

BOHEMIAN UPPER HESSENBERG TOEPLITZ MATRICES

EUNICE Y. S. CHAN*, ROBERT M. CORLESS*, LAUREANO GONZALEZ-VEGA†, J. RAFAEL SENDRA‡, JUANA SENDRA§, AND STEVEN E. THORNTON*

Abstract. We look at Bohemian matrices, specifically those with entries from $\{-1, 0, +1\}$. More, we specialize the matrices to be upper Hessenberg, with subdiagonal entries 1. Even more, we consider Toeplitz matrices of this kind. Many properties remain after these specializations, some of which surprised us. Focusing on only those matrices whose characteristic polynomials have maximal height allows us to explicitly identify these polynomials and give a lower bound on their height. This bound is exponential in the order of the matrix.

1. Introduction. A matrix family is called **Bohemian** if its entries come from a fixed finite discrete (and hence bounded) set, usually integers. The name is a mnemonic for **B**ounded **H**eight **M**atrix of **I**ntegers. Such families arise in many applications (e.g. compressed sensing) and the properties of matrices selected “at random” from such families are of practical and mathematical interest. An overview of some of our original interest in Bohemian matrices can be found in [4].

We began our study by considering Bohemian upper Hessenberg matrices. We proved two recursive formulae for the characteristic polynomials of upper Hessenberg matrices (see [3] for details). During the course of our computations, we encountered “maximal polynomial height” characteristic polynomials when the matrices were not only upper Hessenberg, but Toeplitz ($h_{i,j}$ constant along diagonals $j - i = k$). Further restrictions to this class allowed identification of key results including explicit formulae for the characteristic polynomials of maximal height, which motivates this paper. In what follows, we lay out definitions and prove several facts of interest about characteristic polynomials and their respective height for these families.

In Figure 1, we see all the eigenvalues of all 14×14 upper Hessenberg Toeplitz matrices with subdiagonal entries equal to 1 and all other entries from the population $\{-1, 0, +1\}$. We see a wide irregularly hexagonal shape. In contrast, upper Hessenberg Bohemian matrices that are *not* Toeplitz generate an irregular octagonal shape (see [3]). More, the density of eigenvalues (here, a darker colour indicates higher density of eigenvalues) is quite irregular, with high-density flecks dispersed throughout. In some ways the picture is reminiscent of seeds in a cotton ball, if the cotton ball has been flattened. The conjugate symmetry and $z \rightarrow -z$ symmetry are evident; to save space, we could have plotted only the first quadrant, but for completeness have included all four. This helps to show that there is a slightly lower density of eigenvalues near (not on) the real line. The density of eigenvalues actually *on* the real line is quite high, although this is not evident from the picture.

The one thing that is easily explained about that figure is the wide flat top (and bottom). To do this, consider eigenvalues of Bohemian Upper Hessenberg Toeplitz matrices with *zero diagonal*. Figure 2 is a picture of the set of eigenvalues of all 14×14 upper Hessenberg Toeplitz matrices with subdiagonal entries equal to 1, diagonal entries equal to 0, and all other entries from the population $\{-1, 0, +1\}$. Here, we

*Department of Applied Mathematics, Western University (echan295@uwo.ca, rcorless@uwo.ca, sthornt7@uwo.ca).

†Departamento de Matematicas, Estadística y Computación, Universidad de Cantabria (laureano.gonzalez@unican.es).

‡Research Group ASYNACS, Departamento de Física y Matemáticas, University of Alcalá (rafael.sendra@uah.es).

§Universidad Politécnica de Madrid (jsendra@etsist.upm.es).

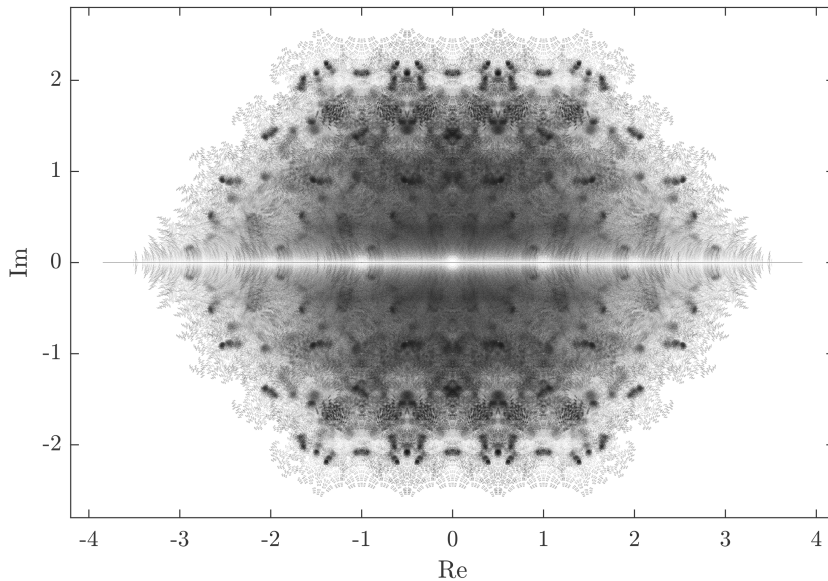


FIG. 1. *The set of eigenvalues of all 14×14 upper Hessenberg Toeplitz matrices with subdiagonal entries equal to 1, and all other entries from the set $\{-1, 0, +1\}$. A more detailed image can be found at assets.bohemianmatrices.com/gallery/UHT_14x14.png*

also see a hexagonal shape, but this time, it is not as wide. The matrices B giving rise to Figure 1 are exactly the matrices $B = A$, $B = A + I$ and $B = A - I$ where the matrices A give rise to Figure 2; thus the eigenvalues of each A occur three times, once with zero shift, once with -1 shift, and once with 1 shift. That is, Figure 1 is simply three copies of Figure 2 placed side by side, giving the appearance of a flat (or mostly flat) top and bottom.

In Figure 2 we see more clearly that the high-density “flecks” occur moderately near to the edge of the eigenvalue inclusion region. We have no explanation for this. We also see that the eigenvalues fit into a rough diamond shape; one wonders if the eigenvalues $\lambda = x + iy$ fit into a region of shape $|x| + |y| \leq O(\sqrt{n})$. Again, we have no explanation for this (or even much data; we do not know if this guess is even correct experimentally).

In this paper we seek to explain some other features of these pictures, and to learn more about Bohemian upper Hessenberg Toeplitz matrices. We provide supplementary material through a git repository available at <https://github.com/BohemianMatrices/Bohemian.Upper.Hessenberg-Toeplitz.Matrices>. This repository provides all code and data used to generate the results, figures, and tables in this paper.

2. Prior Work. In our sister paper “Bohemian Upper Hessenberg Matrices” [3], we introduced the following theorems, definitions, remarks, and propositions for upper

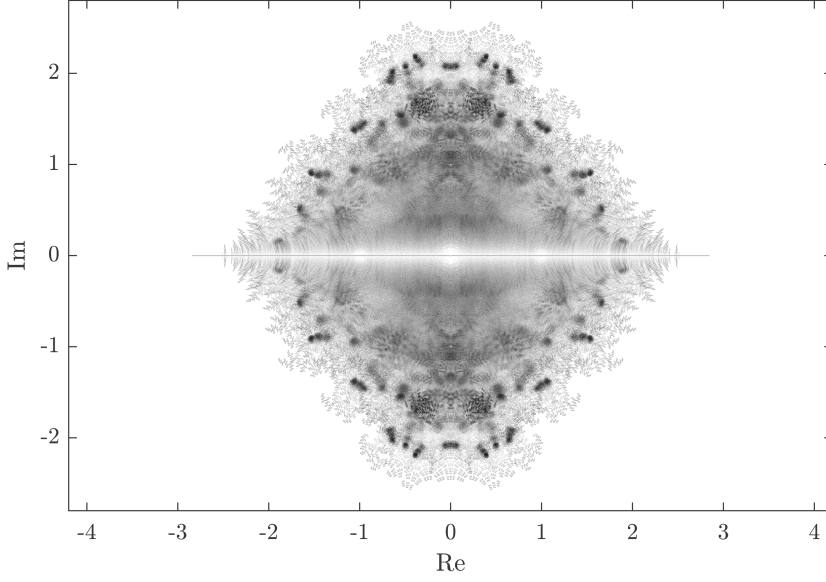


FIG. 2. The set of eigenvalues of all 14×14 upper Hessenberg Toeplitz matrices subdiagonal entries equal to 1, diagonal entries equal to 0, and all other entries from the set $\{-1, 0, +1\}$. A more detailed image can be found at assets.bohemianmatrices.com/gallery/UHT_0-Diag_14x14.png

Hessenberg Bohemian matrices of the form

$$(2.1) \quad \mathbf{H}_n = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,n} \\ s & h_{2,2} & h_{2,3} & \cdots & h_{2,n} \\ 0 & s & h_{3,3} & \cdots & h_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s & h_{n,n} \end{bmatrix}$$

with characteristic polynomial $Q_n(z) \equiv \det(z\mathbf{I} - \mathbf{H}_n)$.

DEFINITION 2.1. The set of all $n \times n$ Bohemian upper Hessenberg matrices with upper triangle population P and subdiagonal population from a discrete set of roots of unity, say $s \in \{e^{i\theta_k}\}$ where $\{\theta_k\}$ is some finite set of angles, is called $\mathcal{H}_{\{\theta_k\}}^{n \times n}(P)$. In particular, $\mathcal{H}_{\{0\}}^{n \times n}(P)$ is the set of all $n \times n$ Bohemian upper Hessenberg matrices with upper triangle entries from P and subdiagonal entries equal to 1 and $\mathcal{H}_{\{\pi\}}^{n \times n}(P)$ is when the subdiagonal entries are -1 .

THEOREM 2.2.

$$(2.2) \quad Q_n(z) = zQ_{n-1}(z) - \sum_{k=1}^n s^{k-1} h_{n-k+1,n} Q_{n-k}(z)$$

with the convention that $Q_0(z) = 1$ ($\mathbf{H}_0 = []$, the empty matrix).

THEOREM 2.3. *Expanding $Q_n(z)$ as*

$$(2.3) \quad Q_n(z) = q_{n,n}z^n + q_{n,n-1}z^{n-1} + \cdots + q_{n,0},$$

we can express the coefficients recursively by

$$(2.4a) \quad q_{n,n} = 1,$$

$$(2.4b) \quad q_{n,j} = q_{n-1,j-1} - \sum_{k=1}^{n-j} s^{k-1} h_{n-k+1,n} q_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(2.4c) \quad q_{n,0} = - \sum_{k=1}^n s^{k-1} h_{n-k+1,n} q_{n-k,0} \quad \text{for } n > 0, \quad \text{and}$$

$$(2.4d) \quad q_{0,0} = 1.$$

DEFINITION 2.4. *The characteristic height of a matrix is the height of its characteristic polynomial.*

PROPOSITION 2.5. *For any matrix \mathbf{A} , $-\mathbf{A}$ has the same characteristic height as \mathbf{A} .*

PROPOSITION 2.6. *The maximal characteristic height of $\mathbf{H}_n \in \mathcal{H}_{\{0,\pi\}}^{n \times n}(\{-1, 0, +1\})$ occurs when $s^{k-1} h_{i,i+k-1} = -1$ for $1 \leq i \leq n-k+1$ and $1 \leq k \leq n$.*

3. Upper Hessenberg Toeplitz Matrices. For the remainder of the paper consider upper Hessenberg matrices with a Toeplitz structure of the form

$$(3.1) \quad \mathbf{M}_n = \begin{bmatrix} t_1 & t_2 & t_3 & \cdots & t_n \\ 1 & t_1 & t_2 & \cdots & t_{n-1} \\ 0 & 1 & t_1 & \cdots & t_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & t_1 \end{bmatrix}$$

with $t_k \in \{-1, 0, +1\}$ for $1 \leq k \leq n$. Let

$$(3.2) \quad P_n(z) \equiv \det(z\mathbf{I} - \mathbf{M}_n) = \sum_{k=0}^n p_{n,k} z^k$$

be the characteristic polynomial of \mathbf{M}_n with $p_{n,n} = 1$.

PROPOSITION 3.1. *The characteristic polynomial recurrence from Theorem 2.2 can be written for upper Hessenberg Toeplitz matrices as*

$$(3.3) \quad P_n(z) = zP_{n-1}(z) - \sum_{k=1}^n t_k P_{n-k}(z)$$

with the convention that $P_0(z) = 1$ ($\mathbf{M}_0 = []$, the empty matrix).

Proof. For a matrix \mathbf{M}_n , the entries at the i th row and the $i+k-1$ -th column for $1 \leq i \leq n-k+1$ (i.e. the $k-1$ -th diagonal) are all equal to t_k . In equation (2.2), we can replace $h_{n-k+1,n}$ with t_k ($i = n-k+1$) recovering equation (3.3). \square

PROPOSITION 3.2. *The characteristic polynomial recurrence from Theorem 2.3 can be written for upper Hessenberg Toeplitz matrices as*

$$(3.4a) \quad p_{n,n} = 1,$$

$$(3.4b) \quad p_{n,j} = p_{n-1,j-1} - \sum_{k=1}^{n-j} t_k p_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(3.4c) \quad p_{n,0} = - \sum_{k=1}^n t_k p_{n-k,0}, \text{ and}$$

$$(3.4d) \quad p_{0,0} = 1.$$

Proof. Performing the same replacement as above (a notational change), we recover equation (3.4). \square

PROPOSITION 3.3. $p_{n,i}$ is independent of t_j for $j > n - i$.

Proof. First, assume $p_{n,\ell}$ is a function of $t_1, \dots, t_{n-\ell}$ for $\ell = i$ and all n . By Proposition 3.2

$$(3.5) \quad p_{n,\ell} = p_{n-1,\ell-1} - \sum_{k=1}^{n-\ell} t_k p_{n-k,\ell}.$$

Isolating the $p_{n-1,\ell-1}$ term, we have

$$(3.6) \quad p_{n-1,\ell-1} = p_{n,\ell} + \sum_{k=1}^{n-\ell} t_k p_{n-k,\ell}$$

The first term, $p_{n,\ell}$, is a function of $t_1, \dots, t_{n-\ell}$. Each term $t_k p_{n-k,\ell}$ in the sum is a function of $t_1, \dots, t_{n-k-\ell}, t_k$. Taking $k = n - \ell$, we have the sum is a function of $t_1, \dots, t_{n-\ell}$. Hence, $p_{n-1,\ell-1}$ is a function of $t_1, \dots, t_{n-1-(\ell-1)} = t_{n-\ell}$.

When $i = 0$, by Proposition 3.2 we have

$$(3.7) \quad p_{n,0} = - \sum_{k=1}^n t_k p_{n-k,0}$$

which is a function of t_1, \dots, t_n . \square

THEOREM 3.4. The set of characteristic polynomials for all matrices \mathbf{M}_n with $t_k \in \{-1, 0, +1\}$ for $1 \leq k \leq n$ has cardinality 3^n .

Proof. Let

$$(3.8) \quad \mathbf{A}_n = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & a_1 \end{bmatrix}$$

with $a_k \in \{-1, 0, +1\}$ for $1 \leq k \leq n$. Let $R_n(z; a_1, \dots, a_n)$ be the characteristic polynomial of \mathbf{A}_n . Assume $P_\ell = R_\ell$ for $\ell < n$. By Proposition 3.1, for \mathbf{A}_n and \mathbf{M}_n to have the same characteristic polynomial we find

$$(3.9) \quad zP_{n-1} - \sum_{k=1}^n t_k P_{n-k} = zR_{n-1} - \sum_{k=1}^n a_k R_{n-k}.$$

Since $P_\ell = R_\ell$ for all $\ell < n$, and the $\sum_{k=1}^n t_k P_{n-k}$ and $\sum_{k=1}^n a_k R_{n-k}$ terms are polynomials of degree $n-1$ in z , we find $P_n = R_n$ only when $t_k = a_k$ for all $1 \leq k \leq n$ (the zP_{n-1} and zR_{n-1} terms are the only terms of degree n in z). Hence, for each combination of t_k , no other upper Hessenberg Toeplitz matrix with $t_k \in \{-1, 0, +1\}$ and subdiagonal 1 has the same characteristic polynomial. \square

4. Maximal Characteristic Height Upper Hessenberg Toeplitz Matrices.

THEOREM 4.1. *The characteristic height of \mathbf{M}_n is maximal when $t_k = -1$ for $1 \leq k \leq n$.*

Proof. Following from Proposition 2.6, the entries in the i th row and $i+k-1$ -th column for $1 \leq i \leq n-k+1$ correspond to t_k , after substituting $s = 1$ we find $t_k = -1$ gives the maximal characteristic height. \square

PROPOSITION 4.2. *Let $F \subset \mathbb{R}$ be a closed and bounded set with $a = \min F$, $b = \max F$ and $\#F \geq 2$. Let \mathbf{M}_n be upper Hessenberg Toeplitz with $t_k \in F$. If $|a| \geq |b|$, \mathbf{M}_n attains maximal characteristic height for $t_k = a$ for all $1 \leq k \leq n$. If $|b| \geq |a|$, \mathbf{M}_n attains maximal characteristic height for $t_k = a$ for k even, and $t_k = b$ for k odd.*

Proof. First, consider the case when $|a| \geq |b|$. Since $a < b$ we find $a < 0$. Let $\bar{t}_k = -t_k$. Writing Proposition 2.5 in terms of \bar{t}_k gives

$$(4.1a) \quad p_{n,n} = 1,$$

$$(4.1b) \quad p_{n,j} = p_{n-1,j-1} + \sum_{k=1}^{n-j} \bar{t}_k p_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(4.1c) \quad p_{n,0} = \sum_{k=1}^n \bar{t}_k p_{n-k,0}, \text{ and}$$

$$(4.1d) \quad p_{0,0} = 1.$$

If all \bar{t}_k are positive then $p_{n,j}$ must be positive for all n and j . Hence, the maximal characteristic height is attained when \bar{t}_k is maximal, or equivalently t_k is minimal and negative. Thus $t_k = \min F = a$ gives maximal characteristic height.

Next, consider when $|b| \geq |a|$. Since $a < b$ we find $b > 0$. By Proposition 2.5 we know that the characteristic height of \mathbf{M}_n is equal to the characteristic height of $-\mathbf{M}_n$. Rewriting Proposition 3.2 for $-\mathbf{M}_n$ by substituting $p_{n,j}$ with $(-1)^{n-j} p_{n,j}$ we find the recurrence for the characteristic polynomial of $-\mathbf{M}_n$:

$$(4.2a) \quad p_{n,n} = 1,$$

$$(4.2b) \quad p_{n,j} = p_{n-1,j-1} + \sum_{k=1}^{n-j} (-1)^{k-1} t_k p_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(4.2c) \quad p_{n,0} = \sum_{k=1}^n (-1)^{k-1} t_k p_{n-k,0}, \text{ and}$$

$$(4.2d) \quad p_{0,0} = 1.$$

Separating out the even and odd values of k in the sums we can write the recurrence as

$$(4.3a) \quad p_{n,n} = 1,$$

$$(4.3b) \quad p_{n,j} = p_{n-1,j-1} + \sum_{k \text{ odd}}^{n-j} t_k p_{n-k,j} - \sum_{k \text{ even}}^{n-j} t_k p_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(4.3c) \quad p_{n,0} = \sum_{k \text{ odd}}^n t_k p_{n-k,0} - \sum_{k \text{ even}}^n t_k p_{n-k,0}, \text{ and}$$

$$(4.3d) \quad p_{0,0} = 1.$$

The odd sums are maximal for $t_k = \max F = b$ and the even sums are maximal for $t_k = \min F = a$. Hence, the maximal characteristic height is attained for $t_k = b$ when k is odd, and $t_k = a$ when k is even.

When $|a| = |b|$, equations (4.1) and (4.3) are equivalent and the maximal height is attained both when $t_k = b$ for all k , and $t_k = b$ for k odd and $t_k = a$ for k even. \natural

PROPOSITION 4.3. \mathbf{M}_n also attains maximal characteristic height when $t_k = (-1)^{k-1}$ for $1 \leq k \leq n$. \blacksquare

Proof. By Proposition 4.2, we have $F = \{-1, 0, +1\}$ with $a = -1$, and $b = +1$. Thus \mathbf{M}_n is also of maximal characteristic height for $t_k = b = +1$ for odd values of k , and $t_k = a = -1$ for even values of k . \natural

PROPOSITION 4.4. *The maximum characteristic height grows at least exponentially in n .*

Proof. When $t_k = -1$, the characteristic height is maximal by Theorem 4.1. Equation (3.4c) from Proposition 3.2 reduces to

$$(4.4) \quad p_{n,0} = \sum_{k=1}^n p_{n-k,0} = 2^{n-1}$$

for $n \geq 1$ with $p_{0,0} = 1$ by equation (3.4d). Thus, the maximal characteristic height must grow at least exponentially in n . \natural

CONJECTURE 4.5. *The maximum characteristic height approaches $C(1 + \varphi)^n$ as $n \rightarrow \infty$ for some constant C where φ is the golden ratio.*

Remark 4.6. This limit is illustrated in Figure 3, motivating this conjecture.

PROPOSITION 4.7. *Let $\overline{\mathbf{M}}_n$ be of maximal characteristic height and let μ_n be the degree of the term of the characteristic polynomial of $\overline{\mathbf{M}}_n$ corresponding to the height. The characteristic height of $\overline{\mathbf{M}}_n$ is independent of t_j for $j > n - \mu_n$.*

Proof. Let P_n be the characteristic polynomial of $\overline{\mathbf{M}}_n$. By Proposition 3.3, p_{n,μ_n} is independent of t_j for $j > n - \mu_n$. Thus, t_j for $j > n - \mu_n$ only affects $p_{n,k}$ for $k < \mu_n$. Since $\overline{\mathbf{M}}_n$ is of maximal height, $|p_{n,k}| \leq |p_{n,\mu_n}|$ for $k < \mu_n$ for all $t_j \in \{-1, 0, +1\}$ with $j > n - \mu_n$. \natural

n	μ_n	τ_n
2	1	2
3	1	5
4	1	12
5	1	27
6	2	66
7	2	168
8	2	416
9	2	1,008
10	3	2,528

TABLE 1

Maximum height τ_n and degree of term of characteristic polynomial corresponding to maximum height μ_n upper Hessenberg Toeplitz matrices for n from 2 to 10.

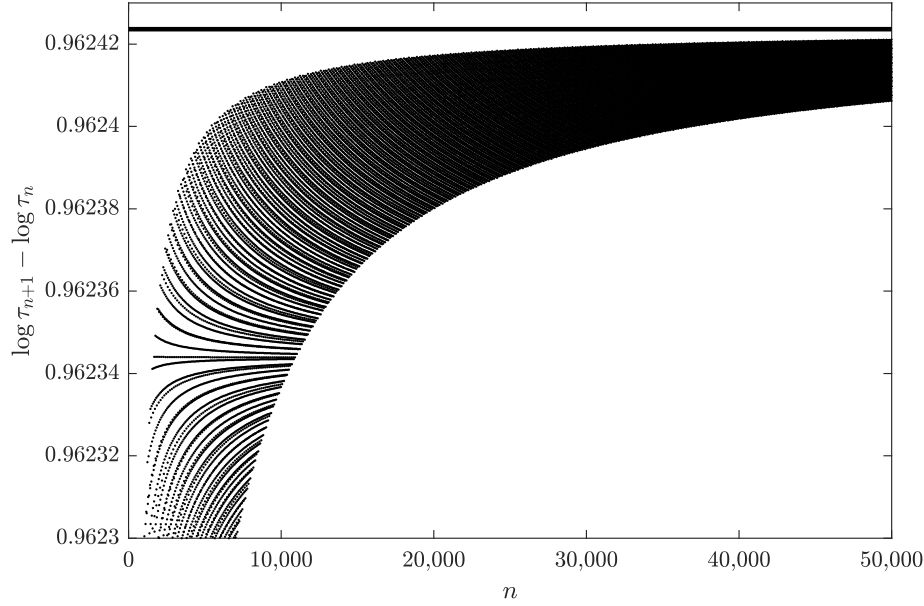


FIG. 3. The points are $\log \tau_{n+1} - \log \tau_n$ for n from 0 to 50,000 where τ_n is the maximal characteristic height of \mathbf{M}_n (i.e. when $t_k = -1$, for example). The solid line is $\log(1 + \varphi)$ where φ is the golden ratio.

PROPOSITION 4.8. For fixed n , μ_n is the same for all matrices $\overline{\mathbf{M}}_n$ of maximal characteristic height.

Proof. The characteristic polynomial of \mathbf{M}_n when $t_k = -1$ has the same coefficients as the characteristic polynomial of \mathbf{M}_n for $t_k = (-1)^{k-1}$ up to a sign change. By Proposition 4.7, changing any of the entries t_j of $\overline{\mathbf{M}}_n$ for $j > n - \mu_n$ does not affect the value of μ_n . Therefore μ_n is fixed. \square

THEOREM 4.9. The number of upper Hessenberg Toeplitz matrices of dimension n with $t_k \in \{-1, 0, +1\}$ for $1 \leq k \leq n$ of maximal characteristic height is $2 \cdot 3^{\mu_n}$.

Proof. By Theorem 4.1 and Proposition 4.3, there are two matrices that attain maximal characteristic height. By Proposition 4.7, any combination of $t_j \in \{-1, 0, +1\}$ for $j > n - \mu_n$ will not affect the characteristic height. Thus there are 3^{μ_n} combinations of t_j that result in the same characteristic height for each of the two choices of t_k that give maximal characteristic height. \square

Remark 4.10. We have found that μ_n remains constant for 3 or 4 subsequent values of n followed by an increment by 1. We have verified this pattern experimentally up to degree 50,000. Figure 4 shows the pattern for matrix dimension up to 100.

Remark 4.11. The sequence $\mu_{n+1} - \mu_n$ is nearly equivalent to the sequence for the generalized Fibonacci word $f^{[3]}$

$$(4.5) \quad a(n) = \left\lfloor \frac{n+2}{\varphi+2} \right\rfloor - \left\lfloor \frac{n+1}{\varphi+2} \right\rfloor$$

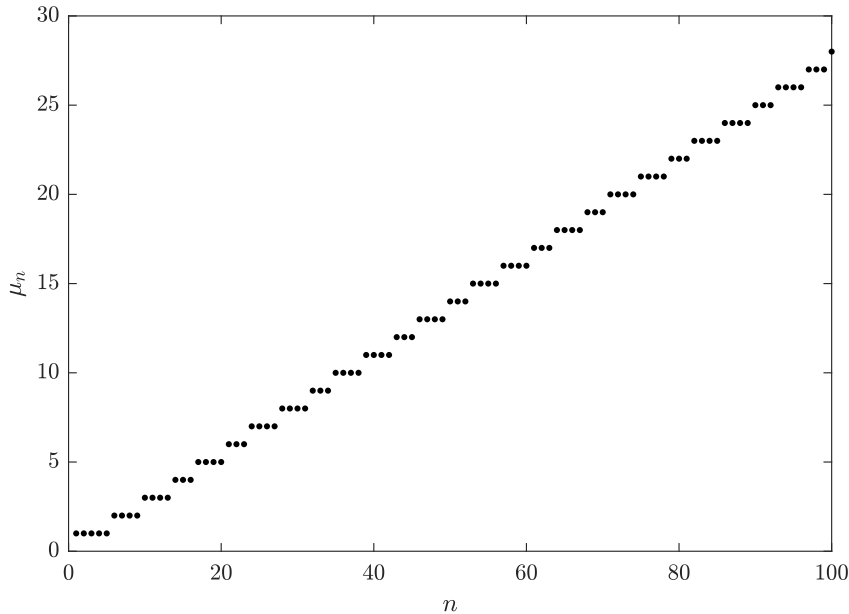


FIG. 4. Degree of the term corresponding to the height of the characteristic polynomial of an $n \times n$ upper Hessenberg Toeplitz matrix of maximal characteristic height.

(A221150 on the OEIS). We have found that up to at least degree 50,000, $\mu_{n+1} - \mu_n = a(n + 326)$ except when $n \in \{0, 2, 24, 148, 24, 149\}$.

Remark 4.12. The sequence μ_n is nearly equivalent to the sequence

$$(4.6) \quad \left\lfloor \frac{n + 327}{\varphi + 2} \right\rfloor - 90$$

for $n > 2$. The two sequences are equal for all values up to $n = 50,000$ except when $n = 24, 149$.

The sequences presented in the previous remarks are examples of *high-precision fraud* [2] requiring evaluation up to dimension 25,000 and nearly 25,000 digits of precision to identity.

5. Maximal Height Characteristic Polynomials. In this section we restrict our analysis to specific upper Hessenberg Toeplitz matrices of maximal characteristic height, that is $t_k = -1$ for all k . We denote a dimension n matrix of this form by $\widetilde{\mathbf{M}}_n$. $\widetilde{\mathbf{M}}_n$ is of maximal height by Proposition 4.3.

PROPOSITION 5.1. *The characteristic polynomial of $\widetilde{\mathbf{M}}_n$ is of the form*

$$(5.1) \quad P_n = z^n + p_{n,n-1}z^{n-1} + \cdots + p_{n,0}$$

where $p_{n,j}$ is positive for all n and j .

Proof. When $t_k = -1$ for $1 \leq k \leq n$, Proposition 3.2 reduces to

$$(5.2a) \quad p_{n,n} = 1,$$

$$(5.2b) \quad p_{n,j} = p_{n-1,j-1} + \sum_{k=1}^{n-j} p_{n-k,j} \quad \text{for } 1 \leq j \leq n-1,$$

$$(5.2c) \quad p_{n,0} = \sum_{k=1}^n p_{n-k,0}, \text{ and}$$

$$(5.2d) \quad p_{0,0} = 1.$$

Since $p_{0,0}$ is positive, and all coefficients in the above equations are positive, $p_{n,j}$ must be positive for all n and j . \dagger

PROPOSITION 5.2. *The generating function of the sequence $(p_{i,i}, p_{i+1,i}, \dots)$ for all $i \geq 0$ is*

$$(5.3) \quad G_i(x) = \left(\frac{1-x}{1-2x} \right)^{i+1}.$$

Proof. First we will prove the $i = 0$ case. Let

$$(5.4) \quad G_0(x) = \sum_{\ell=0}^{\infty} p_{\ell,0} x^{\ell}.$$

Then,

$$(5.5) \quad (1-2x)G_0(x) = p_{0,0} + \sum_{\ell=1}^{\infty} (p_{\ell,0} - 2p_{\ell-1,0})x^{\ell}.$$

From equation (5.2c),

$$(5.6) \quad (1-2x)G_0(x) = p_{0,0} + (p_{1,0} - 2p_{0,0})x + \sum_{\ell=2}^{\infty} (p_{\ell,0} - 2p_{\ell-1,0})x^{\ell}$$

$$(5.7) \quad = p_{0,0} + (p_{1,0} - 2p_{0,0})x + \sum_{\ell=2}^{\infty} \left(\sum_{k=1}^{\ell} p_{\ell-k,0} - 2 \sum_{k=1}^{\ell-1} p_{\ell-1-k,0} \right) x^{\ell}$$

$$(5.8) \quad = p_{0,0} + (p_{1,0} - 2p_{0,0})x + \sum_{\ell=2}^{\infty} \left(\sum_{k=1}^{\ell} p_{\ell-k,0} - 2 \sum_{k=2}^{\ell} p_{\ell-k,0} \right) x^{\ell}$$

$$(5.9) \quad = p_{0,0} + (p_{1,0} - 2p_{0,0})x + \sum_{\ell=2}^{\infty} \left(p_{\ell-1,0} - \sum_{k=2}^{\ell} p_{\ell-k,0} \right) x^{\ell}.$$

Since $p_{0,0} = p_{1,0} = 1$,

$$(5.10) \quad (1-2x)G_0(x) = 1 - x + \sum_{\ell=2}^{\infty} \left(p_{\ell-1,0} - \sum_{k=1}^{\ell-1} p_{\ell-1-k,0} \right) x^{\ell}$$

$$(5.11) \quad = 1 - x.$$

Therefore

$$(5.12) \quad G_0(x) = \frac{1-x}{1-2x}.$$

Next we prove the general case for $i > 0$. Assume inductively that

$$(5.13) \quad G_i(x) = \left(\frac{1-x}{1-2x} \right)^{i+1} = \sum_{\ell=0}^{\infty} p_{i+\ell, i} x^\ell.$$

$$\begin{aligned} \sum_{\ell=0}^{\infty} p_{i+\ell+1, i+1} x^\ell &= \left(\frac{1-2x}{1-2x} \right) \sum_{\ell=0}^{\infty} p_{i+\ell+1, i+1} x^\ell \\ &= \left(\frac{1}{1-2x} \right) \left[\sum_{\ell=0}^{\infty} p_{i+\ell+1, i+1} x^\ell - 2x \sum_{\ell=0}^{\infty} p_{i+\ell+1, i+1} x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[p_{i+1, i+1} + \sum_{\ell=1}^{\infty} (p_{i+\ell+1, i+1} - 2p_{i+\ell, i+1}) x^\ell \right] \end{aligned}$$

Because $p_{i+1, i+1} = 1 = p_{i, i}$

$$\begin{aligned} &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} (p_{i+\ell+1, i+1} - 2p_{i+\ell, i+1}) x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} \left(p_{i+\ell+1, i+1} - p_{i+\ell, i+1} - p_{i+\ell, i+1} \right) x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} \left(p_{i+\ell+1, i+1} - p_{i+\ell, i+1} - \sum_{k=0}^{\ell-1} p_{i+\ell-k, i+1} + \sum_{k=1}^{\ell-1} p_{i+\ell-k, i+1} \right) x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} \left(p_{i+\ell+1, i+1} - p_{i+\ell, i+1} - \sum_{k=0}^{\ell-1} p_{i+\ell-k, i+1} + \sum_{k=1}^{\ell-1} p_{i+\ell-k, i+1} \right) x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} \left(\left(p_{i+\ell+1, i+1} - \sum_{k=1}^{\ell} p_{i+\ell+1-k, i+1} \right) - \left(p_{i+\ell, i+1} - \sum_{k=1}^{\ell-1} p_{i+\ell-k, i+1} \right) \right) x^\ell \right]. \blacksquare \end{aligned}$$

Rewriting equation (5.2b) as

$$(5.14) \quad p_{n, j} = p_{n+1, j+1} - \sum_{k=1}^{n-j} p_{n+1-k, j+1},$$

we find

$$\begin{aligned} \sum_{\ell=0}^{\infty} p_{i+\ell+1, i+1} x^\ell &= \left(\frac{1}{1-2x} \right) \left[p_{i, i} + \sum_{\ell=1}^{\infty} (p_{i+\ell, i} - p_{i+\ell-1, i}) x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[\sum_{\ell=0}^{\infty} p_{i+\ell, i} x^\ell - \sum_{\ell=1}^{\infty} p_{i+\ell-1, i} x^\ell \right] \\ &= \left(\frac{1}{1-2x} \right) \left[\sum_{\ell=0}^{\infty} p_{i+\ell, i} x^\ell - \sum_{\ell=0}^{\infty} p_{i+\ell, i} x^{\ell+1} \right] \\ &= \left(\frac{1-x}{1-2x} \right) \sum_{\ell=0}^{\infty} p_{i+\ell, i} x^\ell \\ &= \left(\frac{1-x}{1-2x} \right)^{i+2} \end{aligned}$$

†

PROPOSITION 5.3. *The coefficients $p_{n,k}$ are given by the OEIS sequence [A105306](#) for the “number of directed column-convex polynomials of area n , having the top of the right-most column at height k .” We have $p_{n,k} = T_{n+1,k+1}$ where*

$$(5.15) \quad T_{n,k} = \begin{cases} \sum_{j=0}^{n-k-1} \binom{k+j}{k-1} \binom{n-k-1}{j} & \text{if } k < n \\ 1 & \text{if } k = n \end{cases}$$

Maple “simplifies” this to

$$(5.16) \quad T_{n,k} = \begin{cases} {}_kF\left(\begin{matrix} k+1, k+1-n \\ 2 \end{matrix} \middle| -1\right) & \text{if } n \neq k \\ 1 & \text{if } n = k \end{cases}$$

where $F(\cdot)$ is the hypergeometric function defined as

$$(5.17) \quad F\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{a^{\bar{n}} b^{\bar{n}} z^n}{c^{\bar{n}} n!}$$

where $q^{\bar{n}}$ is $q \cdot (q+1) \cdots (q+n-1)$.

Proof. We will show that

$$(5.18) \quad p_{i+n,i} = T_{n+i+1,i+1} = \begin{cases} \sum_{j=0}^{n-1} \binom{i+j+1}{i} \binom{n-1}{j} & \text{if } n > 0 \\ 1 & \text{if } n = 0. \end{cases}$$

By Proposition 5.2

$$(5.19) \quad p_{i+n,i} = \frac{1}{n!} \frac{d^n}{dx^n} G_i(x) \Big|_{x=0}$$

where

$$(5.20) \quad G_i(x) = \left(\frac{1-x}{1-2x}\right)^{i+1} = f_i(g(x))$$

with

$$(5.21) \quad f_i(x) = x^{i+1}, \text{ and}$$

$$(5.22) \quad g(x) = \frac{1-x}{1-2x} = \frac{1}{1-2x} - \frac{x}{1-2x}.$$

Differentiating $f_i(x)$ and $g(x)$ with respect to x ,

$$(5.23) \quad \frac{d^n}{dx^n} f_i(x) = \begin{cases} (i+1)(i) \cdots (i-n+2)x^{i+1-n} & \text{for } n \leq i+1 \\ 0 & \text{for } n > i+1 \end{cases}$$

$$(5.24) \quad = \binom{i+1}{n} n! x^{i+1-n}$$

and

$$(5.25) \quad \frac{d^n}{dx^n} g(x) = \frac{d^n}{dx^n} \frac{1}{1-2x} + \frac{d^n}{dx^n} \frac{x}{1-2x}$$

$$(5.26) \quad = \frac{2^n n!}{(1-2x)^{n+1}} + \frac{2^{n-1} n!}{(1-2x)^n} + \frac{2^n n! x}{(1-2x)^{n+1}}$$

$$(5.27) \quad = \frac{2^n n! (1-x)}{(1-2x)^{n+1}} - \frac{2^{n-1} n!}{(1-2x)^n}$$

with

$$(5.28) \quad \left. \frac{d^n}{dx^n} g(x) \right|_{x=0} = \begin{cases} n! 2^{n-1} & \text{for } n > 0 \\ 1 & \text{for } n = 0. \end{cases}$$

When $n = 0$,

$$(5.29) \quad p_{i+n,i} = p_{i,i} = G_i(0) = 1.$$

For $n > 0$, by Faà di Bruno's formula we have

$$(5.30) \quad \frac{d^n}{dx^n} G_i(x) = \frac{d^n}{dx^n} f_i(g(x))$$

$$(5.31) \quad = \sum_{k=1}^n f_i^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x))$$

and therefore

$$(5.32) \quad \left. \frac{d^n}{dx^n} G_i(x) \right|_{x=0} = \sum_{k=1}^n f_i^{(k)}(g(0)) B_{n,k}(g'(0), g''(0), \dots, g^{(n-k+1)}(0))$$

$$(5.33) \quad = \sum_{k=1}^n f_i^{(k)}(1) B_{n,k}(1, 4, 24, \dots, (n-k+1)! 2^{n-k}).$$

By Theorem 6 of [1],

$$(5.34) \quad B_{n,k}(1, 4, 24, \dots, (n-k+1)! 2^{n-k}) = B_{n,k}(q_0(1), q_1(2), \dots, q_{n-k}(n-k+1))$$

$$(5.35) \quad = \binom{n-1}{k-1} \frac{n!}{k!} 2^{n-k}$$

because the function

$$(5.36) \quad q_n(x) = \frac{x!}{(x-n)!} 2^n$$

satisfies

$$(5.37) \quad q_n(x+y) = \sum_{k=0}^n \binom{n}{k} q_k(y) q_{n-k}(x).$$

Returning to the proof,

$$(5.38) \quad p_{i+n,i} = \frac{1}{n!} \left. \frac{d^n}{dx^n} G_i(x) \right|_{x=0}$$

$$(5.39) \quad = \frac{1}{n!} \sum_{k=1}^n \binom{i+1}{k} \binom{n-1}{k-1} k! \frac{n!}{k!} 2^{n-k}$$

$$(5.40) \quad = \sum_{k=1}^n \binom{i+1}{k} \binom{n-1}{k-1} 2^{n-k}$$

$$(5.41) \quad = \sum_{k=0}^{n-1} \binom{i+1}{k+1} \binom{n-1}{k} 2^{n-k-1}$$

$$(5.42) \quad = \sum_{k=0}^{n-1} \binom{i+1}{k+1} \binom{n-1}{k} \sum_{j=0}^{n-k-1} \binom{n-k-1}{j}$$

$$(5.43) \quad = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{k} \binom{i+1}{k+1} \binom{n-k-1}{j}$$

$$(5.44) \quad = \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} \binom{n-1}{k} \binom{i+1}{k+1} \binom{n-k-1}{j}$$

$$(5.45) \quad = \sum_{j=0}^{n-1} \sum_{k=0}^j \binom{n-1}{k} \binom{i+1}{k+1} \binom{n-k-1}{n-j-1}$$

$$(5.46) \quad = \sum_{j=0}^{n-1} \sum_{k=0}^j \binom{n-1}{n-j-1} \binom{j}{k} \binom{i+1}{k+1}$$

$$(5.47) \quad = \sum_{j=0}^{n-1} \binom{n-1}{j} \sum_{k=0}^j \binom{j}{k} \binom{i+1}{k+1}$$

$$(5.48) \quad = \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{i+j+1}{j+1}$$

$$(5.49) \quad = \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{i+j+1}{i}$$

PROPOSITION 5.4. *The characteristic polynomial of $\widetilde{\mathbf{M}}_n$ is*

$$P_n(z) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n}{2\ell} \left(\frac{z}{2}+1\right)^{n-2\ell} \left(1+\frac{z^2}{4}\right)^\ell + \frac{z}{2} \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2\ell+1} \left(\frac{z}{2}+1\right)^{n-2\ell-1} \left(1+\frac{z^2}{4}\right)^\ell.$$

This proposition can be proved in several ways. We choose below to think of $z \in \mathbb{C} \setminus \{\pm 2i\}$, for a reason that will become clear. Since the end result is a polynomial in z , proving the formula for $z \neq \pm 2i$ will recover the exceptional cases by continuity.

Another equally valid approach would be to think of z as being transcendental and noting that the characteristic polynomial of $\widetilde{\mathbf{M}}_n$ has integer coefficients.

Proof. From Proposition 3.1

$$(5.50) \quad P_n(z) = zP_{n-1}(z) - \sum_{k=1}^n t_k P_{n-k}(z)$$

$$(5.51) \quad = zP_{n-1}(z) - \sum_{k=0}^{n-1} t_{n-k} P_k(z).$$

If $t_k = -1$ for $1 \leq k \leq n$,

$$(5.52) \quad P_n(z) = zP_{n-1}(z) + \sum_{k=0}^{n-1} P_k(z).$$

Let $T_j(z) = \sum_{k=0}^j P_k(z)$. $T_n(z) = T_{n-1}(z) + P_n(z)$, so

$$(5.53) \quad P_n(z) = zP_{n-1}(z) + T_{n-1}(z)$$

$$(5.54) \quad T_n(z) = zP_{n-1}(z) + 2T_{n-1}(z)$$

or

$$(5.55) \quad \begin{bmatrix} P_n(z) \\ T_n(z) \end{bmatrix} = \begin{bmatrix} z & 1 \\ z & 2 \end{bmatrix}^n \begin{bmatrix} P_0(z) \\ T_0(z) \end{bmatrix}$$

$$(5.56) \quad = \begin{bmatrix} z & 1 \\ z & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

since $P_0(z) = 1$ and $T_0(z) = \sum_{j=0}^0 P_0(z) = 1$. The eigenvalues of this matrix are

$$(5.57) \quad \lambda_+ = 1 + \frac{z}{2} + \Delta$$

$$(5.58) \quad \lambda_- = 1 + \frac{z}{2} - \Delta$$

$$(5.59) \quad \Delta = \sqrt{1 + z^2/4}.$$

If $z = \pm 2i$ the eigenvalues are multiple and our approach would have to be modified. We ignore this and recover the true result at the end. The eigenvectors are

$$(5.60) \quad \mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1 - \frac{z}{2} + \Delta & 1 - \frac{z}{2} - \Delta \end{bmatrix}$$

and

$$(5.61) \quad \mathbf{V}^{-1} = \frac{-1}{2\Delta} \begin{bmatrix} 1 - \frac{z}{2} - \Delta & -1 \\ -1 + \frac{z}{2} - \Delta & 1 \end{bmatrix}$$

hence

$$(5.62) \quad \mathbf{V}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-1}{2\Delta} \begin{bmatrix} \frac{-z}{2} - \Delta \\ \frac{z}{2} - \Delta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{z}{4\Delta} \\ \frac{1}{2} - \frac{z}{4\Delta} \end{bmatrix}.$$

Therefore

$$(5.63) \quad \begin{bmatrix} P_n(z) \\ T_n(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 - \frac{z}{2} + \Delta & 1 - \frac{z}{2} - \Delta \end{bmatrix} \begin{bmatrix} \lambda_+^n \left(\frac{1}{2} + \frac{z}{4\Delta} \right) \\ \lambda_-^n \left(\frac{1}{2} - \frac{z}{4\Delta} \right) \end{bmatrix}$$

and in particular

$$(5.64) \quad P_n(z) = \lambda_+^n \left(\frac{1}{2} + \frac{z}{4\Delta} \right) + \lambda_-^n \left(\frac{1}{2} - \frac{z}{4\Delta} \right).$$

Now

$$(5.65) \quad \lambda_+^n = \left(\frac{z}{2} + 1 + \Delta \right)^n$$

$$(5.66) \quad = \sum_{k=0}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right) \Delta^k$$

and

$$(5.67) \quad \lambda_-^n = \left(\frac{z}{2} + 1 - \Delta \right)^n$$

$$(5.68) \quad = \sum_{k=0}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right) (-\Delta)^k .$$

$$(5.69) \quad \therefore P_n(z) = \sum_{k=0}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right)^{n-k} \left(\frac{1}{2} \Delta^k + \frac{1}{2} (-\Delta)^k \right) \\ + \frac{z}{4\Delta} \sum_{k=0}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right)^{n-k} (\Delta^k - (-\Delta)^k) .$$

Every odd term drops out of the first, and every even out of the second.

$$\therefore P_n(z) = \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right)^{n-k} \Delta^k + \frac{z}{4\Delta} \sum_{\substack{k=0 \\ k \text{ odd}}}^n \binom{n}{k} \left(\frac{z}{2} + 1 \right)^k \cdot 2\Delta^k \\ = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \binom{n}{2\ell} \left(\frac{z}{2} + 1 \right)^{n-2\ell} \left(1 + \frac{z^2}{4} \right)^\ell + \frac{z}{2} \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2\ell+1} \left(\frac{z}{2} + 1 \right)^{n-2\ell-1} \left(1 + \frac{z^2}{4} \right)^\ell .$$

At this point the difficulty with $\Delta = 0$ has been resolved by continuity. We see that $P_n(z)$ is a polynomial of degree n . \square

6. A Connection with Compositions. Consider the case with symbolic entries t_i , and subdiagonals -1 for convenience with minus signs in the formulae. For instance, the 5 by 5 example upper Hessenberg Toeplitz matrix is

$$(6.1) \quad \mathbf{M}_5 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ -1 & t_1 & t_2 & t_3 & t_4 \\ 0 & -1 & t_1 & t_2 & t_3 \\ 0 & 0 & -1 & t_1 & t_2 \\ 0 & 0 & 0 & -1 & t_1 \end{bmatrix} .$$

In this section we consider what happens when we take determinants $P_n(z) = \det(z\mathbf{I} - \mathbf{M}_n)$. Examining $P_0(0)$, $P_1(0)$, $P_2(0)$, $P_3(0)$, and $P_4(0)$, and in particular $P_k(0)$ (i.e. $\det(-\mathbf{M}_k)$) we see that

$$(6.2) \quad P_0(0) = 1 \text{ by convention}$$

$$(6.3) \quad P_1(0) = t_1$$

$$(6.4) \quad P_2(0) = t_1^2 + t_2$$

$$(6.5) \quad P_3(0) = t_1^3 + 2t_1t_2 + t_3$$

$$(6.6) \quad P_4(0) = t_1^4 + 3t_1^2t_2 + 2t_1t_3 + t_2^2 + t_4.$$

One may interpret these (looking at the subscripts) as *compositions*: $2 = 1 + 1 = 2$; $3 = 1 + 1 + 1 = 1 + 2 = 2 + 1 = 3$; $4 = 1 + 1 + 1 + 1 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 3 = 3 + 1 = 2 + 2 = 4$. The number of compositions of n is 2^{n-1} , which we get if all $t_j = 1$.

From the Wikipedia entry on composition (combinatorics), “a composition of an integer n is a way of writing n as the sum of a sequence of strictly positive integers.”

One may interpret the recurrence relation

$$(6.7) \quad p_{n,0} = \sum_{k=1}^n t_k p_{n-k,0}$$

from Proposition 3.2 as saying that to generate a composition of n , you get the composition of $n - k$ and then add the number “ k ” to them; adding these together gives all compositions. For example, when $n = 5$ we have $p_{0,0} = 1$, $p_{1,0} = t_1$, $p_{2,0} = t_1^2 + t_2$, $p_{3,0} = t_1^3 + 2t_1t_2 + t_3$, and $p_{4,0} = t_1^4 + 3t_1^2t_2 + 2t_1t_3 + t_2^2 + t_4$. Then

$$\begin{aligned} p_{5,0} &= t_1 p_{4,0} + t_2 p_{3,0} + t_3 p_{2,0} + t_4 p_{1,0} + t_5 p_{0,0} \\ &= t_1^5 + 3t_1^3t_2 + 2t_1^2t_3 + t_1t_2^2 + t_1t_4 + t_2t_1^3 + 2t_1t_2^2 + t_2t_3 + t_1^2t_3 + t_2t_3 + t_4t_1 + t_5 \\ &= t_1^5 + 4t_1^3t_2 + 3t_1^2t_3 + 3t_1t_2^2 + 2t_1t_4 + 2t_2t_3 + t_5. \end{aligned}$$

Remark 6.1. This determinant also contains the whole characteristic polynomial. Simply replace t , with $t_1 - z$ and we get $\det(\mathbf{M}_n - z\mathbf{I}) = (-1)^n P_n$. This suggests that “compositions with all parts bigger than 1” can be used to generate all compositions. This fact is well-known. The combinatorial analysis of this recurrence formula is not quite trivial.

7. Concluding Remarks. The class of upper Hessenberg Bohemian matrices, and the much smaller class of Bohemian upper Hessenberg Toeplitz matrices, give a useful way to study Bohemian matrices in general. This is an instance of Polya’s adage “find a useful specialization.” [5, p. 190] Because these classes are simpler than the general case, we were able to establish several theorems.

In this paper we have introduced two new formulae for computing the characteristic polynomials of upper Hessenberg Toeplitz matrices. Our first formula, Proposition 3.1, computes the characteristic polynomials recursively. Our second formula, Proposition 3.2, computes the coefficients recursively. Finally, we show the number of upper Hessenberg Toeplitz matrices of maximal characteristic height which is at least 2^n and we conjecture $\mathcal{O}((1 + \varphi)^n)$ in Theorem 4.9.

Many puzzles remain. Perhaps the most striking is the angular appearance of the set of eigenvalues $\Lambda(\mathbf{M}_n)$, such as in Figures 1, and 2. General matrices have eigenvalues asymptotic to a (scaled) disc [6]; our computations suggest that as $n \rightarrow \infty$, $\Lambda(\mathbf{M}_n)/n^{1/2}$ tends to an irregular hexagonal shape, rather than a disk. More, the density does not appear to be approaching uniformity. Further, the boundary is irregular, with shapes suggestive of what is popularly known as the “dragon curve” (in reverse—these delineate where the eigenvalues are absent, near the edge). We have no explanation for this.

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