TIME-CHANGES PRESERVING ZETA FUNCTIONS

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To Graham Everest (1957–2010), in memoriam

Abstract. We associate to any dynamical system with finitely many periodic orbits of each length a collection of possible time-changes of the sequence of periodic point counts that preserve the property of counting periodic points. Intersecting over all dynamical systems gives a monoid of time-changes that have this property for all such systems. We show that the only polynomials lying in this 'universally good' monoid are the monomials, and that this monoid is uncountable. Examples give some insight into how the structure of the collection of maps varies for different dynamical systems.

1. Introduction

We are concerned with operations (time-changes) that act on integer sequences preserving the following property. An integer sequence (a_n) is called *realisable* if there is a map $T\colon X \to X$ (a 'dynamical system') with the property that

$$a_n = \mathrm{Fix}_T(n) = |\{x \in X \mid T^n x = x\}|$$

for all $n \geqslant 1$. This defines the same class of integer sequences if we require T to be a homeomorphism and X a compact metric space, or indeed if T is required to be a C^∞ diffeomorphism of the 2-torus, by work of Puri and the last author [8] or Windsor [10] respectively. Certain operations on integer sequences preserve this property for trivial reasons: if (a_n) is 'realised' by (X, T) and (b_n) by (Y, S), then $(a_n b_n)$ is realised by the Cartesian product $T \times S\colon X \times Y \to X \times Y$ and $(a_n + b_n)$ is realised by the disjoint union $T \sqcup S\colon X \sqcup Y \to X \sqcup Y$, where $T \sqcup S$ is defined to be T on X and S on Y. All these statements may be expressed in terms of the dynamical zeta function of $T\colon X \to X$, formally defined as $\zeta_T(z) = \exp\left(\sum_{n\geqslant 1} \mathrm{Fix}_T(n)\frac{z^n}{n}\right)$. Thus, for example, the space of zeta functions is closed under multiplication because the sum of two realisable sequences is realisable, and is closed under a Hadamard-like formal multiplication because the product is. We refer to work of Carnevale and Voll [1] or Pakapongpun and the last author [6, 7] for more on the combinatorial and analytic properties of these 'functorial' operations.

A different kind of operation on sequences (or on zeta functions) is a 'time-change': a function $h\colon \mathbb{N} \to \mathbb{N}$ defines an operation on integer sequences by sending (a_n) to $(a_{h(n)})$. This may be thought of as replacing the usual sequence of iterates $T, T^2, T^3, \dots$ with the time-changed sequence $T^{h(1)}, T^{h(2)}, T^{h(3)}, \dots$.

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Definition 1. For a map $T\colon X \to X$ with $\mathrm{Fix}_T(n) < \infty$ for all $n \geqslant 1$, define

$$\mathscr{P}(X,T) = \{h\colon \mathbb{N} \to \mathbb{N} \mid \big(\mathrm{Fix}_T(h(n))\big) \text{ is a realisable sequence}\}$$

to be the set of good time-changes for (X,T). Also define

$$\mathscr{P} = \bigcap_{\{(X,T)\}} \mathscr{P}(X,T)$$

to be the monoid of universally good time-changes, where the intersection is taken over all such maps.

It is not obvious that any non-trivial maps could have this property, but the results below show that many do. Clearly the identity $h(n) = n$ has this property, and if functions h_1, h_2 lie in $\mathscr{P}$, then their composition $h_1 \circ h_2$ does, because by definition if (a_n) is a realisable sequence then $(a_{h_2(n)})$ is also realisable, and so $(a_{h_1(h_2(n))})$ is too. Thus $\mathscr{P}$ is a monoid inside the monoid of all maps $\mathbb{N} \to \mathbb{N}$ under composition. Here we prove two results about the structure of $\mathscr{P}$.

Theorem 2. *A polynomial lies in $\mathscr{P}$ if and only if it is a monomial.*

We illustrate what is going on in Theorem 2 via some simple examples.

Example 3. (a) Let T denote the 'golden mean' shift, so $\mathrm{Fix}_T(n)$ is the nth Lucas number and $\zeta_T(z) = \frac{1}{1-z-z^2}$ (when we invoke a specific dynamical system, an adequate reference is [2, Ch. 11]). The Cartesian square $T \times T$ is of course also a shift of finite type, and a calculation shows that $\zeta_{T\times T}(z) = \frac{1}{(1+z)(1-2z-2z^2+z^3)}$. The time-change obtained by sampling along the squares, in contrast, is a map with periodic point count $(1, 7, 76, 2207, \dots)$. Theorem 2 asserts in part that there is a smooth map S with this periodic point data. Such a map clearly cannot be conjugate to a shift of finite type, because $\limsup_{n\to\infty} \frac{1}{n^2} \log \mathrm{Fix}_S(n) = \frac{1+\sqrt{5}}{2} > 0$.
(b) In the reverse direction, Theorem 2 says that there must be *some* map with the property that time-changing by sampling the periodic point counts along the polynomial $n^2 + 1$ produces an integer sequence which cannot be the periodic point count of *any* map. Calculations suggest that the golden mean shift T has the property that the sequence $(\mathrm{Fix}_T(n^2 + 1))$ *does* count periodic points for some map (though the sequence $(\mathrm{Fix}_T(n^2 + n))$ does *not*). However (for example), there is a map U with $\mathrm{Fix}_U(n) = \sigma(n)$ (the sum of divisors of n), since it may be built up simply as the union of one orbit of length k for every $k \in \mathbb{N}$. Time-changing this map along the polynomial $n^2 + 1$ gives the sequence $(3, 6, 18, 18, 42, \dots)$ which cannot count the periodic points of any map, as such a map would need to have $\frac{6-3}{2}$ closed orbits of length 2.
(c) A Lehmer–Pierce sequence, with nth term $|\det(A^n - I)|$ for some integer matrix A, counts periodic points for an ergodic toral endomorphism (if it is positive for all $n \geqslant 1$). Time-changing it along the squares then gives a sequence that counts periodic points for some map, and this sequence has a characteristic quadratic-exponential growth rate, resembling a 'bilinear' or 'elliptic' divisibility sequence. However, it has fundamentally different arithmetic properties and so cannot be an elliptic sequence by work of Luca and the last author [4].

Theorem 2 suggests that $\mathscr{P}$ is (unsurprisingly) small, but we also use work of the second author to show that there are many other maps in it, resulting in the following.

Theorem 4. *The monoid $\mathscr{P}$ is uncountable.*

2. Proofs of Theorem 2

First we recall from [8] that an integer sequence (a_n) is realisable if and only if

$$n\big|(\mu * a)_n = \sum_{d\,|n} \mu(n/d)a_d = \sum_{d\,|n} \mu(d)a_{n/d} \geqslant 0 \tag{1}$$

(that is, $(\mu * a)(n)$ is non-negative and divisible by n) for all $n \geqslant 1$, (where μ denotes the Möbius function and $*$ denotes Dirichlet convolution). This is because we have $a_n = \mathrm{Fix}_T(n)$ for all $n \geqslant 1$ if and only if $\mathrm{Orb}_T(n) = \frac{1}{n}(\mu * a)(n)$ is the number of closed orbits of length n under T for all $n \geqslant 1$.

Proof of 'if' in Theorem 2: monomials preserve realisability. We follow the method of the thesis [5] of the second author. Assume that $h(n) = cn^k$ for some $c \in \mathbb{N}$ and $k \in \mathbb{N}_0$. If $k = 0$ then the result is clear, as the constant sequence $(a_c, a_c, a_c, \dots)$ is realised by the space comprising a_c points all fixed by a map. If (a_n) is realised by (X, T) then (a_{cn}) is realised by (X, T^c) for any $c \in \mathbb{N}$, so it is enough to consider the case $h(n) = n^k$ for some $k \geqslant 1$. Assume therefore that (a_n) is realisable — which for this argument we think of as satisfying (1) rather than in terms of maps — and write $b_n = a_{n^k}$ for $n \geqslant 1$. We wish to show property (1) for the sequence (b_n). Fix $n \in \mathbb{N}$, and let

$$n = p_1^{n_1} \cdots p_r^{n_r}$$

be its prime decomposition, with $n_j \geqslant 1$ for $j = 1, \dots, r$. Then

$$(\mu * b)_n = a_{n^k} - \sum_{p_i} a_{n^k/p_i^k} + \sum_{p_i,p_j} a_{n^k/p_i^k p_j^k} - \cdots + (-1)^r a_{n^k/p_1^k \cdots p_r^{k_r}} \tag{2}$$

where $p_i, p_j, \dots$ are distinct members of $\{p_1, \dots, p_r\}$. Let

$$\delta = n^k/p_1^{k-1} \cdots p_r^{k-1},$$

so in particular $n\big|\delta$. Let

$$e = \sum_{\substack{m\,|n^k \\ \delta\,|m}} \sum_{d\,|m} \mu(m/d)a_d. \tag{3}$$

Since (a_n) is realisable, we have by (1) that

$$m\big|\sum_{d\,|m} \mu(m/d)a_d \geqslant 0,$$

so in particular $e \geqslant 0$ and $n\big|e$. Thus it is enough to show that $e = (\mu * b)_n$. Let $m\big|n^k$ with $\delta\big|m$, so that we may write

$$m = p_1^{k(n_1-1)+j_1} \cdots p_r^{k(n_r-1)+j_r} \tag{4}$$

with $1 \leqslant j_1, \dots, j_r \leqslant k$. Thus by (3) we have

$$e = \sum_{j_1=1}^{k} \cdots \sum_{j_r=1}^{k} \sum_{d\,|m} \mu(d)a_{m/d}$$

with m given by (4). Let

$$m_1 = m/p_1^{k(n_1-1)+j_1} = p_2^{k(n_2-1)+j_2} \cdots p_r^{k(n_r-1)+j_r}. \tag{5}$$

Then we have

$$\sum_{d\,|m}\mu(d)a_{m/d}=\sum_{d\,|m_1}\mu(d)\bigl(a_{m/d}-a_{m/p_1 d}\bigr).$$

Thus, because m_1 is independent of j_1,

$$\sum_{j_1=1}^{k}\sum_{d\,|m}\mu(d)a_{m/d}=\sum_{d\,|m_1}\sum_{j_1=1}^{k}\mu(d)\bigl(a_{m/d}-a_{m/p_1 d}\bigr)$$

and hence

$$\sum_{j_1=1}^{k}\sum_{d\,|m}\mu(d)a_{m/d}=\sum_{d\,|m_1}\mu(d)\bigl(a_{p_1^{kn_1}m_1/d}-a_{p_1^{k(n_1-1)}m_1/d}\bigr).$$

It follows from (3) that

$$e=\sum_{j_2=1}^{k}\cdots\sum_{j_r=1}^{k}\sum_{d\,|m_1}\mu(d)\bigl(a_{p_1^{kn_1}m_1/d}-a_{p_1^{kn_1}m_1/p_1^k d}\bigr),$$

where m_1 is given by (5). The same procedure may be repeated, first setting

$$m_2=m_1/p_2^{k(n_2-1)+j_2},$$

to obtain $e=e_1-e_2$, where

$$e_1=\sum_{j_3=1}^{k}\cdots\sum_{j_r=1}^{k}\sum_{d\,|m_2}\mu(d)\bigl(a_{p_1^{kn_1}p_2^{kn_2}m_2/d}-a_{p_1^{kn_1}p_2^{kn_2}m_2/p_2^k d}\bigr)$$

and

$$e_2=\sum_{j_3=1}^{k}\cdots\sum_{j_r=1}^{k}\sum_{d\,|m_2}\mu(d)\bigl(a_{p_1^{kn_1}p_2^{kn_2}m_2/p_1^k d}-a_{p_1^{kn_1}p_2^{kn_2}m_2/p_1^k p_2^k d}\bigr).$$

Continuing inductively shows that each expression obtained matches up with a term in (2), as required. □

Proof of 'only if' in Theorem 2: only monomials preserve realisability. This argument proceeds rather differently, because we are free to construct dynamical systems with convenient properties to constrain what the polynomial can be. So assume that

$$h(n)=c_k+c_{k-1}n+c_{k-2}n^2+\cdots+c_0n^k$$

is a polynomial in $\mathscr{P}$ with $c_0\neq 0$, $k\geqslant 1$, and $h(\mathbb{N})\subset\mathbb{N}$. For completeness we recall the following well-known result.

Lemma 5. *The coefficients of h are rational, and the set of primes dividing some $h(n)$ with $n\in\mathbb{N}$ is infinite.*

Proof. We have

$$\begin{pmatrix}h(1)\\h(2)\\h(3)\\\vdots\\h(k+1)\end{pmatrix}=\begin{pmatrix}1&1&1&\cdots&1\\1&2&4&\cdots&2^k\\1&3&9&\cdots&3^k\\\vdots&&&&\\1&(k+1)&(k+1)^2&\cdots&(k+1)^k\end{pmatrix}\begin{pmatrix}c_k\\c_{k-1}\\c_{k-2}\\\vdots\\c_0\end{pmatrix},$$

and the determinant $\prod_{1\leqslant i<j\leqslant k+1}(j-i)$ of this matrix (a so-called 'Vandermonde' determinant, an instance of Stigler's law [11]) is non-zero, so the coefficients are all rational.

Turning to the prime divisors of the values of h, if $c_k=0$ the claim is clear, and if $k=1$ we may for example write c_1+c_0n as $\gcd(c_1,c_0)\big(c_1'+c_0'n\big)$ with $\gcd(c_1'c_0')=1$ to see this, so assume that $c_k\neq 0$ and $k>1$. Then we may write $h(n)=np(n)+c_k$ for some polynomial p of positive degree. We may not have $p(\mathbb{N})\subset\mathbb{N}$ of course, but h (and hence p) certainly has rational coefficients. Then we have

$$\frac{m!c_k^2p(m!c_k^2)+c_k}{c_k}=m!c_kp(m!c_k^2)+1=\frac{h(m!c_k^2)}{c_k}.$$

If m is large then $p(m!c_k^2)$ is an integer because p has rational coefficients and c_k is rational, so $h(m!c_k^2)$ must be divisible by some prime greater than m. $\square$

Using Lemma 5, we let q be a very large prime dividing some value of h, let n_0 be the smallest value of n such that $q\big|h(n)$, and let (X,T) consist of a single orbit of length q. (Looking further ahead, it is here that we are failing to solve question (4) from Section 5, in that we choose the system using information from the candidate polynomial.) Then

$$a_n=\mathrm{Fix}_T(n)=\begin{cases}0 & \text{if } q\not|\, n;\\ q & \text{if } q\big|n.\end{cases} \tag{6}$$

Thus $(a_{h(n)})$ is a realisable sequence that only takes on the values 0 and q. Since q is prime, we have

$$a_{h(1)}\equiv a_{h(q)}\pmod{q}$$

by (1). Since $(a_{h(n)})$ only takes the values 0 and q by construction, we deduce that n_0 is the smallest n such that $a_{h(n)}=q$. Thus the sequence $(a_{h(n)})$ starts

$$(a_{h(n)})=(0,\dots,0,q,\dots) \tag{7}$$

with the first q in the $h(n_0)$th place. Now $(a_{h(n)})$ is by hypothesis realisable by some dynamical system (Y,S), so (7) says that S has no fixed points, no points of period 2, and so on, but it has q points of period $h(n_0)$. By (1) this is only possible if $h(n_0)\big|q$, so we deduce that

$$h(n_0)=q. \tag{8}$$

Now consider the points of period $2n_0$ in (Y,S). There are $a_{h(2n_0)}$ of these points, and of course any point fixed by S^{n_0} is also fixed by S^{2n_0}, so

$$a_{h(2n_0)}\geqslant a_{h(n_0)}=q.$$

On the other hand, the sequence $(a_{h(n)})$ only takes on the values 0 and q, so in fact

$$a_{h(2n_0)}=q.$$

The same argument shows that $a_{h(jn_0)} = q$ for all $j \geqslant 1$. By (6), it follows that $q \big| h(jn_0)$ for all $j \geqslant 1$. Thus we have

$$\begin{aligned} h(n_0) &= c_k + c_{k-1}n_0 + \cdots + c_0 n_0^k \equiv 0, \\ h(2n_0) &= c_k + c_{k-1}2n_0 + \cdots + c_0 2^k n_0^k \equiv 0, \\ &\vdots \\ h((k+1)n_0) &= c_k + c_{k-1}(k+1)n_0 + \cdots + c_0(k+1)^k n_0^k \equiv 0 \end{aligned}$$

modulo q. That is,

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 4 & \cdots & 2^k \\ 1 & 3 & 9 & \cdots & 3^k \\ \vdots & & & & \\ 1 & (k+1) & (k+1)^2 & \cdots & (k+1)^k \end{pmatrix} \begin{pmatrix} c_k \\ c_{k-1}n_0 \\ c_{k-2}n_0^2 \\ \vdots \\ c_0 n_0^k \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

modulo q. Since k is fixed and q is large, the determinant $\prod_{1 \leqslant i < j \leqslant k+1}(j-i)$ of this matrix is non-zero modulo q, so we deduce that it is invertible modulo q, and hence

$$c_{k-j} n_0^j \equiv 0 \pmod{q} \tag{9}$$

for $j = 0, \ldots, k$.

Now, by definition, n_0 is the smallest n with $q \big| h(n)$, which tells us nothing about the size of n_0. However, we have seen in (8) that the realisability preserving property shows that $h(n_0) = q$. It follows that for large q we have

$$n_0 \approx \left(\frac{q}{c_0}\right)^{1/k} \ll q$$

since $c_0 \neq 0$. So (9) shows that

$$c_{k-j} n_0^j \approx c_{k-j}\left(\frac{q}{c_0}\right)^{j/k} \ll q$$

for $j \leqslant k-1$, and therefore the congruence (9) implies that

$$c_k = c_{k-1} = \cdots = c_1 = 0$$

because we can choose q to be as large as we please. It follows that $h(n) = c_0 n^k$ as claimed. We can of course deduce nothing about c_0, because $c_0 n_0^k \approx q$. □

3. Examples and Proof of Theorem 4

The forward part of Theorem 2, stating that monomials preserve realisability, gives families of results concerning congruences in the spirit of Fermat's little theorem and positivity statements from (1).

Example 6. Let (a_n) denote any realisable sequence, for example:

- the Bernoulli numerators (τ_n) or denominators (β_n) defined by

$$\left|\frac{B_{2n}}{2n}\right| = \frac{\tau_n}{\beta_n}$$

in lowest terms for all $n \geqslant 1$, where

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

(see A27641, shown to be realisable in [3]; A2445 shown to be realisable in [5], respectively); or

- the Euler numbers $\big((-1)^n E_{2n}\big)$, where

$$\frac{2}{\mathrm{e}^t + \mathrm{e}^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

(see A364, shown to be realisable in [5]); or

- the Lucas sequence $(1, 3, 4, 7, 11, \dots)$ (see A000204 and [9] for its special status as a realisable sequence); or
- the divisor sequence $(\sigma(n)) = (1, 3, 4, 7, 6, 12, 8, \dots)$ (realisable as it corresponds to a single orbit of each integer length),

then

$$0 \leqslant \sum_{d\,|n} \mu(d) a_{n/d}^k \equiv 0 \pmod{n}$$

for any $k \geqslant 1$ and $n \geqslant 1$.

We also find integrality of some related sequences via the Euler product expansion of the dynamical zeta function as follows. Recall that we write $\mathrm{Fix}_T(n)$ for the number of points fixed by the nth iterate of a map $T\colon X \to X$, and $\mathrm{Orb}_T(n)$ for the number of closed orbits of length n under T. By thinking of the collection of all periodic orbits as a disjoint union of individual orbits, it is clear that

$$\zeta_T(z) = \exp\left(\sum_{n \geqslant 1} \mathrm{Fix}_T(n) \tfrac{z^n}{n} \right) = \prod_{n \geqslant 1} \big(1 - z^n\big)^{-\mathrm{Orb}_T(n)},$$

so the Taylor expansion of $\zeta_T(z)$ at $z = 0$ automatically has integer coefficients.

Example 7. The following sequences of coefficients are integral, answering questions raised in the relevant Online Encyclopedia of Integer Sequences entry.

- The sequence A166168 is the sequence of Taylor coefficients of the zeta function of the dynamical system with periodic point data given by sampling the Lucas sequence along the squares, and so is integral as conjectured there. More generally, the same property holds for the Lucas sequence sampled along any integer power.
- Clearly there is the relation

$$\exp\left(\sum_{n \geqslant 1} \sigma(n) \tfrac{z^n}{n} \right) = \sum_{n \geqslant 0} p(n) z^n,$$

where p is the partition function A41; sampling along the squares gives as Taylor coefficients the Euler transform of the Dedekind ψ function. The argument here shows that sampling along any power also gives integral Taylor coefficients.

- The full shift on A symbols shows that the Taylor coefficients of
$$\exp\left(\sum_{n\geqslant 1} A^{n^k}\frac{z^n}{n}\right)$$
are integral for and $A,k\in\mathbb{N}$ (see A155200).

Because of the diversity of integer sequences satisfying the condition (1), it is clear that the property of 'preserving realisability' is extremely onerous. Indeed, the forward direction of Theorem 2 is a little surprising, and one might ask if there are any further functions with this property. In fact Moss [5] has constructed many such maps.

Lemma 8. *Let p be a prime, and define $g_p\colon\mathbb{N}\to\mathbb{N}$ by*
$$g_p(n)=\begin{cases} n & \text{if } p\nmid n;\\ pn & \text{if } p\big|n.\end{cases}$$
Then g_p lies in $\mathscr{P}$.

Proof. Let (a_n) be a realisable sequence and write $(b_n)=(a_{g_p(n)})$. We need to show that (b_n) satisfies (1) whenever (a_n) does. Fix n and write $n=p^{\mathrm{ord}_p(n)}m$ with $\gcd(m,p)=1$.

Assume first that $\mathrm{ord}_p(n)=0$. Then $p\nmid n$ and so
$$\sum_{d\,|n}\mu(n/d)b_d=\sum_{d\,|n}\mu(n/d)a_d$$
and so (b_n) satisfies (1) at n.

Next assume that $\mathrm{ord}_p(n)=1$, so that $n=pm$ and $p\nmid m$. Then
$$\begin{aligned}(\mu*b)(n)=\sum_{d\,|pm}\mu(d)b_{pm/d}&=\sum_{d\,|m}\mu(d)b_{n/d}+\mu(p)\sum_{d\,|m}\mu(d)b_{m/d}\\ &=\sum_{d\,|m}\mu(d)a_{p^2m/d}-\sum_{d\,|}\mu(d)a_{m/d}\end{aligned}\tag{10}$$
since μ is multiplicative. Now
$$\begin{aligned}(\mu*a)(pn)=(\mu*a)(p^2m)&=\sum_{d\,|p^2m}\mu(d)a_{p^2m/d}\\ &=\sum_{d\,|m}\mu(d)a_{p^2m/d}-\sum_{d\,|m}\mu(d)a_{pm/d}\end{aligned}\tag{11}$$
and
$$\begin{aligned}(\mu*a)(n)=(\mu*a)(pm)&=\sum_{d\,|pm}\mu(d)a_{pm/d}\\ &=\sum_{d\,|m}\mu(d)a_{pm/d}-\sum_{d\,|m}\mu(d)a_{m/d}.\end{aligned}\tag{12}$$
Adding (11) and (12) gives
$$(\mu*a)(pn)+(\mu*a)(n)=\sum_{d\,|m}\mu(d)a_{p^2m/d}-\sum_{d\,|m}\mu(d)a_{m/d}=(\mu*b)(n)$$
by (10), so (b_n) satisfies (1) at n.

Finally, assume that $\mathrm{ord}_p(n) \geqslant 2$. Then

$$\sum_{d\,|n} \mu(n/d) b_d = \underbrace{\sum_{d\,|m} \mu(n/d) a_d}_{\Sigma_0} + \sum_{j=1}^{\mathrm{ord}_p(n)} \underbrace{\sum_{d\,|m} \mu(n/p^j d) a_{pd}}_{\Sigma_j}$$

Now $\mu\big(\frac{n}{d}\big) = 0$ for all d dividing m, so $\Sigma_0 = 0$.

Similarly, $\mu\big(\frac{n}{p^j d}\big) = 0$ for $j \leqslant \mathrm{ord}_p(n) - 2$, so $\Sigma_j = 0$ for $1 \leqslant j \leqslant \mathrm{ord}_p(n) - 2$.

For the two remaining terms, we have

$$\begin{aligned} \Sigma_{\mathrm{ord}_p(n)} + \Sigma_{\mathrm{ord}_p(n)-1} &= \sum_{d\,|m} \mu(m/d) a_{pd} + \sum_{d\,|m} \mu(pm/d) a_{pd} \\ &= \sum_{d\,|m} \mu(m/d) a_{pd} - \sum_{d\,|m} \mu(m/d) a_{pd} = 0, \end{aligned}$$

so (1) holds trivially for (b_n) at n.

We deduce that (b_n) satisfies (1) for all $n \geqslant 1$, as required. □

Proof of Theorem 4. Let $S = \{p_1, p_2, \dots\} \subseteq \{2, 3, 5, 7, 11, \dots\}$ be any set of primes, and define $g_S \colon \mathbb{N} \to \mathbb{N}$ formally by $g_S = g_{p_1} \circ g_{p_2} \circ \cdots$ in the notation of Lemma 8. (For definiteness, we write a set of primes as $\{p_{j_1}, p_{j_2}, \dots\}$ with $p_{j_1} < p_{j_2} < \cdots$.) More precisely, the map g_S is defined as follows. For $n \in \mathbb{N}$ the set

$$\{p_j \mid p_j \text{ divides } n\} = \{p_{j_1}, \dots, p_{j_t}\}$$

is finite, and then we define

$$g_S(n) = g_{p_{j_1}} \circ \cdots \circ g_{p_{j_t}}(n).$$

If S and T are different subsets of the primes, then there is a prime p in the symmetric difference of S and T, and clearly $g_S(p) \neq g_T(p)$. It follows that there are uncountably many different functions g_S.

Formally, we also need to slightly improve the simple observation in Section 1 as follows. If $(h_1, h_2, \dots)$ is a sequence of functions in $\mathscr{P}$ with the property that

$$\{j \in \mathbb{N} \mid h_j(n) \neq n\} = \{j_n^{(1)}, j_n^{(2)}, \dots, j_n^{(r_n)}\}$$

is finite for any $n \in \mathbb{N}$, then the infinite composition $h = h_1 \circ h_2 \circ \cdots$ defined by

$$h(n) = h_{j_n^{(1)}} \circ \cdots \circ h_{j_n^{(r_n)}}(n)$$

for any $n \in \mathbb{N}$ is also in $\mathscr{P}$. This is clear, because for any given n checking (1) only involves evaluating h on finitely many terms. We deduce that there are uncountably many different elements of $\mathscr{P}$ from Lemma 8. □

4. Dynamical systems with additional polynomial time-changes

As mentioned earlier, if X simply comprises finitely many fixed points for T then $\mathscr{P}(X, T) = \mathbb{N}^{\mathbb{N}}$. Less trivial maps will have fewer maps that preserve realisability, and the complex way in which properties of a map relate to the structure of its associated set of maps are illustrated by examples of systems (X, T) with

$$\mathscr{P} \subsetneq \mathscr{P}(X, T) \subsetneq \mathbb{N}^{\mathbb{N}}. \tag{13}$$

Example 9. Let $T\colon X\to X$ be the full shift on $a\geqslant 2$ symbols, so that we have $\mathrm{Fix}_T(n)=a^n$ for all $n\geqslant 1$. Then we claim (this is an observation from the thesis of the second named author [5]) that if $h(n)=c_0+c_1n+\cdots+c_kn^k$ is any polynomial with non-negative integer coefficients, then $h\in\mathscr{P}(X,T)$. By Theorem 2, we know that the sequence (a^{n^j}) is realised by some map T_j for any $j=1,\dots,k$. Certainly the constant sequence $(a,a,\dots)$ is realised by the identity map T_0 on a set with a elements. Then the Cartesian product

$$S=\underbrace{T_0\times\cdots\times T_0}_{c_0\text{ copies}}\times\underbrace{T_1\times\cdots\times T_1}_{c_1\text{ copies}}\times\cdots\times\underbrace{T_k\times\cdots\times T_k}_{c_k\text{ copies}}$$

has

$$\mathrm{Fix}_S(n)=a^{c_0}\left(a^n\right)^{c_1}\cdots\left(a^{n^k}\right)^{c_k}=a^{h(n)}$$

for $n\geqslant 1$, by construction. Thus $h\in\mathscr{P}(X,T)$, showing that this is strictly larger than $\mathscr{P}$. On the other hand, if the map that exchanges 1 and 2 (and fixes all other elements of $\mathbb{N}$) lies in $\mathscr{P}(X,T)$, then we must be able to find some dynamical system (Y,S) with $\mathrm{Fix}_S(1)=a^2$ and $\mathrm{Fix}_S(2)=a$. This forces $a^2\leqslant a$, so $a\leqslant 1$. It follows that $\mathscr{P}(X,T)$ is strictly smaller than $\mathbb{N}^{\mathbb{N}}$.

In general it is not at all easy to describe $\mathscr{P}(X,T)$ — indeed with the exception of the trivial case $\mathbb{N}^{\mathbb{N}}$ which arises for the identity map on a finite set, we have no examples with a complete description. Example 9 relies on the accidental fact that $a^na^m=a^{n+m}$, allowing us to translate Cartesian products of systems into addition in the time-change. The next example of a system satisfying (13) relies on a different arithmetic trick, as well as the result from Example 9.

Example 10. Let $T\colon X\to X$ be the map $x\mapsto -ax$ modulo 1 on the additive circle $X=\mathbb{R}/\mathbb{Z}$ for some integer $a\geqslant 2$. Then we have $\mathrm{Fix}_T(n)=a^n-(-1)^n$ for $n\geqslant 1$, and we claim that if $h(n)=n^2+1$ then $h\in\mathscr{P}(X,T)$. (In fact, the same argument shows the same property for any polynomial with non-negative coefficients, but for simplicity of notation we consider this specific example.) To prove this, we first show that

$$\eta(n)=\sum_{d\,|n}(-1)^d\mu(n/d)=0$$

for all $n>2$. Writing $\boldsymbol{\mu}(s)=\sum_{n\geqslant 1}\frac{\mu(n)}{n^s}$, $\boldsymbol{\zeta}$ for the Riemann zeta function, and $\boldsymbol{\eta}$ for the Dirichlet η-function $\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n^s}$ it is clear that $\boldsymbol{\eta}(s)=(1-2^{-s})\boldsymbol{\zeta}(s)$ by splitting into odd and even terms, and $\boldsymbol{\zeta\mu}=1$, so $\boldsymbol{\mu}(s)\boldsymbol{\eta}(s)=(1-2^{1-s})$ for $\Re(s)>1$. It follows that $\eta(1)=-1$, $\eta(2)=2$, and $\eta(n)=0$ for $n>2$.

As all our other arguments are elementary, for completeness we also show this directly by separating out the power of 2 dividing n, as follows.

(1) If $n>2$ is odd, then

$$\eta(n)=-\sum_{d\,|n}\mu(n/d)=-\sum_{d\,|n}\mu(d)=0.$$

(2) If $n=2^k$ for some $k>1$, then

$$\eta(n)=\sum_{d\,|2^k}(-1)^d\mu(2^k/d)=\mu(1)+\mu(2)=0.$$

(3) If $n = 2m$ with $m > 2$ odd, then

$$\eta(n) = \sum_{d\,|2m} \mu(d)(-1)^{2m/d} = \sum_{d\,|m} \mu(d)\big((-1)^{2m/d} - (-1)^{m/d}\big) = 2\sum_{d\,|m} \mu(d) = 0.$$

(4) Finally, if $n = 2^k m$ with $k, m > 1$ and m odd, then

$$\eta(n) = \sum_{d\,|2^k m} \mu(d)(-1)^{2^k m/d} = \sum_{d\,|m} \mu(d)\big((-1)^{2^k m/d} - (-1)^{2^{k-1} m/d}\big) = 0.$$

We now show that $h \in \mathscr{P}(X,T)$ using the basic relation (1). That is, we need to show the congruence and positivity properties in (1) for the sequence (a_n) defined by $a_n = a^{n^2+1} + (-1)^n$ for $n \geqslant 1$ (since $(-1)^{n^2+1} = -(-1)^n$). Then $(a*\mu)_1 = a^2 - 1$ and $(a*\mu)_2 = a^2(a^3-1)+2$, so we see that $(a*\mu)_n$ is non-negative and divisible by n for $n = 1, 2$ as desired. For $n > 2$, we have

$$(a*\mu)_n = \sum_{d\,|N} \mu(n/d)a^{d^2+1} + \sum_{d\,|n} (-1)^d \mu(n/d) = \sum_{d\,|N} \mu(n/d)a^{d^2+1} \tag{14}$$

since $\eta(n) = 0$. Now a special case of Example 9 shows that the sequence (a^{n^2+1}) is realisable, so by (1) the last sum in (14) must be non-negative and divisible by n for all $n > 2$. This shows that (a_n) is a realisable sequence, and hence $h \in \mathscr{P}(X,T)$. To see that $\mathscr{P}(X,T)$ is not everything, notice that if the map exchanging 1 and 3 lies in $\mathscr{P}(X,T)$, then $a^3 \leqslant a$, which is impossible.

5. Questions

(1) The simple arguments showing that realisable sequences can be added and multiplied may be seen using disjoint unions and products of dynamical systems. Is there a similar argument showing that monomials preserve realisability? For example, from a system (X,T) with $a_n = \mathrm{Fix}_T(n)$ for all $n \geqslant 1$, is there a simple construction of a map $(X^{(2)}, T^{(2)})$ with the property that $\mathrm{Fix}_{T^{(2)}}(n) = a_{n^2}$ for all $n \geqslant 1$? Of course the proof above notionally 'constructs' such a system because it contains a 'formula' for how many orbits of each length such a map must have, but in a far from natural or geometric way.
(2) There is no *a priori* reason for any given $\mathscr{P}(X,T)$ to be a monoid under composition of functions, though $\mathscr{P}$ clearly is. For cases with $\mathscr{P}(X,T) \supsetneq \mathscr{P}$, what combinatorial properties of $(\mathrm{Fix}_T(n))$ determine the property that $\mathscr{P}(X,T)$ is a monoid?
(3) Is there a sequence of maps $\big((X_n,T_n)\big)_{n\geqslant 1}$ with the property that

$$\mathscr{P}(X_n,T_n) \supsetneq \mathscr{P}(X_{n+1},T_{n+1})$$

for all $n \geqslant 1$?
(4) Is there a map $T\colon X \to X$ with the property that the only polynomials in $\mathscr{P}(X,T)$ are monomials?
(5) Is there a map $T\colon X \to X$ with the property that $\mathscr{P}(X,T) = \mathscr{P}$?

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