \mathbb{F}_2$. The functor $\overline{I_{\mathbb{F}_2}}$ is uniserial, with unique composition series given by the polynomial filtration such that $p_n \overline{I_{\mathbb{F}_2}} / p_{n-1} \overline{I_{\mathbb{F}_2}}$ is zero for $n = 0$ and isomorphic to Λ^n for $n > 0$.

To simplify the exposition, for the remainder of this section $\Bbbk = \mathbb{F}_p$. These results extend *mutatis mutandis* to the case of an arbitrary finite field.

Notation 2.20. For $d \in \mathbb{N}$ and $\Bbbk = \mathbb{F}_p$, denote by
  (1) $\mathsf{cr}_d : \mathscr{F}_d \to \mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod}$ the functor $\mathrm{Hom}_{\mathscr{F}}(T^d, -)$ restricted to $\mathscr{F}_d$;
  (2) $\alpha_d : \mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod} \to \mathscr{F}_d$ the functor $M \mapsto T^d \otimes_{\mathfrak{S}_d} M$;
  (3) $\beta_d : \mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod} \to \mathscr{F}_d$ the functor $M \mapsto (T^d \otimes M)^{\mathfrak{S}_d}$,

where T^d is consider as a right (respectively left) $\mathfrak{S}_d$-module by place permutations and $T^d \otimes M$ is given the diagonal structure in (3).

Proposition 2.21. [Kuh02] *For $d \in \mathbb{N}$ and $\Bbbk = \mathbb{F}_p$, the functor cr_d is exact and fits into a recollement diagram of abelian categories:*

$$\mathscr{F}_{d-1} \overset{q_{d-1}}{\underset{p_{d-1}}{\rightleftarrows}} \mathscr{F}_d \overset{\alpha_d}{\underset{\beta_d}{\rightleftarrows}} \mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod}$$

In particular, there is an equivalence of categories $\mathscr{F}_d / \mathscr{F}_{d-1} \overset{\cong}{\to} \mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod}$.

Remark 2.22. The functor cr_d is related to the classical Schur functor, via the relationship between $\mathscr{F}_d$ and the category $\mathscr{P}_d$ of strict polynomial functors (see Proposition 3.6).

2.5. **Stratification by rank.**

Definition 2.23. For $c, d \in \mathbb{N}$, let $\mathscr{F}^{\geq c} \subset \mathscr{F}$ be the kernel of the evaluation functor $\mathrm{ev}_{c-1} : F \mapsto F(\Bbbk^{c-1})$ and $\mathscr{F}_d^{\geq c}$ denote the intersection of the full subcategories $\mathscr{F}_d$ and $\mathscr{F}^{\geq c}$ in $\mathscr{F}$.

There is a decreasing filtration:
$$\ldots \subset \mathscr{F}^{\geq c+1} \subset \mathscr{F}^{\geq c} \subset \ldots \subset \mathscr{F}^{\geq 0} = \mathscr{F}$$
and one has:

Proposition 2.24. [Kuh94b, Remark 2.9] *For $c \in \mathbb{N}$, there is a recollement diagram:*

$$\mathscr{F}^{\geq c+1} \rightleftarrows \mathscr{F}^{\geq c} \overset{\mathrm{ev}_{c-1}}{\rightleftarrows} \mathbb{F}_p[GL_c] - \mathrm{mod}.$$

There is the analogous stratification of $\mathscr{F}_d$ by the subcategories and $\mathscr{F}_d^{\geq c}$ and this has finite length by the following result:

Proposition 2.25. *For $c, d \in \mathbb{N}$, the difference functor restricts to $\overline{\Delta_{\Bbbk}} : \mathscr{F}_d^{\geq c} \to \mathscr{F}_{d-1}^{\geq c-1}$ that is faithful if $c > 0$. In particular, if $c > d \in \mathbb{N}$, then $\mathscr{F}_d^{\geq c}$ is 0.*

Proof. The key ingredient is that, if $F \in \mathscr{F}^{\geq 1}$, then $F = 0$ if and only if $\overline{\Delta_{\Bbbk}} F = 0$. $\square$

3. Strict polynomial functors

This section reviews the basic theory of strict polynomial functors, working over a field $\Bbbk$. The exposition is largely based upon that of Krause [Kra13, Kra17b] (where the more general case of $\Bbbk$ a commutative ring is considered) and of Kuhn (cf. [Kuh02] for example). The notation is inspired by that of Friedlander and Suslin (cf. [FFSS99]).

3.1. **Basic structure.**

Definition 3.1. For $d \in \mathbb{N}$, let
  (1) $\Gamma^d \mathcal{V}^f$ be the $\Bbbk$-linear category with objects $V \in \mathsf{Ob}\ \mathcal{V}^f$ and morphisms $\mathrm{Hom}_{\Gamma^d \mathcal{V}^f}(V, W) := \Gamma^d(\mathrm{Hom}_{\mathcal{V}^f}(V, W))$;
  (2) $\mathscr{P}_d$, the category of degree d strict polynomial functors, be the category of $\Bbbk$-linear functors from $\Gamma^d \mathcal{V}^f$ to $\mathcal{V}$;
  (3) $\mathscr{P}_d^{\mathrm{fd}} \subset \mathscr{P}_d$ be the full subcategory of functors taking values in $\mathcal{V}^f$.
The category $\mathscr{P}$ of strict polynomial functors is $\bigoplus_{d \in \mathbb{N}} \mathscr{P}_d$.

Proposition 3.2. [FFSS99] *For $d \in \mathbb{N}$, the category $\mathscr{P}_d$ is abelian with enough projectives and enough injectives. There is an exact, faithful forgetful functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}$ that takes values in $\mathscr{F}_d$.*

There is important additional structure (for $d, e \in \mathbb{N}$):
  (1) The (external) tensor product: $\otimes : \mathscr{P}_d \times \mathscr{P}_e \to \mathscr{P}_{d+e}$ such that the forgetful functor $\mathscr{O}$ is (strict) symmetric monoidal, in the obvious sense.
  (2) Composition $\circ : \mathscr{P}_d \times \mathscr{P}_e^{\mathrm{fd}} \to \mathscr{P}_{de}$, $(P, Q) \mapsto P \circ Q$ so that $\mathscr{O}(P \circ Q)(V) = \mathscr{O}P(\mathscr{O}Q(V))$ for $V \in \mathsf{Ob}\ \mathcal{V}^f$.
  (3) Duality $D : \mathscr{P}_d^{\mathrm{op}} \to \mathscr{P}_d$ that is compatible with $D : \mathscr{F}^{\mathrm{op}} \to \mathscr{F}$.
  (4) The Frobenius twist for $\Bbbk$ a field of characteristic p. For $r \in \mathbb{N}$, the rth iterated Frobenius defines $I^{(r)} \in \mathsf{Ob}\ \mathscr{P}_{p^r}$ and the Frobenius twist functor $(-)^{(r)} : \mathscr{P}_d \to \mathscr{P}_{dp^r}$ is given by precomposition $- \circ I^{(r)}$. This is compatible with the Frobenius twist on $\mathscr{F}$ when $\Bbbk$ is a finite field.

Example 3.3. For $d \in \mathbb{N}$, T^d, S^d, Γ^d and Λ^d are canonically strict polynomial of weight d, as is $\overline{S}^d$ when $\Bbbk$ has characteristic p.

Remark 3.4. For $n \geq d \in \mathbb{N}$, $\Gamma^d \circ \mathrm{Hom}(\Bbbk^n, -)$ is a projective generator of $\mathscr{P}_d$ and $S^d \circ \mathrm{Hom}(\Bbbk^n, -)$ an injective cogenerator. This provides the link with the classical Schur algebras: the Schur algebra $S(n, d)$ is the endomorphism ring $\mathrm{End}_{\mathscr{P}_d}(\Gamma^d \circ \mathrm{Hom}(\Bbbk^n, -))$ and, for $n \geq d$, $\mathscr{P}_d$ is equivalent to the category of $S(n, d)$-modules.

Definition 3.5. For $d \in \mathbb{N}$, the Schur functor is the exact functor
$$\mathrm{Hom}_{\mathscr{P}_d}(T^d, -) : \mathscr{P}_d \to \Bbbk[\mathfrak{S}_d] - \mathrm{mod},$$
where the $\mathfrak{S}_d$-action is induced by the place permutation action on T^d.

This is related to the dth cross-effect functor of Notation 2.20 (for simplicity, working over the prime field $\mathbb{F}_p$):

Proposition 3.6. *For $d \in \mathbb{N}$ and $\Bbbk = \mathbb{F}_p$, the following diagram commutes up to natural isomorphism:*

$$
\begin{array}{ccc}
\mathscr{P}_d & \xrightarrow{\ \mathscr{O}\ } & \mathscr{F}_d \\
& \searrow{\scriptstyle \mathrm{Hom}_{\mathscr{P}_d}(T^d, -)} & \downarrow{\scriptstyle \mathrm{cr}_d} \\
& & \Bbbk[\mathfrak{S}_d] - \mathrm{mod}.
\end{array}
$$

Remark 3.7. The above Proposition makes explicit how the functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}_d$ can be viewed as an enriched Schur functor, retaining the information that is lost upon passage to the quotient category $\mathscr{F}_d \to \mathscr{F}_d/\mathscr{F}_{d-1} \cong \Bbbk[\mathfrak{S}_d] - \mathrm{mod}$ (see Proposition 2.21).

3.2. Interlude on sequences and partitions.
Partitions of d index the simple objects of $\mathscr{P}_d$ and arise in describing the highest weight structure of the category. It is convenient here to index sequences of non-negative integers by $\mathbb{N}$.

Notation 3.8.
  (1) For $\lambda \in \mathbb{N}^{\mathbb{N}}$, $|\lambda| := \sum_{i \in \mathbb{N}} \lambda_i$.
  (2) Let $\mathsf{Seq} \subset \mathbb{N}^{\mathbb{N}}$ denote the subset of sequences λ such that $|\lambda| < \infty$ and $\mathsf{Part} \subset \mathsf{Seq}$ the subset of partitions, namely λ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \in \mathbb{N}$.
  (3) For $\lambda \in \mathsf{Part}$, the length $l(\lambda)$ of λ is zero if $\lambda = 0$, otherwise $l(\lambda) = 1 + \sup\{i | \lambda_i \neq 0\}$.
  (4) For $\lambda \in \mathsf{Part}$, $\lambda' \in \mathsf{Part}$ denotes the conjugate partition.
  (5) For $d \in \mathbb{N}$, let $\mathsf{Seq}_d \subset \mathsf{Seq}$ denote the set of λ such that $|\lambda| = d$; similarly $\mathsf{Part}_d := \mathsf{Seq}_d \cap \mathsf{Part}$ is the set of partitions of d.

Definition 3.9.
  (1) For $d \in \mathbb{N}$, the dominance order $\trianglelefteq$ on Part_d defined by $\mu \trianglelefteq \lambda$ if and only if $\sum_{i=0}^{t} \mu_i \leq \sum_{i=0}^{t} \lambda_i$ for all $t \in \mathbb{N}$.
  (2) The left lexicographical order $\preceq_\ell$ on Seq defined by $\mu \preceq_\ell \lambda$ if and only if $\mu_t \leq \lambda_t$ for all $t \in \mathbb{N}$ such that $\mu_i = \lambda_i$ for $i < t$.

Lemma 3.10. *The left lexicographical order $\preceq_\ell$ is a total order on Seq. Moreover, for $d \in \mathbb{N}$, the following hold.*
  (1) For $\mu, \lambda \in \mathsf{Part}_d$, if $\mu \trianglelefteq \lambda$, then $\mu \preceq_\ell \lambda$.
  (2) The minimal (respectively maximal) element of Part_d for $\preceq_\ell$ is (1^d) (resp. (d)).

Recall the following standard definition:

Definition 3.11. For p a prime and $\lambda \in \mathsf{Part}$,
  (1) λ is p-regular if $\lambda_i > \lambda_{i+p-1}$ for all $i \in \mathbb{N}$ such that $\lambda_i > 0$;
  (2) λ is p-restricted if λ' is p-regular (equivalently if $\lambda_i - \lambda_{i+1} < p$ for all $i \in \mathbb{N}$).
The set of p-restricted partitions is denoted $\mathsf{Part}^{[p-\mathrm{res}]}$.

Lemma 3.12. *Let p be a prime. For $\lambda \in \mathsf{Part}$, there is a unique set of p-restricted partitions $\lambda[i]$, $i \in \mathbb{N}$, such that $\lambda = \sum_{i \in \mathbb{N}} p^i \lambda[i]$, where the sum and scalar multiplication is formed termwise. In particular, $\lambda[i] = 0$ for $i \gg 0$.*

Proof. The p-restricted partitions $\lambda[i]$ are uniquely determined by

$$\lambda_j - \lambda_{j+1} = \sum_i (\lambda[i]_j - \lambda[i]_{j+1}) p^i,$$

for $i, j \in \mathbb{N}$, so that the right hand side corresponds to the p-adic expansion of $\lambda_j - \lambda_{j+1}$. $\square$

Notation 3.13. Let $\{F^n \in \mathsf{Ob}\ \mathscr{P}_n | n \in \mathbb{N}\}$ be an $\mathbb{N}$-graded strict polynomial functor with $F^0 = \Bbbk \in \mathscr{P}_0$. Then, for $\lambda \in \mathsf{Seq}$, set $F^\lambda := \bigotimes_{i \in \mathbb{N}} F^{\lambda_i}$, so that $F^\lambda \in \mathsf{Ob}\ \mathscr{P}_{|\lambda|}$.

3.3. Exponential strict polynomial functors.
The theory of exponential functors (see Section 2.2) also applies in the context of strict polynomial functors, as in [FFSS99].

Remark 3.14. Using the definition of the categories $\mathscr{P}_d$ given here, one uses the fact that $(\mathcal{V}^f, \oplus)$ induces a (graded) symmetric monoidal structure on the categories $\Gamma^* \mathcal{V}^f$, so that $\oplus : \Gamma^d \mathcal{V}^f \times \Gamma^e \mathcal{V}^f \to \Gamma^{d+e} \mathcal{V}^f$ for $d, e \in \mathbb{N}$.

The main properties and applications of exponential functors carry over, *mutatis mutandis.*

Proposition 3.15. *Let* $\Bbbk$ *be a field of characteristic* $p > 0$. *For* $d \in \mathbb{N}$ *and* $\lambda, \mu \in \mathsf{Seq}_d$, *the forgetful functor* $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}$ *induces an isomorphism:*

$$\mathscr{O} : \mathrm{Hom}_{\mathscr{P}_d}(\overline{S}^\lambda, \overline{S}^\mu) \overset{\cong}{\to} \mathrm{Hom}_{\mathscr{F}}(\overline{S}^\lambda, \overline{S}^\mu).$$

Proof. The forgetful functor $\mathscr{O}$ induces a monomorphism of $\Bbbk$-vector spaces, hence it suffices to show that they are finite-dimensional of the same dimension.

Using the exponentiality of $\overline{S}^*$, this reduces to the fact that $\mathscr{O}$ induces an isomorphism

$$\mathrm{Hom}_{\mathscr{P}}(\overline{S}^m, \overline{S}^n) \cong \mathrm{Hom}_{\mathscr{F}}(\overline{S}^m, \overline{S}^n) \cong \left\{ \begin{array}{ll} \Bbbk & m = n \\ 0 & \text{otherwise}, \end{array} \right.$$

which follows from the simplicity of the functors $\overline{S}^n$ in the respective categories, with endomorphism ring $\Bbbk$. $\qquad\square$

4. Highest weight categories and Strict polynomial functors

This section recalls the highest weight structure on the category of strict polynomial functors. The theory of highest weight categories is reviewed in Section 4.1, including the Ringel duality that is applied in Section 5.

4.1. **Highest weight categories.** This section is based upon Dlab and Ringel's work [DR92] together with the papers of Krause [Kra17b, Kra17a] (Krause works more generally, over a commutative ring).

Definition 4.1. Let $\mathscr{C}$ be a $\Bbbk$-linear abelian category such that $\dim_\Bbbk \mathrm{Ext}_{\mathscr{C}}^i(X, Y) < \infty$ for all X, Y and $i \in \{0, 1\}$. Then $\mathscr{C}$ is a highest weight category if there exists a set of *standard objects* $\Delta := \{\Delta_\lambda | \lambda \in \Lambda\}$, where $(\Lambda, \leq)$ is a finite poset such that the following conditions are satisfied:
  (1) $\mathrm{End}_{\mathscr{C}}(\Delta_\lambda)$ is a division ring;
  (2) $\mathrm{Hom}_{\mathscr{C}}(\Delta_\lambda, \Delta_\mu) = 0$ if $\lambda > \mu$;
  (3) $\mathrm{Ext}_{\mathscr{C}}^1(\Delta_\lambda, \Delta_\mu) = 0$ if $\lambda \geq \mu$;
  (4) a projective generator of $\mathrm{Filt}(\Delta)$ is a projective generator of $\mathscr{C}$.
A $\Bbbk$-algebra A is quasi-hereditary if the category of left A-modules satisfies all the above conditions.

The highest weight structure determines the set of *costandard objects* $\nabla := \{\nabla_\lambda | \lambda \in \Lambda\}$ which plays a dual rôle to Δ. These objects arise conceptually via the following result.

Proposition 4.2. *For* $\mathscr{C}$ *a* $\Bbbk$-*linear highest weight category (as in Definition 4.1) the full subcategory* $\mathrm{Filt}(\nabla)$ *of* ∇-*filtered objects identifies as*

$$\mathrm{Filt}(\nabla) = \{Y \in \mathrm{Ob}\,\mathscr{C} | \mathrm{Ext}_{\mathscr{C}}^1(\Delta_\lambda, Y) = 0 \; \forall \lambda \in \Lambda\}.$$

Notation 4.3. For X an object of $\mathscr{C}$, let $\mathrm{add}\, X \subset \mathscr{C}$ be the full subcategory whose objects are direct summands of finite direct sums of copies of X.

The results of Ringel [Rin91] give the following (cf. [Kra17b, Corollary 1.2.6]):

Proposition 4.4. *Let* $\mathscr{C}$ *be a* $\Bbbk$-*linear highest weight category. For an object* T *of* $\mathscr{C}$, *the following are equivalent*
  (1) T *is a projective generator of* $\mathrm{Filt}(\nabla)$;
  (2) T *is an injective generator of* $\mathrm{Filt}(\Delta)$;
  (3) $\mathrm{add}\, T = \mathrm{Filt}(\Delta) \cap \mathrm{Filt}(\nabla)$.

The indecomposable objects of $\mathrm{Filt}(\Delta) \cap \mathrm{Filt}(\nabla)$ are described by the following result:

Proposition 4.5. [Rin91, Proposition 2] [DR92, Proposition 3.1] *Let* $(\mathscr{C}, \Lambda, \leq)$ *be a* $\Bbbk$-*linear highest weight category. For* $\lambda \in \Lambda$, *there exists an indecomposable object* $T(\lambda)$ *in* $\mathrm{Filt}(\Delta) \cap \mathrm{Filt}(\nabla)$ *and short exact sequences*

$$0 \to \Delta_\lambda \to T(\lambda) \to X_\lambda \to 0$$
$$0 \to Y_\lambda \to T(\lambda) \to \nabla_\lambda \to 0,$$

where $X_\lambda \in \mathrm{Filt}(\Delta_\mu | \mu < \lambda)$ *and* $Y_\lambda \in \mathrm{Filt}(\nabla_\mu | \mu < \lambda)$.

Moreover $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$ *satisfies the equivalent conditions of Proposition 4.4.*

Remark 4.6.
(1) The *characteristic object* $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$ is a generalized tilting and cotilting module in the sense of Auslander and Reiten (see the references in [Rin91]).
(2) The *minimal tilting object* $T(\lambda)$ is indecomposable in $\mathrm{Filt}(\Delta) \cap \mathrm{Filt}(\nabla)$, not necessarily in $\mathscr{C}$, and is uniquely determined up to isomorphism by the above.
(3) The characteristic object T determines both $\mathrm{Filt}(\Delta)$ and $\mathrm{Filt}(\nabla)$ by [Rin91, Corollary 4]:

$$
\begin{aligned}
\mathrm{Filt}(\Delta) &= \{ X \in \mathsf{Ob}\ \mathscr{C} | \mathrm{Ext}^i_\mathscr{C}(X, T) = 0\ \forall i \geq 1 \} \\
\mathrm{Filt}(\nabla) &= \{ Y \in \mathsf{Ob}\ \mathscr{C} | \mathrm{Ext}^i_\mathscr{C}(T, Y) = 0\ \forall i \geq 1 \}.
\end{aligned}
$$

This leads to Ringel duality, based on the following:

Theorem 4.7. [Rin91, Theorem 6] *Let* $\mathscr{C}$ *be a* $\Bbbk$-*linear highest weight category with characteristic object* T. *Then*

(1) the $\Bbbk$-*algebra* $A_T := \mathrm{End}_\mathscr{C}(T)$ *is quasi-hereditary, in particular* $A_T - \mathrm{mod}$ *is a highest weight category;*

(2) the functor $F_T := \mathrm{Hom}_\mathscr{C}(T, -) : \mathscr{C} \to A_T - \mathrm{mod}$ *induces an equivalence between* $\mathrm{Filt}(\nabla)$ *and* $\mathrm{Filt}(\Delta')$, *where the standard objects* Δ'_λ *are given by* $\Delta'_\lambda := F_T(\Delta_\lambda)$.

The category $\mathscr{C}' := A_T - \mathrm{mod}$ *is the Ringel dual of* $\mathscr{C}$.

Remark 4.8. The characteristic object T is not in general projective in $\mathscr{C}$, hence F_T is not in general an equivalence of categories.

Corollary 4.9. *Let* $\mathscr{C}$ *be a* $\Bbbk$-*linear highest weight category satisfying the hypotheses of Definition 4.1. Then the double Ringel dual* $(\mathscr{C}')'$ *is equivalent to* $\mathscr{C}$.

In particular, for two such $\Bbbk$-*linear highest weight categories* $\mathscr{C}_1$, $\mathscr{C}_2$, *with characteristic modules* T_1, T_2 *respectively, if* $\mathrm{End}_{\mathscr{C}_1}(T_1)$ *and* $\mathrm{End}_{\mathscr{C}_2}(T_2)$ *are Morita equivalent, then* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *are equivalent.*

4.2. **Weyl functors and the highest weight structure.** The general theory of highest weight categories applies to strict polynomial functors over a field $\Bbbk$. The standard objects are given by the Weyl functors.

Notation 4.10. For $\lambda \in \mathsf{Part}_d$, let $W_\lambda \in \mathsf{Ob}\ \mathscr{P}_d$ denote the associated Weyl functor (see [Kra17b, Section 2.3] and the indications below).

Remark 4.11. The Weyl functor W_λ is the image of an explicit morphism $\Gamma^\lambda \twoheadrightarrow W_\lambda \hookrightarrow \Lambda^{\lambda'}$ and there is an explicit presentation in $\mathscr{P}_d$:

$$\bigoplus_{i \in \mathbb{N}} \bigoplus_{t=1}^{\lambda_{i+1}} \Gamma^{\lambda(i,t)} \to \Gamma^\lambda \to W_\lambda \to 0,$$

where the morphism $\Gamma^{\lambda(i,t)} \to \Gamma^\lambda$ is given by a Koszul differential-type morphism that is defined using the exponential structure of Γ^* and $\lambda(i,t) \in \mathsf{Seq}$ is the sequence with $\lambda(i,t)_i = \lambda_i + t$, $\lambda(i,t)_{i+1} = \lambda_{i+1} - t$ and $\lambda(i,t)_j = \lambda_j$ for $j \notin \{i, i+1\}$.

The following result corresponds to the fact that the Schur algebra $S(n, d)$ is quasi-hereditary for $n \geq d$. (See [Kra17b] for the more general case over $\Bbbk$ a commutative ring. Note also that Krause works with the left lexicographical order $\preceq_\ell$ rather than the dominance order $\trianglelefteq$.) For $\Bbbk$ a field, these results are due to Donkin (cf. [Don93]).

Theorem 4.12. *For $d \in \mathbb{N}$, the category $\mathscr{P}_d$ is a highest weight category with weights $(\mathsf{Part}_d, \trianglelefteq)$ and standard objects $\Delta_\lambda := W_\lambda$. In particular, for $\lambda, \mu \in \mathsf{Part}_d$,*

(1) $\mathrm{End}_{\mathscr{P}_d}(W_\lambda) = \Bbbk$;
(2) $\mathrm{Hom}_{\mathscr{P}_d}(W_\lambda, W_\mu) = 0$ *if* $\lambda \ntrianglelefteq \mu$, *in particular, if* $\lambda \rhd \mu$.
(3) There is a short exact sequence

$$0 \to U(\lambda) \to \Gamma^\lambda \to W_\lambda \to 0$$

where $U(\lambda) \in \mathrm{Filt}(W_\mu | \mu \rhd \lambda) \subseteq \mathrm{Filt}(W_\mu | \mu \ntrianglelefteq \lambda)$.
(4) $\bigoplus_{\lambda \in \mathsf{Part}_d} \Gamma^\lambda$ *is a projective generator of* $\mathscr{P}_d$.

Moreover, the minimal tilting object $T(\lambda)$ is $\Lambda^{\lambda'}$.

Proof. (Indications based upon the proof of [Kra17b, Theorem 2.6.1].) The first two statements follow from [Kra17b, Proposition 2.4.2]. The property of the Weyl filtration of Γ^λ follows from [Kra17b, Corollary 2.5.3]. It is then straightforward to check that $\mathscr{P}_d$ satisfies the hypotheses of Definition 4.1. $\square$

Notation 4.13. For $d \in \mathbb{N}$, let $\mathrm{Filt}_d(\Delta)$ denote the full subcategory $\mathrm{Filt}(W_\mu | \mu \in \mathsf{Part}_d) \subset \mathscr{P}_d$.

The simple objects of $\mathscr{P}_d$ are described as follows in terms of the highest weight structure:

Proposition 4.14. *(Cf. [Kra17b].) For $d \in \mathbb{N}$ and $\lambda \in \mathsf{Part}_d$, W_λ has simple cosocle denoted L_λ and $\{L_\lambda | \lambda \in \mathsf{Part}_d\}$ is a set of representatives of the isomorphism classes of simple objects of $\mathscr{P}_d$. Moreover, the simple functors are self-dual: i.e. $DL_\lambda \cong L_\lambda$.*

Remark 4.15. The convention on indexing means that L_λ is the highest weight composition factor of $\Lambda^{\lambda'}$. In particular $L_\lambda(\Bbbk^t)$ is zero for $t < \lambda_0'$ and non-zero for $t = \lambda_0'$.

Example 4.16.

(1) The partition (1^d) corresponds to the exterior power functor Λ^d (i.e. $L_{(1^d)} \cong \Lambda^d$).
(2) For p a prime and n a positive integer, the simple functor $\overline{S}^n$ is indexed by the partition $((p-1)^a, b)$, where $n = a(p-1) + b$, with $0 \leq b < p - 1$.

An important fact is the stability of the categories $\mathrm{Filt}_d(\Delta)$ under the (exterior) tensor product (see [Kra17b, Proposition 2.7.1] which is based on the results of Boffi [Bof88]).

Theorem 4.17. [Bof88, Theorem 3.7 and Section 4] *For partitions λ, μ, $W_\lambda \otimes W_\mu \in \mathrm{Ob}\ \mathscr{P}_{|\lambda|+|\mu|}$ lies in $\mathrm{Filt}_{|\lambda|+|\mu|}(\Delta)$. Moreover the multiplicity of W_ν, $\nu \in \mathsf{Part}_{|\lambda|+|\mu|}$, in the associated graded of such a Weyl filtration is given by the Littlewood-Richardson coefficient $c(\lambda, \mu; \nu)$.*

4.3. **The Steinberg tensor product theorem.** In this section, take $\Bbbk$ a field of characteristic $p > 0$. Kuhn [Kuh02] proved a Steinberg tensor product theorem for the category $\mathscr{F}$ of functors over a finite field $\Bbbk$ which also applies in the context of strict polynomial functors. This establishes how the simple functors L_λ indexed by the p-restricted partitions generate all simple functors via the Frobenius twist and the (external) tensor product.

Recall from Lemma 3.12 that, for $\lambda \in \mathsf{Part}$, there is a unique set of p-restricted partitions $\lambda[i] \in \mathsf{Part}$, for $i \in \mathbb{N}$, such that $\lambda = \sum_{i \in \mathbb{N}} p^i \lambda[i]$.

Theorem 4.18. *For $\Bbbk$ a field of characteristic $p > 0$ and $\lambda \in \mathsf{Part}$, there is an isomorphism in $\mathscr{P}_{|\lambda|}$:*

$$L_\lambda \cong \bigotimes_{i \in \mathbb{N}} L_{\lambda[i]}^{(i)}.$$

In particular, the tensor product contains only one component not isomorphic to $\Bbbk$ if and only if $\lambda = p^i \lambda[i]$ for a p-restricted partition $\lambda[i]$.

Corollary 4.19. *Let $\Bbbk$ be a field of characteristic $p > 0$ and $\lambda \in \mathsf{Part}$ be a partition. Then $\mathscr{O}(L_\lambda)$ has polynomial degree $\sum_i |\lambda[i]|$; this is equal to d if and only if λ is p-restricted.*

5. Representation stability for strict polynomial functors

The fundamental ingredient to this paper, namely representation stability for certain highly-connected subcategories of strict polynomial functors is developed in this Section. Harman's stability theorem is given in Section 5.2 and the following Sections develop the understanding of the tensor product with respect to representation stability, leading to the crucial weak form of representation stability for $\mathscr{F}$ given in Theorem 5.21.

5.1. Stratifying $\mathscr{P}_d$.

Definition 5.1. For $c, d \in \mathbb{N}$, define the full subcategory $\mathscr{P}_d^{\geq c} \subset \mathscr{P}_d$ as the pullback:

$$
\begin{array}{ccc}
\mathscr{P}_d^{\geq c} & \xrightarrow{\;\mathscr{O}\;} & \mathscr{F}_d^{\geq c} \\
\big\uparrow & \lrcorner & \big\uparrow \\
\downarrow & & \downarrow \\
\mathscr{P}_d & \xrightarrow[\;\mathscr{O}\;]{} & \mathscr{F}.
\end{array}
$$

Explicitly, P lies in $\mathscr{P}_d^{\geq c}$ if and only if $\mathscr{O}P(\Bbbk^{c-1}) = 0$.

Lemma 5.2. *For $c, d \in \mathbb{N}$,*

(1) *$\mathscr{P}_d^{\geq c} = 0$ if $c > d$;*
(2) *$\mathscr{P}_d^{\geq 0} = \mathscr{P}_d$;*
(3) *$\mathscr{P}_d^{\geq d}$ is equivalent to $\mathcal{V}$, the category of $\Bbbk$-vector spaces.*

The simple objects of $\mathscr{P}_d^{\geq c}$ are identified as follows:

Lemma 5.3. *For $c \leq d \in \mathbb{N}$ and $\mu \in \mathsf{Part}_d$, the following conditions are equivalent:*

(1) *$L_\mu \in \mathsf{Ob}\, \mathscr{P}_d^{\geq c}$;*
(2) *$\mu'_0 \geq c$;*
(3) *$\mu_c \neq 0$;*
(4) *$\mu \trianglelefteq (d - c + 1, 1^{c-1})$.*

The stratification of $\mathscr{P}_d$ by the categories $\mathscr{P}_d^{\geq c}$ can be compared with that of $\mathscr{F}_d$ by $\mathscr{F}_d^{\geq c}$ (see Proposition 2.24), in particular these categories give a filtration

$$0 \subsetneq \mathscr{P}_d^{\geq d} \subsetneq \mathscr{P}_d^{\geq d-1} \subsetneq \ldots \subsetneq \mathscr{P}_d^{\geq 2} \subsetneq \mathscr{P}_d^{\geq 1} = \mathscr{P}_d.$$

The external tensor product behaves well with respect to these subcategories:

Proposition 5.4. *For $c, d, e \in \mathbb{N}$, the (external) tensor product restricts to $\otimes : \mathscr{P}_d^{\geq c} \times \mathscr{P}_e \to \mathscr{P}_{d+e}^{\geq c}$. If $e < c$ then the following diagram is a pullback of categories:*

$$
\begin{array}{ccc}
\mathscr{P}_d^{\geq c} \times \mathscr{P}_e & \longrightarrow & \mathscr{P}_{d+e}^{\geq c} \\
\cap \downarrow & \lrcorner & \cap \downarrow \\
\mathscr{P}_d \times \mathscr{P}_e & \xrightarrow{\otimes} & \mathscr{P}_{d+e}.
\end{array}
$$

Notation 5.5. For $c, d \in \mathbb{N}$, let $\mathsf{Part}_d^{\geq c} \subset \mathsf{Part}_d$ denote the subset of partitions μ such that $\mu_0' \geq c$ (respectively $\mathsf{Part}_d^{[p-\mathrm{res}] \geq c} \subset \mathsf{Part}^{[p-\mathrm{res}]}{}_d$ for p-restricted partitions).

Proposition 5.6. *(Cf. [Har15].) For $c, d \in \mathbb{N}$, the category $\mathscr{P}_d^{\geq c}$ is a highest weight category with weights $(\mathsf{Part}_d^{\geq c}, \trianglelefteq)$ and with standard objects $\{W_\mu | \mu \in \mathsf{Part}_d^{\geq c}\}$. In particular, the minimal tilting object containing W_μ is $\Lambda^{\mu'}$.*

Remark 5.7. Harman [Har15, Section 2.3] works with the category of modules over the appropriate Schur algebra. The category $\mathscr{P}_d^{\geq c}$ here corresponds to $S(N, d)^{\leq d-c}$ in the notation of *loc. cit.*, where N is an integer $N \geq d$.

5.2. **Representation stability.** When $\Bbbk$ is a field of characteristic $p > 0$, Harman [Har15] has shown that the categories $\mathscr{P}_d^{\geq d-t}$ (for fixed t and varying d) exhibit a form of representation stability, by exploiting Ringel duality for highest weight categories.

First observe the following combinatorial stability lemma:

Lemma 5.8. *For integers $t \leq d \leq e \in \mathbb{N}$, the map of sets*

$$
\begin{array}{ccc}
\mathsf{Part}_d^{\geq d-t} & \to & \mathsf{Part}_e^{\geq e-t} \\
\lambda & \mapsto & \lambda \bullet 1^{e-d}
\end{array}
$$

given by concatenation of partitions is an injection and is a bijection if $d \geq 2t$. For a prime p, this restricts to $\mathsf{Part}_d^{[p-\mathrm{res}] \geq d-t} \to \mathsf{Part}_d^{[p-\mathrm{res}] \geq e-t}$ that is a bijection if $d \geq 2t$.

It is therefore convenient to introduce the following terminology:

Definition 5.9. A pair of natural numbers $(d, t) \in \mathbb{N}^{\times 2}$ is *stable* if $d \geq 2t$ and *strictly stable* if $d > 2t$.

Theorem 5.10. [Har15, Theorem 2.8] *Let $\Bbbk$ be a field of characteristic $p > 0$, $(d, t) \in \mathbb{N}^{\times 2}$ be a stable pair and $e > d$ an integer such that $d \equiv e \mod p^{\lceil \log_p t \rceil}$.*

The categories $\mathscr{P}_d^{\geq d-t}$ and $\mathscr{P}_e^{\geq e-t}$ are equivalent as highest weight categories with respect to the bijection of weights $\mathsf{Part}_d^{\geq d-t} \cong \mathsf{Part}_e^{\geq e-t}$ of Lemma 5.8. In particular, the simple object $L_\lambda \in \mathsf{Ob}\, \mathscr{P}_d^{\geq d-t}$ is sent under this equivalence to $L_{\lambda \bullet 1^{e-d}} \in \mathsf{Ob}\, \mathscr{P}_e^{\geq e-t}$.

Proof. (Indications.) The proof uses Ringel duality, in particular Corollary 4.9. It suffices to show respective endomorphism rings of the characteristic modules are isomorphic:

$$
\mathrm{End}\Big(\bigoplus_{\lambda \in \mathsf{Part}_d^{\geq d-t}} \Lambda^{\lambda'} \Big) \cong \mathrm{End}\Big(\bigoplus_{\mu \in \mathsf{Part}_e^{\geq e-t}} \Lambda^{\mu'} \Big).
$$

The hypothesis upon $d - e$ ensures that this is the case, as can be checked by direct calculation. $\square$

Remark 5.11. [Har15, Theorem 2.8] uses the hypothesis $d > 2t$; this can be weakened as above. However, for the main applications here in characteristic two, the strict hypothesis is required. This is due to the exceptional case occurring in Proposition 5.16.

Notation 5.12. For d, e, t satisfying the hypotheses of Theorem 5.10, let

$$\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-t} \overset{\cong}{\to} \mathscr{P}_e^{\geq e-t}$$

denote the equivalence of highest weight categories provided by Theorem 5.10.

5.3. **Tensor products and representation stability.** The tensor product is compatible with representation stability by the following result:

Proposition 5.13. *Let* $\Bbbk$ *be a field of characteristic* $p > 0$ *and* $d, e, t \in \mathbb{N}$ *such that* (d, t) *is stable and* $d \equiv e \mod p^{\lceil \log_p t \rceil}$. *Consider* $d', d'' \in \mathbb{N}$ *such that* $d' + d'' = d$ *with* $d'' \leq t$ *and set* $t' := t - d''$ *and* $e' = e - d''$.

Then the pair (d', t') *is stable and the periodicity equivalences of Theorem 5.10 fit into the following diagram that commutes up to natural isomorphism:*

$$
\begin{array}{ccc}
\mathscr{P}_{d'}^{\geq d'-t'} \times \mathscr{P}_{d''} & \overset{\otimes}{\longrightarrow} & \mathscr{P}_d^{\geq d-t} \\
{\scriptstyle \mathrm{Per}_{d',e'} \times \mathrm{Id}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \mathrm{Per}_{d,e}} \\
\mathscr{P}_{e'}^{\geq e'-t'} \times \mathscr{P}_{d''} & \underset{\otimes}{\longrightarrow} & \mathscr{P}_e^{\geq e-t}.
\end{array}
$$

Lemma 5.14. *For* $d, t \in \mathbb{N}$ *and natural numbers* d', d'' *such that*

$$
\begin{aligned}
d &= d' + d'' \\
d'' &\leq t,
\end{aligned}
$$

set $t' := t - d''$. *Then* $t' \in \mathbb{N}$ *and the following conditions are equivalent:*

(5.1) $$d \geq 2t$$
(5.2) $$d' \geq 2t' + d''.$$

That is (d, t) *is stable if and only if* (d', t') *is stable and* $d'' \leq d' - 2t'$.

Proof of Proposition 5.13. As in Lemma 5.14, $t \geq t' \geq 0$ and (d', t') is a stable pair. Theorem 5.10 therefore provides the periodicity equivalences: $\mathrm{Per}_{d',e'} : \mathscr{P}_{d'}^{\geq d'-t'} \overset{\cong}{\to} \mathscr{P}_{e'}^{\geq e'-t'}$ and $\mathscr{P}_d^{\geq d-t} \overset{\cong}{\to} \mathscr{P}_e^{\geq e-t}$ and hence the vertical functors of the diagram. Moreover, all the functors are exact.

Since the respective categories $\mathrm{Filt}(\Delta)$ contain the projective objects, it is sufficient to prove the result after restricting to the subcategories $\mathrm{Filt}(\Delta) \subset \mathscr{P}$. Now, Theorem 4.17 implies that the tensor product of objects with good filtrations has a good filtration. Similarly, by construction, the periodicity equivalences preserve the subcategories $\mathrm{Filt}(\Delta)$. Thus one has the following diagram of functors:

$$
\begin{array}{ccc}
\mathrm{Filt}_{d'}^{\geq d'-t'}(\Delta) \times \mathrm{Filt}_{d''}(\Delta) & \overset{\otimes}{\longrightarrow} & \mathrm{Filt}_d^{\geq d-t}(\Delta) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
\mathrm{Filt}_{e'}^{\geq e'-t'}(\Delta) \times \mathrm{Filt}_{d''}(\Delta) & \underset{\otimes}{\longrightarrow} & \mathrm{Filt}_e^{\geq e-t}(\Delta)
\end{array}
$$

(where the notation should be self-explanatory) and it suffices to show that this commutes up to natural isomorphism.

Any $X \in \mathrm{Filt}(\Delta)$ admits a finite resolution by direct summands of tilting objects by [Rin91, Lemma 6] (see also [Don98, Proposition A4.4]). In particular, there is a copresentation

$$0 \to X \to T_0 \to T_1$$

where $T_0, T_1 \in \mathrm{add}\,(T)$ (using the notation of Proposition 4.4).

Hence, it suffices to prove the result after restricting to the respective full subcategories $\mathrm{Filt}(\Delta) \cap \mathrm{Filt}(\nabla)$. Here the result follows by inspection, using the fact that the periodicity isomorphism is constructed using the behaviour on such tilting objects. $\square$

Remark 5.15. An alternative viewpoint exploiting Theorem 4.7 is as follows. Namely, rather than restricting to $\mathrm{Filt}(\Delta)$, one can use $\mathrm{Filt}(\nabla)$ and consider the behaviour of the functor $\mathrm{Hom}(T, -)$. This allows the exponential property of T to be applied (arising from the fact that the respective twisting objects $T(\lambda)$ are components of a (multigraded) exponential functor). It is straightforward to use this to check that the periodicity isomorphism is compatible with the tensor product, as in the proof of Theorem 5.10.

5.4. **The Steinberg tensor product theorem and representation stability.** The behaviour of the simple objects under the equivalence of Theorem 5.10 over a field of characteristic $p > 0$, in particular with respect to the Steinberg tensor product theorem (Theorem 4.18), is of interest.

Proposition 5.16. *Let p be a prime, $d \geq t \in \mathbb{N}$ and λ be a partition $\lambda \in \mathsf{Part}_{\bar{d}}^{\geq d-t}$ with associated decomposition $\lambda = \sum_i \lambda[i] p^i$ for p-restricted partitions $\lambda[i]$, $i \in \mathbb{N}$. Set*

$$
\begin{aligned}
d' &:= |\lambda[0]| \\
d'' &:= d - d' \geq 0.
\end{aligned}
$$

Suppose that (d, t) is a stable pair, then exactly one of the following holds:

(1) *$\lambda[0] \in \mathsf{Part}_{d'}^{\geq d'-t'} = \mathsf{Part}_{d'}^{\geq d-t}$ whereas, for $i > 0$, $\lambda[i] \notin \mathsf{Part}_{|\lambda[i]|}^{\geq d-t}$.*

(2) *$p = 2$, $d = 2t$ and $\lambda = 2\lambda[1]$ with $\lambda[1] = (1^t)$.*

In case (1):

(i) *$t' := t - d'' \geq 0$ and $d' \geq 2t' + d''$, in particular (d', t') is a stable pair.*

(ii) *For $d \leq e \in \mathbb{N}$, the partition $\lambda \bullet 1^{e-d}$ decomposes as p-restricted partitions as $\sum_i (\lambda \bullet 1^{e-d})[i] p^i$ where*

$$
(\lambda \bullet 1^{e-d})[i] = \begin{cases} \lambda[0] \bullet 1^{e-d} & i = 0 \\ \lambda[i] & \text{otherwise.} \end{cases}
$$

Proof. If $t = 0$, then the result follows easily from Lemma 5.2, hence suppose that $t > 0$; since (d, t) is a stable pair, this implies that $d > 0$ and $d - t > 0$.

By hypothesis, using the equivalent conditions of Lemma 5.3, $\lambda_{d-t} \neq 0$. We first show that, apart from the exceptional case (2), if $i > 0$, $\lambda[i]_{d-t} = 0$, hence $\lambda[0]_{d-t} \neq 0$.

Suppose that $\lambda[i]_{d-t} \neq 0$ for some $i > 0$, hence $|\lambda[i]| \geq (d - t)$. Then $p(d - t) \leq p^i(d - t) \leq d$, so that $(p - 1)d \leq pt$. This inequality is strict unless $i = 1$ and $\lambda[0] = 0$.

By stability, $d \geq 2t$, so the inequality gives $2(p-1)t \leq pt$; this is a contradiction unless $p = 2$. In the case $p = 2$, one must have $d = 2t$ and $\lambda[0] = 0$, so this cannot fall in case (1). The only possible solution is that given in case (2).

Consider the case (1). By the above, $\lambda[0]_{d-t} \neq 0$. By definition $d' = |\lambda[0]|$ and $d' - t' = d - t$; it follows that $d' - t' \leq d'$, hence $t' \geq 0$. The fact that (d', t') is stable follows from Lemma 5.14.

The statement concerning $\lambda \bullet 1^{e-d}$ follows from elementary properties of the concatenation of partitions, using the fact that $\lambda[0]_{d-t} \neq 0$ whereas $\lambda[i]_{d-t} = 0$ for $i > 0$. $\square$

Corollary 5.17. *Let $\Bbbk$ be a field of characteristic $p > 0$ and L_λ be a simple object of $\mathsf{Part}_{\bar{d}}^{\geq d-t}$, where (d, t) is a stable pair, and write $L_\lambda \cong \bigotimes_{i \in \mathbb{N}} L_{\lambda[i]}^{(i)}$ as in Theorem 4.18.*

Suppose that λ satisfies case (1) of Proposition 5.16 (i.e. $L_\lambda \not\cong (\Lambda^t)^{(1)}$ at $p = 2$). Then $L_{\lambda[0]} \in \mathrm{Ob}\ \mathscr{P}_{d'}^{\geq d-t} = \mathscr{P}_{d'}^{\geq d'-t'}$, where $d' = |\lambda[0]|$ and $t' = t - d'' \geq 0$, for $d' + d'' = d$, with (d', t') a stable pair and $d'' \equiv 0 \mod p$.

In particular, L_λ lies in the image of the composite

$$\mathscr{P}_{d'}^{\geq d'-t'} \times \mathscr{P}_{d''/p} \xrightarrow{\mathrm{Id} \times (-)^{(1)}} \mathscr{P}_{d'}^{\geq d'-t'} \times \mathscr{P}_{d''} \xrightarrow{\otimes} \mathscr{P}_{d}^{\geq d-t}$$

with (d', t') a stable pair and $d'' \leq t$.

5.5. **Comparison via $\mathscr{O}$.** Let $\Bbbk = \mathbb{F}_p$ and $c, d \in \mathbb{N}$.

Notation 5.18. Write $G_0(\mathscr{P}_d^{\geq c})$ (respectively $G_0(\mathscr{F}_d^{\geq c})$) for the Grothendieck group of finite objects in $\mathscr{P}_d^{\geq c}$ (resp. $\mathscr{F}_d^{\geq c}$), so that $G_0(\mathscr{P}_d^{\geq c})$ is the free abelian group on $\mathsf{Part}_d^{\geq c}$ and $G_0(\mathscr{F}_d^{\geq c})$ is the free abelian group on $\mathsf{Part}_d^{[p-\mathrm{res}]\geq c}$.

Lemma 5.19. *The functor $\mathscr{O} : \mathscr{P}_d^{\geq c} \to \mathscr{F}_d^{\geq c}$ induces a surjective morphism of abelian groups*

$$G_0(\mathscr{O}) : G_0(\mathscr{P}_d^{\geq c}) \twoheadrightarrow G_0(\mathscr{F}_d^{\geq c}).$$

Notation 5.20. For $d < e \in \mathbb{N}$, let $\bullet 1^{e-d} : G_0(\mathscr{F}_d^{\geq c}) \to G_0(\mathscr{F}_e^{\geq e+c})$ denote the morphism of abelian groups induced by applying the free abelian group functor to the operation of concatenation of partitions of Lemma 5.8.

Theorem 5.21. *Suppose that $\Bbbk = \mathbb{F}_p$ and that (d, t) is a stable pair of integers (strictly stable if $p = 2$). For $d < e \in \mathbb{N}$ such that $d \equiv e \mod p^{\lceil \log_p t \rceil}$, the periodicity equivalence $\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-t} \to \mathscr{P}_e^{\geq e-t}$ induces a commutative diagram of abelian groups:*

$$
\begin{array}{ccc}
G_0(\mathscr{P}_d^{\geq d-t}) & \xrightarrow{G_0(\mathscr{O})} & G_0(\mathscr{F}_d^{\geq d-t}) \\
{\scriptstyle G_0(\mathrm{Per}_{d,e})} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \bullet 1^{e-d}} \\
G_0(\mathscr{P}_e^{\geq e-t}) & \xrightarrow{G_0(\mathscr{O})} & G_0(\mathscr{F}_e^{\geq e-t})
\end{array}
$$

in which the vertical morphisms are isomorphisms.

Proof. The proof is by induction upon d. For $1 \leq d < p$ the result holds by inspection.

For the inductive step, since it suffices to check this on the generators $\mathsf{Part}_d^{\geq d-t}$ of $G_0(\mathscr{P}_d^{\geq d-t})$. On p-restricted partitions, commutativity is clear, using the behaviour of $\mathrm{Per}_{d,e}$ on the simple objects given in Theorem 5.10.

If λ is not p-restricted, Corollary 5.17 shows that $\mathscr{O}L_\lambda \cong \mathscr{O}(L_{\lambda[0]} \otimes \bigotimes_{i>0} L_{\lambda[i]}^{(i-1)})$ where $L_{\lambda[0]} \in \mathsf{Ob}\ \mathscr{P}_{d'}^{\geq d'-t'}$ with (d', t') stable and $\bigotimes_{i>0} L_{\lambda[i]}^{(i-1)} \in \mathsf{Ob}\ \mathscr{P}_{d''/p}$ where $d'' \neq 0$. In particular

$$L_{\lambda[0]} \otimes \bigotimes_{i>0} L_{\lambda[i]}^{(i-1)} \in \mathsf{Ob}\ \mathscr{P}_{d'+d''/p}$$

where $d' + d''/p < d$.

This allows the inductive hypothesis to be applied. To conclude, one uses the behaviour of the periodicity isomorphism on tensor products that is given by Proposition 5.13 in conjunction with the identification of the behaviour of concatenation given by Proposition 5.16. $\square$

Remark 5.22. The above results should be compared with those of Harman [Har15], for example [Har15, Proposition 3.1]. Harman uses the Schur functor to pass to the category $\mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod}$ rather than $\mathscr{O}$ to pass to $\mathscr{F}_d$. Recall from Proposition 2.21 that the former is equivalent to the category $\mathscr{F}_d/\mathscr{F}_{d-1}$ over the prime field, hence the quotient functor $\mathscr{F}_d \to \mathscr{F}_d/\mathscr{F}_{d-1}$ induces a surjection

$$G_0(\mathscr{F}_d) \twoheadrightarrow G_0(\mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod})$$

to the Grothendieck group of $\mathbb{F}_p[\mathfrak{S}_d] - \mathrm{mod})$. This sends the generators indexed by partitions λ with $|\lambda| < d$ to zero.

6. Natural transformations between symmetric power functors

This short section serves to recall the relationship between natural transformations between symmetric power functors and Steenrod operations. The relationship with unstable modules over the Steenrod algebra is important in Section 9.

6.1. **Natural transformations and Steenrod operations.** In this section, $\Bbbk = \mathbb{F}_p$.

Notation 6.1. As in [Kuh94a], let $\mathscr{A}(p)$ denote the algebra of Steenrod pth powers, with gradings divided by 2 if p is odd. Hence $\mathscr{A}(p)$ is generated by the reduced powers P^i, where $|P^i| = i(p-1)$ with this grading convention; for $p = 2$, P^i is the ith Steenrod square Sq^i.

Let $\mathscr{U}(p)$ denote the category of unstable $\mathscr{U}(p)$-modules (this category is essentially defined in Remark 6.4 below).

Proposition 6.2. [Ste62] *The algebra $\mathscr{A}(p)$ is generated by the operations P^{p^i} for $i \in \mathbb{N}$.*

The algebra $\mathscr{A}(p)$ is a connected, cocommutative $\mathbb{F}_p$-Hopf algebra (no Koszul signs), in particular is equipped with conjugation χ and the tensor product of two $\mathscr{A}$-modules is an $\mathscr{A}$-module with module structure induced by the diagonal.

We review the relationship between the natural transformations $\mathrm{Hom}_{\mathscr{F}}(S^*, S^*)$ and $\mathscr{A}(p)$ following [Kuh94a], exploiting the exponential structure of S^*:

Proposition 6.3. [Kuh94a] *For $\Bbbk = \mathbb{F}_p$*

$$\mathrm{Hom}_{\mathscr{F}}(S^1, S^n) = \left\{ \begin{array}{ll} \Bbbk & n = p^t \\ 0 & \text{otherwise} \end{array} \right.$$

with generators the iterated Frobenius maps $S^1 \to S^{p^t}$, $x \mapsto x^{p^t}$, for $t \in \mathbb{N}$.

The underlying bigraded commutative algebra of $\mathrm{Hom}_{\mathscr{F}}(S^*, S^*)$ is the polynomial algebra $\mathbb{F}_p[\mathrm{Hom}_{\mathscr{F}}(S^1, S^*)]$. In particular, $\mathrm{Hom}_{\mathscr{F}}(S^m, S^n)$ is zero if $m > n$ and $\mathbb{F}_p$ for $m = n$.

Moreover, the functor S^* takes values in $\mathbb{N}$-graded $\mathscr{A}(p)$-modules.

Remark 6.4. Kuhn [Kuh94a] proves the stronger result that the representation category associated to $\{S^n | n \in \mathbb{N}\}$ is precisely the category $\mathscr{U}(p)$ of unstable modules over $\mathscr{A}(p)$.

Definition 6.5. [Kuh94a] Let $r : \mathscr{F} \to \mathscr{U}(p)$ be the right exact functor given by $r(F)^n := \mathrm{Hom}_{\mathscr{F}}(\Gamma^n, F)$, where Steenrod operations act via natural transformations between divided powers.

Example 6.6. (Cf. [Kuh98].) The symmetric power functors S^* are recovered as the functor $V \mapsto r(I_{\mathbb{F}_p} \circ (V \otimes -))$, where $I_{\mathbb{F}_p}$ is the functor introduced in Notation 2.11, which has the structure of a (non-graded) exponential functor (see Lemma 2.12). The diagonal of S^* is induced by the diagonal $I_{\mathbb{F}_p} \to I_{\mathbb{F}_p} \otimes I_{\mathbb{F}_p}$ and the product of S^* by that on $I_{\mathbb{F}_p}$. This makes transparent the fact that the diagonal $S^*(V) \to S^*(V) \otimes S^*(V)$ and product $S^*(V) \otimes S^*(V) \to S^*(V)$ are morphisms of $\mathscr{U}(p)$, since the functor r is monoidal with respect to $\otimes$.

Remark 6.7. Studying the natural transformations between symmetric powers in strict polynomial functors is not *a priori* interesting, since $\mathrm{Hom}_{\mathscr{P}}(S^m, S^n)$ is zero unless $m = n$.

Working over a finite field, one of the key techniques developed in [FFSS99] uses the extension of scalars to allow information for $\mathscr{F}$ to be extracted from strict polynomial functors. This can be used to explain the key fact, exploited here, that the forgetful functor $\mathscr{O}$ induces an isomorphism $\mathscr{O} : \mathrm{Hom}_{\mathscr{P}_{p^r}}((S^1)^{(r)}, S^{p^r}) \to \mathrm{Hom}_{\mathscr{F}}(S^1, S^{p^r})$ for all $r \in \mathbb{N}$. This is applied here by exploiting the polynomial filtration of S^*.

7. Symmetric powers and the polynomial filtration

This section spells out how the polynomial filtration can be studied by using the theory of strict polynomial functors. A crucial point is that we work over the prime field $\mathbb{F}_p$, so that Frobenius twist functors can be neglected; it is, however, important to retain information on the grading.

7.1. Preliminaries.

Definition 7.1. Let p be a prime.
   (1) For $\lambda \in \mathsf{Seq}$, define $||\lambda||_p := \sum_{i \in \mathbb{N}} \lambda_i p^i$.
   (2) For p a prime and $r \in \mathbb{N}$, let $\mathsf{Seq}^p(r) \subset \mathsf{Seq}$ denote the set of λ such that $||\lambda||_p = r$ and $\mathsf{Seq}^p_d(r) := \mathsf{Seq}_d \cap \mathsf{Seq}^p(r)$.

Lemma 7.2. *Let p be a prime and $r \in \mathbb{N}$.*
   (1) $\mathsf{Seq}^p(r)$ is a finite set.
   (2) For $d \in \mathbb{N}$, $\mathsf{Seq}^p_d(r) = \emptyset$ if $d \not\equiv r \mod (p-1)$ (equivalently, for $\omega \in \mathsf{Seq}$, $|\omega| \equiv ||\omega||_p$ mod $(p-1)$).

The following is analogous to F^λ introduced in Notation 3.13

Notation 7.3. For $\{F^n \in \mathsf{Ob}\,\mathscr{P}_n | n \in \mathbb{N}\}$ be an $\mathbb{N}$-graded strict polynomial functor with $F^0 = \mathbb{k} \in \mathscr{P}_0$, where $\mathbb{k}$ is a field of characteristic $p > 0$, let $F^{[\lambda]} \in \mathsf{Ob}\,\mathscr{P}_{||\lambda||_p}$ be the functor $F^{[\lambda]} := \bigotimes_{i \in \mathbb{N}} (F^{\lambda_i})^{(i)}$.

Lemma 7.4. *If $\mathbb{k} = \mathbb{F}_p$, for $\lambda \in \mathsf{Seq}$, the functors $\mathscr{O}(F^\lambda)$ and $\mathscr{O}(F^{[\lambda]})$ are isomorphic and belong to $\mathscr{F}_{|\lambda|} \subset \mathscr{F}$.*

7.2. Filtering S^r in $\mathscr{P}_r$.
Let $\mathbb{k}$ be a field of characteristic p, $r \in \mathbb{N}$, and consider the rth symmetric power S^r as an object of $\mathscr{P}_r$.

Definition 7.5.
   (1) For $\lambda \in \mathsf{Seq}^p(r)$, let $m^\lambda : S^{[\lambda]} \to S^r$ be the morphism of $\mathscr{P}_r$ given by the composite
   $$S^{[\lambda]} = \bigotimes_{i \in \mathbb{N}} (S^{\lambda_i})^{(i)} \to \bigotimes_{i \in \mathbb{N}} S^{p^i \lambda_i} \to S^r,$$
   where the first map is induced by the iterated Frobenius pth power maps and the second is multiplication of symmetric powers.
   (2) For $d \in \mathbb{N}$, let $S^r_{\leq d} \subset S^r$ be:
   $$S^r_{\leq d} := \sum_{\substack{\lambda \in \mathsf{Seq}^p(r) \\ |\lambda| \leq d}} \mathrm{image}(m^\lambda).$$

The following is well-known (cf. [Kuh02, Kuh97] for example).

Proposition 7.6. *For $r \in \mathbb{N}$ there is an increasing filtration of S^r in $\mathscr{P}_r$:*
$$S^r_{\leq 1} \subset S^r_{\leq 2} \subset \ldots \subset S^r_{\leq r-1} \subset S^r_{\leq r} = S^r.$$
The subquotients, for $d \in \mathbb{N}$, are given by
$$S^r_{\leq d} / S^r_{\leq d-1} \cong \bigoplus_{\lambda \in \mathsf{Seq}^p_d(r)} \overline{S}^{[\lambda]}.$$

Moreover, there is a natural isomorphism $\mathscr{O}(S^r_{\leq d}) \cong p_d S^r$, where $\mathscr{O} : \mathscr{P}_r \to \mathscr{F}$, hence the above filtration induces the polynomial filtration $p_\bullet S^r$ of S^r.

7.3. **The polynomial filtration of S^*.** Let $\Bbbk = \mathbb{F}_p$ throughout this subsection. The fact that the Frobenius twist is then canonically equivalent to the identity functor of $\mathscr{F}$ is exploited below.

For $d \in \mathbb{N}$, consider the functor p_d of Definition 2.18 as an endofunctor of $\mathscr{F}$. By Proposition 7.6, the polynomial filtration $p_d S^r$ of S^r arises from $\mathscr{P}_r$ and there is an isomorphism:

$$p_d S^r / p_{d-1} S^r \cong \bigoplus_{\lambda \in \mathsf{Seq}_d^p(r)} \mathscr{O}(\overline{S}^{[\lambda]}).$$

Over $\Bbbk$, the Frobenius twist is neglected, but it is important to retain the information on the grading (corresponding to r). To emphasize this, the above is rewritten as

$$(7.1) \qquad p_d S^r / p_{d-1} S^r \cong \bigoplus_{\lambda \in \mathsf{Seq}_d^p(r)} \bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \lambda_i},$$

where the term indexed by λ has underlying functor $\overline{S}^\lambda$ in $\mathscr{F}$ and Φ is as in Notation 2.6. (Heuristically, the Frobenius twist $-^{(i)}$ is replaced by the functor Φ^i.)

Proposition 7.7. *Let $\Bbbk = \mathbb{F}_p$ and $d \in \mathbb{N}$. The $\mathbb{N}$-graded functor $p_d S^* / p_{d-1} S^*$ has the structure of an $\mathscr{A}(p)$-module and lies in the image of the forgetful functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}_d$.*

Proof. The fact that the $\mathbb{N}$-graded functor lies in the image of $\mathscr{O}$ follows from the discussion above. The functoriality of p_d implies that a natural transformation $f : S^m \to S^n$ induces $p_d f : p_d S^m \to p_d S^n$ and this passes to the subquotients of the filtration, giving the $\mathscr{A}(p)$-module structure. The fact that this action arises from $\mathscr{P}_d$ follows from Proposition 3.15. $\square$

The following immediate Corollary opens the way for exploiting the structure of the categories $\mathscr{P}_d$ when studying the $\mathscr{A}(p)$-indecomposables of S^*:

Corollary 7.8. *Let $\Bbbk$ be the prime field $\mathbb{F}_p$ and $d \in \mathbb{N}$. The components of the $\mathbb{N}$-graded functor $\mathbb{F}_p \otimes_{\mathscr{A}(p)} (p_d S^* / p_{d-1} S^*)$ lie in the image of the forgetful functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}$.*

To understand the action of $\mathscr{A}(p)$ upon $p_d S^* / p_{d-1} S^*$, by Proposition 3.15 it suffices to consider it as a functor of $\mathscr{F}^{\mathbb{N}}$. Using the results of [Kuh98], this is succinctly encoded using the functor r (see Definition 6.5).

Proposition 7.9. *The bigraded functor $(d, n) \mapsto p_d S^n / p_{d-1} S^n \in \mathscr{F}$ is isomorphic to the bigraded exponential functor:*

$$\bigotimes_{i \geq 0} (\Phi^i \overline{S})^*,$$

using the graded tensor product, where the d degree is given by the polynomial degree.

This is isomorphic to the functor to $\mathbb{N}$-graded unstable modules $V \mapsto r(\overline{S}^\bullet \circ (V \otimes -))$ and, via this isomorphism, the exponential structure on $p_\bullet S^* / p_{\bullet-1} S^*$ is induced by the exponential structure of $\overline{S}^\bullet$.

Proof. The result follows by unravelling the definitions and the identifications underlying Proposition 6.3, and by appealing to [Kuh98] to identify the underlying graded unstable module. Namely, there is an isomorphism $r(V \otimes -) \cong V \otimes F(1)$ in $\mathscr{U}(p)$, where $F(1)$ is the free unstable module on a generator of degree one, which is isomorphic to $\mathrm{Hom}_{\mathscr{F}}(S^1, S^*)$ as a graded vector space. Then [Kuh98, Theorem 1.3(3)] implies that $r(\overline{S}^* \circ (V \otimes -))$ is isomorphic to $\overline{S}^*(V \otimes F(1))$ with induced $\mathscr{A}(p)$-action. Using the fact that $\overline{S}^*$ is exponential, the result follows. $\square$

Remark 7.10.

(1) The functor $I_{\Bbbk}$ has polynomial filtration with associated graded $\bigoplus_{n\in\mathbb{N}}\overline{S}^n$ (see [Kuh97] for example). By comparing with Example 6.6, this gives a heuristic explanation of the origin of the identification of the bigraded functor $p_\bullet S^*/p_{\bullet-1}S^*$ as $V\mapsto r(\overline{S}^*\circ(V\otimes-))$. However, since r is not exact, this does not constitute a proof.

(2) The functor Φ on $\mathbb{N}$-graded $\mathbb{F}_p$-vector enriches to a functor $\Phi:\mathscr{U}(p)\to\mathscr{U}(p)$ (see [Sch94] for example) and this functor is monoidal with respect to the tensor product $\otimes$ of $\mathscr{U}(p)$. Hence, by using the Cartan formula, to understand the $\mathscr{A}(p)$-action on $\bigotimes_{i\geq0}(\Phi^i\overline{S})^*$, it suffices to consider the action on the subspace $\overline{S}^*$. The operation P^i acts via the diagonal $\overline{S}^n\to\overline{S}^{n-i}\otimes\overline{S}^i$, which corresponds to:
$$P^i:\overline{S}^n\to\overline{S}^{n-i}\otimes(\Phi\overline{S})^{pi}$$
when taking into account the grading.

(3) The filtration of $F(1)$ by degree has associated graded $\bigoplus_{i\geq1}\Phi^i\mathbb{F}_p$. Thus the exponential property of $\overline{S}^*$ explains the relationship between $\overline{S}^*(V\otimes F(1))$ and $\bigotimes_{i\geq0}(\Phi^i\overline{S})^*(V)$.

Remark 7.11. An alternative description of the natural unstable module structure on $p_dS^*/p_{d-1}S^*$ is given by using the functor $\tilde{m}_d:\mathscr{P}_d\to\mathscr{U}$ introduced by Nguyen D.H. Hai in [Hai10]. Namely, $p_dS^*/p_{d-1}S^*$ is isomorphic to the functor
$$V\mapsto\tilde{m}_d\big(\overline{S}^d(V\otimes-)\big).$$

This can be seen using the proof of Proposition 7.9, which gives the identification with $\overline{S}^d(V\otimes F(1))$, together with the description of $\tilde{m}_d$ in [Hai10].

This gives an alternative argument for showing that the Steenrod operations act via natural transformations of $\mathscr{P}$ (cf. Proposition 7.7).

8. THE FUNCTORS Q^*

The main characters are introduced here, namely the indecomposables Q^* for the action of $\mathscr{A}(p)$ on S^*, together with their subquotients and approximations, using the material introduced above. Throughout, $\Bbbk=\mathbb{F}_p$.

8.1. Indecomposables.

Definition 8.1. For $n\in\mathbb{N}$, let Q^n be the cokernel of the natural transformation:
$$\bigoplus_{\substack{m<n\\f\in\mathrm{Hom}_{\mathscr{F}}(S^m,S^n)}}S^m\xrightarrow{f}S^n.$$

Lemma 8.2. *For $n\in\mathbb{N}$, Q^n is a finite functor and has polynomial degree exactly n, with cosocle* $\mathrm{cosoc}\,Q^n\cong\overline{S}^n$.

Recall from Section 6 that $\mathscr{A}(p)$ is an $\mathbb{F}_p$-Hopf algebra, in particular an augmented $\mathbb{F}_p$-algebra.

Proposition 8.3. *For $\mathscr{A}(p)$-modules M,N, there is a surjective natural transformation*
$$\mathbb{F}_p\otimes_{\mathscr{A}(p)}(M\otimes N)\twoheadrightarrow(\mathbb{F}_p\otimes_{\mathscr{A}(p)}M)\otimes(\mathbb{F}_p\otimes_{\mathscr{A}(p)}N).$$

Proposition 6.3 implies that Q^* is isomorphic to $\mathbb{F}_p\otimes_{\mathscr{A}(p)}S^*$, as $\mathbb{N}$-graded functors. Combining this with Proposition 8.3 gives:

Proposition 8.4. *The exponential structure of S^* induces a cocommutative $\mathbb{N}$-graded coproduct $Q^*\to Q^*\otimes Q^*$, so that Q^* is a functor to graded cocommutative coalgebras.*

The following result gives a useful tool for comparing Q^{n+a} with Q^n, for $a,n\in\mathbb{N}$:

Proposition 8.5. *For $a, n \in \mathbb{N}$, the coproduct $Q^{n+a} \to Q^a \otimes Q^n$ together with the morphism induced by the surjection $Q^a \twoheadrightarrow \overline{S}^a$ induces a surjection $(Q^{n+a}; \overline{S}^a) \twoheadrightarrow Q^n$.*

Proof. The existence of the morphism is clear; surjectivity follows by considering the cosocle, using Lemma 2.17. $\qquad\square$

8.2. **The filtration of Q^* induced by the polynomial filtration of S^*.**

Definition 8.6. For $d, n \in \mathbb{N}$, let $Q^n[d] \subset Q^n$ be the image of $p_d S^n$ in Q^n, so that there is a filtration (for $0 < n$):

$$0 = Q^n[0] \subset Q^n[1] \subset \ldots \subset Q^n[n-1] \subset Q^n[n] = Q^n.$$

Notation 8.7. For $d, n \in \mathbb{N}$, let

(1) Q^n_d denote the subquotient $Q^n[d]/Q^n[d-1]$;
(2) $\mathfrak{Q}^n_d$ denote the degree n part of $\mathbb{F}_p \otimes_{\mathscr{A}(p)} (p_d S^*/p_{d-1} S^*)$.

Remark 8.8. By Corollary 7.8, the graded functor $\mathbb{F}_p \otimes_{\mathscr{A}(p)} (p_d S^*/p_{d-1} S^*)$ is a graded object of $\mathscr{P}_d$. Hence $\mathfrak{Q}^n_d$ can be considered as an object of $\mathscr{P}_d$ and also, via the forgetful functor $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}_d \subset \mathscr{F}$, as a functor of $\mathscr{F}$. The context should always make clear what is intended, so the forgetful functor $\mathscr{O}$ will usually be omitted from the notation.

Lemma 8.9. *For $d, n \in \mathbb{N}$,*

(1) $Q^n_d = \mathfrak{Q}^n_d = 0$ *if $d > n$;*
(2) *there is a natural inclusion $Q^n[d] \subset p_d Q^n$;*
(3) *there is a natural surjection $\mathfrak{Q}^n_d \twoheadrightarrow Q^n_d$.*

Proof. The first two statements are clear. The final statement follows from the right exactness of $\mathbb{F}_p \otimes_{\mathscr{A}(p)} -$. $\qquad\square$

Remark 8.10. The canonical morphism $\mathfrak{Q}^n_d \to Q^n_d$ is not always an isomorphism (see Corollary 9.27). This stems from the fact that $\mathbb{F}_p \otimes_{\mathscr{A}(p)} -$ is right exact but is not exact.

The functors considered here are finite length objects of $\mathscr{F}$, hence it is convenient to use the Grothendieck group $G_0(\mathscr{F})$ of $\mathscr{F}$:

Notation 8.11. For F a finite functor of $\mathscr{F}$, let $[F]$ denote the class of F in the Grothendieck group $G_0(\mathscr{F})$.

The Grothendieck group $G_0(\mathscr{F})$ is isomorphic to the free abelian group on the set of p-restricted partitions, hence is equipped with the usual lattice structure induced by $(\mathbb{Z}, \leq)$, which gives the operation $\wedge$ used below:

Proposition 8.12. *For $1 \leq d \leq n \in \mathbb{N}$, writing K^n_d for the kernel of $\mathfrak{Q}^n_d \twoheadrightarrow Q^n_d$, in $G_0(\mathscr{F})$:*

$$[K^n_d] \leq [\mathfrak{Q}^n_d] \wedge \Big[\bigoplus_{i=0}^{[\log_p(n/p)]} S^{n-p^i(p-1)}/p_d S^{n-p^i(p-1)} \Big].$$

Proof. Since $\mathscr{A}(p)$ is generated by the operations P^{p^i} (see Proposition 6.2) and $|P^{p^i}| = p^i(p-1)$, it suffices to consider the image of these operations. The instability condition from $\mathscr{U}(p)$ gives the upper bound on the i. $\qquad\square$

The following is a standard consequence of the definition of $\mathfrak{Q}^*_d$ as $\mathscr{A}(p)$-indecomposables:

Lemma 8.13. *For $d, n \in \mathbb{N}$, there is an inequality in $G_0(\mathscr{F})$:*

$$\big[p_d S^n/p_{d-1} S^n\big] \leq \sum_{m \leq n} \dim_{\mathbb{F}_p}(\mathscr{A}(p)^{n-m})[\mathfrak{Q}^m_d].$$

Proposition 8.12 then has the Corollary:

Corollary 8.14. *For $1 \leq d \leq n \in \mathbb{N}$, the surjection $\mathfrak{Q}_d^n \to Q_d^n$ is an isomorphism if*

$$\left[\mathfrak{Q}_d^n\right] \wedge \left[\mathfrak{Q}_e^m\right] = 0$$

for all $0 \leq m < n$ and $e > d$. In particular, this holds if the composition factors of $\mathfrak{Q}_e^m$ are all indexed by p-restricted partitions.

Remark 8.15. Corollary 8.14 can be viewed as being a refinement of the following standard 'first occurrence' argument used in the study of Q: if λ is a p-restricted partition and n is the least integer such that $\mathscr{O}L_\lambda$ is a composition factor of S^n, then $\mathscr{O}L_\lambda$ is a composition factor of Q^n.

As an application, one has:

Proposition 8.16. *Let $p = 2$ and suppose that $(d, n - d)$ is strictly stable (i.e. $d > 2(n - d)$). Then $\mathfrak{Q}_d^n \twoheadrightarrow Q_d^n$ is an isomorphism if one of the following holds:*

(1) $n - d \leq 5$;
(2) $n - d = 6$ and $n \leq 18$.

Proof. The pair $(d, n - d)$ is assumed to be strictly stable: this eliminates in particular the exceptional cases (d, d) for $n = 3d$.

A standard calculation shows that the composition factors of $\mathfrak{Q}_d^n$ are all 2-restricted if $n - d \leq 3$ whereas, for $n - d = 4$, the sequence $(n - 6, 1, 1)$ means that $\mathfrak{Q}_{n-4}^n$ contains a composition factor $L_{(3,1^{n-7})}$.

Hence, Corollary 8.14 immediately gives the first statement. The second statement also follows from the Corollary, by an elementary analysis of the strict stability condition. $\qquad \square$

Remark 8.17. The first potential failure in Proposition 8.16 of $\mathfrak{Q}_d^n \twoheadrightarrow Q_d^n$ to be an isomorphism is for $\mathfrak{Q}_{13}^{19} \twoheadrightarrow Q_{13}^{19}$, due to the presence of the composition factor $L_{(3,1^{11})}$ in $\mathfrak{Q}_{14}^{18}$. Now $\mathscr{O}(L_{(3,1^{11})}) \cong L_{(2,1^{11})} \oplus L_{(1^{13})}$ in $\mathscr{F}$. In particular, the possible failure of injectivity of the above map can only be verified on $\mathbb{F}_2^{\oplus 13}$. At best, it could be detected on $\mathbb{F}_2^{\oplus 12}$.

9. Exploiting instability

The main objective of this Section is to prove Corollary 9.23, which gives a vanishing criterion for $\mathfrak{Q}_d^n$. This result is proved by exploiting the fact that, for p a prime and $d \in \mathbb{N}$, the functor $\mathfrak{Q}_d^*$ takes values in the category $\mathscr{U}(p)$ of unstable $\mathscr{A}(p)$-modules. Although the results will only be applied for the prime $p = 2$, the general case is treated here.

9.1. **Basic functions at a prime p.** Throughout this section, p denotes a fixed prime. The material exposed here is well-known at the prime 2 (cf. [WW18a]).

Definition 9.1. Let $n \in \mathbb{N}$ and define:
(1) $\alpha_p(n)$, the sum of the coefficients of the p-adic expansion of n;
(2) $\gamma_p : \mathbb{N} \to \mathbb{N}$ by $\gamma_p(n) := \frac{p^n - 1}{p - 1}$;
(3) $\mu_p : \mathbb{N} \to \mathbb{N}$ the function $\mu_p(0) = 0$ and, for $n > 0$:

$$\mu_p(n) := \inf\{k \in \mathbb{N} | \exists a_1, \dots, a_k \in \mathbb{N} \text{ such that } n = \sum_j \gamma_p(a_j)\}$$

(4) $f_p(n) := \mu_p(n) + pn$.

Proposition 9.2. *Let $k, n \in \mathbb{N}$.*
(1) $f_p(n) \equiv \mu_p(n) \equiv n \mod p$ and $\gamma_p(n) \equiv 1 \mod p$;
(2) $\mu_p(n + 1) - \mu_p(n) \in \{1, 1 - p\}$ and hence the function f_p is strictly increasing;

(3) $n \geq \mu_p(n)$ for all n and $\mu_p(n) = n$ if and only if $0 \leq n \leq p$;
(4) $\mu_p(n) \leq k$ if and only if $\alpha_p(n(p-1) + k) \leq k$;
(5) $\mu_p(n) \leq k$ if and only if $\mu_p(pn + k) \leq k$;
(6) the function μ_p satisfies the recursive formula for $n > 0$:

$$\mu_p(n) = 1 + \mu_p(n - \gamma_p([\log_p(n(p-1) + 1)]));$$

(7) $\mu_p(n) = \mu_p(f_p(n))$.

Proof. The first three statements are elementary. (4) is proved by the argument of [WW18a, Proposition 2.4.4] and (5) then follows (cf. [WW18a, Proposition 2.4.5]) by using the equality $(pn + k)(p - 1) + k = p((p - 1)n + k)$.

(6) is proved by using the argument used in the proof of [WW18a, Theorem 5.4.2]: in the definition of $\mu_p(n)$ one can also require the following two conditions on the sequence of natural numbers (a_j):

(1) $a_j \geq a_{j+1}$;
(2) if $a_j > 0$, then $a_j > a_{j+p-1}$ unless $a_{j+p} = 0$,

as seen by using the equality for $a, b \in \mathbb{N}$: $p(p^a - 1) + (p^{b+1} - 1) = (p^{a+1} - 1) + p(p^b - 1)$. Moreover, under these conditions, the sequence (a_j) is unique and $a_1 = [\log_p(n(p-1) + 1)]$. This also gives the stated recursive formula.

Finally, (7) follows from (6), as in [WW18a, Proposition 5.4.4]. $\qquad\square$

The recursive expression for μ_p makes it straightforward to calculate the function μ_p. For instance, at $p = 2$, one has the following:

Proposition 9.3. *Let $p = 2$ and k, n be natural numbers.*

(1) *For $1 \leq k \leq n$, $\mu_2(2^n - k) = k$.*
(2) *$1 \leq \mu_2(n) \leq \log_2(n) + 1$ for $n \geq 1$.*

Proof. The first statement is [WW18a, Proposition 2.4.6] (and follows easily from the fact that $\mu_2(2^n - n) = n$ and $\mu_2(2^n - 1) = 1$). The second follows from Proposition 9.2 (6). $\qquad\square$

Remark 9.4. Various characterizations of the sequence of integers $\mu_2(n)$ are known; see [OEI, A100661,A045412] for example.

9.2. **Instability and conjugate reduced powers.** Recall that P^i denotes the ith Steenrod reduced power and that χ denotes the conjugation of the Hopf algebra $\mathscr{A}(p)$, so that χP^i is the associated conjugate operation.

The polynomial algebra $\mathbb{F}_p[x]$ (with the conventions here, $|x| = 1$) lies in $\mathscr{U}(p)$ and, moreover, has the structure of an unstable algebra over $\mathscr{A}(p)$ (see [Sch94]), so that the action of the operations χP^i is determined by their action on x, by multiplicativity. This action is given by the total conjugate reduced power, which acts via:

$$x \mapsto \sum_{i \in \mathbb{N}} (-1)^i x^{p^i}.$$

In particular, χP^j acts trivially on x unless $j = \gamma_p(i)$ for some i, when

$$\chi P^{\gamma_p(i)}(x) = (-1)^i x^{p^i}.$$

Proposition 9.5. *Let $M \in \mathsf{Ob}\,\mathscr{U}(p)$ be an unstable $\mathscr{A}(p)$-module, then χP^n acts trivially on M^i if $i < \mu_p(n)$.*

Proof. In classical terminology, (see [Sch94] or [Ste62]), adapted to $\mathscr{A}(p)$, this is the assertion that χP^n has excess $\mu_p(n)$. For a proof at $p = 2$, see [WW18a, Proposition 2.4.10]. The general case is sketched below.

By the theory of unstable modules, one reduces to considering the cases $M = \mathbb{F}_p[x]^{\otimes i}$ (for $i \in \mathbb{N}$) and the action on the product of the corresponding i algebra generators. The conjugate total Steenrod power acts multiplicatively, so the result follows from the discussion above. $\qquad\square$

9.3. Essential sequences.
The instability property allows one to reduce to considering a subset of the sequences Seq^p, the *essential sequences* introduced below (the prime p is left implicit).

Notation 9.6. For $\omega \in \mathsf{Seq}$, let
   (1) ω^- denote the truncated shifted sequence given by $\omega^-_{i+1} = \omega_i$, so that $|\omega| = |\omega^-| + \omega_0$ and $||\omega||_p = p||\omega^-||_p + \omega_0$. ;
   (2) $\omega + 1$ the sequence with $(\omega+1)_0 = \omega_0 + 1$ and $(\omega+1)_i = \omega_i$ for $i > 0$.

Definition 9.7. The set $\overline{\mathsf{Seq}^p} \subset \mathsf{Seq}$ of essential sequences is defined recursively by $\omega \in \overline{\mathsf{Seq}^p}$ if one of the following holds:
   (1) $\omega^- = 0$;
   (2) $\omega^- \neq 0$ and $\omega^- \in \overline{\mathsf{Seq}^p}$ and $\omega_0 \geq \mu_p(||\omega^-||_p)$.

For $d, n \in \mathbb{N}$, set $\overline{\mathsf{Seq}^p}_d := \overline{\mathsf{Seq}^p} \cap \mathsf{Seq}_d$ and $\overline{\mathsf{Seq}^p}_d(n) := \overline{\mathsf{Seq}^p} \cap \mathsf{Seq}_d(n)$.

Lemma 9.8. *Suppose that $0 \neq \omega \in \overline{\mathsf{Seq}^p}$, then*
   (1) $\omega_i \neq 0$ for $0 \leq i < l(\omega)$;
   (2) for $d \in \mathbb{N}$, the set $\overline{\mathsf{Seq}^p}_d$ is finite;
   (3) $\omega + 1 \in \overline{\mathsf{Seq}^p}$;
   (4) $||\omega||_p \leq \gamma_p(|\omega|)$ with equality if and only if $\omega = (1^{|\omega|})$;
   (5) if $0 \neq \omega \in \overline{\mathsf{Seq}^p}(n)$, then $\omega_0 \geq \mu_p(n)$.

Proof. The first point is proved by induction upon $l(\omega)$ and the second follows readily. The third point is clear.

For (4), by induction on the length one checks that the sequence $\omega := (1^{|\omega|})$ is essential, with $l(\omega) = |\omega|$ and $||\omega||_p = \gamma_p(\omega)$. It is clear that, if $\omega' \neq \omega$ is another essential sequence with $|\omega'| = |\omega|$, then $||\omega'||_p < ||\omega||_p$.

Finally, suppose that $0 \neq \omega \in \overline{\mathsf{Seq}^p}(n)$. If $\omega^- = 0$, then $\omega_0 = n \geq \mu_p(n)$; otherwise, $\mu_p(||\omega^-||_p) \leq \omega_0$, by definition of essential sequences, hence $\mu_p(||\omega||_p) = \mu_p(p||\omega^-|| + \omega_0) \leq \omega_0$, by Proposition 9.2 (5). $\qquad\square$

Definition 9.9. For $d, n \in \mathbb{N}$, define
   (1) $\sigma : \overline{\mathsf{Seq}^p}_d(n) \hookrightarrow \overline{\mathsf{Seq}^p}_{d+1}(n+1)$ by $\sigma(\omega) := \omega + 1$;
   (2) $\varphi : \overline{\mathsf{Seq}^p}_d(n) \hookrightarrow \overline{\mathsf{Seq}^p}_{d+\mu_p(n)}(pn + \mu_p(n))$ by $\varphi(\omega) := (\mu_p(n), \omega)$.

These operators give a recursive description of $\overline{\mathsf{Seq}^p}(n)$ by the following result:

Proposition 9.10. *For $0 < d \leq n \in \mathbb{N}$,*

$$\overline{\mathsf{Seq}^p}_d(n) = \begin{cases} \sigma\left(\overline{\mathsf{Seq}^p}_{d-1}(n-1)\right) \amalg \varphi\left(\overline{\mathsf{Seq}^p}_{d-\mu_p(n)}\left(\frac{n-\mu_p(n)}{p}\right)\right) & \mu_p(n) = \mu_p\left(\frac{n-\mu_p(n)}{p}\right) \\ \sigma\left(\overline{\mathsf{Seq}^p}_{d-1}(n-1)\right) & \text{otherwise.} \end{cases}$$

Proof. Consider an essential sequence $\omega \in \overline{\mathsf{Seq}^p}_d(n)$, so that $\omega_0 \geq \mu_p(||\omega^-||_p)$. In the case $\omega_0 > \mu_p(||\omega^-||_p)$, the sequence $(\omega_0 - 1, \omega^-)$ is essential and maps under σ to ω.

In the remaining case, $\omega_0 = \mu_p(||\omega^-||_p)$ and $0 \neq \omega^- \in \overline{\mathsf{Seq}^p}_{d-\omega_0}\left(\frac{n-\omega_0}{p}\right)$. Moreover $\varphi(\omega^-) = \omega$, since $\frac{n-\omega_0}{p} = ||\omega^-||_p$ so that $\mu_p\left(\frac{n-\omega_0}{p}\right) = \omega_0$. Thus ω lies in the image of φ; it remains to show that, in this case, $\omega_0 = \mu_p(n)$.

Now $\mu_p(n) = \mu_p(p||\omega^-||_p + \mu_p(||\omega^-||_p)) = \mu_p(f_p(||\omega^-||_p)$ and the latter is $\mu_p(||\omega^-||_p)$ by Proposition 9.2 (7), so that $\omega_0 = \mu_p(n)$ in this case. $\qquad\square$

Remark 9.11.
  (1) The condition $\mu_p(n) = \mu_p\left(\frac{n-\mu_p(n)}{p}\right)$ is equivalent to n lying in the image of f_p.
  (2) See Figure 9.7 for the essential sequences ω with $||\omega||_p \leq 15$ at the prime $p = 2$. The image of φ is highlighted in red.

Further understanding of the behaviour of $\overline{\mathsf{Seq}^p}(n)$ for fixed n is given by the following (noting that $\mathsf{Seq}^p_d(n) = \emptyset$ if $d \not\equiv n \mod (p-1)$).

Proposition 9.12. *Let $d < n \in \mathbb{N}$ be positive integers such that $n - d \equiv 0 \mod (p-1)$. If $\overline{\mathsf{Seq}^p}_d(n) \neq \emptyset$, then $\overline{\mathsf{Seq}^p}_{d+(p-1)}(n) \neq \emptyset$.*

Proof. Consider $\omega \in \overline{\mathsf{Seq}^p}_d(n)$; the hypothesis $d < n$ implies that $\omega^- \neq 0$.

Now, if $l(\omega^-) > 1$ then $|\omega^-| < ||\omega^-||_p$, so that the evident induction on $|-|$ provides an $\omega' \in \overline{\mathsf{Seq}^p}$ such that $||\omega'||_p = ||\omega^-||_p$ and $|\omega'| = |\omega^-| + (p-1)$. It is clear that the sequence (ω_0, ω') is essential.

In the remaining case, ω is of the form (ω_0, ω_1) with $\omega_0 \geq \mu_p(\omega_1)$ and $\omega_1 > 0$. Consider the sequence $(\omega_0 + p, \omega_1 - 1)$. The inequality $\mu_p(\omega_1 - 1) \leq \mu_p(\omega_1) + p - 1$ implies that this is essential, as required. □

9.4. **Minimal essential sequences.**

Definition 9.13. For $n \in \mathbb{N}$ set $\deg^{\mathrm{ess}}_{\min,p}(n) := \inf\{d \in \mathbb{N} | \overline{\mathsf{Seq}^p}_d(n) \neq \emptyset\}$.

Proposition 9.14. *The function $\deg^{\mathrm{ess}}_{\min,p} : \mathbb{N} \to \mathbb{N}$ is determined recursively by $\deg^{\mathrm{ess}}_{\min,p}(0) = 0$ and, for $n > 0$:*

$$\deg^{\mathrm{ess}}_{\min,p}(n) = \begin{cases} \inf\{\deg^{\mathrm{ess}}_{\min,p}(n-1) + 1, \deg^{\mathrm{ess}}_{\min,p}\left(\frac{n-\mu_p(n)}{p}\right) + \mu_p(n)\} & \mu_p(n) = \mu_p\left(\frac{n-\mu_p(n)}{p}\right) \\ \deg^{\mathrm{ess}}_{\min,p}(n-1) + 1 & \text{otherwise.} \end{cases}$$

In particular, $\deg^{\mathrm{ess}}_{\min,p}(n) \leq \deg^{\mathrm{ess}}_{\min,p}(n-1) + 1$.

Moreover, $\deg^{\mathrm{ess}}_{\min,p}(n) \geq [\log_p((p-1)n + 1)]$, with equality when n is of the form $\frac{p^k - 1}{p-1}$.

Proof. The first statement follows from Proposition 9.10.

The lower bound follows from the inequality $||\omega||_p \leq \frac{p^{|\omega|} - 1}{p-1}$ given by Lemma 9.8; the sequence $\omega = (1^k)$ has $|\omega| = k$ and $||\omega||_p = \frac{p^k - 1}{p-1}$ □

This result can be made more precise. For simplicity of exposition, the case $p = 2$ is presented below:

Proposition 9.15. *Let $p = 2$. Then, if $n \in \mathrm{image} f_2$,*

$$\deg^{\mathrm{ess}}_{\min,2}(n) = \deg^{\mathrm{ess}}_{\min,2}\left(\frac{n - \mu_2(n)}{2}\right) + \mu_2(n).$$

Moreover, if $n = f_2(k)$, one of the following two cases occurs:
  (1) *$\mu_2(k+1) = \mu_2(k) - 1$ (so that $f_2(k+1) = n + 1$) and, in this case,*
$$\deg^{\mathrm{ess}}_{\min,2}(n+1) \leq \deg^{\mathrm{ess}}_{\min,2}(n).$$
  (2) *$\mu_2(k+1) = \mu_2(k) + 1$ (so that $f_2(k+1) = n + 3$) and, in this case,*
$$\deg^{\mathrm{ess}}_{\min,2}(n+3) \leq \deg^{\mathrm{ess}}_{\min,2}(n) + 2 < \deg^{\mathrm{ess}}_{\min,2}(n+2) + 1.$$

Proof. The result is proved by induction upon n. The inductive step follows from the more precise results stated in the second part.

Suppose that $n = f_2(k)$, thus $k = \frac{n-\mu_2(n)}{2}$ and $\mu_2(k) = \mu_2(n)$. Moreover, by Proposition 9.2 (2), $\mu_2(k+1) = \mu_2(k) \pm 1$ giving that $f_2(k) \in \{n+1, n+3\}$.

In the first case, since $\deg^{\mathrm{ess}}_{\min,2}(k+1) \leq \deg^{\mathrm{ess}}_{\min,2}(k)+1$ by Proposition 9.14, $\deg^{\mathrm{ess}}_{\min,2}(k+1) + \mu_2(k+1) \leq \deg^{\mathrm{ess}}_{\min,2}(k) + \mu_2(k) = \deg^{\mathrm{ess}}_{\min,2}(n)$, by the inductive hypothesis. In particular, this is less that $\deg^{\mathrm{ess}}_{\min,2}(n)+1$.

The second case is similar. Namely $\deg^{\mathrm{ess}}_{\min,2}(k+1) + \mu_2(k+1) \leq \deg^{\mathrm{ess}}_{\min,2}(k) + \mu_2(k) + 2 = \deg^{\mathrm{ess}}_{\min,2}(n) + 2$, whereas $\deg^{\mathrm{ess}}_{\min,2}(n+2) + 1 = \deg^{\mathrm{ess}}_{\min,2}(n) + 3$.

These establish the inductive step and hence the proof. $\qquad\square$

Proposition 9.2 implies that $\deg^{\mathrm{ess}}_{\min,p}(n) \to \infty$ as $n \to \infty$. However, for $p = 2$, the function is bounded above by a quadratic function in $\log_2$, as shown by the following result:

Proposition 9.16. *Let $p = 2$. Then, for $1 \leq n \in \mathbb{N}$:*

$$\deg^{\mathrm{ess}}_{\min,2}(n) \leq \frac{1}{2}(\log_2(n)+1)(\log_2(n)+6).$$

Hence $\frac{\deg^{\mathrm{ess}}_{\min,2}(n)}{n} \to 0$ as $n \to \infty$.

Proof. The result is proved by a straightforward induction upon n, by applying the logarithmic bound on $\mu_2(n)$ given by Proposition 9.3. $\qquad\square$

9.5. **σ-stability.** For $d \leq n \in \mathbb{N}$, the inclusions σ give:

$$\overline{\mathsf{Seq}^p}_d(n) \hookrightarrow \overline{\mathsf{Seq}^p}_{d+1}(n+1) \hookrightarrow \ldots \hookrightarrow \overline{\mathsf{Seq}^p}_{d+t}(n+t) \hookrightarrow \overline{\mathsf{Seq}^p}_{d+t+1}(n+t+1) \hookrightarrow \ldots.$$

The purpose of this section is to show that this stabilizes.

Definition 9.17. For $D \in \mathbb{N}$, define $\mu_p^{\sup}(D) := \sup\{\mu_p(d) | d \leq D\}$.

Lemma 9.18. *For $d \leq n \in \mathbb{N}$ and $\omega \in \mathsf{Seq}^p_d(n)$, the following inequalities hold:*

$$\omega_0 \geq \frac{pd-n}{p-1} = d - \frac{n-d}{p-1}$$

$$|\omega^-| \leq ||\omega^-||_p \leq \frac{n-d}{p-1}.$$

Proof. By hypothesis, $|\omega| = \omega_0 + |\omega^-| = d$ and $||\omega||_p = \omega_0 + p||\omega^-||_p = n$. Now $|\omega^-| \leq ||\omega^-||_p$, so that $n \geq \omega_0 + p(d - \omega_0)$, which gives the first inequality, which implies the second by using $||\omega^-||_p = \frac{n-\omega_0}{p}$. $\qquad\square$

Proposition 9.19. *For $d \leq n \in \mathbb{N}$, the stabilization map*

$$\sigma : \overline{\mathsf{Seq}^p}_d(n) \hookrightarrow \overline{\mathsf{Seq}^p}_{d+1}(n+1)$$

is an isomorphism if

$$d \geq \frac{n-d}{p-1} + \mu_p^{\sup}\left(\frac{n-d}{p-1}\right).$$

In particular, this holds if $d \geq 2\left(\frac{n-d}{p-1}\right)$.

Proof. By Proposition 9.10, it suffices to show that, for $\omega \in \overline{\mathsf{Seq}^p}_{d+1}(n+1)$, $\omega_0 > \mu_p(||\omega^-||_p)$.

Lemma 9.18 implies that $\omega_0 \geq d + 1 - \frac{n-d}{p-1}$, whereas $||\omega^-||_p \leq \frac{n-d}{p-1}$. By definition of $\mu_p[-]$, it follows that $\mu_p(||\omega^-||_p) \leq \mu_p\left[\frac{n-d}{p-1}\right]$. The hypothesis therefore implies the required strict inequality.

The final statement is given by using the inequality $d \geq \mu_p(d)$ given by Proposition 9.2, which implies that $D \geq \mu_p^{\sup}(D)$. $\qquad\square$

9.6. **Instability reduces to essentials.** By construction, for $d, n \in \mathbb{N}$, there is a surjection:

$$\bigoplus_{\omega \in \mathsf{Seq}^p_d(n)} \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega_i} \right) \twoheadrightarrow \mathfrak{Q}^n_d.$$

The purpose of this section is to show that one can reduce to using the terms indexed by the essential sequences, thus establishing their relevance for the study of $\mathfrak{Q}^n_d$. The ideas used in the proof (at the prime $p = 2$) go back to Singer [Sin91] (cf. [WW18a, Theorem 6.3.12]).

Theorem 9.20. *For $d, n \in \mathbb{N}$ there is a surjection*

$$\bigoplus_{\omega \in \overline{\mathsf{Seq}^p}_d(n)} \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega_i} \right) \twoheadrightarrow \mathfrak{Q}^n_d.$$

Thus $\mathfrak{Q}^n_d = 0$ if $d < \deg^{\mathrm{ess}}_{\min, p}(n)$.

The proof of Theorem 9.20 uses the left lexicographical order $\preceq_\ell$ (restricted to Seq^p_d) by exploiting the following Lemma:

Lemma 9.21. *Consider $p_d S^* / p_{d-1} S^* \cong \bigoplus_{\omega \in \mathsf{Seq}_d} \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega_i} \right)$ as an $\mathscr{A}(p)$-module.*
If $\theta \in \mathscr{A}(p)$ has positive degree and $\omega \in \mathsf{Seq}_d$, then

$$\theta \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega_i} \right) \subset \bigoplus_{\substack{\lambda \in \mathsf{Seq}_d \\ \lambda \prec_\ell \omega \\ ||\lambda||_p = ||\omega||_p + |\theta|}} \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \lambda_i} \right).$$

Proof. This is a straightforward consequence of the explicit form of the $\mathscr{A}(p)$-action as given in Remark 7.10. $\qquad\square$

The other ingredient is the *conjugation trick*, which is a basic tool in studying the functor $\mathbb{F}_p \otimes_{\mathscr{A}(p)} -$.

Proposition 9.22. *For $\mathscr{A}(p)$-modules M, N, the following two composites are equal*

$$\mathscr{A}(p) \otimes M \otimes N \rightrightarrows M \otimes N \twoheadrightarrow \mathbb{F}_p \otimes_{\mathscr{A}(p)} (M \otimes N)$$

for the maps $\theta \otimes m \otimes n \mapsto (\theta m) \otimes n$ and $\theta \otimes m \otimes n \mapsto m \otimes \chi(\theta) n$, where $\theta \in \mathscr{A}(p)$, $m \in M$ and $n \in N$, where $M \otimes N \twoheadrightarrow \mathbb{F}_p \otimes_{\mathscr{A}(p)} (M \otimes N)$ is the canonical surjection to the $\mathscr{A}$-module indecomposables.

Proof. This is a straightforward generalization of [WW18a, Proposition 2.5.1], which relies only upon the formal properties of the conjugation associated to a connected, graded cocommutative bialgebra. (Note that the conventions used here imply that there are no Koszul signs.) $\qquad\square$

Proof of Theorem 9.20. The result is proved by descending induction on $\lambda \in \mathsf{Seq}_d(n)$ with respect to $\preceq_\ell$, showing that

$$\bigoplus_{\omega \in \overline{\mathsf{Seq}^p}_d(n) \cup \{\mu \in \mathsf{Seq}^p_d(n) | \mu \preceq_\ell \lambda\}} \left(\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega_i} \right) \twoheadrightarrow \mathfrak{Q}^n_d$$

is surjective, with initial case λ the maximal element of $\mathsf{Seq}^p_d(n)$.

For the inductive step, passing from λ to its predecessor for λ^- (if λ is minimal in $\mathsf{Seq}^p_d(n)$ the argument is modified accordingly), if λ is essential, then the indexing set is unchanged. Otherwise there are two mutually exclusive cases:

(1) $\lambda_0 < \mu(||\lambda^-||_p)$;
(2) λ^- is not essential and $\lambda_0 \geq \mu(||\lambda^-||_p)$.

Suppose that $\lambda_0 < \mu_p(||\lambda||_p)$, then the conjugation trick is used to show that $\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \lambda_i}$ maps to zero in $\mathfrak{Q}^n_d$, as follows:

Using the multiplicative structure of $p_\bullet S^* / p_{\bullet-1} S^*$, an element of $\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \lambda_i}$ can be written as a linear combination of elements of the form $x P^{|y|} y$, where $x \in \overline{S}^{\lambda_0}$ and $|y| = ||\lambda^-||_p$. Now, Proposition 9.5 implies that the conjugate operation $\chi P^{|y|}$ acts trivially on x, since $|x| = |\lambda_0|$ and $||\lambda^-||_p = |y|$, by hypothesis. The conclusion follows by applying Proposition 9.22.

In the second case, an element of $\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \lambda_i}$ can be written as a linear combination of elements of the form $x \Phi y$, considering y as lying in the term indexed by λ^-, which is non essential, by hypothesis. Moreover, $||\lambda^-||_p < ||\lambda||_p$.

By an evident induction upon n, there is an expression
$$y = \sum_s P^{n_s} y_s + \sum_t z_t,$$
where $P^{n_s} \in \mathscr{A}(p)$ are operations of positive degree and each z_t lies in a term $\bigotimes_{i \in \mathbb{N}} (\Phi^i \overline{S})^{p^i \omega(t)_i}$ indexed by an essential sequence $\omega(t)$ with $||\omega(t)||_p = ||\lambda^-||_p$.

By Lemma 9.21, modulo terms of lower lexicographical order:
$$x \Phi(P^{n_s} y_s) \equiv P^{p n_s}(x \Phi y_s)$$
hence, modulo lower terms for $\prec_\ell$, $x\Phi(P^{n_s} y_s)$ is $\mathscr{A}$-decomposable.

Now consider a term $x z_t$ that corresponds to the sequence $(\lambda_0, \omega(t))$. If $\lambda_0 < \mu_p(||\omega(t)||_p)$ then the first step applies to show that $x z_t$ maps to zero in $\mathfrak{Q}^n_d$. Otherwise $(\lambda_0, \omega(t))$ is essential, i.e. $x z_t$ is indexed by an essential sequence.

This completes the inductive step and hence the proof. $\qquad\square$

Corollary 9.23. *Let $d \leq n \in \mathbb{N}$ be integers. If $d < \deg^{\mathrm{ess}}_{\min, p}(n)$, then $Q^n[d] = 0$.*

Proof. Using the definition of $\deg^{\mathrm{ess}}_{\min, p}(n)$, this follows from Theorem 9.20, since $Q^n[d]$ has a finite filtration with filtration quotients of the from Q^n_e ($e \leq d$) and Q^n_e is a quotient of $\mathfrak{Q}^n_e$. $\qquad\square$

Remark 9.24. As well as giving the connectivity result Corollary 9.23, Theorem 9.20 reduces the calculational and conceptual complexity significantly. Moreover, the σ-stabilization property of Proposition 9.19 hints at a regularity to the structure of Q^n. (The reader is encouraged to consult Figure 9.7 and consider this structure.)

9.7. **Low-dimensional calculations.** The following result gives the low-dimensional behaviour of the functors $\mathfrak{Q}^n_d \in \mathrm{Ob}\, \mathscr{P}_d$.

Proposition 9.25. *Let $\Bbbk = \mathbb{F}_2$. The non-zero functors $\mathfrak{Q}^n_d$ for $n \leq 8$ are:*

d	$\mathfrak{Q}^n_d$							
8								Λ^8
7							Λ^7	$L_{(2,1^5)}$
6						Λ^6	$L_{(2,1^4)} \cdot \Lambda^6$	$L_{(2^2,1^2)} \cdot \Lambda^6 \cdot L_{(2,1^4)}$
5					Λ^5	$L_{(2,1^3)}$	$L_{(2^2,1)} \cdot \Lambda^5$	$L_{(2,1^3)}$
4				Λ^4	$L_{(2,1^2)} \cdot \Lambda^4$	$L_{(2^2)} \cdot L_{(2,1^2)}$	Λ^4	$\mathfrak{Q}^8_4$
3			Λ^3	$L_{(2,1)}$			$L_{(3)} \cdot \Lambda^3$	
2		Λ^2	$L_{(2)} \cdot \Lambda^2$					
1	Λ^1							
n	1	2	3	4	5	6	7	8

in which the highlighted terms are (contain, in the case of $\mathfrak{Q}^8_4$) the simples that are indexed by partitions that are not 2-restricted.

Moreover, in $G_0(\mathscr{P}_4)$:

$$[\mathfrak{Q}_4^8] = [L_{(3,1)}] + [L_{(2^2)}] + [\Lambda^4] + [L_{(2,1^2)}].$$

Proof. (Indications.) Theorem 9.20 implies that one only has contributions from essential sequences, and these can be read off from Table 9.7. The result is then proved by standard methods. In particular, the calculations of $\mathfrak{Q}_n^n$, $\mathfrak{Q}_{n-1}^n$ and $\mathfrak{Q}_{n-2}^n$ fall into periodic families that are well understood (see Section 10 below).

This leaves the cases of $\mathfrak{Q}_4^7$, $\mathfrak{Q}_3^7$ and $\mathfrak{Q}_5^8$, $\mathfrak{Q}_4^8$ which are straightforward (but illustrate the salient features nicely). The reader is encouraged to provide the details for themselves. $\square$

Remark 9.26. The composition factors highlighted in red play a rôle in considering the associated functors Q_d^n as follows.

The forgetful $\mathscr{O} : \mathscr{P}_d \to \mathscr{F}_d$ sends a simple object L_λ to a simple object of Eilenberg-MacLane polynomial degree exactly d if and only if λ is 2-restricted, otherwise $\mathscr{O}L_\lambda$ has polynomial degree less that d and, in general, is not simple.

Hence Proposition 8.12 and its corollaries implies that it is the presence of composition factors indexed by non 2-restricted partitions that can lead to the surjection $\mathscr{O}\mathfrak{Q}_d^n \to Q_d^n$ failing to be an isomorphism.

Corollary 9.27. *Let* $\Bbbk = \mathbb{F}_2$. *For* $n \le 8$ *the canonical morphism* $\mathscr{O}(\mathfrak{Q}_d^n) \twoheadrightarrow Q_d^n$ *is an isomorphism except in the case* $(n, d) = (7, 3)$, *when there is a short exact sequence*

$$0 \to \Lambda^2 \to \mathscr{O}\mathfrak{Q}_3^7 \to Q_3^7 \to 0.$$

Proof. Proposition 8.12 together with inspection of the result of Proposition 9.25 show that the only case for which the surjection is potentially not an isomorphism is $(n, d) = (7, 3)$.

Now, $\mathfrak{Q}_3^7 \cong S^3$ and $\mathscr{O}\mathfrak{Q}_3^7$ is a uniserial functor with socle series:

$$\Lambda^2, \Lambda^1, \Lambda^2, \Lambda^3.$$

Hence, to prove the result, it suffices to show that Λ^1 is in the socle of Q_3^7; the factor Λ^2 corresponds to the composition factor $L_{(2^2)}$ of $\mathfrak{Q}_4^6$. Moreover, it then suffices to show that Λ^1 is in the socle of Q^7. This is a case of Proposition 9.28 below. $\square$

Proposition 9.28. *Let* $\Bbbk = \mathbb{F}_2$. *For* $n \in \mathbb{N}$, *the functor* Q^n *has a composition factor* Λ^1 *if and only if* n *is of the form* $2^a - 1$, *where* a *is a positive integer. Moreover,* $\Lambda^1 \subset \mathrm{soc}\, Q^{2^a-1}$ *and is the unique composition factor of* Λ^1 *in* Q^{2^a-1}.

Proof. For $n > 0$, the symmetric power functor S^n has a unique composition factor of Λ^1, as follows from $S^n(\mathbb{F}_2) \cong \mathbb{F}_2$ and $S^n(0) = 0$, whereas $S^0 \cong \mathbb{F}_2$ has no such factor. The 'Peterson conjecture' [WW18a, Theorem 2.5.5] implies that Q^n has a composition factor of Λ^1 if and only if n is of the form $2^a - 1$ for $0 < a \in \mathbb{N}$.

It remains to show that the composition factor Λ^1 lies in the socle of Q^{2^a-1}. This is proved by using the action of Sq^1 on S^*. This induces a non-trivial natural transformation $S^n \to S^{n+1}$ (for $0 < n \in \mathbb{N}$) and, since $Sq^1Sq^1 = 0$, this gives a complex $S^1 \to S^2 \to S^3 \to \dots$ which is acyclic. (Since S^* is an exponential functor, by the Künneth theorem, acyclicity reduces to the fact that the algebra $\mathbb{F}_2[x]$ equipped with differential $dx = x^2$ (extended as a derivation) has homology concentrated in degree zero.)

It follows that $S^{2^a-1}/\mathrm{Im}Sq^1$ embeds in S^{2^a} as the kernel of $Sq^1 : S^{2^a} \to S^{2^a+1}$. The iterated Frobenius embeds Λ^1 in this kernel, so Λ^1 lies in the socle of $S^{2^a-1}/\mathrm{Im}Sq^1$. The result follows, since Q^{2^a-1} is a quotient of $S^{2^a-1}/\mathrm{Im}Sq^1$. $\square$

FIGURE 1. Essential sequences ω for $||\omega|| \leq 15$ at the prime $p = 2$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mu(n)$	0	1	2	1	2	3	2	1	2	3	2	3	4	3	2	1
																15
															14	(13,1)
														13	(12,1)	(11,2)
													12	(11,1)	(10,2)	(9,3)
												11	(10,1)	(9,2)	(8,3)	(9,1,1),(7,4)
											10	(9,1)	(8,2)	(7,3)	(8,1,1),(6,4)	(7,2,1),(5,5)
										9	(8,1)	(7,2)	(6,3)	(7,1,1),(5,4)	(6,2,1),(4,5)	(5,3,1),(3,6)
									8	(7,1)	(6,2)	(5,3)	(6,1,1),(4,4)	(5,2,1),(3,5)	(4,3,1),(2,6)	(3,4,1),(1,7)
								7	(6,1)	(5,2)	(4,3)	(5,1,1),(3,4)	(4,2,1)	(3,3,1)	(2,4,1)	(3,2,2),(1,5,1)
							6	(5,1)	(4,2)	(3,3)	(4,1,1),(2,4)	(3,2,1)			(2,2,2)	(1,3,2)
						5	(4,1)	(3,2)	(2,3)	(3,1,1)	(2,2,1)					(1,1,3)
					4	(3,1)	(2,2)	(1,3)	(2,1,1)							(1,1,1,1)
				3	(2,1)		(1,1,1)									
			2	(1,1)												
		1														
	0															

The generators under σ are indicated in red; these are given by the image of φ (except for 0).

10. REPRESENTATION STABILITY FOR Q^*

The aim of this section is to use representation stability for the categories $\mathscr{P}_d^{\geq d-t}$ to explain the regularity (or rather periodicity) in the structure of Q^* that was hinted at in Remark 9.24. This requires that, throughout this section, $\Bbbk$ be taken to be the prime field $\mathbb{F}_2$.

10.1. Representation stability at the prime 2.

The following result summarizes basic connectivity results, obtained by applying the inequality for ω_0 given by Lemma 9.18.

Lemma 10.1. *For* $a, d, n \in \mathbb{N}$ *with* $d \leq n$.

$$
\begin{aligned}
\Lambda^a &\in & \mathsf{Ob}\ \mathscr{P}_a^{\geq a} \\
\mathfrak{Q}_d^n,\ p_d S^n/p_{d-1}S^n &\in & \mathsf{Ob}\ \mathscr{P}_d^{\geq d-(n-d)} \\
Q^n/Q^n[d-1],\ S^n/p_{d-1}S^n &\in & \mathsf{Ob}\ \mathscr{F}_n^{\geq d-(n-d)} \\
Q^n &\in & \mathsf{Ob}\ \mathscr{F}_n^{\geq d-(n-\deg_{\min,2}^{\mathrm{ess}}(n))}.
\end{aligned}
$$

Theorem 10.2. *For natural numbers* $d \leq n$ *such that* $(d, n-d)$ *is stable, if* $e > d \in \mathbb{N}$ *such that* $e \equiv d \mod 2^{\lceil \log_2(n-d) \rceil}$, *under the equivalence of categories of Theorem 5.10,*

$$
\begin{aligned}
\mathrm{Per}_{d,e} : \mathscr{P}_d^{\geq d-(n-d)} &\overset{\cong}{\longrightarrow} & \mathscr{P}_e^{\geq e-(n-d)} \\
\mathfrak{Q}_d^n &\mapsto & \mathfrak{Q}_e^{n+e-d}.
\end{aligned}
$$

Proof. This is a straightforward application of Theorem 5.10. It suffices to observe that the diagram giving rise to $\mathfrak{Q}_d^n$ as a colimit lies entirely within the category $\mathscr{P}_d^{\geq d-(n-d)}$ and corresponds under the equivalence of categories to the respective diagram for $\mathfrak{Q}_e^{n+e-d}$. $\square$

Theorem 10.2 together with Theorem 5.21 imply:

Corollary 10.3. *Let* $d, e, n \in \mathbb{N}$ *satisfy the hypotheses of Theorem 10.2 and, in addition, suppose that the pair* $(d, n-d)$ *is strictly stable (i.e.* $d > 2(n-d)$*). Then, under the isomorphism* $\bullet 1^{e-d}$ *between Grothendieck groups:*

$$
\begin{aligned}
\bullet 1^{e-d} : G_0(\mathscr{F}_d^{\geq d-(n-d)}) &\overset{\cong}{\to} & G_0(\mathscr{F}_d^{\geq e-(n-d)}) \\
[\mathfrak{Q}_d^n] &\mapsto & [\mathfrak{Q}_e^{n+e-d}].
\end{aligned}
$$

Remark 10.4. Corollary 10.3 has to be stated for Grothendieck groups, since the categories $\mathscr{F}_d^{\geq d-(n-d)}$ and $\mathscr{F}_d^{\geq e-(n-d)}$ are not currently known to be equivalent, whereas the Grothendieck groups are isomorphic.

10.2. Conjectural periodicity for $Q^*/Q^*[d-1]$.

For $0 < d \leq n \in \mathbb{N}$, Lemma 10.1 shows that $Q^n/Q^n[d-1]$ lies in $\mathscr{F}_n^{\geq d-(n-d)}$. Similarly, if $e > d$, $Q^{n+e-d}/Q^{n+e-d}[e-1]$ lies in $\mathscr{F}_{n+e-d}^{\geq e-(n-d)}$. Corollary 10.3 suggests the following:

Conjecture 10.5. Suppose that $d, e, n \in \mathbb{N}$ satisfy the hypotheses of Corollary 10.3, then

$$
\begin{aligned}
\bullet 1^{e-d} : G_0(\mathscr{F}_n^{\geq d-(n-d)}) &\overset{\cong}{\to} & G_0(\mathscr{F}_{n+e-d}^{\geq e-(n-d)}) \\
\big[Q^n/Q^n[d-1]\big] &\mapsto & \big[Q^{n+e-d}/Q^{n+e-d}[e-1]\big].
\end{aligned}
$$

Remark 10.6. Corollary 10.3 implies that Conjecture 10.5 holds if $Q_i^n \cong \mathfrak{Q}_i^n$ for all $i > d$ and $Q_j^{n+e-d} \cong \mathfrak{Q}_j^{n+e-d}$ for all $e > d$. Corollary 9.27 shows that $\mathfrak{Q}_3^7 \not\cong Q_3^7$; however, this does not satisfy the stability hypothesis.

In particular, Proposition 8.16 implies the following (corresponding to a zone in which the representation theory is well understood):

Proposition 10.7. *Conjecture 10.5 holds if $n - d \leq 5$.*

It is possible to strengthen the conjecture, bearing in mind Remark 10.4, as follows:

Conjecture 10.8. Suppose that $d, e, n \in \mathbb{N}$ satisfy the hypotheses of Corollary 10.3, then the lattices of subobjects of $Q^n/Q^n[d-1]$ and $Q^{n+e-d}/Q^{n+e-d}[e-1]$ are isomorphic, compatibly with the identification of Conjecture 10.5.

10.3. **The stable zone.** Recall that Proposition 9.19 implies that the operation σ on essential sequences is a bijection at $p = 2$ if the pair $(d, n-d)$ is stable; this is an avatar of the stabilization property and also gives the stability condition $d \geq \frac{2n}{3}$.

Lemma 10.9. *For $t, m \in \mathbb{N}$,*

  (1) $(m + 2t, t)$ is stable;
  (2) $(m + t, t)$ is stable if and only if $m \geq t$ (say that (m, t) is semi-stable*).*

Remark 10.10. The minimal d for which $\mathfrak{Q}_d^n$ can be non-zero is $d = \deg_{\min,2}^{\mathrm{ess}}(n)$. Unfortunately, the corresponding pair $(d, n - \deg_{\min,2}^{\mathrm{ess}}(n))$ is not in general stable or even semi-stable.

Indeed, $(d, n - \deg_{\min,2}^{\mathrm{ess}}(n))$ is semi-stable if and only if $\deg_{\min,2}^{\mathrm{ess}}(n) \geq \frac{n}{2}$. This fails for $n = 14$, since $\deg_{\min,2}^{\mathrm{ess}}(14) = 6$. In fact, Proposition 9.16 shows that the semi-stable condition is almost never satisfied.

Remark 10.11. Conjecture 10.8 suggests the following strategy for studying the structure of Q^n, for $n \in \mathbb{N}$ in general (i.e. without supposing that $(d, n - \deg_{\min,2}^{\mathrm{ess}}(n))$ is either stable or semi-stable.)

  (1) Observe that $Q^n \cong Q^n/Q^n[\deg_{\min,2}^{\mathrm{ess}}(n) - 1]$; hence the expected periodicity is $2^{\pi(n)}$, where $\pi(n) := \lceil \log_2(n - \deg_{\min,2}^{\mathrm{ess}}(n)) \rceil$. Observe that $\pi(n) \leq \tilde{\pi}(n) := \lceil \log_2(n-1) \rceil$, as used in the introduction.
  (2) To pass into the (conjecturally) stable range, by Lemma 10.9, consider
$$n_{\mathrm{stab}} := n + 2^{\pi(n)+1}.$$
  (In the introduction, $\tilde{\pi}(n)$ was used to simplify the exposition.) Here one considers the quotient (in the stable range):
$$Q^{n_{\mathrm{stab}}}/Q^{n_{\mathrm{stab}}}[\deg_{\min,2}^{\mathrm{ess}}(n) + 2^{\pi(n)+1} - 1].$$
  (3) Use the division functor $(-; \Lambda^{2^{\pi(n)}} \otimes \Lambda^{2^{\pi(n)}})$ (which corresponds to the division functor $(-; \Lambda^{2^{\pi(n)}})$ applied twice, by Proposition 2.14), together with Proposition 8.5 to analyse the structure of Q^n.

Remark 10.12. Proposition 9.16 considers the asymptotic behaviour of $\deg_{\min,2}^{\mathrm{ess}}(n)$. In particular, it shows that
$$\lim_{n \to 0} \frac{\deg_{\min,2}^{\mathrm{ess}}(n)}{n} = 0.$$
Hence $\deg_{\min,2}^{\mathrm{ess}}(n)$ can be neglected asymptotically.

Thus, for large n, one should consider that the above procedure breaks splits the induced filtration of Q^n into the 'stable' half $Q^n[\lceil \frac{2n}{3} \rceil]$ and the 'unstable' part $Q^n[\lceil \frac{2n}{3} \rceil]$.

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