WEAK ORDER AND DESCENTS FOR MONOTONE TRIANGLES

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Abstract. Monotone triangles are a rich extension of permutations that biject with alternating sign matrices. The notions of weak order and descent sets for permutations are generalized here to monotone triangles, and shown to enjoy many analogous properties. It is shown that any linear extension of the weak order gives rise to a shelling order on a poset, recently introduced by Terwilliger, whose maximal chains biject with monotone triangles; among these shellings are a family of EL-shellings.

The weak order turns out to encode an action of the 0-Hecke monoid of type A on the monotone triangles, generalizing the usual bubble-sorting action on permutations. It also leads to a notion of descent set for monotone triangles, having another natural property: the surjective algebra map from the Malvenuto-Reutenauer Hopf algebra of permutations into quasisymmetric functions extends in a natural way to an algebra map out of the recently-defined Cheballah-Giraudo-Maurice algebra of alternating sign matrices.

1. Introduction

Permutations in the symmetric group $\mathfrak{S}_n$ on n letters, when thought of as $n \times n$ permutation matrices, are special cases of fascinating objects known as *alternating sign matrices* (*ASMs*). The latter have been intensely studied since their introduction by Mills, Robbins and Rumsey [12], and turn out to be connected with such areas as statistical mechanics, representation theory, and number theory– see Bressoud [6] and Brubaker, Bump and Friedberg [7] for more history and context. We recall their definition here, as well as their bijection with the equivalent objects known as *monotone triangles*.

A vector in $\{0, \pm 1\}^n$ is called *alternating* if its ± 1 values alternate in sign, beginning and ending with $+1$. Denote by Alt_n the set of all such alternating vectors of length n. An $n \times n$ *alternating sign matrix* is one whose row and column vectors all lie in Alt_n. Denote by ASM_n the set of all such matrices. For example, we depict here on the left a matrix A in ASM_6, abbreviating "$+$" and $-$ for entries $+1$ and -1:

$$\begin{bmatrix} 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ + & - & + & - & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & + & 0 & - & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{bmatrix} = A \qquad \leftrightarrow \qquad T = \begin{array}{c} 2 \\ 2\ 4 \\ 1\ 3\ 6 \\ 1\ 3\ 4\ 6 \\ 1\ 2\ 3\ 5\ 6 \end{array} \tag{1}$$

There is a simple bijection between ASM_n and the set MT_n of *monotone triangles of size n*. A monotone triangle of size n is a sequence $T = (T_0, T_1, \ldots, T_{n-1}, T_n)$ of subsets of $[n] := \{1, 2, \ldots, n\}$ where $\#T_m = m$, with the extra property that T_{m+1} *interlaces* T_m in this sense: if one list entries of T_m, T_{m+1} in increasing order as

$$\begin{aligned} T_m &= \{i_1 < i_2 < \cdots < i_m\}, \\ T_{m+1} &= \{j_1 < j_2 < \cdots < j_m < j_{m+1}\}, \end{aligned}$$

then one has

$$j_1 \leq i_1 \leq j_2 \leq i_2 \leq j_3 \leq \cdots \leq j_m \leq i_m \leq j_{m+1}. \tag{2}$$

One depicts T as a triangular array having T_m as its m^{th} row from the top, omitting $T_0 = \varnothing, T_n = [n]$. For example, $T = (\varnothing, \{2\}, \{2,4\}, \{1,3,6\}, \{1,3,4,6\}, \{1,2,3,5,6\}, [6]) \in \mathrm{MT}_6$ is shown on the right in (1). For the sake of defining the bijections $\mathrm{ASM}_n \leftrightarrow \mathrm{MT}_n$, first introduce the *indicator vector* $\mathbb{1}_S$ in $\{0,1\}^n$ for a subset $S \subseteq [n]$, having coordinates $(\mathbb{1}_S)_i = 1$ for $i \in S$ and $(\mathbb{1}_S)_i = 0$ for $i \notin S$. Then given A in ASM_n,

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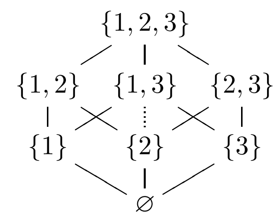
one maps $A \mapsto T = (T_0, \ldots, T_n)$ in MT_n whose m^{th} row T_m is the unique subset for which $\mathbb{1}_{T_m}$ is the sum of the first m rows of A. The inverse bijection sends $T \mapsto A$ where the m^{th} row of A is $\mathbb{1}_{T_m} - \mathbb{1}_{T_{m-1}}$. For example, the matrix A in ASM_6 shown on the left in (1) above has corresponding monotone triangle T in MT_6 shown to its right.

It is not hard to check (see Terwilliger [17, Thm. 3.2]) that an $(m+1)$-subset $J \subset [n]$ interlaces an m-set $I \subset [n]$ if and only if the difference of the indicator vectors $\mathbb{1}_J - \mathbb{1}_I$ lies in Alt_n. Thus MT_n is in bijection with the maximal chains of a partial order on the subsets of $[n]$ that is the transitive closure of the relation $I < J$ when J interlaces I; Terwilliger denotes this partial order Φ_n. Note that this partial order Φ_n is stronger than the usual *Boolean algebra poset* $2^{[n]}$, whose order relation is given by inclusion $\subseteq$, and whose maximal chains are the monotone triangles of the form $T(w) := (\varnothing, \{w_1\}, \{w_1, w_2\}, \ldots, \{w_1, w_2, \ldots, w_{n-1}\}, [n])$, which correspond to the permutations $w = (w_1, w_2, \ldots, w_n)$ in $\mathfrak{S}_n$. This monotone triangle $T(w)$ also corresponds to the usual permutation matrix of w^{-1}, thinking of permutation matrices as a subset of ASM_n. The Hasse diagram for the poset Φ_3 on subsets of [3] is shown below, with solid edges indicating the weaker Boolean algebra $2^{[3]}$ ordering, and the unique extra order relation $\{2\} < \{1, 3\}$ from Φ_3 shown dotted:

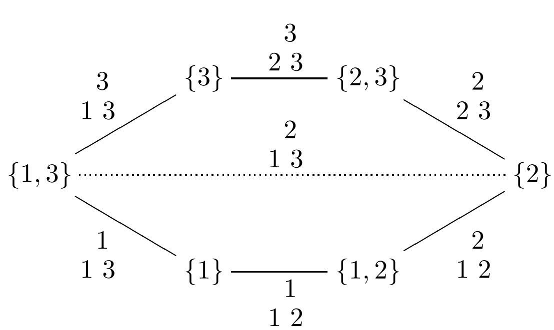


Section 2 explores properties of the order Φ_n, including characterizing it via a generalization of interlacing.

One of our original goals was to show that Φ_n is a *shellable* poset, a notion that we review here. Say that an abstract simplicial complex Δ is *pure* if all of its *facets* (=inclusion-maximal simplices) have the same number of vertices. In this case, say that an ordering $F_1, F_2, \ldots$ of the facets of Δ is a (pure) *shelling* if for every $j \geq 2$, the intersection of the boundary of F_j with the subcomplex generated by the facets $F_1, \ldots, F_{j-1}$ forms a pure subcomplex of codimension one within the boundary of F_j; said differently, for any pair $1 \leq i < j$, there exists $k < j$ such that $F_i \cap F_j \subseteq F_k \cap F_j$ with $\#F_k \cap F_j = \#F_j - 1$. Having a shelling for Δ imposes strong topological properties for its *geometric realization* $\|\Delta\|$, and strong algebraic properties for its *Stanley-Reisner ring* $k[\Delta]$; see Björner [1, Appendix] and [3, §1]. Here we are starting with a partially ordered set P having both a bottom element $\hat{0}$ and top element $\hat{1}$, such as the *Boolean algebra* $2^{[n]}$ with inclusion order on subsets of $[n]$, or the order Φ_n on subsets, where in either case, $\hat{0} = \varnothing$ and $\hat{1} = [n]$. In this setting, one often removes the bottom and top elements, and associates an abstract simplicial complex called the *order complex* to its *proper part*, so that Δ has vertex set $P \setminus \{\hat{0}, \hat{1}\}$, and simplices for each totally ordered subset of $P \setminus \{\hat{0}, \hat{1}\}$. This means that facets of Δ biject with maximal chains of P.

As mentioned above, for $P = \Phi_n$ and its subposet the Boolean algebra $2^{[n]}$, these facets or maximal chains are naturally labeled by the monotone triangles MT_n and permutations $\mathfrak{S}_n$, respectively. We illustrate this here for $n = 3$, depicting the order complex $\Delta(\Phi_3 \setminus \{\hat{0}, \hat{1}\})$, with one extra facet (edge) shown dotted, whose removal gives the subcomplex $\Delta(2^{[3]} \setminus \{\hat{0}, \hat{1}\})$.

(3)



For the Boolean algebra $2^{[n]}$, this order complex $\Delta(2^{[n]} \setminus \{\hat{0}, \hat{1}\})$ is isomorphic to the *Coxeter complex* of type A_{n-1}, and a result of Björner [3, Thm. 2.1] shows that it is shellable, with a shelling order on its facets provided by any linear ordering on the permutations $\mathfrak{S}_n$ that extends the *(right) weak order* $<_W$. This weak order is the transitive closure of the relation in which $ws_i <_W w$ if $w = (w_1, \ldots, w_n)$ has $w_i > w_{i+1}$, where $s_i = (i, i+1)$ is an adjacent transposition. One can view this weak order as induced from the action of the *bubble-sorting operators* $\pi_1, \ldots, \pi_{n-1}$ on $\mathfrak{S}_n$

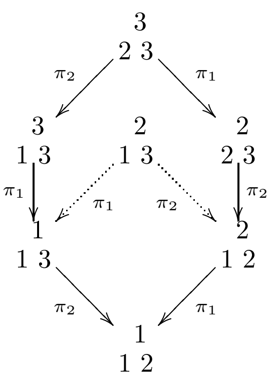
$$w \times \pi_i = \begin{cases} ws_i & \text{if } w_i > w_{i+1}, \\ w & \text{if } w_i < w_{i+1}, \end{cases} \tag{4}$$

which satisfy the relations of the *0-Hecke monoid* of type A_{n-1}:

$$\begin{aligned} \pi_i \pi_j &= \pi_j \pi_i \text{ if } |j - i| \geq 2, \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1}, \\ \pi_i^2 &= \pi_i. \end{aligned} \tag{5}$$

Note that π_i acts on right. This notational choice highlights the relationship between the application of π_i and multiplication on the right by s_i. One may then define the (right) weak order by $w \leq_W w'$ if and only if w lies in the 0-Hecke orbit of w'.

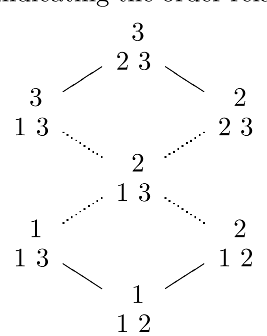
Section 3 extends this 0-Hecke action from $\mathfrak{S}_n$ to MT_n, by letting $T \times \pi_i$ replace the i^{th}-row of the monotone triangle T with the componentwise smallest row that still forms a monotone triangle with the remaining rows. One can then extend the weak order $<_W$ from $\mathfrak{S}_n$ to MT_n by setting $T \leq T'$ whenever T lies in the 0-Hecke orbit of T'. For $n = 3$, these actions of $H_3(0)$ on $\mathfrak{S}_3$ and MT_3 look as follows, illustrating the weak order posets $<_W$ on both:



Section 4 then uses this to prove our first main result.

Theorem 1.1. *Linear extensions of* $<_W$ *on* MT_n *give shelling orders on* Φ_n.

There is another sense in which the terminology *weak order* is appropriate. Lascoux and Schützenberger [10] showed that the componentwise order on MT_n is a *distributive lattice*, one that turns out to be the *MacNeille completion* of the *(strong) Bruhat order* $<_B$ on $\mathfrak{S}_n$; we therefore refer to this componentwise order on MT_n as its *(strong) Bruhat order* $< B$. Depicted below is the the poset $(\mathrm{MT}_3, <_B)$, with the usual Bruhat order $(\mathfrak{S}_3, <_B)$ as a subposet, and dotted edges indicating the order relation to the unique element T in $\mathrm{MT}_3 \setminus \mathfrak{S}_3$:



It turns out (see Remark 3.5) that this Bruhat order $<_B$ on MT_n is stronger than the weak order $<_W$ defined above; in particular, any linear extension of the componentwise order gives rise to a shelling of Φ_n.

The weak order shellings provided by Theorem 1.1 have another tight analogy to the weak order shellings of the Boolean posets $(2^{[n]}, \subseteq)$, in that they contain as a special case certain *EL-shellings*, a notion which we recall here. Given a poset P, with $C(P) = \{x \lessdot y : x, y \in P\}$ its set of *cover relations* ($x \lessdot y$ means $x < y$ but $\nexists z$ with $x < z < y$), an *EL-labeling* of P is a function $\lambda : C(P) \to \Lambda$ where $(\Lambda, <_\Lambda)$ is any poset, having these properties:

- for every interval $[x, y] \subset P$, there is a unique maximal chain $(x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y)$, that has *weakly rising* labels
$$\lambda(x_0, x_1) \leq_\Lambda \lambda(x_1, x_2) \leq_\Lambda \cdots \leq_\Lambda \lambda(x_{k-1}, x_{x_k})$$
- if $x \lessdot z < y$, with $z \neq x_1$, then $\lambda(x, x_1) <_\Lambda \lambda(x, z)$.

For example, the Boolean algebras $(2^{[n]}, \subseteq)$ have a very simple EL-labeling. It assigns a covering relation between subsets $I \subset J$ with $\#J = \#I + 1$ the unique integer $\lambda(I, J) := j$ such that $J = I \cup \{j\}$; here the labels come from the poset $\Lambda = \{1, 2, \ldots, n\}$ with the usual ordering on integers. A poset is *EL-shellable* or *lexicographically shellable* if it admits an EL-labeling. Björner [1, Thm. 2.3] showed that for a poset with an EL-labeling, one obtains a shelling order on its maximal chains via any linear extension of the lexicographic extension of Λ to sequences of edge labels. In Section 5, we prove the following.

Theorem 1.2. *There is a partial order on* Alt_n *so that the edge-labeling* λ *which assigns* $\lambda(I \lessdot J) = \mathbb{1}_J - \mathbb{1}_I$ *in* Alt_n *becomes an EL-labeling of* Φ_n*. Furthermore, any of the EL-shelling orders associated with this labeling will be a linear order on* MT_n *that extends the weak order* $<_W$.

The weak order shellings and EL-shellings in Theorems 1.1, 1.2 show that Φ_n is a *Cohen-Macaulay* poset, and allow one to combinatorially re-interpret its *flag f-vector* $f(\Phi_n) := (f_J)_{J \subset [n-1]}$; here f_J is the number of chains in Φ_n that pass through the ranks in J. One can instead consider the *flag h-vector* $h(\Phi_n) = (f_J)_{J \subset [n-1]}$, defined by an inclusion-exclusion relation:

$$f_J = \sum_{I \subseteq J} h_I, \qquad \text{or equivalently,}$$
$$h_J = \sum_{J \subseteq I} (-1)^{\#J \setminus I} f_I.$$

General shelling theory then implies this combinatorial interpretation for h_J:

$$h_J(\Phi_n) = \#\{T \in \mathrm{MT}_n : \mathrm{Des}(T) = J\}.$$

Here one is led to define the *descent set* $\mathrm{Des}(T)$ for a monotone triangle T as follows via the following generalization of the usual descent set $\mathrm{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}$, that is, $w \times \pi_i \neq w\}$ for permutations w in $\mathfrak{S}_n$:

$$\mathrm{Des}(T) := \{i \in [n-1] : T \times \pi_i \neq T\}.$$

Section 6 discusses this descent set $\mathrm{Des}(T)$, and collects some data on its distribution over MT_n.

There is a further way in which this notion of a descent set for monotone triangles extends a pleasant property of descents for permutations. Recall that Malvenuto and Reutenauer [11] defined a graded Hopf algebra, sometimes denoted $\mathrm{FQSym} = \bigoplus_{n \geq 0} \mathrm{FQSym}_n$, where FQSym_n has $\mathbb{Z}$-basis elements $\mathbf{w}$ indexed by permutations w in $\mathfrak{S}_n$. The ring structure is determined by a *shuffle product* for u, v in $\mathfrak{S}_n, \mathfrak{S}_m$ defined as

$$\mathbf{uv} = \sum_{w \in u \sqcup v[n]} \mathbf{w}$$

in which the sum runs over all shuffles w of $u = (u_1, \ldots, u_n)$, and $v[n] = (v_1 + n, \ldots, v_m + n)$. This shuffle product was introduced in such a way as to make a ring (and Hopf algebra) morphism into the *quasisymmetric functions* QSym, defined by

$$\begin{array}{rcl} \mathrm{FQSym} & \longrightarrow & \mathrm{QSym} \\ \mathbf{w} & \longmapsto & L_{\alpha(\mathrm{Des}(w))}. \end{array} \tag{6}$$

Here L_α denotes Gessel's *fundamental quasisymmetric function* associated to a composition α, and $\alpha(\mathrm{Des}(w))$ is the composition whose partial sums give the elements of $\mathrm{Des}(w)$; see [16, §7.19] and Section 7 below. Recently, Cheballah, Giraudo and Maurice embedded FQSym inside a larger graded Hopf algebra $\mathcal{ASM}$ whose n^{th}-graded component has a basis $\{\mathbf{A}\}$ indexed by A in ASM_n [8], and whose product and coproduct extend that of FQSym. Section 7 proves the following.

Theorem 1.3. *The map* FQSym $\to$ QSym *in* (6) *extends to an algebra (but not a coalgebra) morphism*

$$\begin{array}{rcl} \mathcal{ASM} & \longrightarrow & \mathrm{QSym} \\ \boldsymbol{A} & \longmapsto & L_{\alpha(\mathrm{Des}(A))} \end{array}$$

where $\mathrm{Des}(A) = \mathrm{Des}(T(A))$ *for an alternating sign matrix A is the descent set of its monotone triangle $T(A)$.*

Sections 8 concludes by comparing poset properties of the weak order on MT_n with analogous properties for the weak order on $\mathfrak{S}_n$, including a conjecture for the homotopy type of open intervals in $(\mathrm{MT}_n, <_W)$.

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2. Interlacing, monotone trapezoids, and the order Φ_n

The goal here is to relate Terwilliger's order Φ_n with the notions of interlacing and monotone trapezoids.

Definition 2.1.
Start with the *componentwise* order $<_{\mathrm{comp}}$ on subsets $I, I' \subset [n]$ of the same cardinality k for $0 \leq k \leq n$,

$$I = \{i_1 < i_2 < \cdots < i_k\},$$
$$I' = \{i'_1 < i'_2 < \cdots < i'_k\},$$

defined by setting $I \leq_{\mathrm{comp}} I'$ if $i_m \leq i'_m$ for $m = 1, 2, \ldots, k$.

For $J = \{j_1 < \cdots < j_\ell\} \subset [n]$ with $\#J = \ell \geq k = \#I$, say that J *interlaces* I, written $I \leq_{\mathrm{lace}} J$, if

$$\{j_1, j_2, \ldots, j_k\} \leq_{\mathrm{comp}} I \leq_{\mathrm{comp}} \{j_{\ell-k+1}, j_{\ell-k+2}, \ldots, j_{\ell-1}, j_\ell\}.$$

Note that when $\#J = k+1 = \#I + 1$, this condition $I \leq_{\mathrm{lace}} J$ is the usual definition of J interlacing I, as given in (2) earlier. One then has the following proposition which is easily checked (or see [17, §3]).

Proposition 2.2. *If $\#J = \#I + 1$, then $I \leq_{\mathrm{lace}} J$ if and only if $\mathbb{1}_J - \mathbb{1}_I$ lies in Alt_n* $\square$.

One can also readily check that $\leq_{\mathrm{lace}}$ is a partial order, that is, $I \leq_{\mathrm{lace}} J \leq_{\mathrm{lace}} K$ implies $I \leq_{\mathrm{lace}} K$. This partial order $\leq_{\mathrm{lace}}$ is closely related to *monotone trapezoids* and Terwilliger's order Φ_n, as we now explain.

Definition 2.3.
An (I, J)-*monotone trapezoid* is a sequence of subsets $T = (I_k, I_{k+1}, \ldots, I_{\ell-1}, I_\ell)$ of $\{1, 2, \ldots\}$ with

- $I_k = I, I_\ell = J$,
- $\#I_m = m$, and
- $I_m \leq_{\mathrm{lace}} I_{m+1}$ for $k \leq m < \ell$.

In other words, an (I, J)-monotone trapezoid is a saturated chain in $\leq_{\mathrm{lace}}$ from I to J. When $(I, J) = (\varnothing, [n])$, one calls T a *monotone triangle* of size n.

Proposition 2.4. *The following are equivalent for subsets $I, J \subseteq [n]$:*
(a) There exists at least one (I, J)-monotone trapezoid.
(b) $I \leq_{\Phi_n} J$.
(c) $I \leq_{\mathrm{lace}} J$.

In proving this proposition, and in the sequel, the following construction will be useful.

Definition 2.5.
For $I \leq_{\text{lace}} J$ with $\#I = k$ and $\#J \geq k+2$, define $H_{\min}(I,J) := \{h_1, h_2, \ldots, h_{k+1}\}$ by the rule

$$h_m := \max(i_{m-1}, j_m), \tag{7}$$

and convention $i_p := 0$ for $p = 0$. Thus when $k = 0$, so that $I = \varnothing$, then $H_{\min}(\varnothing, J) = \{j_1\}$.

Lemma 2.6. *The set $H_{\min}(I,J)$ has these properties:*
(i) *It is a $(k+1)$-subset, that is, $h_1 < \cdots < h_{k+1}$.*
(ii) *It lies in the family $\{H \in \binom{[n]}{k+1} : I \leq_{\text{lace}} H \leq_{\text{lace}} J\}$.*
(iii) *Every H' in this family has $H_{\min}(I,J) \leq_{\text{comp}} H'$.*

Proof. Assertion (i). The definition of $H_{\min}(I,J)$ implies $h_m < h_{m+1}$ since

$$h_m = \max(i_{m-1}, j_m) \leq \max(i_m - 1, j_{m+1} - 1) = \max(i_m, j_{m+1}) - 1 = h_{m+1} - 1.$$

Assertion (ii). We must show two $\leq_{\text{lace}}$-inequalities, or equivalently, four $\leq_{\text{comp}}$-inequalities.

- Two of the four come from $i_{m-1}, j_m \leq \max(i_{m-1}, j_m) = h_m$ for $m = 1, 2, \ldots, k+1$, which shows both that $I \leq_{\text{comp}} \{h_2, \ldots, h_{k+1}\}$ and also that $\{j_1, \ldots, j_{k+1}\} \leq_{\text{comp}} H_{\min}(I,J)$.
- The inequality $\{h_1, \ldots, h_k\} \leq_{\text{comp}} I$ comes from

$$h_m = \max(i_{m-1}, j_m) \leq \max(i_m, j_m) = i_m$$

which uses $i_{m-1} < i_m$ and the fact that $\{j_1, \ldots, j_m\} \leq_{\text{comp}} I$ since $I \leq_{\text{lace}} J$.
- The last inequality $H_{\min}(I,J) \leq \{j_{\ell-k}, j_{\ell-k+1}, \ldots, j_{\ell-1}, j_\ell\}$ comes from

$$h_m = \max(i_{m-1}, j_m) \leq j_{\ell-k+(m-1)}$$

which uses $j_m < j_{\ell-k+(m-1)}$ (as $\ell - k \geq 2$) and $i_{m-1} \leq j_{\ell-k+(m-1)}$ (as $I \leq_{\text{lace}} J$).

Assertion (iii). Any such $H' = \{h'_1 < \cdots < h'_{k+1}\}$ has $I \leq_{\text{lace}} H' \leq_{\text{lace}} J$, implying for $1 \leq m \leq k+1$ that

- $h'_m \geq i_{m-1}$, coming from $I \leq_{\text{comp}} \{h'_2, h'_3, \ldots, h'_{k+1}\}$,
- $h'_m \geq j_m$, coming from $\{j_1, \ldots, j_m\} \leq_{\text{comp}} H'$.

Thus $h'_m \geq \max(i_{m-1}, j_m) = h_m$, that is, $H_{\min}(I,J) \leq_{\text{comp}} H'$, as desired. ☐

With the construction $H_{\min}(I,J)$ and its properties in hand, one can now prove Proposition 2.4.

Proof of Proposition 2.4. Note (a) ⇔ (b) via Proposition 2.2 and definition of Φ_n. Then (a) ⇒ (c) from the transitivity of $\leq_{\text{lace}}$, while (c) ⇒ (a) follows by induction on $\#J - \#I$ via Lemma 2.6. ☐

Remark 2.7.
It is worth pointing out an involutive poset symmetry in Φ_n, coming from the action of the longest permutation $w_0 = (n, n-1, \ldots, 2, 1)$ in $\mathfrak{S}_n$. This permuation w_0 acts on subsets as follows:

$$I = \{i_1 < i_2 < \cdots < i_k\} \quad \overset{w_0}{\longmapsto} \quad w_0(I) := \{n+1-i_k < \cdots < n+1-i_2 < n+1-i_1\}.$$

Since $i \leq j$ if and only if $n+1-i \geq n+1-j$, this action of w_0 preserves the interlacing inequalities (2) that define the covering relations $I \lessdot_{\Phi_n} J$. Thus it is an involutive automorphism of the poset Φ_n, and therefore also gives an involution on monotone triangles

$$T = (T_0, T_1, \ldots, T_n) \quad \overset{w_0}{\longmapsto} \quad w_0(T) := (w_0(T_0), w_0(T_1), \ldots, w_0(T_n)).$$

Passing through the bijection $\text{ASM}_n \leftrightarrow \text{MT}_n$, the corresponding involution w_0 acting on a matrix $A = (a_{ij})$ in ASM_n simply reflects it through a vertical axis: $w_0(A) := (a_{i,n+1-j})$.

Due to this w_0-symmetry, for $I <_{\text{lace}} J$ with $\#J - \#I \geq 2$, instead of defining the set $H_{\min}(I,J)$ as in Definition 2.5, we could have defined a set $H_{\max}(I,J) = \{h'_1 < h'_2 < \cdots < h'_{k+1}\}$ via two equivalent formulas:

$$\begin{aligned} h'_m &= \min(i_m, j_{m-1+\ell-k}) \text{ for } m = 1, 2, \ldots, k+1, \text{ with convention } i_{k+1} := \infty, \text{ or} \\ H_{\max}(I,J) &= w_0(H_{\min}(w_0(I), w_0(J))). \end{aligned} \tag{8}$$

One would then have the corresponding properties as in Lemma 2.6, namely that $H_{\max}(I,J)$ is actually a $(k+1)$-subset, that it lies between I and J in the order $<_{\text{lace}}$, and that it is the componentwise *maximum* among all such $(k+1)$-subsets between I and J. We simply chose here to use $H_{\min}(I,J)$, not $H_{\max}(I,J)$.

The key property that we will need for shellability of Φ_n is that, for any pair $I \leq_{\text{lace}} J$, there is a componentwise smallest (I,J)-monotone trapezoid, and that it can be characterized *locally*.

Lemma 2.8. *Fixing $I \leq_{\text{lace}} J$, the following are equivalent for an (I,J)-monotone trapezoid*

$$T := ((I =)I_k, I_{k+1}, \ldots, I_{\ell-1}, I_\ell(= J)) :$$

(a) $I_m = H_{\min}(I_{m-1}, J)$ for $m = k+1, k+2, \ldots, \ell-1$.
(b) $I_m = H_{\min}(I_{m-1}, I_{m+1})$ for $m = k+1, k+2, \ldots, \ell-1$.
(c) The elements of $I_m = \{h_1^{(m)} < h_2^{(m)} \cdots < h_m^{(m)}\}$ are $h_p^{(m)} = \max(j_p, i_{p+k-m})$ with $i_q = 0$ for $q \leq 0$.
(d) T is the componentwise smallest among all (I,J)-monotone trapezoids.

Proof. First check that if T satisfies (a), then its entries have the formula from (c), using induction on m. The base case $m = k+1$ comes from the definition of $H_{\min}(I_k, J)$. The inductive step is this calculation:

$$h_p^{(m)} = \max(j_p, h_{p-1}^{(m-1)}) = \max(j_p, \max(j_p, i_{p-1+k-(m-1)})) = \max(j_p, i_{p+k-m}).$$

Next check that if T satisfies (b), then its entries obey the formula from (c), this time using induction on $\#J - \#I = \ell - k$. Assume that (b) holds for the trapezoid T, so

$$I_m = \{h_1^{(m)} < h_2^{(m)} \cdots < h_m^{(m)}\} = H_{\min}(I^{(m-1)}, I^{(m+1)}).$$

This means that

$$h_p^{(m)} = \max(h_p^{(m+1)}, h_{p-1}^{(m-1)}). \tag{9}$$

By restriction, condition (b) also holds for the smaller trapezoid $(I_m, I_{m+1}, \ldots, I_{\ell-1}, I_\ell = J)$, and hence by induction, one has $h_p^{(m+1)} = \max(j_p, h_{p-1}^{(m)})$. Similarly, by restriction, condition (b) also holds for the smaller trapezoid $(I = I_k, I_{k+1}, \ldots, I_{m-1}, I_m)$, and hence by induction, one has $h_{p-1}^{(m-1)} = \max(h_{p-1}^{(m)}, i_{p-1+k-(m-1)})$. Plugging these last two expressions into (9), one concludes that

$$\begin{aligned} h_p^{(m)} &= \max(\max(j_p, h_{p-1}^{(m)}), \max(h_{p-1}^{(m)}, i_{p-1+k-(m-1)})) \\ &= \max(j_p, h_{p-1}^{(m)}, i_{p+k-m})) = \max(j_p, i_{p+k-m}) \end{aligned}$$

since $h_{p-1}^{(m)} < h_p^{(m)}$. This last expression is the one from (c), as desired.

Thus since (a) does define a monotone trapezoid having I, J as its bottom, top rows, then T satisfiying (b) or (c) is equivalent to T being the one defined by (a).

To see (c) $\Leftrightarrow$ (d), let $T' = ((I =)I'_k, I'_{k+1}, \ldots, I'_{\ell-1}, I'_\ell(= J))$ be an (I,J)-monotone trapezoid, with $I'_m = \{i'_1 < \ldots < i'_m\}$. Then $i'_p \geq \max(j_p, i_{p+k-m})$ by the inequalities defining monotone trapezoids. Since the sets defined using (c) form an (I,J)-monotone trapezoid, we see they must form the minimal (I,J)-monotone trapezoid and vice versa. ☐

Remark 2.9.
It should not be surprising that there exists a componentwise smallest (I,J)-monotone trapezoid, as in Lemma 2.8, since Lascoux and Schützenberger [10, §5] showed that the componentwise order on MT_n has meet and join operations given by componentwise minimum and maximum. Similarly, there is a componentwise largest such (I,J)-monotone trapezoid, having similar properties, which can be built in a analogous fashion by iterating the $H_{\max}(I,J)$ construction from Remark 2.7.

3. Action of $\mathcal{H}_n(0)$ and the weak order

Recall from the Introduction (5) that the 0-*Hecke monoid* $\mathcal{H}_n(0)$ for the symmetric group $\mathfrak{S}_n$ (or type A_{n-1}) is the monoid with $n-1$ generators $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the usual *braid relations*

$$\begin{aligned} \pi_i\pi_j &= \pi_j\pi_i && \text{for } |i-j| \geq 2, \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} && \text{for } i = 1, 2, \ldots, n-2, \end{aligned} \tag{10}$$

together with the *quadratic relations*

$$\pi_i^2 = \pi_i \text{ for } i = 1, 2, \ldots, n-1. \tag{11}$$

See Norton [13] for background on $\mathcal{H}_n(0)$ and the associated monoid algebra, called a 0-*Hecke algebra*.

Definition 3.1.
Define maps $\pi_i : \mathrm{MT}_n \longrightarrow \mathrm{MT}_n$ for $i = 1, 2, \ldots, n-1$ sending $T \mapsto T \times \pi_i$, where $T \times \pi_i$ is obtained from T by replacing its i^{th} row T_i with $H_{\min}(T_{i-1}, T_{i+1})$.

Proposition 3.2. *The operators π_i on MT_n satisfy the braid and quadratic relations* (10), (11), *and hence define an action of $\mathcal{H}_n(0)$ on MT_n.*

Proof. The relations $\pi_i^2 = \pi_i$ and $\pi_i\pi_j = \pi_j\pi_i$ for $|i-j| \geq 2$ should be clear; only $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ requires verification. We can check this locally in rows $i-1, i, i+1, i+2$ of a monotone triangle T, by tracking two generic entries in rows $i, i+1$ shown in bold below. Here, we are using concatenation of sets of entries to abbreviate their maximum:

$$\begin{array}{ccc} & a \quad b & \\ c & d & \mathbf{e} \\ & f \quad \mathbf{g} & \\ h & i & \end{array} \overset{\pi_i}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ ? & af & \mathbf{bg} \\ & f \quad \mathbf{g} & \\ h & i & \end{array} \overset{\pi_{i+1}}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ ? & af & \mathbf{bg} \\ & ? \quad \mathbf{afi} & \\ h & i & \end{array} \overset{\pi_i}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ ? & ? & \mathbf{abfi} \\ & ? \quad \mathbf{afi} & \\ h & i & \end{array}$$

$$\begin{array}{ccc} & a \quad b & \\ c & d & \mathbf{e} \\ & f \quad \mathbf{g} & \\ h & i & \end{array} \overset{\pi_{i+1}}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ c & d & \mathbf{e} \\ & ch \quad \mathbf{di} & \\ h & i & \end{array} \overset{\pi_i}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ ? & ach & \mathbf{bdi} \\ & ch \quad \mathbf{di} & \\ h & i & \end{array} \overset{\pi_{i+1}}{\longmapsto} \begin{array}{ccc} & a \quad b & \\ ? & ach & \mathbf{bdi} \\ & ? \quad \mathbf{achi} & \\ h & i & \end{array}$$

Thus it only remains to check these equalities

$$\max(a, b, f, i) \overset{?}{=} \max(b, d, i), \tag{12}$$

$$\max(a, f, i) \overset{?}{=} \max(a, c, h, i), \tag{13}$$

which both follow, since

- $a \leq d \leq b$ and $f \leq i$ implies that the two sides in (12) are both equal to $\max(b, i)$,
- $c, h \leq f \leq i$ implies that the two sides in (13) are both equal to $\max(a, i)$. $\square$

Once one knows that the operators π_i satisfy the braid relations, one can define operators π_w for every permutation w in $\mathfrak{S}_n$ as follows: pick any factorization $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w that is shortest possible (i.e., *reduced*) as a product of the adjacent transpositions $\{s_1, s_2, \ldots, s_{n-1}\} =: S$, and then let

$$\pi_w := \pi_{i_1}\pi_{i_2} \ldots \pi_{i_\ell}.$$

As a consequence of satisfying the relations of $\mathcal{H}_n(0)$, one could equivalently define π_w recursively as follows:

$$\pi_w\pi_i := \begin{cases} \pi_{ws_i} & \text{if } w(i) < w(i+1), \text{ that is, if } i \notin \mathrm{Des}(w), \\ \pi_w & \text{if } w(i) > w(i+1), \text{ that is, if } i \in \mathrm{Des}(w), \end{cases} \tag{14}$$

starting with the initial condition $\pi_e := 1$.

Remark 3.3.
It is worth noting in the case where T has $T_i \subset T_{i+1}$ for all i, so that

$$T = T(w) := (\varnothing, \{w_1\}, \{w_1, w_2\}, \ldots, \{w_1, w_2, \ldots, w_{n-1}\}, [n])$$

for some permutation $w = (w_1, w_2, \ldots, w_n)$ in $\mathfrak{S}_n$, then one has

$$T(w) \times \pi_i = \begin{cases} T(w) & \text{if } w_i < w_{i+1}, \text{ that is, if } i \notin \mathrm{Des}(w), \\ T(ws_i) & \text{if } w_i > w_{i+1}, \text{ that is, if } i \in \mathrm{Des}(w). \end{cases}$$

Here $s_i = (i, i+1)$ is the adjacent transposition, so that

$$ws_i = (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_{n-1}, w_n).$$

Thus the action of $\mathcal{H}_n(0)$ on MT_n extends its action on $\mathfrak{S}_n$ via *(bubble-)sorting operators* as mentioned in the Introduction. We let $w \times \pi_i$ denote the permutation corresponding to $T(w) \times \pi_i$, so that $T(w) \times \pi_i = T(w \times \pi_i)$.

Definition 3.4.
Extend the *weak order* $<_W$ on the symmetric group $\mathfrak{S}_n$ to a weak order $<_W$ on monotone triangles MT_n as the transitive closure of the relations $T \times \pi_i \leq T$ where i is any index in the range $1, 2, \ldots, n-1$. Equivalently, $T \leq_W T'$ means that T lies in the $\mathcal{H}_n(0)$-orbit of T'.

Remark 3.5.
The name *weak order* is appropriate here, since $(\mathrm{MT}_n, <_W)$ is indeed weaker than the *componentwise* order $(\mathrm{MT}_n, <_B)$, and we view the latter as the appropriate extension of (strong) Bruhat order on $\mathfrak{S}_n$ to a strong Bruhat order on MT_n, via MacNeille completion. To see that $(\mathrm{MT}_n, <_W)$ is weaker than the componentwise order, note that it is the transitive closure of the relations $T \times \pi_i \leq_W T$, where $T \times \pi_i$ is obtained from T by replacing the i^{th} row of T with $H_{\min}(T_{i-1}, T_{i+1})$, the latter being componentwise smaller by Lemma 2.6.

4. Proof of Theorem 1.1

Recall the statement of the theorem.

Theorem 1.1. *Any linear extension $T^{(1)}, T^{(2)}, \cdots, T^{(N)}$ of $<_W$ on MT_n gives a shelling order on Φ_n.*

Before proving the theorem, we note in the next proposition a useful reinterpretation of Lemma 2.8, generalizing the definition of the $T \times \pi_i$ on monotone triangles. Given any subset $J \subseteq S := \{s_1, \ldots, s_{n-1}\}$, recall there is a unique longest permutation $w_0(J)$ in the *(Young or parabolic) subgroup* $\langle J \rangle$ of $\mathfrak{S}_n$ generated by J. This $w_0(J)$ is an involution, characterized within $\langle J \rangle$ by the property that

$$J = \mathrm{Des}(w_0(J)) (= \mathrm{Des}(w_0(J)^{-1})) \tag{15}$$

(here we identify $J = \{s_{j_1}, \ldots, s_{j_k}\}$ with $\{j_1, \ldots, j_k\}$). For example, if $n = 9$ and $J = \{s_1, s_2, s_4, s_5, s_6, s_8\} \subset \{s_1, s_2, \ldots, s_8\} = S$, then the parabolic subgroup $\langle J \rangle$ inside $\mathfrak{S}_9$ is the subgroup isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_4 \times \mathfrak{S}_2$ that stabilizes the blocks of the partition $\{1, 2, 3\}, \{4, 5, 6, 7\}, \{8, 9\}$. Its longest permutation is $w_0(J) = (3, 2, 1,\ 7, 6, 5, 4,\ 9, 8)$.

Proposition 4.1. *Given any monotone triangle T and $J \subseteq S$, then $T \times \pi_{w_0(J)}$ is the unique componentwise smallest monotone triangle $T^{\min}$ having the same rows T_m as T for all $s_m \notin J$.*

Proof. Lemma 2.8(b) shows that this componentwise smallest triangle $T^{\min}$ is uniquely characterized by

$$T^{\min}_m = \begin{cases} T_m & \text{for } s_m \notin J, \\ H_{\min}(T^{\min}_{m-1}, T^{\min}_{m+1}) & \text{for } s_m \in J, \end{cases}$$

On the other hand, we claim that the triangle $T' = T \times \pi_{w_0(J)}$ has these same properties:

- $T' = T \times \pi_{w_0(J)}$ shares the same rows $T'_m = T_m$ for $s_m \notin J$ since $w_0(J)$ lies in $\langle J \rangle$.
- For any $s_m \in J$ one has $T' \times \pi_m = T \times \pi_{w_0(J)} \pi_m = T \times \pi_{w_0(J)} = T'$ combining (14) with the fact that s_m lies in $J = \mathrm{Des}(w_0(J))$ by (15). This means that $T'_m = H_{\min}(T'_{m-1}, T'_{m+1})$. $\square$

Proof of Theorem 1.1. Thinking of each monotone triangle $T^{(i)}$ as corresponding to a facet, we identify it with its subset of $n+1$ vertices, namely

$$T^{(i)} = \{\varnothing = T_0^{(i)}, T_1^{(i)}, \ldots, T_{n-1}^{(i)}, T_n^{(i)} = [n]\}.$$

Shellability, as defined in the Introduction, requires that for each pair i,j with $1 \leq i < j \leq N$, we must exhibit some $k < j$ satisfying $\#T^{(k)} \cap T^{(j)} = n$ (including $\varnothing$ and $[n]$) and $T^{(i)} \cap T^{(j)} \subseteq T^{(k)} \cap T^{(j)}$.

Given $i < j$, let $J := \{m : T_m^{(i)} \neq T_m^{(j)}\}$. We claim that $T^{(j)} \times \pi_m \neq T^{(j)}$ for at least one m in J, otherwise Proposition 4.1 implies the two equalities here

$$T^{(j)} = T^{(j)} \times \pi_{w_0(J)} = T^{(i)} \times \pi_{w_0(J)} \leq_W T^{(i)},$$

but then the inequality $T^{(j)} \leq_W T^{(i)}$ would contradict $i < j$.

Given such an m, one checks that the index k defined by $T^{(j)} \times \pi_m = T^{(k)}$ does the job:

- $T^{(k)} = T^{(j)} \times \pi_m <_W T^{(j)}$ implies that $k < j$.
- $\#\left(T^{(k)} \cap T^{(j)}\right) = \#\left(T^{(j)} \times \pi_m\right) \cap T^{(j)} = n-1$, since $T^{(j)} \times \pi_m \neq T^{(j)}$.
- $T^{(i)} \cap T^{(j)} \subseteq T^{(k)} \cap T^{(j)}$ because s_m lies in J. $\square$

We close this section with two remarks about the above shelling.

Remark 4.2.
Since the π_i operators on MT_n restrict to the usual bubble-sorting operators on the symmetric group $\mathfrak{S}_n$ embedded inside MT_n via $w \mapsto T(w)$, one finds that the subposet $(\mathfrak{S}_n, <_W)$ is actually an *order ideal* inside $(\mathrm{MT}_n, <_W)$; it is even the *principal order ideal* below $T(w_0)$ where $w_0 = (n, n-1, \ldots, 2, 1)$.

As a consequence, it is possible to pick a linear extension of $<_W$ on MT_n which contains all of the elements of the order ideal $\mathfrak{S}_n$ as an initial segment. This then gives a shelling order on the facets of $\Delta(\Phi_n \setminus \{\hat{0}, \hat{1}\})$ which shells the Coxeter complex $\Delta(2^{[n]} \setminus \{\hat{0}, \hat{1}\})$ *first*, before continuing on to shell the remaining facets of $\Delta(\Phi_n \setminus \{\hat{0}, \hat{1}\})$ that do not correspond to permutations.

Remark 4.3.
Shellability implies that the $(n-2)$-dimensional simplicial complex $\Delta(\Phi_n \setminus \{\hat{0}, \hat{1}\})$ has the homotopy type of a bouquet of $(n-2)$-spheres. The Coxeter complex $\Delta(2^{[n]} \setminus \{\hat{0}, \hat{1}\})$ inside it is homeomorphic to a single $(n-2)$-sphere, and this sphere has well-known easy embeddings into $\mathbb{R}^{n-1}$. For example, it is isomorphic to the *barycentric subdivision* of the boundary of a simplex with vertex set $\{1, 2, \ldots, n\}$. Alternatively one can embed it within the hyperplane $x_1 + \cdots + x_n = 0$ inside $\mathbb{R}^n$ by extending piecewise-linearly the map that sends its vertices to the $\mathfrak{S}_n$-images of the fundamental dominant weights of type A_{n-1}: the vertex indexed by a subset I with $\varnothing \subsetneq I \subseteq [n]$ is sent to the vector $\sum_{i \in I} e_i - \frac{\#I}{n}(e_1 + \cdots + e_n)$ where e_i is the i^{th} standard basis vector of $\mathbb{R}^n$.

After looking at the picture (3) of $\Delta(\Phi_3 \setminus \{\hat{0}, \hat{1}\})$, which embeds it in $\mathbb{R}^2$, one might wonder whether $\Delta(\Phi_n \setminus \{\hat{0}, \hat{1}\})$ embeds in some simple way into $\mathbb{R}^{n-1}$. We are doubtful. For example, when $n = 4$, one can check that if one takes either of the two vertex coordinates for embedding $\Delta(2^{[4]} \setminus \{\hat{0}, \hat{1}\})$ into $\mathbb{R}^3$ as described in the previous paragaph, when one extends this piecewise-linearly over the extra simplices in $\Delta(\Phi_4 \setminus \{\hat{0}, \hat{1}\})$, it leads to self-intersections, and not an embedding.

5. EL-labeling and proof of Theorem 1.2

Recall the statement of the theorem.

Theorem 1.2. *There is a partial order on* Alt_n *so that the edge-labeling* λ *which assigns* $\lambda(I \lessdot J) = \mathbb{1}_J - \mathbb{1}_I$ *in* Alt_n *becomes an EL-labeling of* Φ_n. *Furthermore, any of the EL-shelling orders associated with this EL-labeling is a linear order on* MT_n *which extends the weak order* $<_W$.

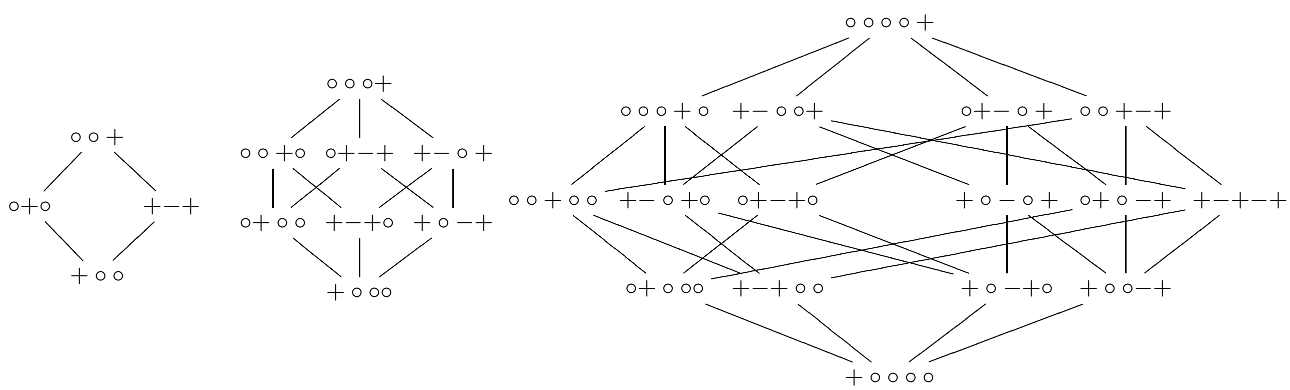
We will define the partial order on Alt_n via its identification with a Boolean algebra $2^{[n-1]}$. Note that a vector v in $\{0, \pm 1\}^n$ lies in Alt_n exactly when each of its *tail sums* $v \cdot \mathbb{1}_{[i,n]} = v_i + v_{i+1} + \cdots + v_n$ lies in $\{0, +1\}$, with $\sum_{i=1}^n v_i = +1$. The following proposition is straightforward to verify.

Proposition 5.1. *One has mutually-inverse bijections*

$$\begin{array}{rcl} \mathrm{Alt}_n & \xrightarrow{\varphi} & 2^{[n-1]} \\ v & \overset{\varphi}{\longmapsto} & S(v) := \{i \in [n{-}1] : v \cdot \mathbb{1}_{[i+1,n]} = +1\} \\ e_1 + \sum_{i \in S}(e_{i+1} - e_i) & \overset{\varphi^{-1}}{\longleftarrow\!\shortmid} & S. \end{array}$$

Definition 5.2.
Put a partial order $<_{EL}$ on Alt_n that pulls back the inclusion order on $2^{[n-1]}$ via the above bijection φ, that is, $v \leq_{EL} w$ if and only if $S(v) \subseteq S(w)$. Equivalently, $v \leq_{EL} w$ if and only for every $i = 1, 2, \ldots, n$ one has dot product $(w - v) \cdot (e_i + e_{i+1} + \cdots + e_n) \geq 0$.

Example 5.3.
Here is the order $<_{EL}$ on Alt_n for $n = 3, 4, 5$:



Next, we show that $\lambda : C(\Phi_n) \to \mathrm{Alt}_n$ defined by $\lambda(I \lessdot J) := \mathbb{1}_J - \mathbb{1}_I$ is an EL-labeling of Φ_n with respect to $<_{EL}$ on Alt_n. For the rest of this section, fix a pair $I <_{\mathrm{lace}} J$ in Φ_n with $H_{\min}(I, J)$ as in Definition 2.5.

Lemma 5.4. *Assume $I <_{\mathrm{lace}} H <_{\mathrm{lace}} J$ with $\#H = \#I + 1$. Then $\mathbb{1}_{H_{\min}(I,J)} - \mathbb{1}_I \ \leq_{EL} \ \mathbb{1}_H - \mathbb{1}_I$.*

Proof. Recall $\leq_{EL}$ can be rephrased as follows: $A \leq_{EL} B$ if and only if $(\mathbb{1}_B - \mathbb{1}_A) \cdot \mathbb{1}_{[\ell,n]} \geq 0$ for all ℓ.
Thus, since $H_{\min}(I, J) \leq_{\mathrm{comp}} H$ according to Lemma 2.6(iii), for all ℓ one will have

$$\big((\mathbb{1}_H - \mathbb{1}_I) - (\mathbb{1}_{H_{\min}(I,J)} - \mathbb{1}_I)\big) \cdot \mathbb{1}_{[\ell,n]} = (\mathbb{1}_H - \mathbb{1}_{H_{\min}(I,J)}) \cdot \mathbb{1}_{[\ell,n]} \geq 0. \quad \square$$

It turns out that one can characterize $H_{\min}(I, J)$ in terms of $\leq_{EL}$.

Lemma 5.5. *Assume $I <_{\mathrm{lace}} H <_{\mathrm{lace}} J$ with $\#J = \#I + 2$. Then*

$$\mathbb{1}_H - \mathbb{1}_I \ \leq_{EL} \ \mathbb{1}_J - \mathbb{1}_K \quad \textit{if and only if} \quad H = H_{\min}(I, J).$$

Proof. Name the elements of I, H, J as follows:

$$\begin{aligned} I &= \{i_1 < \cdots < i_p\}, \\ H &= \{h_1 < \cdots < h_p < h_{p+1}\}, \\ J &= \{j_1 < \cdots < j_p < j_{p+1} < j_{p+2}\}. \end{aligned}$$

($\Leftarrow$): Assume $H = H_{\min}(I, J)$. We check for each ℓ that $(\mathbb{1}_H - \mathbb{1}_I) \cdot \mathbb{1}_{[\ell,n]} \leq (\mathbb{1}_J - \mathbb{1}_H) \cdot \mathbb{1}_{[\ell,n]}$, or equivalently,

$$\#J \cap [\ell, n] + \#I \cap [\ell, n] - 2\#H \cap [\ell, n] \ \geq \ 0.$$

If $H\cap[\ell,n]=\varnothing$, this is clear. Otherwise, let $H\cap[\ell,n]=\{h_k,h_{k+1},\dots,h_{p+1}\}$, so that $\#H\cap[\ell,n]=p+2-k$. Then the interlacing $I<_{\text{lace}}H<_{\text{lace}}J$ along with $h_k=\max(i_{k-1},j_k)$ imply that

$$I\cap[\ell,n]=\begin{cases}\{i_k,i_{k+1},\dots,i_p\} & \text{if } h_k>i_{k-1},\\ \{i_{k-1},i_k,i_{k+1},\dots,i_p\} & \text{if } h_k=i_{k-1},\end{cases}$$

$$J\cap[\ell,n]=\begin{cases}\{j_{k+1},j_{k+2},\dots,i_{p+1}\} & \text{if } h_k>j_k,\\ \{j_k,j_{k+1},j_{k+2},\dots,j_{p+2}\} & \text{if } h_k=j_k.\end{cases}$$

From this one can calculate that

$$\#J\cap[\ell,n]+\#I\cap[\ell,n]-2\#H\cap[\ell,n]=\begin{cases}0 & \text{if } h_k=j_k>i_{k-1} \text{ or } h_k=i_{k-1}>j_k,\\ +1 & \text{if } h_k=i_{k-1}=j_k.\end{cases}$$

($\Rightarrow$): Assume $\mathbb{1}_H-\mathbb{1}_I\ \leq_{EL}\ \mathbb{1}_J-\mathbb{1}_H$.

Claim: One cannot have both strict inequalities $i_{k-1}<h_k<i_k$, nor a strict inequality $i_p<h_{p+1}$.

To see this claim, note that in either case ($i_{k-1}<h_k<i_k$ or $i_p<h_{p+1}$), it would imply $h_k\in H\setminus I$. Then since $I<_{\text{lace}}H$, this would imply $(\mathbb{1}_H-\mathbb{1}_I)\cdot\mathbb{1}_{[h_k,n]}=+1$. But then $h_k\in H$ and $H<_{\text{lace}}J$ implies $(\mathbb{1}_J-\mathbb{1}_H)\cdot\mathbb{1}_{[h_k,n]}=0<+1=(\mathbb{1}_H-\mathbb{1}_I)\cdot\mathbb{1}_{[h_k,n]}$, a contradiction to our assumption.

By Lemma 2.8 (c) and (d), $I<_{\text{lace}}H<_{\text{lace}}J$ implies $h_k\geq\max(i_{k-1},j_k)$ for $k=1,2,\dots,p+1$. We must now show that these are all equalities, not inequalities. For the sake of contradiction, assume not and pick k *maximal* such that $h_k>\max(i_{k-1},j_k)$.

The Claim above then forces $k\leq p$ and $h_k=i_k$ (else $i_{k-1}<h_k<i_k$ or $k=p+1$ and $i_p<h_{p+1}$). Then $h_{k+1}>h_k=i_k$ and the maximality of k forces $h_{k+1}=\max(i_k,j_{k+1})=\max(h_k,j_{k+1})=j_{k+1}$. And again the Claim forces $k+1\leq p$ and $i_{k+1}=h_{k+1}(=j_{k+1})$.

We now repeat this argument to show by induction that for all $m=k+1,k+2,\dots$, one has both $m\leq p$ and this triple coincidence $j_m=h_m=i_m$; this would contradict finiteness of p. The inductive step again notes that $h_{m+1}>h_m=i_m$ and maximality of k forces $h_{m+1}=\max(i_m,j_{m+1})=\max(h_m,j_{m+1})=j_{m+1}$. But then the Claim forces $m+1\leq p$ and $i_{m+1}=h_{m+1}(=j_{m+1})$, recreating the inductive hypothesis. □

Proof of Theorem 1.2. We first check our edge-labeling λ satisfies the two conditions for an EL-labeling:

- for every interval $[x,y]\subset\Phi_n$, there is a unique maximal chain $(x=x_0\lessdot x_1\lessdot\cdots\lessdot x_k=y)$, that has *weakly rising* labels $\lambda(x_0,x_1)\leq_\Lambda\lambda(x_1,x_2)\leq_\Lambda\cdots\leq_\Lambda\lambda(x_{k-1},x_{x_k})$
- if $x\lessdot z<y$, with $z\neq x_1$, then $\lambda(x,x_1)<_\Lambda\lambda(x,z)$.

The first condition follows by combining Lemma 2.8(b) and Lemma 5.5, which show that for any $I<_{\text{lace}}J$, the unique maximal chain in the interval $[I,J]$ corresponds to the (I,J)-monotone trapezoid $T_{\min}(I,J)$. Then the second condition comes from Lemma 5.4.

For the second assertion of the theorem, it suffices to check that if T,T' are monotone triangles with $T'<_W T$, then any of the above EL-shellings, which come from linearly extending the lexicographic ordering of $<_{EL}$ on edge labels, will have T' earlier than T. By definition of the weak order $<_W$, it suffices to check this holds when $T'=T\times\pi_i$ for some i. In this case, it follows because Lemma 5.4 shows that T will have lexicographically earlier edge label sequence than T': the two sequences first differ in replacing the label $\mathbb{1}_{T_{i+1}}-\mathbb{1}_{T_i}$ with the $<_{EL}$-smaller label $\mathbb{1}_{H_{\min}(T_i,T_{i+2})}-\mathbb{1}_{T_i}$. □

6. Descents, h-vectors and flag h-vectors

Recall from the Introduction the usual *descent set* for a permutation $w=(w_1,\dots,w_n)$ in $\mathfrak{S}_n$

$$\mathrm{Des}(w):=\{k\in[n-1]:w_k>w_{k+1}\}=\{k\in[n-1]:w\times\pi_k=ws_k<_W w\}.$$

It has a natural extension to monotone triangles T, motivated by the weak order $<_W$ and our shelling results.

Definition 6.1.
Define the *descent set* $\mathrm{Des}(T)$ for $T = (T_0, T_1, \ldots, T_n)$ in MT_n by

$$\begin{aligned}\mathrm{Des}(T) &:= \{k \in [n-1] : T \times \pi_k <_W T\} \\ &= \{k \in [n-1] : T_k \neq H_{\min}(T_{k-1}, T_{k+1})\}.\end{aligned}$$

There is another way to define $\mathrm{Des}(T)$.

Lemma 6.2. *For T in MT_n, one has*

$$\mathrm{Des}(T) := \{k \in [n-1] : \text{ there does not exist some } T' \neq T \text{ with } T' \times \pi_k = T\}.$$

In particular, T is one of the maximal elements of the weak order $<_W$ if and only if $\mathrm{Des}(T) = [n-1]$.

Proof. Since $\pi_k^2 = \pi_k$, if there exists T' with $T' \times \pi_k = T$, then $T \times \pi_k = T' \times = T' \times \pi_k = T$, so $k \notin \mathrm{Des}(T)$.

Conversely, if $k \notin \mathrm{Des}(T)$, so that $T \times \pi_k = T$, we wish to exhibit at least one $T' \neq T$ having $\pi_k(T') = T$. From $T \times \pi_k = T = (T_0, T_1, \ldots, T_n)$ we know that $T_k = H_{\min}(I, J)$ where $I := T_{k-1}, J := T_{k+1}$, so that if we construct T' from T by replacing T_k with $H_{\max}(I, J)$ as defined in (8), then it will certainly have $T' \times \pi_k = T$.

It only remains to show that $T' \neq T$, that is $H_{\max}(I, J) \neq H_{\min}(I, J)$. To check this, name elements:

$$\begin{aligned} I &= \{i_1 < i_2 < \cdots < i_{k-1}\}, \\ H_{\min}(I, J) &= \{h_1 < h_2 < \cdots < h_{k-1} < h_k\}, \\ H_{\max}(I, J) &= \{h'_1 < h'_2 < \cdots < h'_{k-1} < h'_k\}, \\ J &= \{j_1 < j_2 < \cdots < j_{k-1} < j_k < j_{k+1}\} \end{aligned}$$

Then the formulas defining $H_{\min}(I, J), H_{\max}(I, J)$ are $h_m = \max(i_{m-1}, j_m), h'_m = \min(i_m, j_{m+1})$, implying that $h'_m = h_m$ if and only if $i_m = j_m$ or $i_{m-1} = j_{m+1}$. Since $\#I \cap J \leq \#I = k-1$, such an equality occurs at most $k-1$ times, and hence $h'_m \neq h_m$ for at least one $m = 1, 2, \ldots, k$. $\square$

Remark 6.3.
Embedded in the previous proof are operators $\pi'_k : T \mapsto T'$ on MT_n for $k = 1, 2, \ldots, n-1$, where T' is obtained from T by replacing T_k with $T'_k = H_{\max}(T_{k-1}, T_{k+1})$. Because of the relation between the $H_{\min}$ and $H_{\max}$ constructions described in Remark 2.7, the operators $\{\pi'_k\}_{k=1,2,\ldots,n-1}$ satisfy the same braid and quadratic relations as $\{\pi_k\}$, giving a (different) action of the 0-Hecke monoid $\mathcal{H}_n(0)$ on MT_n.

One can check that this other action, in fact, extends the *(right-)regular action* of $\mathcal{H}_n(0)$ on itself, when one identifies the monotone triangle $T(w)$ in MT_n with π'_w in $\mathcal{H}_n(0)$. One could use it to define a different version of a weak order on MT_n, having a unique top element $T(w_0)$, but several different minimal elements. One reason that we instead chose the action by $\{\pi_k\}$ and their resulting weak order $<_W$ is so that the monotone triangle $T(e)$ corresponding to $e = (1, 2, \ldots, n)$ in S_n labels the first facet in all of the shellings.

As mentioned in the Introduction, descent sets conveniently encode the *flag f-vector* $f(\Phi_n) := (f_J)_{J \subset [n-1]}$, where f_J counts the number of chains that pass through the ranks in J. One instead considers the *flag h-vector* $h(\Phi_n) = (h_J)_{J \subset [n-1]}$, defined by these inclusion-exclusion relations:

$$f_J = \sum_{I : I \subseteq J} h_I, \quad \text{or equivalently,} \quad h_J = \sum_{I : I \subseteq J} (-1)^{\#J \setminus I} f_I.$$

General shelling theory (e.g., Björner [3, §1(B)]) then implies this combinatorial interpretation for h_J:

$$h_J(\Phi_n) = \#\{T \in \mathrm{MT}_n : \mathrm{Des}(T) = J\}$$

The usual f-vector $f = (f_{-1}, f_0, f_1, \ldots, f_{n-2})$ and h-vector $h = (h_0, h_1, \ldots, h_{n-1})$ for $\Delta(\Phi_n \setminus \{\hat{0}, \hat{1}\})$ can then be obtained by grouping the terms in $(f_J), (h_J)$ as follows:

$$f_i = \sum_{J \in \binom{[n-1]}{i+1} :} f_J, \quad \text{and} \quad h_i = \sum_{J \in \binom{[n-1]}{i} :} h_J.$$

n	$h(\Phi_n) = (h_0, h_1, \ldots, h_{n-1})$
2	$(1, 1)$
3	$(1, 4, 2)$
4	$(1, 11, 21, 9)$
5	$(1, 26, 130, 192, 80)$
6	$(1, 57, 638, 2318, 3101, 1321)$
7	$(1, 120, 2773, 21472, 67340, 87616, 39026)$
8	$(1, 247, 11264, 172222, 1108243, 3260759, 4280764, 2016716)$

n	$\sum_{J \subset [n-1]} h_J(\Phi_n)\ x_J$ where $x_J := \prod_{i \in J} x_i$
2	$x_1 + 1$
3	$2x_1x_2 + 2x_1 + 2x_2 + 1$
4	$9x_1x_2x_3 + 7x_1x_2 + 7x_1x_3 + 7x_2x_3 + 3x_1 + 5x_2 + 3x_3 + 1$
5	$80x_1x_2x_3x_4 + 52x_1x_2x_3 + 44x_1x_2x_4 + 44x_1x_3x_4 + 52x_2x_3x_4 + 16x_1x_2 + 26x_1x_3 + 32x_2x_3$ $+14x_1x_4 + 26x_2x_4 + 16x_3x_4 + 4x_1 + 9x_2 + 9x_3 + 4x_4 + 1$
6	$1321x_1x_2x_3x_4x_5 + 745x_1x_2x_3x_4 + 562x_1x_2x_3x_5 + 487x_1x_2x_4x_5 + 562x_1x_3x_4x_5 + 745x_2x_3x_4x_5$ $+180x_1x_2x_3 + 251x_1x_2x_4 + 298x_1x_3x_4 + 405x_2x_3x_4 + 120x_1x_2x_5 + 215x_1x_3x_5 + 298x_2x_3x_5$ $+120x_1x_4x_5 + 251x_2x_4x_5 + 180x_3x_4x_5 + 30x_1x_2 + 65x_1x_3 + 92x_2x_3 + 58x_1x_4 + 125x_2x_4$ $+92x_3x_4 + 23x_1x_5 + 58x_2x_5 + 65x_3x_5 + 30x_4x_5 + 5x_1 + 14x_2 + 19x_3 + 14x_4 + 5x_5 + 1$

TABLE 1. The h-vectors of Φ_n for $n \leq 8$ and flag h-polynomials of Φ_n for $n \leq 6$. All data computed using SAGE.

In particular, $h_i(\Phi_n) = \#\{T \in \mathrm{MT}_n : \#\mathrm{Des}(T) = i\}$. See Table 1 for the h-vector $h(\Phi_n)$ and flag h-polynomial for small values of n.

We remark on some features of this data. Note the sequence of values $1, 2, 9, 80, 1321, 39026, 2016716$ for

$$h_{n-1} = \#\{T \in \mathrm{MT}_n : \mathrm{Des}(T) = [n-1]\} = \#\{\text{ maximal elements in the poset } (\mathrm{MT}_n, <_W)\},$$

appearing at the right in Table 1, which is not in the Online Encyclopedia of Integer Sequences (OEIS).

The data invites comparison with the Boolean algebra $2^{[n]}$, which has h-vector $h(2^{[n]}) = (h_0, h_1, \ldots, h_{n-1})$ given by the *Eulerian numbers*, that is, $h_i(2^{[n]}) = \#\{w \in \mathfrak{S}_n : \#\mathrm{Des}(w) = i\}$. The Eulerian numbers are well-behaved in many ways (see Petersen [14]). For example, they satisfy recurrences and have the *symmetry* $h_i = h_{n-1-i}$. They also have the very strong property that the *h-polynomial*

$$h(2^{[n]}, t) := \sum_{i=0}^{n-1} h_i t^i = \sum_{w \in \mathfrak{S}_n} t^{\#\mathrm{Des}(w)}$$

has only real zeroes. This implies *log-concavity* $h_i^2 \geq h_{i+1}h_{i-1}$, which then implies *unimodality*, meaning that there is some k (in this case $k = \lfloor \frac{n-1}{2} \rfloor$ works) for which $h_0 \leq h_1 \leq \cdots h_k \geq \cdots \geq h_{n-2} \geq h_{n-1}$. From the data in Table 1, the reader can check that for Φ_n, the h-polynomial

$$h(\Phi_n, t) := \sum_{i=0}^{n-1} h_i t^i = \sum_{T \in \mathrm{MT}_n} t^{\#\mathrm{Des}(T)}$$

is irreducible in $\mathbb{Q}[t]$ with only real zeroes for $n \leq 8$, hence is log-concave for those values.

Question 6.4. *Does $h(\Phi_n, t)$ have only real zeroes? If not, is its coefficient sequence log-concave, or at least unimodal?*

Question 6.5. *What is the largest entry in the h-vector of Φ_n? Is it always h_{n-2}?*

7. Descents as a map to QSYM, and proof of Theorem 1.3

As described in the Introduction, the map $w \mapsto \mathrm{Des}(w)$ that sends a permutation w in $\mathfrak{S}_n$ to its descent set was pleasingly reinterpreted in the work of Malvenuto and Reutenauer [11] as a morphism of Hopf algebras. We wish to explain here how this extends to the map $T \mapsto \mathrm{Des}(T)$ sending a monotone triangle to its descent set, giving at least an algebra (but not coalgebra) morphism out of the Hopf algebra of ASMs recently defined by Cheballah, Giraudo and Maurice [8].

Let us start by recalling the algebra structures on quasisymmetric functions, permutations, and ASMs.

Definition 7.1.
The *ring of quasisymmetric functions* QSym can be defined as the subalgebra of the algebra $\mathbb{Z}[[x_1, x_2, \ldots]]$ of formal power series that has $\mathbb{Z}$-basis given by the *monomial quasisymmetric functions*

$$M_\alpha := \sum_{1 \le i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

as $\alpha = (\alpha_1, \ldots, \alpha_k)$ runs through all (ordered) compositions having $\alpha_i \in \{1, 2, \ldots\}$ and any length $k \ge 0$.

The ring QSym was introduced by Gessel [9]. He observed that if one defines the unitriangularly related $\mathbb{Z}$-basis of *fundamental quasisymmetric functions*

$$L_\alpha := \sum_{\beta : \beta \text{ coarsens } \alpha} M_\beta \tag{16}$$

then results from Stanley's theory of *P-partitions* [16, Cor. 7.19.5] imply the following expansion for products of L_α's. Given a subset $J = \{j_1 < \cdots < j_\ell\} \subseteq [n-1]$, define its *associated composition* of n to be

$$\alpha(J) := (j_1, j_2 - j_1, j_3 - j_2, \ldots, j_\ell - j_{\ell-1}, n - j_\ell).$$

In other words, $\alpha(J)$ is the composition whose partial sums are the elements of J.

For u, v in $\mathfrak{S}_a, \mathfrak{S}_b$, let $u \sqcup\!\sqcup\, v[a]$ be the set of all shuffles $w = (w_1, w_2, \ldots, w_{a+b})$ of the sequences $u = (u_1, \ldots, u_a)$, and $v[a] = (a + v_1, a + v_2, \ldots, a + v_b)$. In other words, $w \in \mathfrak{S}_{a+b}$ is in $u \sqcup\!\sqcup\, v[a]$ if $(u_1, \ldots, u_a)$ and $(a + v_1, \ldots, a + v_b)$ are subsequences of w.

Proposition 7.2. *Given u, v in $\mathfrak{S}_a, \mathfrak{S}_b$,*

$$L_{\alpha(\mathrm{Des}(u))} \cdot L_{\alpha(\mathrm{Des}(v))} = \sum_{w \,\in\, u \,\sqcup\!\sqcup\, v[a]} L_{\alpha(\mathrm{Des}(w))}. \tag{17}$$

This was part of Malvenuto and Reutenauer's motivation for the following definition.

Definition 7.3.
The *Malvenuto-Reutenauer (Hopf) algebra of permutations* is a graded free abelian group

$$\mathrm{FQSym} = \bigoplus_{n \ge 0} \mathrm{FQSym}_n,$$

in which FQSym_n has $\mathbb{Z}$-basis elements $\{\mathbf{w}\}_{w \in \mathfrak{S}_n}$. As an algebra, its multiplication is extended $\mathbb{Z}$-linearly from this rule: for u, v in $\mathfrak{S}_a, \mathfrak{S}_b$,

$$\mathbf{u} \cdot \mathbf{v} = \sum_{w \,\in\, u \,\sqcup\!\sqcup\, v[a]} \mathbf{w} \tag{18}$$

in which the sum runs over the same set of w as in (17).

Thus the algebra structure on FQSym was defined so that this map is a (surjective) algebra morphism:

$$\begin{array}{ccc} \mathrm{FQSym} & \xrightarrow{\varphi} & \mathrm{QSym} \\ \mathbf{w} & \longmapsto & L_{\alpha(\mathrm{Des}(w))} \end{array} \tag{19}$$

Definition 7.4.
Cheballah, Giraudo and Maurice [8] embedded FQSym inside a larger graded Hopf algebra

$$\mathcal{ASM} = \bigoplus_{n \geq 0} \mathcal{ASM}_n, \tag{20}$$

whose n^{th}-graded component $\mathcal{ASM}_n$ has a $\mathbb{Z}$-basis $\{\mathbf{A}\}$ indexed by A in ASM_n. Its algebra structure generalizes that of FQSym to the following *row-shuffle*[1] product. Given ASMs A, B of sizes $a \times a$ and $b \times b$, define $A \circ b$ to be the $a \times (a+b)$ matrix with first a columns A and last b columns all 0-vectors. Likewise, $a \circ B$ is the $b \times (a+b)$ matrix with last b columns B and first a columns all 0-vectors. Then define

$$\mathbf{A} \cdot \mathbf{B} = \sum_{C \,\in\, (A \circ b) \,\sqcup\!\sqcup\, (a \circ B)} \mathbf{C} \tag{21}$$

where C runs through all the $(a+b) \times (a+b)$ matrices obtained by shuffling the rows of $A \circ b$ and of $a \circ B$.

Example 7.5.
If $A = \begin{bmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$, then $A \circ b = \begin{bmatrix} \mathbf{0} & + & \mathbf{0} & 0 & 0 \\ + & - & + & 0 & 0 \\ \mathbf{0} & + & \mathbf{0} & 0 & 0 \end{bmatrix}$, and $a \circ B = \begin{bmatrix} 0 & 0 & 0 & \mathbf{0} & + \\ 0 & 0 & 0 & + & \mathbf{0} \end{bmatrix}$. One then has

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= \begin{bmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&+&0&0&0 \\ 0&0&0&0&+ \\ 0&0&0&+&0 \end{bmatrix} + \begin{bmatrix} 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&0&0&0&+ \\ 0&+&0&0&0 \\ 0&0&0&+&0 \end{bmatrix} + \begin{bmatrix} 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&0&0&0&+ \\ 0&0&0&+&0 \\ 0&+&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&+&0&0&0 \\ 0&0&0&0&+ \\ +&-&+&0&0 \\ 0&+&0&0&0 \\ 0&0&0&+&0 \end{bmatrix} + \begin{bmatrix} 0&+&0&0&0 \\ 0&0&0&0&+ \\ +&-&+&0&0 \\ 0&0&0&+&0 \\ 0&+&0&0&0 \end{bmatrix} \\
&+ \begin{bmatrix} 0&+&0&0&0 \\ 0&0&0&0&+ \\ 0&0&0&+&0 \\ +&-&+&0&0 \\ 0&+&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&+ \\ 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&+&0&0&0 \\ 0&0&0&+&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&+ \\ 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&0&0&+&0 \\ 0&+&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&+ \\ 0&+&0&0&0 \\ 0&0&0&+&0 \\ +&-&+&0&0 \\ 0&+&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&+ \\ 0&0&0&+&0 \\ 0&+&0&0&0 \\ +&-&+&0&0 \\ 0&+&0&0&0 \end{bmatrix}.
\end{aligned}$$

Note that when one restricts the product formula (21) to the elements of the form $\mathbf{w} := \mathbf{A(w)}$ where $A(w)$ is the permutation matrix corresponding to w^{-1}, it agrees with the multiplication rule for $\mathbf{u} \cdot \mathbf{v}$ given in (18). We also wish to recast the formula (21) in terms of monotone triangles. The following proposition is straightforward using the bijection $\mathrm{ASM}_n \to \mathrm{MT}_n$ described in the Introduction.

Proposition 7.6. *Fix A, B in $\mathrm{ASM}_a, \mathrm{ASM}_b$, with corresponding monotone triangles $T(A), T(B)$ in $\mathrm{MT}_a, \mathrm{MT}_b$. Let C in $(A \circ b) \sqcup\!\sqcup (a \circ B)$ have*

- *$S \subset [a+b]$ the a-element subset indexing the rows of C that come from $A \circ b$, and*
- *$[a+b] \setminus S$ the b-element subset indexing the rows of C that come from $a \circ B$.*

Then $T(C)$ in MT_{a+b} has as its k^{th} row the set

$$T(C)_k = T(A)_i \sqcup (a + T(B))_j,$$

where

- *$i = \#S \cap [k]$, and*
- *$j = \#([a+b] \setminus S) \cap [k]$ $(= k - i)$.*

Example 7.7.
For $A = \begin{bmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ as in Example 7.5, one has $T(A) = \begin{matrix} & 2 & \\ 1 & & 3 \\ 1 & 2 & 3 \end{matrix}$ and $a + T(B) = \begin{matrix} & 5 & \\ 4 & & 5 \end{matrix}$ Hence the terms $\mathbf{C}$ appearing in the product $\mathbf{A} \cdot \mathbf{B}$ correspond to these monotone triangles $T(C)$:

[1]Actually, in [8] the algebra structure uses column shuffles, but this is equivalent to what is described here after transposing the alternating sign matrices $A \mapsto A^t$.

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,4\}$	$\{1,3,5\}$
$T(C)$	2 1 3 1 2 3 1 2 3 5	2 1 3 1 3 5 1 2 3 5	2 1 3 1 3 5 1 3 4 5	5 2 5 1 3 5 1 2 3 5	5 2 5 1 3 5 1 3 4 5

S	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,4,5\}$	$\{3,4,5\}$
$T(C)$	2 2 5 2 4 5 1 3 4 5	5 2 5 1 3 5 1 2 3 5	5 2 5 1 3 5 1 3 4 5	2 2 5 2 4 5 1 3 4 5	2 4 5 2 4 5 1 3 4 5

Recall the statement of the theorem.

Theorem 1.3. The map $\mathrm{FQSym} \xrightarrow{\varphi} \mathrm{QSym}$ in (6) extends to an algebra (but not a coalgebra) morphism

$$\begin{array}{rcl} \mathcal{ASM} & \xrightarrow{\varphi} & \mathrm{FQSym} \\ \mathbf{A} & \longmapsto & L_{\alpha(\mathrm{Des}(A))} \end{array}$$

where $\mathrm{Des}(A) = \mathrm{Des}(T(A))$ for A in ASM_n is the descent set of its monotone triangle $T(A)$.

Proof of Theorem 1.3. Given A, B in $\mathrm{ASM}_a, \mathrm{ASM}_b$, we claim that the multiset of descent sets $\{\mathrm{Des}(T(C))\}$ as C runs through the elements of $(A \circ b) \sqcup\!\sqcup (a \circ B)$ depends only upon the descent sets $\mathrm{Des}(T(A)), \mathrm{Des}(T(B))$, not on A, B themselves. Assuming this claim for the moment, one finishes the proof by picking arbitrary u, v in $\mathfrak{S}_a, \mathfrak{S}_b$ having $\mathrm{Des}(u) = \mathrm{Des}(T(A))$ and $\mathrm{Des}(v) = \mathrm{Des}(T(B))$, and calculating

$$\begin{aligned} \varphi(\mathbf{A} \cdot \mathbf{B}) = \sum_{C \in (A \circ b) \sqcup\!\sqcup (a \circ B)} L_{\alpha(\mathrm{Des}(T(C)))} &= \sum_{w \in u \sqcup\!\sqcup v} L_{\alpha(\mathrm{Des}(w))} \\ &= L_{\alpha(\mathrm{Des}(u))} L_{\alpha(\mathrm{Des}(v))} = L_{\alpha(\mathrm{Des}(A))} L_{\alpha(\mathrm{Des}(B))} = \varphi(\mathbf{A})\varphi(\mathbf{B}). \end{aligned}$$

Here the second equality used the claim, while the third equality used (19).

To prove the claim, note that each C in $(A \circ b) \sqcup\!\sqcup (a \circ B)$ is determined by the a-subset $S \subset [a+b]$ indexing the rows of C that come from $A \circ b$. We give rules in cases below that decide whether some $k \in [a+b-1]$ lies in $\mathrm{Des}(T(C))$, based only on the subset S and the descent sets $\mathrm{Des}(T(A))$ and $\mathrm{Des}(T(B))$, not on A, B themselves. As notation, let $i := \#S \cap [k-1]$, $j := \#(([a+b] \setminus S) \cap [k-1]))$, and name these elements:

$$\begin{aligned} T(A)_{i+2} &= \{a_1 < \cdots < a_i < a_{i+1} < a_{i+2}\}, \\ (a + T(B))_{j+2} &= \{b_1 < \cdots < b_j < b_{j+1} < b_{j+2}\}. \end{aligned}$$

Note that deciding whether k lies in $\mathrm{Des}(T(C))$ simply means checking whether any of the entries of T'_k, where $T' := (T(C) \times \pi_k)_k = H_{\min}(T(C)_{k-1}, T(C)_{k+1})$, *differs* from the corresponding entry of $T(C)_k$, when computed via the formula (7) as the maximum of its two neighboring entries to the northwest and southwest.

Case 1. Both $k, k+1$ lie in S.
In this case, Proposition 7.6 implies that $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

$$\begin{array}{ccccccccccc} & & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & \\ & a_1 & \cdots & a_i & & a_{i+1} & & b_1 & \cdots & b_j & \\ a_1 & \cdots & a_i & & a_{i+1} & & a_{i+2} & & b_1 & \cdots & b_j \end{array}$$

Each of the entries b_m in $T(C)_k$ equals its northwest neighbor, so is unchanged in $T(C) \times \pi_k$. This implies that $k \in \mathrm{Des}(T(C))$ if and only if $k \in \mathrm{Des}(T(A))$.

Case 2. Both $k, k+1$ lie in $[a+b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

$$\begin{array}{ccccccccccc}
 & & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & \\
 & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & b_{j+1} & \\
a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & b_{j+1} & & b_{j+2}
\end{array}$$

Similarly to Case 1, each entry a_m in $T(C)_k$ equals its southwest neighbor, so is unchanged in $T(C) \times \pi_k$. This implies $k \in \mathrm{Des}(T(C))$ if and only if $k \in \mathrm{Des}(T(B))$.

Case 3. k lies in S, but $k+1$ lies in $[a+b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

$$\begin{array}{ccccccccccc}
 & & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & \\
 & a_1 & \cdots & a_i & & a_{i+1} & & b_1 & \cdots & b_j & \\
a_1 & \cdots & a_i & & a_{i+1} & & b_1 & \cdots & b_j & & b_{j+1}
\end{array}$$

We claim that in this case, $k \notin \mathrm{Des}(T(C))$, since each entry a_m of $T(C)_k$ equals its southwest neighbor, while each entry b_m of $T(C)_k$ equals its northwest neighbor.

Case 4. $k+1$ lies in S, but k lies in $[a+b] \setminus S$.
Here $(T(C)_{k-1}, T(C)_k, T(C)_{k+1})$ look like this:

$$\begin{array}{ccccccccccc}
 & & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & \\
 & a_1 & \cdots & a_i & & b_1 & \cdots & b_j & & b_{j+1} & \\
a_1 & \cdots & a_i & & a_{i+1} & & b_1 & \cdots & b_j & & b_{j+1}
\end{array}$$

In this case $k \in \mathrm{Des}(T(C))$, since the entry b_1 of $T(C)_k$ has $b_1 > a \geq \max(a_i, a_{i+1})$.

To see that $\mathbf{A} \overset{\varphi}{\longmapsto} L_{\alpha(\mathrm{Des}(A))}$ is not a coalgebra morphism, for example, one can check from the coproduct formula of Cheballah, Giraudo and Maurice [8, (1.3.5)] that the alternating sign matrix $A = \begin{bmatrix} 0 & +1 & 0 \\ +1 & -1 & +1 \\ 0 & +1 & 0 \end{bmatrix}$ has coproduct $\Delta(\mathbf{A}) = 1 \otimes \mathbf{A} + \mathbf{A} \otimes 1$, that is, $\mathbf{A}$ is primitive. Meanwhile, its image $\varphi(\mathbf{A}) = L_{(1,1,1)}$ has

$$\Delta(\varphi(\mathbf{A})) = \Delta(L_{(1,1,1)}) = 1 \otimes L_{(1,1,1)} + L_{(1)} \otimes L_{(1,1)} + L_{(1,1)} \otimes L_{(1)} + L_{(1,1,1)} \otimes 1,$$

which is not the same as $(\varphi \otimes \varphi)(\Delta(\mathbf{A})) = 1 \otimes L_{(1,1,1)} + L_{(1,1,1)} \otimes 1$. That is, $\varphi(\mathbf{A})$ is *not* primitive. □

Remark 7.8.
It is well-known, and not hard to see (e.g., as a special case of [16, Thm. 7.19.7]), that applying φ to the sum of all of the basis elements $\{\mathbf{w}\}_{w \in S_n}$ for FQSym_n gives a readily-identifiable symmetric function

$$\varphi\left(\sum_{w \in \mathfrak{S}_n} \mathbf{w}\right) = \sum_{w \in \mathfrak{S}_n} L_{\alpha(\mathrm{Des}(w))} = (x_1 + x_2 + \cdots)^n.$$

This fails for ASM_n, e.g., the data in Table 1 for $n = 4$ together with (16) shows that

$$\begin{aligned}
\varphi\left(\sum_{A \in \mathrm{ASM}_n} \mathbf{A}\right) &= \sum_{A \in \mathrm{ASM}_4} L_{\alpha(\mathrm{Des}(T(A)))} \\
&= L_{(4)} + 3L_{(1,3)} + 5L_{(2,2)} + 3L_{(3,1)} + 7L_{(1,1,2)} + 7L_{(1,2,1)} + 7L_{(2,1,1)} + 9L_{(1,1,1,1)} \\
&= M_{(4)} + 4M_{(1,3)} + 6M_{(2,2)} + 4M_{(3,1)} + 16M_{(1,1,2)} + 14M_{(1,2,1)} + 16M_{(2,1,1)} + 42M_{(1,1,1,1)}
\end{aligned}$$

which is not a symmetric function, because its coefficient on M_α is not constant for all compositions α within the same rearrangement class. It would be interesting to find natural subcollections $\{A\}$ of ASM_n, not contained entirely in $\mathfrak{S}_n$, for which $\varphi\left(\sum_A \mathbf{A}\right)$ is a symmetric function.

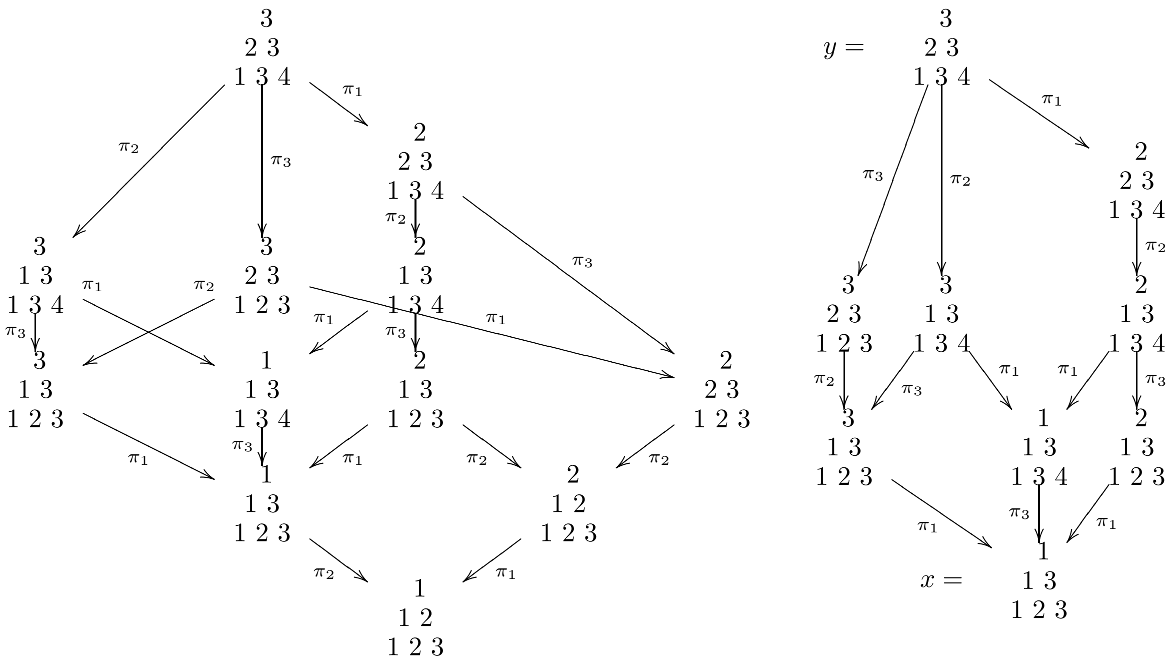


Figure 1. An interval of weak order in MT_4 that is not a lattice, and a subinterval within it.

8. Poset properties of weak order on MT_n

The weak order $<_W$ on the symmetric group $\mathfrak{S}_n$ has many pleasant poset-theoretic properties:

- It has bottom and top elements $\hat{0} = e = (1, 2, \ldots, n-1, n)$ and $\hat{1} = w_0 = (n, n-1, \ldots, 2, 1)$.
- It is a *lattice*.
- It is *ranked* with rank function given by the cardinality $\#\mathrm{Inv}(w)$ of the *(left-)inversion set* of w:

$$\mathrm{Inv}(w) = \{(w_j, w_i) : 1 \le i < j \le n \text{ and } w_i > w_j\}.$$

- It has an encoding via *inclusion* of these (left-)inversion sets: $u <_W v$ if and only if $\mathrm{Inv}(u) \subset \mathrm{Inv}(v)$.
- The *Möbius function* $\mu(u, v)$ for $u <_W v$ only takes on values in $\{0, +1, -1\}$.
- More precisely, the *homotopy type* of the order complex $\Delta(u, v)$ of any of its open intervals (u, v) is *contractible* or *homotopy-spherical*. Specifically, one can phrase this in terms of $\mathcal{H}_n(0)$-action on S_n as follows: $\Delta(u, v)$ is contractible unless $u = v \times \pi_{w_0(J)}$ for some subset $J \subset \mathrm{Des}(u)$, in which case, $\Delta(u, v)$ is homotopy-equivalent to a $(\#J - 2)$-dimensional sphere; see Björner [2, Theorem 6].

Only a few of these properties extend to the weak order $<_W$ to MT_n. It is still true that $(\mathrm{MT}_n, <_W)$ has a bottom element $\hat{0} = T(e) = (\varnothing, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, [n])$, but it no longer has a top element $\hat{1}$, as there are many maximal elements.

Since MT_n is finite, and has no top element, it cannot be a *lattice*, but it is also true that its intervals fail to be lattices. For example, the lower interval shown on the left in Figure 1 is not a lattice, because, for example,

$$\begin{array}{c}1\\1\ 3\\1\ 2\ 3\end{array} \quad\text{and}\quad \begin{array}{c}2\\1\ 2\\1\ 2\ 3\end{array} \quad \text{do not have a least upper bound since both} \quad \begin{array}{c}2\\1\ 3\\1\ 2\ 3\end{array} \quad\text{and}\quad \begin{array}{c}3\\2\ 3\\1\ 2\ 3\end{array}$$

are minimal upper bounds. Note that this same lower interval is not ranked since there are maximal chains of lengths four and five.

Alternating sign matrices $A = (a_{ij})$ have a well-established *inversion number* $\mathrm{inv}(A) := \sum_{i<k,\ j>\ell} a_{ij} a_{k\ell}$, introduced by Mills, Robbins and Rumsey [12, p344], which generalizes the rank function $\#\mathrm{Inv}(w)$ for

$(\mathfrak{S}_n, <_W)$ of permutations. However, it is not clear that it relates to chains in the weak order $(\mathrm{MT}_n, <_W)$. For example, one might hope that the length of the shortest saturated chain from $\hat{0}$ to T in weak order might correspond to the inversion number of the alternating sign matrix of T. However, Roger Behrend noted that this fails for the first time in MT_4, where one can check that

$$T = \begin{array}{c} 3 \\ 2\ 4 \\ 1\ 3\ 4 \end{array} \quad \leftrightarrow \quad A = \begin{bmatrix} 0 & 0 & + & 0 \\ 0 & + & - & + \\ + & - & + & 0 \\ 0 & + & 0 & 0 \end{bmatrix}, \quad \mathrm{inv}(A) = 5$$

but the shortest saturated chain from $\hat{0}$ to T has length 4. Additionally, in MT_5 one can check that

$$T = \begin{array}{c} 3 \\ 3\ 4 \\ 1\ 4\ 5 \\ 1\ 2\ 4\ 5 \end{array} \quad \leftrightarrow \quad A = \begin{bmatrix} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & + & 0 \\ + & 0 & - & 0 & + \\ 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 \end{bmatrix}, \quad \mathrm{inv}(A) = 5$$

but all saturated chains in the weak order from $\hat{0}$ to T have length at least 6.

Question 8.1. *Is there a generalization of the notion of the (left-)inversion set* $\mathrm{Inv}(w)$ *for permutations to an inversion set* $\mathrm{Inv}(T)$ *for monotone triangles, encoding the weak order* $(\mathrm{MT}_n, <_W)$ *via inclusion, that is,* $T <_W T'$ *if and only if* $\mathrm{Inv}(T) \subset \mathrm{Inv}(T')$*?*

In spite of some of the above shortcomings, the Möbius function and homotopy type of open intervals in $(\mathrm{MT}_n, <_W)$ may be just as simple to describe as for weak order on $\mathfrak{S}_n$.

Conjecture 8.2. *For two monotone triangles $T' \leq_W T$, the order complex $\Delta(T', T)$ of their open interval in $<_W$ is contractible unless $T' = T \times \pi_{w_0(J)}$ for some $J \subset \mathrm{Des}(T)$, namely, $J := \{m : T'_m \neq T_m\}$, in which case $\Delta(T', T)$ is homotopy equivalent to a $(\#J - 2)$-dimensional sphere.*

Conjecture 8.2 would imply that $\mu(T', T) = 0$ in the contractible case, and $(-1)^{\#J}$ when $T' = T \times \pi_{w_0(J)}$.

Example 8.3.
An interesting example is the non-lattice lower interval $[\hat{0}, y]$ on the left in Figure 1, which has the order complex $\Delta(\hat{0}, y)$ of its open interval homotopy equivalent to a 1-sphere (circle). Meanwhile, its subinterval $[x, y]$ shown to its right has $\Delta(x, y)$ contractible.

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