Linear compactness and combinatorial bialgebras

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Abstract

We present an expository overview of the monoidal structures in the category of linearly compact vector spaces. Bimonoids in this category are the natural duals of infinite-dimensional bialgebras. We classify the relations on words whose equivalence classes generate linearly compact bialgebras under shifted shuffling and deconcatenation. We also extend some of the theory of combinatorial Hopf algebras to bialgebras that are not connected or of finite graded dimension. Finally, we discuss several examples of quasi-symmetric functions, not necessarily of bounded degree, that may be constructed via terminal properties of combinatorial bialgebras.

Contents

1 Introduction

The graded dual $\mathbf{W}_{\mathsf{P}}$ of the Hopf algebra of *word quasi-symmetric functions* has a basis given by the set of *packed words*, i.e., finite sequences $w = w_1 w_2 \cdots w_n$ with $\{w_1, w_2, \dots, w_n\} = \{1, 2, \dots, m\}$ for some $m \geq 0$. The product for this Hopf algebra is a shifted shuffling operation, while the coproduct is a variant of deconcatenation; for the precise definitions, skip to Section 2.3.

A fruitful method of constructing Hopf algebras of interest in combinatorics is to choose an equivalence relation $\sim$ on packed words and then form the subspace $\mathbf{K}_{\mathsf{P}}^{(\sim)} \subset \mathbf{W}_{\mathsf{P}}$ spanned by the sums over each $\sim$-equivalence class $\kappa_E := \sum_{w \in E} w$. A long list of well-known Hopf algebras can be realized as a subalgebra of $\mathbf{W}_{\mathsf{P}}$ in this way: for example, the *noncommutative symmetric functions* NSym [11], the *Poirier-Reutenaurer algebra* PR [35], the *K-theoretic Poirier-Reutenaurer algebra* KPR [33], the *small multi-Malvenuto-Reutenauer Hopf algebra* $\mathfrak{m}$MR [21], the *Loday-Ronco algebra* [4, 23], and the *Baxter Hopf algebra* [13], among others.

The subspace $\mathbf{K}_{\mathsf{P}}^{(\sim)} \subset \mathbf{W}_{\mathsf{P}}$ is not necessarily a sub-bialgebra, and one of the aims of this paper is to describe precisely when this occurs. The Hopf algebra $\mathbf{W}_{\mathsf{P}}$ is a quotient of a larger bialgebra $\mathbf{W}$ with a basis given by arbitrary words. We will also consider the problem of classifying the word relations that span sub-bialgebras $\mathbf{K}^{(\sim)} \subset \mathbf{W}$ in a similar manner.

For homogeneous relations, versions of these problems have been studied in a few places previously, e.g., [13, 17, 31, 36]. Less has been written about the cases when $\sim$ is allowed to relate words of different lengths. For inhomogeneous relations of this kind, various complications arise when one tries to interpret $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ as an algebra or a coalgebra. To start, such relations may have equivalence classes with infinitely many elements, in which case $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ contains infinite linear combinations of packed words so is not technically a subspace of $\mathbf{W}_{\mathsf{P}}$. One can still try to evaluate the product and coproduct of $\mathbf{W}_{\mathsf{P}}$ on elements of $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ when this happens. However, products may result in infinite linear combinations of the basis elements κ_E, and even if these infinite sums are adjoined to $\mathbf{K}_{\mathsf{P}}^{(\sim)}$, coproducts may have too many terms to belong to $\mathbf{K}_{\mathsf{P}}^{(\sim)} \otimes \mathbf{K}_{\mathsf{P}}^{(\sim)}$.

Nevertheless, some interesting "Hopf algebras" that can be identified with $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ when $\sim$ is inhomogeneous have appeared in the literature [14, 21, 32, 33]. A second, expository goal of this paper is to describe explicitly the monoidal category containing such objects, which in general is not the usual category of bialgebras over a field. This point is often glossed over in the relevant combinatorial literature, though authors tend to indicate correctly that its resolution is topological in nature.

In detail, to make sense of sub-bialgebras of $\mathbf{W}_{\mathsf{P}}$ "spanned" by inhomogeneous word relations, one should first consider the larger vector space $\hat{\mathbf{W}}_{\mathsf{P}}$ consisting of arbitrary (rather than just finite) linear combinations of packed words. This object is naturally viewed as a *linearly compact* topological space. The full subcategory of such spaces, within the category of all topological spaces, has a symmetric monoidal structure which leads to notions of *linearly compact algebras, coalgebras*, and *bialgebras*, of which $\hat{\mathbf{W}}_{\mathsf{P}}$ is an example. In this language, our original classification problem becomes the question: for which word relations $\sim$ is the subspace $\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)}$, whose elements are the arbitrary linear combinations of the sums κ_E, a linearly compact sub-bialgebra of $\hat{\mathbf{W}}_{\mathsf{P}}$?

After some preliminaries in Section 2, we review the main properties of linearly compact vector spaces in Section 3. This background material is semi-classical but perhaps not so widely known in combinatorics. Section 4 goes on to discuss some novel generalizations of the monoidal structures on $\mathbf{W}$ and $\mathbf{W}_{\mathsf{P}}$. In Section 5, we answer the question in the previous paragraph. Our general results about word relations recover a number of specific constructions of (linearly compact) Hopf algebras and bialgebras; we discuss some relevant examples in Section 6.

One application of all this formalism is to extend Aguiar, Bergeron, and Sottile's theory of combinatorial Hopf algebras from [1]. Ignoring some technical details which will be clarified in Section 7, a *combinatorial Hopf algebra* over a field $\Bbbk$ is a graded Hopf algebra H with an algebra morphism $\zeta : H \to \Bbbk$. A morphism $(H, \zeta) \to (H', \zeta')$ of combinatorial Hopf algebras is a graded Hopf algebra morphism $\phi : H \to H'$ with $\zeta = \zeta' \circ \phi$. The Hopf algebra of *quasi-symmetric functions* QSym with the homomorphism $\zeta_{\mathsf{QSym}} : \mathsf{QSym} \to \Bbbk$ setting $x_1 = 1$ and $x_2 = x_3 = \cdots = 0$ is a fundamental example.

It is shown in [1] that if (H, ζ) is a combinatorial Hopf algebra in which H is connected and of finite graded dimension, then there is a unique morphism $(H, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. This morphism supplies a uniform construction of many independent definitions of quasi-symmetric generating functions attached to Hopf algebras. In Section 7, we prove that the same result holds without the assumption that H be connected or have finite graded dimension. This extension is not unexpected; the authors mention in [1, Remark 4.2] that their assumption of finite graded dimension may be dropped, and note work in preparation where this will be proved. The paper cited in [1, Remark 4.2], however, does not seem to have ever appeared in the literature. We hope that our exposition fills this gap.

In Section 8 we illustrate some more applications. We discuss several examples of families of symmetric and quasi-symmetric functions, not necessarily of bounded degree, that can be realized as the images of canonical morphisms from what we call *(linearly compact) combinatorial bialgebras.* For appropriate word relations, the space $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is an object of this type and is therefore equipped with a canonical morphism to a certain linearly compact "completion" of QSym. Our last results give a partial classification of the relations $\sim$ for which the image of this morphism consists entirely of symmetric functions.

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2 Preliminaries

Let $\mathbb{Z} \supset \mathbb{N} \supset \mathbb{P}$ denote the respective sets of all integers, all nonnegative integers, and all positive integers. For $m, n \in \mathbb{N}$, define $[m, n] = \{i \in \mathbb{Z} : m \leq i \leq n\}$ and $[n] = [1, n]$.

2.1 Monoidal structures

Our reference for the background material in this section is [2, Chapter 1]. Suppose $\mathscr{C}$ is a braided monoidal category with tensor product $\bullet$, unit object I, and braiding β.

Definition 2.1. A *monoid* in $\mathscr{C}$ is a triple (A, ∇, ι) where $A \in \mathscr{C}$ is an object and $\nabla : A \bullet A \to A$

and $\iota : I \to A$ are morphisms (referred to as the *product* and *unit*) making these diagrams commute:

$$\begin{array}{ccccc} I \bullet A & \xrightarrow{\iota \bullet \mathrm{id}} & A \bullet A & \xleftarrow{\mathrm{id} \bullet \iota} & A \bullet I \\ & \searrow^{\cong} & \downarrow{\scriptstyle \nabla} & {}^{\cong}\swarrow & \\ & & A & & \end{array} \qquad \begin{CD} A \bullet A \bullet A @>{\nabla \bullet \mathrm{id}}>> A \bullet A \\ @V{\mathrm{id} \bullet \nabla}VV @VV{\nabla}V \\ A \bullet A @>{\nabla}>> A \end{CD} \tag{2.1}$$

Definition 2.2. A *comonoid* in $\mathscr{C}$ is a triple (A, Δ, ϵ) where $A \in \mathscr{C}$ is an object and $\Delta : A \to A \bullet A$ and $\epsilon : A \to I$ are morphisms (referred to as the *coproduct* and *counit*) making the diagrams (2.1), with ∇ and ι replaced by Δ and ϵ and with the directions of all arrows reversed, commute.

A monoid is *commutative* if $\nabla \circ \beta = \nabla$. A comonoid is *cocommutative* if $\beta \circ \Delta = \Delta$.

Definition 2.3. A *bimonoid* in $\mathscr{C}$ is a tuple $(A, \nabla, \iota, \Delta, \epsilon)$ where (A, ∇, ι) is a monoid, (A, Δ, ϵ) is a comonoid, the composition $\epsilon \circ \iota$ is the identity morphism $I \to I$, and these diagrams commute:

$$\begin{array}{ccccc} A \bullet A & \xrightarrow{\nabla} & A & \xrightarrow{\Delta} & A \bullet A \\ \downarrow{\scriptstyle \Delta \bullet \Delta} & & & & \uparrow{\scriptstyle \nabla \bullet \nabla} \\ A \bullet A \bullet A \bullet A & & \xrightarrow{\mathrm{id} \bullet \beta \bullet \mathrm{id}} & & A \bullet A \bullet A \bullet A \end{array} \qquad \begin{CD} I @>{\iota}>> A \\ @V{\cong}VV @VV{\Delta}V \\ I \bullet I @>{\iota \bullet \iota}>> A \bullet A \end{CD} \qquad \begin{CD} A \bullet A @>{\epsilon \bullet \epsilon}>> I \bullet I \\ @V{\nabla}VV @VV{\cong}V \\ A @>{\epsilon}>> I \end{CD} \tag{2.2}$$

A morphism of (bi, co) monoids is a morphism in $\mathscr{C}$ that commutes with the relevant (co)unit and (co)product morphisms. If A is a monoid then $A \bullet A$ is a monoid with product $(\nabla \bullet \nabla) \circ (\mathrm{id} \bullet \beta \bullet \mathrm{id})$ and unit $(\iota \bullet \iota) \circ (I \xrightarrow{\sim} I \bullet I)$. If A is a comonoid then $A \bullet A$ is naturally a comonoid in a similar way. The diagrams (2.2) express that the coproduct and counit of a bimonoid are monoid morphisms, and that the product and unit are comonoid morphisms.

We are exclusively interested in these definitions applied to a few related categories. Let $\Bbbk$ be a field and write $\mathsf{Vec}_{\Bbbk}$ for the usual category of $\Bbbk$-vector spaces with linear maps as morphisms. This category is symmetric monoidal relative to the standard tensor product $\otimes = \otimes_{\Bbbk}$ and braiding map $x \otimes y \mapsto y \otimes x$, with unit object $\Bbbk$. Monoids, comonoids, and bimonoids in this category are the familiar notions of $\Bbbk$-*algebras*, $\Bbbk$-*coalgebras*, and $\Bbbk$-*bialgebras*. In this context, the unit $\iota : \Bbbk \to A$ is completely determined by $\iota(1) \in A$, which we refer to as the *unit element*.

Assume that $\mathscr{C}$ is $\Bbbk$-linear so that the morphisms between any two fixed objects in $\mathscr{C}$ form a $\Bbbk$-vector space. Let $(H, \nabla, \iota, \Delta, \epsilon)$ be a bimonoid in $\mathscr{C}$. The *convolution product* of two morphisms $f, g : H \to H$ is then $f * g = \nabla \circ (f \bullet g) \circ \Delta : H \to H$. The operation $*$ is associative and makes the vector space of morphisms $H \to H$ into a $\Bbbk$-algebra with unit element $\iota \circ \epsilon$, referred to as the *convolution algebra* of H. The bimonoid H is a *Hopf monoid* if the identity morphism $\mathrm{id} : H \to H$ has a left and right inverse $\mathtt{S} : H \to H$ in the convolution algebra. The morphism $\mathtt{S}$ is called the *antipode* of H; if it exists, then $\mathtt{S}$ is the unique morphism $H \to H$ such that $\nabla \circ (\mathrm{id} \bullet \mathtt{S}) \circ \Delta = \nabla \circ (\mathtt{S} \bullet \mathrm{id}) \circ \Delta = \iota \circ \epsilon$. Hopf monoids in $\mathsf{Vec}_{\Bbbk}$ are *Hopf algebras*.

2.2 Graded vector spaces

If I is a set and V_i for $i \in I$ is a $\Bbbk$-vector space, then $\bigoplus_{i \in I} V_i$ is the vector space of sums $\sum_{i \in I} v_i$ where $v_i \in V_i$ for $i \in I$ and $v_i = 0$ for all but finitely many indices $i \in I$. We interpret the direct product $\prod_{i \in I} V_i$ as the vector space of arbitrary formal sums $\sum_{i \in I} v_i$ with $v_i \in V_i$. There is an obvious inclusion $\bigoplus_{i \in I} V_i \subset \prod_{i \in I} V_i$ which is equality if I is finite.

A vector space V is *graded* if it has a direct sum decomposition $V = \bigoplus_{n\in\mathbb{N}} V_n$. A linear map $\phi : U \to V$ between graded vector spaces is *graded* if it has the form $\phi = \bigoplus_{n\in\mathbb{N}} \phi_n$ where each $\phi_n : U_n \to V_n$ is linear. If $U = \prod_{n\in\mathbb{N}} U_n$ and $V = \prod_{n\in\mathbb{N}} V_n$ are direct products of vector spaces, then we also use the term *graded* to refer to the linear maps $\phi : U \to V$ of the form $\phi = \prod_{n\in\mathbb{N}} \phi_n$ where each $\phi_n : U_n \to V_n$ is linear.

An algebra (V, ∇, ι) is *graded* if V is graded, $\nabla(V_i \otimes V_j) \subset V_{i+j}$ for all $i, j \in \mathbb{N}$, and $\iota(\Bbbk) \subset V_0$. Similarly, a coalgebra (V, Δ, ϵ) is *graded* if V is graded, $\Delta(V_n) \subset \bigoplus_{i+j=n} V_i \otimes V_j$ for all $n \in \mathbb{N}$, and $\epsilon(V_n) = 0$ for $n \in \mathbb{P}$. A bialgebra is *graded* if it is graded as both an algebra and a coalgebra. These notions correspond to (co, bi) monoids in the category $\mathsf{GrVec}_\Bbbk$ whose objects are graded $\Bbbk$-vector spaces $V = \bigoplus_{n\in\mathbb{N}} V_n$ and whose morphisms are graded linear maps, in which the tensor product of objects U and V is the graded vector space $U \otimes V = \bigoplus_{n\in\mathbb{N}} (U \otimes V)_n$ with $(U \otimes V)_n = \bigoplus_{i+j=n} U_i \otimes V_j$. The unit object in $\mathsf{GrVec}_\Bbbk$ is the field $\Bbbk$, graded such that all elements have degree zero.

2.3 Word bialgebras

We review the definition of a particular graded bialgebra which will serve as a running example in later sections. Throughout, we use the term *word* to mean a finite sequence of positive integers. If $w = w_1 w_2 \cdots w_n$ is a word with n letters and $I = \{i_1 < i_2 < \cdots < i_k\} \subset [n]$ is a subset of indices, then we set $w|_I = w_{i_1} w_{i_2} \cdots w_{i_k}$. The shuffle product of two words u and v of length m and n is the formal linear combination of words

$$u \sqcup\!\sqcup v = \sum_{\substack{I \subset [m+n] \\ |I| = m}} \sqcup\!\sqcup_I (u, v)$$

where $w = \sqcup\!\sqcup_I(u, v)$ is the unique $(m+n)$-letter word with $w|_I = u$ and $w|_{I^c} = v$. Multiplicities may result in this expression; for example, $12 \sqcup\!\sqcup 21 = 2 \cdot 1221 + 1212 + 2121 + 2 \cdot 2112$.

If $w = w_1 w_2 \cdots w_m$ is a word with $m > 0$ letters, then we set $\max(w) = \max\{w_1, w_2, \ldots, w_m\}$. For the empty word $\emptyset$, we define $\max(\emptyset) = 0$. Let $\mathbb{W}_n$ for $n \in \mathbb{N}$ be the set of pairs $[w, n]$ with $\max(w) \leq n$ and define $\mathbb{W} = \bigcup_{n\in\mathbb{N}} \mathbb{W}_n$. Let $\mathbf{W}_n = \Bbbk\mathbb{W}_n$ be the $\Bbbk$-vector space with $\mathbb{W}_n$ as a basis and define $\mathbf{W} = \bigoplus_{n\in\mathbb{N}} \mathbf{W}_n$.

Denote the word formed by adding $n \in \mathbb{N}$ to each letter of $w = w_1 w_2 \cdots w_m$ by

$$w \uparrow n = (w_1 + n)(w_2 + n) \cdots (w_m + n).$$

Given words $w^1, w^2, \ldots w^l$ with $\max(w^i) \leq n$ and $a_1, a_2, \ldots, a_l \in \Bbbk$, let $\left[\sum_i a_i w^i, n\right] = \sum_i a_i [w^i, n] \in \mathbf{W}_n$. Now define $\nabla_{\sqcup\!\sqcup} : \mathbf{W} \otimes \mathbf{W} \to \mathbf{W}$ to be the linear map with

$$\nabla_{\sqcup\!\sqcup}([v, m] \otimes [w, n]) = [v \sqcup\!\sqcup (w \uparrow m)), n + m] \in \mathbf{W}_{m+n} \tag{2.3}$$

for $[v, m] \in \mathbb{W}_m$ and $[w, n] \in \mathbb{W}_n$. Since v and $w \uparrow m$ are words in disjoint alphabets, there are no multiplicities in the right expression; for example, $\nabla_{\sqcup\!\sqcup}([12, 3] \sqcup\!\sqcup [2, 2]) = [125, 5] + [152, 5] + [512, 5]$. Next let $\epsilon_\odot : \mathbf{W} \to \Bbbk$ and $\Delta_\odot : \mathbf{W} \to \mathbf{W} \otimes \mathbf{W}$ be the linear maps with

$$\epsilon_\odot([w, n]) = \begin{cases} 1 & \text{if } w = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Delta_\odot([w, n]) = \sum_{i=0}^{m} [w_1 \cdots w_i, n] \otimes [w_{i+1} \cdots w_m, n] \tag{2.4}$$

for $[w, n] \in \mathbb{W}_n$ with $w = w_1 w_2 \cdots w_m$. Finally write $\iota_{\sqcup\!\sqcup}$ for the linear map $\Bbbk \to \mathbf{W}$ with $\iota_{\sqcup\!\sqcup}(1) = [\emptyset, 0]$. We consider $\mathbf{W}$ to be a graded vector space in which $[w, n] \in \mathbb{W}_n$ is homogeneous with degree $\ell(w)$, the length of the word w. The following is [28, Theorem 3.5]:

Theorem 2.4. $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ is a graded bialgebra, but not a Hopf algebra.

Let $w = w_1 w_2 \cdots w_n$ be a word. Suppose the set $S = \{w_1, w_2, \dots, w_n\}$ has m distinct elements. If ϕ is the unique order-preserving bijection $S \to [m]$, then the *flattened word* corresponding to w is $\mathrm{fl}(w) = \phi(w_1)\phi(w_2)\cdots\phi(w_n)$.

A *packed word* (also called a *surjective word* [16], *Fubini word* [34], or *initial word* [33]) is a word w with $w = \mathrm{fl}(w)$. Define $\mathbf{I}_{\mathsf{P}}$ to be the subspace of $\mathbf{W}$ spanned by all differences $[v, m] - [w, n]$ where $[v, m], [w, n] \in \mathbb{W}$ have $\mathrm{fl}(v) = \mathrm{fl}(w)$. The following is [28, Proposition 3.7]:

Proposition 2.5. The subspace $\mathbf{I}_{\mathsf{P}}$ is a homogeneous bi-ideal of $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$. The quotient bialgebra $\mathbf{W}_{\mathsf{P}} = \mathbf{W}/\mathbf{I}_{\mathsf{P}}$ is a graded Hopf algebra.

The Hopf algebra $\mathbf{W}_{\mathsf{P}}$ is the graded dual of the algebra of *word quasi-symmetric functions* WQSym [29, 31]. Let $\mathbb{W}_{\mathsf{P}}$ be the set of all packed words. If $[w, n] \in \mathbb{W}$ and w is a word with m distinct letters then $v = \mathrm{fl}(w)$ is the unique packed word such that $[w, n] + \mathbf{I}_{\mathsf{P}} = [v, m] + \mathbf{I}_{\mathsf{P}}$. Identify $v \in \mathbb{W}_{\mathsf{P}}$ with the coset $[v, m] + \mathbf{I}_{\mathsf{P}}$ so that we can view $\mathbb{W}_{\mathsf{P}}$ as a basis for $\mathbf{W}_{\mathsf{P}}$. The unit element of $\mathbf{W}_{\mathsf{P}}$ is then the empty packed word $\emptyset$, and the counit is the linear map $\epsilon_\odot : \mathbf{W}_{\mathsf{P}} \to \Bbbk$ with $\epsilon_\odot(\emptyset) = 1$ and $\epsilon_\odot(w) = 0$ for all $\emptyset \neq w \in \mathbb{W}_{\mathsf{P}}$. For $u, v, w \in \mathbb{W}_{\mathsf{P}}$ with $m = \max(u)$ and $n = \ell(w)$,

$$\nabla_{\sqcup\!\sqcup}(u \otimes v) = u \sqcup\!\sqcup (v \uparrow m) \qquad \text{and} \qquad \Delta_\odot(w) = \sum_{i=0}^{n} \mathrm{fl}(w_1 \cdots w_i) \otimes \mathrm{fl}(w_{i+1} \cdots w_n). \tag{2.5}$$

The subspace of $\mathbf{W}_{\mathsf{P}}$ spanned by the words in $\mathbb{W}_{\mathsf{P}}$ that have no repeated letters is a Hopf subalgebra. This is the well-known *Malvenuto-Poirier-Reutenaurer Hopf algebra* of permutations [3, 26], sometimes also called the Hopf algebra of *free quasi-symmetric functions* FQSym [31].

3 Linearly compact spaces

Let U and V be $\Bbbk$-vector spaces. Define U^* to be the dual space of U, that is, the vector space of all $\Bbbk$-linear maps $\lambda : U \to \Bbbk$. Given a linear map $\phi : U \to V$, define ϕ^* to be the linear map $V^* \to U^*$ with $\phi^*(\lambda) = \lambda \circ \phi$. This makes $*$ into a contravariant functor $\mathsf{Vec}_\Bbbk \to \mathsf{Vec}_\Bbbk$.

We would like to be able to consider "sub-bialgebras" of $\mathbf{W}$ generated by certain infinite linear combinations of basis elements in $\mathbb{W}$. Such linear combinations are not well-defined in $\mathbf{W}$ but are naturally interpreted as elements of $\mathbf{W}^*$. Therefore, we need a way of transferring the monoidal structures on the vector space $\mathbf{W}$ to its dual.

The full dual of an infinite-dimensional $\Bbbk$-algebra is not naturally a $\Bbbk$-coalgebra; see [9, §3.5]. On the other hand, neither the standard form of graded duality nor the more general notion of restricted duality (see [9, §3.5]) suffices for our application, since $\mathbf{W}$ does not have finite graded dimension and since the restricted dual will not permit infinite linear combinations.

The solution to these obstructions is to give the dual space a topology and consider monoidal structures in the category of topological vector spaces rather than $\mathsf{Vec}_\Bbbk$. The topology in question is known as the *linearly compact topology*, whose properties we quickly review. Much of the background material in this section appears in [10, Chapter 1], so we omit some proofs.

A bilinear form $\langle \cdot, \cdot \rangle : U \times V \to \Bbbk$ is *nondegenerate* if $v \mapsto \langle \cdot, v \rangle$ is a bijection $V \to U^*$. For example, the *tautological form* $\langle u, \lambda \rangle := \lambda(u)$ is a nondegenerate bilinear form $U \times U^* \to \Bbbk$. The bilinear form $\langle a, b \rangle := ab$ is likewise a nondegenerate pairing $\Bbbk \times \Bbbk \to \Bbbk$.

Lemma 3.1. Suppose $\langle\cdot,\cdot\rangle : U \times V \to \Bbbk$ is a nondegenerate bilinear form. If there is a direct sum decomposition $U = \bigoplus_{i\in I} U_i$ then $V = \prod_{i\in I} V_i$ where $V_i = \{v \in V : \langle u, v\rangle = 0 \text{ if } u \in U_j \text{ for } i \neq j\}$.

Proof. Identify $\sum_{i\in I} v_i \in \prod_{i\in I} V_i$ with the unique $v \in V$ satisfying $\langle u, v\rangle = \langle u, v_i\rangle$ for $i \in I$ and $u \in U_i$ to get an inclusion $\prod_{i\in I} V_i \hookrightarrow V$. For $v \in V$, the linear map $U \to \Bbbk$ with $u \mapsto \langle u, v\rangle$ for $u \in U_i$ and $u \mapsto 0$ for $u \in \bigoplus_{i\neq j} U_j$ has the form $u \mapsto \langle u, v_i\rangle$ for some $v_i \in V_i$, and $v = \sum_{i\in I} v_i$. □

Suppose $\langle\cdot,\cdot\rangle : U \times V \to \Bbbk$ is a nondegenerate bilinear form and $\{u_i : i \in I\}$ is a basis for U. For each $i \in I$, there exists a unique $v_i \in V$ with $\langle u_j, v_i\rangle = \delta_{ij}$ for all $j \in I$. As $U = \bigoplus_{i\in I} \Bbbk u_i$, Lemma 3.1 implies that $V = \prod_{i\in I} \Bbbk v_i$. Thus each $v \in V$ can be uniquely expressed as the (potentially infinite) sum $v = \sum_{i\in I}\langle u_i, v\rangle v_i$. Following [10], we call $\{v_i : i \in I\}$ a *pseudobasis* for V; this is sometimes also referred to as a *continuous basis* (e.g., in [32, §3]).

View each subspace $\Bbbk v_i$ as a discrete topological space and give $V = \prod_{i\in I} \Bbbk v_i$ the corresponding product topology; this is the *linearly compact topology* on V, also sometimes called the *pseudocompact topology*. This topology depends only on the form $\langle\cdot,\cdot\rangle$ and not on the choice of basis for U. Each proper open subset of V can be expressed as the intersection of a finite collection of sets (each of which is both open and closed) of the form $\left\{\sum_{i\in I} c_i v_i \in V : c_j \in C\right\}$ for fixed choices of $C \subset \Bbbk$ and $j \in I$. If V is finite-dimensional, then the linearly compact topology is the discrete topology.

Definition 3.2. A *linearly compact* $\Bbbk$*-vector space* is a $\Bbbk$-vector space V equipped with the linearly compact topology induced by a nondegenerate bilinear form $U \times V \to \Bbbk$ for some $\Bbbk$-vector space U. Let $\widehat{\mathsf{Vec}_\Bbbk}$ denote the full subcategory of the category of topological $\Bbbk$-vector spaces whose objects are linearly compact vector spaces.

As noted in [10], a topological vector space V belongs to $\widehat{\mathsf{Vec}_\Bbbk}$ if and only if its topology is Hausdorff and linear (i.e., the open affine subspaces form a basis) and any family of closed affine subspaces with the finite intersection property has nonempty intersection. The category $\widehat{\mathsf{Vec}_\Bbbk}$ is closed under arbitrary direct products and finite direct sums, and contains the category of finite-dimensional vector spaces as a full subcategory.

A morphism between linearly compact vector spaces is a linear map that is continuous in the linearly compact topology. We can be more explicit about which linear maps are continuous. Suppose $V, W \in \widehat{\mathsf{Vec}_\Bbbk}$ have pseudobases $\{v_i : i \in I\}$ and $\{w_j : j \in J\}$. Let $\psi : V \to W$ be a linear map and define $\psi_{ij} \in \Bbbk$ to be the coefficient such that $\psi(v_i) = \sum_{j\in J} \psi_{ij} w_j$ for all $i \in I$.

Lemma 3.3. The map $\psi : V \to W$ is continuous in the linearly compact topology if and only if $\{i \in I : \psi_{ij} \neq 0\}$ is finite for each $j \in J$ and $\psi\left(\sum_{i\in I} c_i v_i\right) = \sum_{j\in J}\left(\sum_{i\in I} c_i \psi_{ij}\right) v_j$ for any $c_i \in \Bbbk$.

In other words, ψ is continuous when $\sum_{i\in I} c_i \psi(v_i)$ is always defined and equal to $\psi\left(\sum_{i\in I} c_i v_i\right)$. It is an instructive exercise to work through the proof of this basic lemma.

Proof. If the given properties hold then the inverse image of $\left\{\sum_{i\in J} c_i w_i \in W : c_j \in C\right\}$ under ψ is a union of finite intersections of analogous sets in V and is therefore open. It follows in this case that the inverse image of any open subset of W under ψ is open, so ψ is continuous.

If $\{i \in I : \psi_{ij} \neq 0\}$ is infinite for some $j \in J$, then $\psi^{-1}(\left\{\sum_{i\in J} c_i w_i \in W : c_j = 0\right\})$ cannot be an open subset of V, so ψ is not continuous. Assume ψ is continuous. The map $\phi : V \to W$ given by $\phi\left(\sum_{i\in I} c_i v_i\right) = \sum_{j\in J}\left(\sum_{i\in I} c_i \psi_{ij}\right) v_j$ for any $c_i \in \Bbbk$ is well-defined, linear, and continuous. Since $\psi - \phi$ is then linear and continuous, to deduce that $\psi = \phi$, it suffices to show that the only continuous linear map $V \to W$ with $v_i \mapsto 0$ for all $i \in I$ is zero. This holds as the inverse image of the open set $W - \{0\}$ under such a map does not contain any nonempty open subset of V. □

Suppose we have nondegenerate bilinear forms $\langle \cdot, \cdot \rangle_i : U_i \times V_i \to \Bbbk$ for $i \in \{1, 2\}$. If $\phi : U_2 \to U_1$ is linear, then there exists a unique linear map $\phi^\perp : V_1 \to V_2$ such that $\langle \phi(u_2), v_1 \rangle_1 = \langle u_2, \phi^\perp(v_1) \rangle_2$ for all $u_1 \in U_1$ and $v_2 \in V_2$. If $V_i = U_i^*$ and $\langle \cdot, \cdot \rangle_i$ is the tautological form, then $\phi^* = \phi^\perp$.

Corollary 3.4. In the preceding setup, a linear map $\psi : V_1 \to V_2$ is continuous in the linearly compact topology if and only if $\psi = \phi^\perp$ for some linear map $\phi : U_2 \to U_1$.

The set of continuous linear maps $V \to W$ between linearly compact vector spaces is therefore a $\Bbbk$-vector space. Let $V^\vee$ be the vector space of continuous linear maps $V \to \Bbbk$ for $V \in \widehat{\mathsf{Vec}_\Bbbk}$. This vector space is sometimes called the *continuous dual* of V (for example, in [21, §7.4]).

Corollary 3.5. Suppose $\langle \cdot, \cdot \rangle : U \times V \to \Bbbk$ is a nondegenerate bilinear form. If $\{u_i : i \in I\}$ is a basis for U, then the functions $\langle u_i, \cdot \rangle : V \to \Bbbk$ for $i \in I$ are a basis for $V^\vee$.

If $\psi : V \to W$ is a continuous linear map then $\psi^* : W^* \to V^*$ restricts to a map $W^\vee \to V^\vee$, which we denote $\psi^\vee$. The operation $\vee$ is then a contravariant functor $\widehat{\mathsf{Vec}_\Bbbk} \to \mathsf{Vec}_\Bbbk$. The preceding corollary implies that $U \in \mathsf{Vec}_\Bbbk$ is naturally isomorphic to $(U^*)^\vee$ as a vector space and that $V \in \widehat{\mathsf{Vec}_\Bbbk}$ is naturally isomorphic to $(V^\vee)^*$ as a topological vector space. Thus, if $V \in \widehat{\mathsf{Vec}_\Bbbk}$ then the tautological pairing $V^\vee \times V \to \Bbbk$ is nondegenerate and the linearly compact topology induced by this form recovers the topology on V. We can summarize these observations as follows:

Proposition 3.6. The functors $* : \mathsf{Vec}_\Bbbk \to \widehat{\mathsf{Vec}_\Bbbk}$ and $\vee : \widehat{\mathsf{Vec}_\Bbbk} \to \mathsf{Vec}_\Bbbk$ are dualities of categories.

Define the *completion* of a $\Bbbk$-vector space U with respect to a given basis $\{u_i : i \in I\}$ to be the vector space $\hat{U} = \prod_{i \in I} \Bbbk u_i$ with the product topology, where each subspace $\Bbbk u_i$ is discrete. In other words, $\hat{U}$ is the linearly compact $\Bbbk$-vector space with $\{u_i : i \in I\}$ as a pseudobasis. Of course, if U is finite-dimensional then $U = \hat{U}$. The bilinear form $\langle \cdot, \cdot \rangle : U \times U \to \Bbbk$ with $\langle u_i, u_j \rangle = \delta_{ij}$ extends to a nondegenerate bilinear form $U \times \hat{U} \to \Bbbk$. The space $\hat{U}$ is distinguished from U^* in having a fixed inclusion $U \subset \hat{U}$. Relative to this inclusion, U is a dense subset of $\hat{U}$, which explains why $\hat{U}$ is referred to as a completion.

The category $\widehat{\mathsf{Vec}_\Bbbk}$ has the following monoidal structure. For objects $V, W, V', W' \in \widehat{\mathsf{Vec}_\Bbbk}$ and morphisms $\phi : V \to V'$ and $\psi : W \to W'$, define

$$V \mathbin{\hat{\otimes}} W = (V^\vee \otimes W^\vee)^* \qquad \text{and} \qquad \phi \mathbin{\hat{\otimes}} \psi = (\phi^\vee \otimes \psi^\vee)^*.$$

The object $V \mathbin{\hat{\otimes}} W$ is a linearly compact vector space and the linear map $\phi \mathbin{\hat{\otimes}} \psi$ is continuous in the linearly compact topology. There is a canonical inclusion $V \otimes W \hookrightarrow V \mathbin{\hat{\otimes}} W$ given by the linear map identifying $v \otimes w$ for $v \in V$ and $w \in W$ with the linear function that has $\lambda \otimes \mu \mapsto \lambda(v)\mu(w)$ for $\lambda \in V^\vee$ and $\mu \in W^\vee$. Relative to this inclusion, $V \otimes W$ is a dense subset of the linearly compact space $V \mathbin{\hat{\otimes}} W$, and for this reason one calls $\hat{\otimes}$ the *completed tensor product*. If V and W have pseudobases $\{v_i : i \in I\}$ and $\{w_j : j \in J\}$, then the image of the set $\{v_i \otimes w_j : (i, j) \in I \times J\} \subset V \otimes W$ in $V \mathbin{\hat{\otimes}} W$ is a pseudobasis. We usually identify $V \otimes W$ with its image in $V \mathbin{\hat{\otimes}} W$ without comment.

Let β be the isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$ induced by $x \otimes y \mapsto y \otimes x$. This map uniquely extends to an isomorphism $\hat{\beta} : V \mathbin{\hat{\otimes}} W \to W \mathbin{\hat{\otimes}} V$ for all $V, W \in \widehat{\mathsf{Vec}_\Bbbk}$. Recall that $\Bbbk$ is a linearly compact vector space with the discrete topology. Checking the following is routine:

Proposition 3.7. The category $\widehat{\mathsf{Vec}_\Bbbk}$ is symmetric monoidal relative to the completed tensor product $\hat{\otimes}$, braiding map $\hat{\beta}$, and unit object $\Bbbk$.

Since $\widehat{\mathsf{Vec}}_{\Bbbk}$ is symmetric monoidal, we have corresponding notions of (co, bi, Hopf) monoids in this category. We refer to monoids, comonoids, bimonoids, and Hopf monoids in $\widehat{\mathsf{Vec}}_{\Bbbk}$ respectively as *linearly compact algebras*, *coalgebras*, *bialgebras*, and *Hopf algebras*. A structure of this type consists explicitly of a linearly compact vector space $V \in \widehat{\mathsf{Vec}}_{\Bbbk}$ along with continuous linear maps $V \mathbin{\hat\otimes} V \to V$, $\Bbbk \to V$, $V \to V \mathbin{\hat\otimes} V$, and $V \to \Bbbk$ satisfying the conditions in Section 2.1.

Alternatively, one can define linearly compact (co, bi, Hopf) algebras in $\widehat{\mathsf{Vec}}_{\Bbbk}$ entirely in terms of (co, bi, Hopf) algebras by duality. Let $U \in \mathsf{Vec}_{\Bbbk}$ and $V \in \widehat{\mathsf{Vec}}_{\Bbbk}$ and let $\langle \cdot, \cdot \rangle : U \times V \to \Bbbk$ be a nondegenerate bilinear form. Define $\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle$ for $u_i \in U$ and $v_i \in V$ and extend by continuity and linearity to define a bilinear form $(U \otimes U) \times (V \mathbin{\hat\otimes} V) \to \Bbbk$ that is continuous in the second coordinate. Also let $\langle a, b \rangle = ab$ for $a, b \in \Bbbk$.

Now suppose $\nabla : U \otimes U \to U$, $\iota : \Bbbk \to U$, $\Delta : U \to U \otimes U$, and $\epsilon : U \to \Bbbk$ are linear maps and $\hat\nabla : V \mathbin{\hat\otimes} V \to V$, $\hat\iota : \Bbbk \to V$, $\hat\Delta : V \to V \mathbin{\hat\otimes} V$, and $\hat\epsilon : V \to \Bbbk$ are continuous linear maps such that

$$\langle \nabla(u_1 \otimes u_2), v \rangle = \langle u_1 \otimes u_2, \hat\Delta(v) \rangle \qquad \text{and} \qquad \langle \iota(a), v \rangle = \langle a, \hat\epsilon(v) \rangle$$

for all $u_1, u_2 \in U$, $v \in V$, and $a \in \Bbbk$ and

$$\langle \Delta(u), v_1 \otimes v_2 \rangle = \langle u, \hat\nabla(v_1 \otimes v_2) \rangle \qquad \text{and} \qquad \langle \epsilon(u), b \rangle = \langle u, \hat\iota(b) \rangle$$

for all $u \in U$, $v_1, v_2, \in V$, and $b \in \Bbbk$. Either map in each of the pairs $(\nabla, \hat\Delta)$, $(\iota, \hat\epsilon)$, $(\Delta, \hat\nabla)$ and $(\epsilon, \hat\iota)$ then uniquely determines the other.

In this setup, (U, ∇, ι) is an algebra if and only if $(V, \hat\Delta, \hat\epsilon)$ is a linearly compact coalgebra; (U, Δ, ϵ) is a coalgebra if and only if $(V, \hat\nabla, \hat\iota)$ is a linearly compact algebra; and $(U, \nabla, \iota, \Delta, \epsilon)$ is a bialgebra (respectively, Hopf algebra) if and only if $(V, \hat\nabla, \hat\iota, \hat\Delta, \hat\epsilon)$ is a linearly compact bialgebra (respectively, Hopf algebra). In these cases, we say that the monoidal structure on V is the *(algebraic) dual* of the structure on U via the form $\langle \cdot, \cdot \rangle$.

Example 3.8. Let $\Bbbk[x] = \bigoplus_{n \in \mathbb{N}} \Bbbk x^n$ and $\Bbbk[[x]] = \prod_{n \in \mathbb{N}} \Bbbk x^n$ denote the $\Bbbk$-algebras of polynomials and formal power series in x. The bilinear form $\Bbbk[x] \times \Bbbk[[x]] \to \Bbbk$ with $\langle x^m, \sum_{n \in \mathbb{N}} c_n x^n \rangle = c_m$ is nondegenerate, and restricts to a nondegenerate graded form $\Bbbk[x] \times \Bbbk[x] \to \Bbbk[x]$.

The space $\Bbbk[x]$ is a graded Hopf algebra whose coproduct, counit, and antipode are the algebra morphisms with $\Delta(x) = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$, and $\mathtt{S}(x) = -x$. The space $\Bbbk[[x]]$ is a linearly compact Hopf algebra whose coproduct, counit, and antipode are the linearly compact algebra morphisms with the same formulas.

The Hopf algebra $\Bbbk[x]$ is its own graded dual via the form $\langle \cdot, \cdot \rangle$, but $\Bbbk[[x]]$ is its algebraic dual. The completed tensor product $\Bbbk[[x]] \mathbin{\hat\otimes} \Bbbk[[x]]$ is isomorphic to the vector space of formal power series $\Bbbk[[x, y]]$ in two commuting variables.

Example 3.9. Any graded (co, bi, Hopf) algebra of finite graded dimension extends to a linearly compact (co, bi, Hopf) algebra. In detail, suppose $V = \bigoplus_{n \in \mathbb{N}} V_n$ is a graded $\Bbbk$-vector space where each V_n is finite-dimensional. Let $\hat V = \prod_{n \in \mathbb{N}} V_n$ and give this space the product topology in which each subspace V_n is discrete. Then $\hat V$ is a linearly compact vector space and any graded linear map $V \otimes V \to V$ or $\Bbbk \to V$ or $V \to V \otimes V$ or $V \to \Bbbk$ extends uniquely to a continuous linear map $\hat V \mathbin{\hat\otimes} \hat V \to \hat V$ or $\Bbbk \to \hat V$ or $\hat V \to \hat V \mathbin{\hat\otimes} \hat V$ or $\hat V \to \Bbbk$, respectively. If V has the structure of a graded (bi, co, Hopf) algebra, then these extensions make $\hat V$ into a linearly compact (bi, co, Hopf) algebra; the relevant structure on $\hat V$ is isomorphic to the algebraic dual of the graded dual of V.

Recall that $\mathbb{W}$ is the set of pairs $[w,n]$ where $n \in \mathbb{N}$ and w is a word with letters in $\{1,2,\dots,n\}$, and $\mathbf{W} = \Bbbk\mathbb{W}$. Define $\hat{\mathbf{W}}$ to be the completion of $\mathbf{W}$ with respect to the basis $\mathbb{W}$. For $\sigma \in \hat{\mathbf{W}}$ and $[w,n] \in \mathbb{W}$, let $\sigma(w,n) \in \Bbbk$ denote the coefficient such that $\sigma = \sum_{[w,n]\in\mathbb{W}} \sigma(w,n)[w,n]$. The associated nondegenerate bilinear form $\langle\cdot,\cdot\rangle : \mathbf{W} \times \hat{\mathbf{W}} \to \Bbbk$ is then

$$\langle \sigma, \tau\rangle = \sum_{[w,n]\in\mathbb{W}} \sigma(w,n)\tau(w,n) \qquad \text{for } \sigma \in \mathbf{W} \text{ and } \tau \in \hat{\mathbf{W}}. \tag{3.1}$$

Define $\nabla_\odot : \hat{\mathbf{W}} \mathbin{\hat\otimes} \hat{\mathbf{W}} \to \hat{\mathbf{W}}$ to be the continuous linear map with

$$\nabla_\odot([v,m]\otimes[w,n]) = \begin{cases} [v\odot w, m] = [vw,m] & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

for $[v,m],[w,n] \in \mathbb{W}$. Define $\Delta_{\sqcup\!\sqcup} : \hat{\mathbf{W}} \to \hat{\mathbf{W}} \mathbin{\hat\otimes} \hat{\mathbf{W}}$ to be the continuous linear map with

$$\Delta_{\sqcup\!\sqcup}([w,n]) = \sum_{m=0}^{n} [w \cap \{1,2,\dots,m\}, m] \otimes [(w \downarrow m) \cap \{1,2,\dots,n-m\}, n-m] \tag{3.3}$$

where $w \downarrow m = (w_1 - m)(w_2 - m)\dots(w_n - m)$ and $w \cap S$ denotes the subword formed by omitting all letters not in S. Define $\epsilon_{\sqcup\!\sqcup} : \hat{\mathbf{W}} \to \Bbbk$ and $\iota_\odot : \Bbbk \to \hat{\mathbf{W}}$ to be the continous linear maps with

$$\epsilon_{\sqcup\!\sqcup}([w,n]) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \iota_\odot(1) = \sum_{n\in\mathbb{N}} [\emptyset, n]. \tag{3.4}$$

Theorem 3.10. $(\hat{\mathbf{W}}, \nabla_\odot, \iota_\odot, \Delta_{\sqcup\!\sqcup}, \epsilon_{\sqcup\!\sqcup})$ is a linearly compact bialgebra.

Proof. It is a straightforward exercise to check that $(\hat{\mathbf{W}}, \nabla_\odot, \iota_\odot, \Delta_{\sqcup\!\sqcup}, \epsilon_{\sqcup\!\sqcup})$ is the algebraic dual of the bialgebra $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ via the bilinear form (3.1). □

Define $\hat{\mathbf{W}}_{\mathsf{P}}$ to be the completion of vector space of packed words $\mathbf{W}_{\mathsf{P}}$ with respect to the basis $\mathbb{W}_{\mathsf{P}}$. The natural pairing $\mathbf{W}_{\mathsf{P}} \times \hat{\mathbf{W}}_{\mathsf{P}} \to \Bbbk$ gives $\hat{\mathbf{W}}_{\mathsf{P}}$ the structure of a linearly compact Hopf algebra dual to $(\mathbf{W}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$, which one can realize as a sub-bialgebra of $(\hat{\mathbf{W}}, \nabla_\odot, \iota_\odot, \Delta_{\sqcup\!\sqcup}, \epsilon_{\sqcup\!\sqcup})$. This object is not of much relevance to our discussion, however.

On the other hand, since $\mathbf{W}_{\mathsf{P}}$ has finite graded dimension when graded by word length, the maps $\nabla_{\sqcup\!\sqcup}$, $\iota_{\sqcup\!\sqcup}$, $\Delta_\odot$ and $\epsilon_\odot$ from (2.5) have continuous linear extensions to maps between $\hat{\mathbf{W}}_{\mathsf{P}}$, $\hat{\mathbf{W}}_{\mathsf{P}} \mathbin{\hat\otimes} \hat{\mathbf{W}}_{\mathsf{P}}$, and $\Bbbk$ as appropriate, and the following holds in view of Example 3.9:

Proposition 3.11. $(\hat{\mathbf{W}}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ is a linearly compact Hopf algebra.

Let $\hat{\mathbf{W}}_n$ be the completion of $\mathbf{W}_n$ with respect to $\mathbb{W}_n$. Each subspace $\mathbf{W}_n$ for $n \in \mathbb{N}$ is a sub-coalgebra of $(\mathbf{W}, \Delta_\odot, \epsilon_\odot)$ of finite graded dimension, so $\Delta_\odot$ and $\epsilon_\odot$ extend to continuous linear maps $\hat{\mathbf{W}}_n \to \hat{\mathbf{W}}_n \mathbin{\hat\otimes} \hat{\mathbf{W}}_n$ and $\hat{\mathbf{W}}_n \to \Bbbk$, and the following similarly holds:

Proposition 3.12. For each $n \in \mathbb{N}$, $(\hat{\mathbf{W}}_n, \Delta_\odot, \epsilon_\odot)$ is a linearly compact coalgebra.

Since $\mathbf{W}$ does not have finite graded dimension, the bialgebra structure $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ does not extend to $\hat{\mathbf{W}}$. In particular, the coproduct $\epsilon_\odot$ cannot be evaluated at $\sum_{n\in\mathbb{N}}[\emptyset, n] \in \hat{\mathbf{W}}$. Nevertheless, there is a sense in which $\nabla_{\sqcup\!\sqcup}$ and $\Delta_\odot$ can be interpreted as a compatible morphisms $\hat{\mathbf{W}} \mathbin{\hat\otimes} \hat{\mathbf{W}} \to \hat{\mathbf{W}}$ and $\hat{\mathbf{W}} \to \hat{\mathbf{W}} \mathbin{\hat\otimes} \hat{\mathbf{W}}$. This is the main theme of the next section.

4 Species coalgebroids

Let $\mathsf{Mon}(\mathscr{C})$, $\mathsf{Comon}(\mathscr{C})$, and $\mathsf{Bimon}(\mathscr{C})$ be the categories of monoids, comonoids, and bimonoids in a symmetric monoidal category $\mathscr{C}$. Let FB denote the category of finite sets with bijections as morphisms. A *$\mathscr{C}$-species* is a functor $\mathsf{FB} \to \mathscr{C}$. Such functors form a category, denoted $\mathscr{C}$-Sp, with natural transformations as morphisms. For more background on species, see [2, Chapter 8].

When $\mathscr{F}$ is a $\mathscr{C}$-species and S is a finite set and $\sigma : S \to T$ is a bijection, we write $\mathscr{F}[S]$ for the corresponding object in $\mathscr{C}$ and $\mathscr{F}[\sigma]$ for the corresponding morphism $\mathscr{F}[S] \to \mathscr{F}[T]$, which is necessarily an isomorphism. When $\eta : \mathscr{F} \to \mathscr{G}$ is a natural transformation and S is a finite set, we write η_S for the corresponding morphism $\mathscr{F}[S] \to \mathscr{G}[S]$. We refer to $\mathscr{F}[S]$ and η_S as the *S-component* of $\mathscr{F}$ and η. If S is clear from context and $x \in \mathscr{F}[S]$, then we may write $\eta(x)$ instead of $\eta_S(x)$ for the corresponding element of $\mathscr{G}[S]$. If $\mathscr{C}$ is a small category, then a *subspecies* of a $\mathscr{C}$-species $\mathscr{G}$ is a $\mathscr{C}$-species $\mathscr{F}$ with $\mathscr{F}[S] \subset \mathscr{G}[S]$ for all finite sets S and $\mathscr{F}[\sigma] = \mathscr{G}[\sigma]|_{\mathscr{F}[S]}$ for all bijections $\sigma : S \to T$. We write $\mathscr{F} \subset \mathscr{G}$ to indicate that $\mathscr{F}$ is a subspecies of $\mathscr{G}$.

With these conventions, a *linearly compact coalgebra species* is a functor $\mathscr{V} : \mathsf{FB} \to \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$. Suppose $\mathscr{U}$ and $\mathscr{V}$ are linearly compact coalgebra species and $\alpha : \mathscr{U} \to \mathscr{U}'$ and $\beta : \mathscr{V} \to \mathscr{V}'$ are natural transformations. Define $\mathscr{U} \cdot \mathscr{V} : \mathsf{FB} \to \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$ and $\alpha \cdot \beta : \mathscr{U} \cdot \mathscr{V} \to \mathscr{U}' \cdot \mathscr{V}'$ by

$$(\mathscr{U} \cdot \mathscr{V})[I] = \bigoplus_{S \sqcup T = I} \mathscr{U}[S] \mathbin{\hat\otimes} \mathscr{V}[T] \qquad \text{and} \qquad (\alpha \cdot \beta)_I = \bigoplus_{S \sqcup T = I} \alpha_S \mathbin{\hat\otimes} \beta_T \tag{4.1}$$

for each finite set I, where the sums are over all $2^{|I|}$ ways of writing I as a union of two disjoint sets. Define $(\mathscr{U} \cdot \mathscr{V})[\sigma] : (\mathscr{U} \cdot \mathscr{V})[I] \to (\mathscr{U} \cdot \mathscr{V})[J]$ similarly when $\sigma : I \to J$ is a bijection. The category of linearly compact coalgebra species is symmetric monoidal with respect to this operation, called the *Cauchy product* in [2], with unit object given by the species $\mathbf{1} : \mathbb{N} \to \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$ that has $\mathbf{1}[\varnothing] = \Bbbk$ and $\mathbf{1}[S] = 0$ for all nonempty finite sets S. When $\nabla : \mathscr{V} \cdot \mathscr{V} \to \mathscr{V}$ is a natural transformation and $I = S \sqcup T$, we write $\nabla_{ST} : \mathscr{V}[S] \mathbin{\hat\otimes} \mathscr{V}[T] \to \mathscr{V}[I]$ for the composition of $\nabla_I : (\mathscr{V} \cdot \mathscr{V})[I] \to \mathscr{V}[I]$ with the inclusion $\mathscr{V}[S] \mathbin{\hat\otimes} \mathscr{V}[T] \to (\mathscr{V} \cdot \mathscr{V})[I]$.

Definition 4.1. A *species coalgebroid* is a monoid in the category of linearly compact coalgebra species. Explicitly, suppose $\mathscr{V} : \mathsf{FB} \to \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$ is a functor. Write Δ_I and ϵ_I for the coproduct and counit of $\mathscr{V}[I]$ and let $\Delta = (\Delta_I)$ and $\epsilon = (\epsilon_I)$ denote the corresponding families of linear maps. Suppose $\nabla : \mathscr{V} \cdot \mathscr{V} \to \mathscr{V}$ and $\iota : \mathbf{1} \to \mathscr{V}$ are natural transformations. Then $(\mathscr{V}, \nabla, \iota, \Delta, \epsilon)$ is a species coalgebroid if and only if the following conditions hold:

(a) For all pairwise disjoint finite sets S, T, and U, the following diagrams commute:

$$\begin{array}{ccccc} \Bbbk \mathbin{\hat\otimes} \mathscr{V}[\varnothing] & \xrightarrow{\iota_\varnothing \hat\otimes \mathrm{id}} & \mathscr{V}[\varnothing] \mathbin{\hat\otimes} \mathscr{V}[\varnothing] & \xleftarrow{\mathrm{id} \hat\otimes \iota_\varnothing} & \mathscr{V}[\varnothing] \mathbin{\hat\otimes} \Bbbk \\ & \searrow^{\cong} & \downarrow{\scriptstyle \nabla_{\varnothing\varnothing}} & {}^{\cong}\swarrow & \\ & & \mathscr{V}[\varnothing] & & \end{array}
\qquad
\begin{array}{ccc} \mathscr{V}[S] \mathbin{\hat\otimes} \mathscr{V}[T] \mathbin{\hat\otimes} \mathscr{V}[U] & \xrightarrow{\nabla_{ST} \hat\otimes \mathrm{id}} & \mathscr{V}[S \sqcup T] \mathbin{\hat\otimes} \mathscr{V}[U] \\ {\scriptstyle \mathrm{id} \hat\otimes \nabla_{TU}} \downarrow & & \downarrow {\scriptstyle \nabla_{S \sqcup T, U}} \\ \mathscr{V}[S] \mathbin{\hat\otimes} \mathscr{V}[T \sqcup U] & \xrightarrow{\nabla_{S, T \sqcup U}} & \mathscr{V}[S \sqcup T \sqcup U] \end{array}$$

(b) For all disjoint finite sets S and T, the following diagrams commute:

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\mathrm{id}} & \Bbbk \\ & {}_{\iota_\varnothing}\searrow \quad \nearrow_{\epsilon_\varnothing} & \\ & \mathscr{V}[\varnothing] & \end{array}
\qquad
\begin{array}{ccc} \Bbbk & \xrightarrow{\iota_\varnothing} & \mathscr{V}[\varnothing] \\ {\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \Delta_\varnothing} \\ \Bbbk \mathbin{\hat\otimes} \Bbbk & \xrightarrow{\iota_\varnothing \hat\otimes \iota_\varnothing} & \mathscr{V}[\varnothing] \mathbin{\hat\otimes} \mathscr{V}[\varnothing] \end{array}
\qquad
\begin{array}{ccc} \mathscr{V}[S] \mathbin{\hat\otimes} \mathscr{V}[T] & \xrightarrow{\epsilon_S \hat\otimes \epsilon_T} & \Bbbk \mathbin{\hat\otimes} \Bbbk \\ {\scriptstyle \nabla_{ST}} \downarrow & & \downarrow {\scriptstyle \cong} \\ \mathscr{V}[S \sqcup T] & \xrightarrow{\epsilon_{S \sqcup T}} & \Bbbk \end{array}$$

(c) For all disjoint finite sets S and T, the following diagram commutes:

$$\begin{array}{ccccc}
\mathscr{V}[S]\,\hat{\otimes}\,\mathscr{V}[T] & \xrightarrow{\nabla_{ST}} & \mathscr{V}[S\sqcup T] & \xrightarrow{\Delta_{S\sqcup T}} & \mathscr{V}[S\sqcup T]\,\hat{\otimes}\,\mathscr{V}[S\sqcup T] \\
\downarrow{\scriptstyle \Delta_S\hat{\otimes}\Delta_T} & & & & \uparrow{\scriptstyle \nabla_{ST}\hat{\otimes}\nabla_{ST}} \\
\mathscr{V}[S]\,\hat{\otimes}\,\mathscr{V}[S]\,\hat{\otimes}\,\mathscr{V}[T]\,\hat{\otimes}\,\mathscr{V}[T] & & \xrightarrow{\mathrm{id}\hat{\otimes}\hat{\beta}\hat{\otimes}\mathrm{id}} & & \mathscr{V}[S]\,\hat{\otimes}\,\mathscr{V}[T]\,\hat{\otimes}\,\mathscr{V}[S]\,\hat{\otimes}\,\mathscr{V}[T]
\end{array}$$

We refer to $\nabla : \mathscr{V} \cdot \mathscr{V} \to \mathscr{V}$ and $\iota : \mathbf{1} \to \mathscr{V}$ as the *product* and *unit* of $\mathscr{V}$, and to the families of maps $\Delta = (\Delta_I)$ and $\epsilon = (\epsilon_I)$ as the *coproduct* and *counit* of $\mathscr{V}$. Species coalgebroids form a category, which we denote by $\mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$, whose morphisms are the natural transformations between $\mathsf{Comon}(\widehat{\mathsf{Vec}_{\Bbbk}})$-species that commute with the product and unit morphisms.

If $(\mathscr{V}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$ is a species coalgebroid, then a subspecies $\mathscr{H} \subset \mathscr{V}$ is a *sub-coalgebroid* when $\Delta_S(\mathscr{H}[S]) \subset \mathscr{H}[S] \,\hat{\otimes}\, \mathscr{H}[S]$ for each finite set S and the morphisms ∇ and ι restrict to natural transformations $\mathscr{H} \cdot \mathscr{H} \to \mathscr{H}$ and $\mathbf{1} \to \mathscr{H}$. When these conditions hold, we have $(\mathscr{H}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$.

Remark. If needed, one can introduce a sequence of definitions dual to those above. The natural dual of a linearly compact coalgebra species is an *algebra species*, i.e., a functor $\mathsf{FB} \to \mathsf{Mon}(\mathsf{Vec}_{\Bbbk})$. Such functors form a symmetric monoidal category with unit object $\mathbf{1}$, relative to the Cauchy product defined just as in (4.1) but with the completed tensor product $\hat{\otimes}$ replaced by $\otimes$. The natural dual of a species coalgebroid is then a comonoid in the category of algebra species.

Species coalgebroids generalize linearly compact bialgebras since the latter are monoids in the category of linearly compact coalgebras. We highlight three functors to or from $\mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$:

(i) There is a natural "forgetful" functor

$$\mathcal{F} : \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}}) \to \mathsf{Bimon}(\widehat{\mathsf{Vec}_{\Bbbk}}) \tag{4.2}$$

with $\mathcal{F}(\mathscr{B}) = (\mathscr{V}[\varnothing], \nabla_{\varnothing,\varnothing}, \iota_{\varnothing}, \Delta_{\varnothing}, \epsilon_{\varnothing})$ for each $\mathscr{B} = (\mathscr{V}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$ and with $\mathcal{F}(\eta) = \eta_{\varnothing}$ for each morphism $\eta : \mathscr{B} \to \mathscr{B}'$ in $\mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$.

(ii) For $V \in \mathsf{Vec}_{\Bbbk}$, let $\mathcal{E}(V) : \mathbb{N} \to \mathsf{Vec}_{\Bbbk}$ be the species with $\mathcal{E}(V)[S] = V$ and $\mathcal{E}(V)[\sigma] = \mathrm{id}_V$ for all finite sets S and bijections $\sigma : S \to T$. For any linear map $\phi : V \to V'$, let $\mathcal{E}(\phi) : \mathcal{E}(V) \to \mathcal{E}(V')$ be the natural transformation with $\mathcal{E}(\phi)_S = \phi$ for all finite sets S. This gives a functor

$$\mathcal{E} : \mathsf{Vec}_{\Bbbk} \to \mathsf{Vec}_{\Bbbk}\text{-}\mathsf{Sp}. \tag{4.3}$$

If $B = (V, \mu, i, \delta, e)$ is a linearly compact bialgebra, then define $\mathcal{E}(B) := (\mathcal{E}(V), \nabla, \iota, \Delta, \epsilon)$ to be the species coalgebroid in which $\nabla_{ST} = \mu$, $\iota_I = i$, $\Delta_I = \delta$, and $\epsilon_I = e$ for all disjoint finite sets S, T, and I. This makes $\mathcal{E}$ into a functor $\mathsf{Bimon}(\widehat{\mathsf{Vec}_{\Bbbk}}) \to \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$.

(iii) Suppose $\mathscr{B} = (\mathscr{V}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$ is *finite-dimensional* in that $\dim_{\Bbbk} \mathscr{V}[S] < \infty$ for all finite sets S. For each $n \in \mathbb{N}$, the symmetric group S_n acts as a group of coalgebra automorphisms on $\mathscr{V}[n] := \mathscr{V}[\{1, 2, \ldots, n\}]$ via $x \mapsto \mathscr{V}[\sigma](x)$ for $x \in \mathscr{V}[n]$ and $\sigma \in S_n$. The subspace $\mathbf{I}_n \subset \mathscr{V}[n]$ spanned by all differences $x - \mathscr{V}[\sigma](x)$ for $x \in \mathscr{V}[S]$ and $\sigma \in S_n$ is a coideal and we denote the corresponding quotient coalgebra by $\mathscr{V}[n]_{S_n} = \mathscr{V}[n]/\mathbf{I}_n$. Reuse Δ_n and ϵ_n to denote the coproduct and counit of $\mathscr{V}[n]_{S_n}$. The natural map

$$(\mathscr{V} \cdot \mathscr{V})[n] \to \bigoplus_{i+j=n} \mathscr{V}[i]_{S_i} \otimes \mathscr{V}[j]_{S_j}$$

descends to an isomorphism

$$(\mathscr{V} \cdot \mathscr{V})[n]_{S_n} \xrightarrow{\sim} \bigoplus_{i+j=n} \mathscr{V}[i]_{S_i} \otimes \mathscr{V}[j]_{S_j}$$

and the $[n]$-component of ∇ descends to a linear map

$$\nabla_n : (\mathscr{V} \cdot \mathscr{V})[n]_{S_n} \to \mathscr{V}[n]_{S_n}.$$

The space $V = \bigoplus_{n\in\mathbb{N}} \mathscr{V}[n]_{S_n}$ is a $\Bbbk$-bialgebra with product

$$V \otimes V = \bigoplus_{n\in\mathbb{N}} \bigoplus_{i+j=n} \mathscr{V}[i]_{S_i} \otimes \mathscr{V}[j]_{S_j} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{N}} (\mathscr{V} \cdot \mathscr{V})[n]_{S_n} \xrightarrow{\bigoplus_{n\in\mathbb{N}} \nabla_n} V,$$

and coproduct

$$V \xrightarrow{\bigoplus_{n\in\mathbb{N}} \Delta_n} \bigoplus_{n\in\mathbb{N}} (\mathscr{V}[n]_{S_n} \otimes \mathscr{V}[n]_{S_n}) \hookrightarrow V \otimes V,$$

along with unit $\bigoplus_{n\in\mathbb{N}} \iota_{[n]} = \iota_\varnothing$ and counit $\bigoplus_{n\in\mathbb{N}} \epsilon_n$. Let $\overline{\mathcal{K}}(\mathscr{B})$ denote this bialgebra. When $\eta : \mathscr{B} \to \mathscr{B}'$ in a morphism between finite-dimensional coalgebroids, the direct sum $\bigoplus_{n\in\mathbb{N}} \eta_{[n]}$ descends to a map $\overline{\mathcal{K}}(\mathscr{B}) \to \overline{\mathcal{K}}(\mathscr{B}')$, denoted $\overline{\mathcal{K}}(\eta)$. This makes $\overline{\mathcal{K}}$ into a functor

$$\overline{\mathcal{K}} : \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}}_{Fin}) \to \mathsf{Bimon}(\mathsf{Vec}_\Bbbk) \tag{4.4}$$

where $\mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}}_{Fin})$ is the full subcategory of finite-dimensional species coalgebroids. The functor $\overline{\mathcal{K}}$ is similar to the *bosonic Fock functor* defined in [2, Chapter 15].

We conclude this section by constructing what will be our fundamental example of Definition 4.1. Fix a set S of size n. For each bijection $\lambda : [n] \to S$, let $\mathbb{W}_\lambda$ be the set of pairs $[w, \lambda]$ where w is a word with $\max(w) \leq n$. Define $\hat{\mathbf{W}}_\lambda$ to be the linearly compact $\Bbbk$-vector space with $\mathbb{W}_\lambda$ as a pseudobasis. Write $\mathbb{L}[S]$ for the set of bijections $[n] \to S$ and let

$$\mathscr{W}[S] = \bigoplus_{\lambda\in\mathbb{L}[S]} \hat{\mathbf{W}}_\lambda \in \widehat{\mathsf{Vec}_\Bbbk}.$$

For each bijection $\sigma : S \to T$, define $\mathscr{W}[\sigma]$ to be the continuous linear map $\mathscr{W}[S] \to \mathscr{W}[T]$ with

$$\mathscr{W}[\sigma]([w, \lambda]) = [w, \sigma \circ \lambda] \qquad \text{for } [w, \lambda] \in \mathbb{W}_\lambda. \tag{4.5}$$

These definitions make $\mathscr{W}$ into a functor $\mathsf{FB} \to \widehat{\mathsf{Vec}_\Bbbk}$.

Identify $[w, n] \in \mathbb{W}_n$ with the element $[w, \lambda] \in \mathbb{W}_\lambda$ where λ is the identity map $[n] \to [n]$ and in this way view $\hat{\mathbf{W}}_n$ as a subspace of $\mathscr{W}[n] := \mathscr{W}[\{1, 2, \ldots, n\}]$. We extend $\Delta_\odot : \hat{\mathbf{W}}_n \to \hat{\mathbf{W}}_n \hat{\otimes} \hat{\mathbf{W}}_n$ and

$\epsilon_\odot : \hat{\mathbf{W}}_n \to \Bbbk$ from (2.4) to continuous linear maps $\Delta_\odot : \mathscr{W}[S] \to \mathscr{W}[S] \hat{\otimes} \mathscr{W}[S]$ and $\epsilon_\odot : \mathscr{W}[S] \to \Bbbk$ by requiring that

$$\Delta_\odot|_{\hat{\mathbf{W}}_\lambda} = (\mathscr{W}[\lambda] \hat{\otimes} \mathscr{W}[\lambda]) \circ \Delta_\odot \circ \mathscr{W}[\lambda^{-1}]|_{\hat{\mathbf{W}}_\lambda} \quad \text{and} \quad \epsilon_\odot|_{\hat{\mathbf{W}}_\lambda} = \epsilon_\odot \circ \mathscr{W}[\lambda^{-1}]|_{\hat{\mathbf{W}}_\lambda} \tag{4.6}$$

for each subspace $\hat{\mathbf{W}}_\lambda \subset \mathscr{W}[S]$. It follows from Proposition 3.12 that $\mathscr{W}$ defines a linearly compact coalgebra species $\mathsf{FB} \to \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$.

Given disjoint finite sets S and T with $n = |S|$ and $m = |T|$ and bijections $(\lambda, \mu) \in \mathbb{L}[S] \times \mathbb{L}[T]$, let $\lambda \oplus \mu$ denote the bijection $[n+m] \to S \sqcup T$ with $i \mapsto \lambda(i)$ for $i \in [n]$ and $n+j \mapsto \mu(j)$ for $j \in [m]$. When $S = \varnothing$ and λ is the unique map $[0] = \varnothing \to \varnothing = S$, define $\lambda \oplus \lambda = \lambda$. Write $\nabla_{\sqcup\!\sqcup} : \mathscr{W} \cdot \mathscr{W} \to \mathscr{W}$ for the natural transformation whose I-component $(\mathscr{W} \cdot \mathscr{W})[I] \to \mathscr{W}[I]$ is the direct sum, over all disjoint decompositions $I = S \sqcup T$ and bijections $(\lambda, \mu) \in \mathbb{L}[S] \times \mathbb{L}[T]$, of the maps

$$\mathscr{W}[\lambda \oplus \mu] \circ \nabla_{\sqcup\!\sqcup} \circ \left(\mathscr{W}[\lambda^{-1} \hat{\otimes} \mathscr{W}[\mu^{-1}]\right) : \hat{\mathbf{W}}_\lambda \hat{\otimes} \hat{\mathbf{W}}_\mu \to \hat{\mathbf{W}}_{\lambda \oplus \mu} \tag{4.7}$$

with $\nabla_{\sqcup\!\sqcup} : \hat{\mathbf{W}}_n \hat{\otimes} \hat{\mathbf{W}}_m \to \hat{\mathbf{W}}_{n+m}$ as in (2.3). Let $\iota_{\sqcup\!\sqcup} : \mathbf{1} \to \mathscr{W}$ be the natural transformation whose nontrivial component is the linear map $\mathbf{1}[\varnothing] = \Bbbk \to \mathscr{W}[\varnothing]$ with $1 \mapsto [\emptyset, \mathrm{id}_\varnothing]$.

Remark. We can describe the maps (4.6) and (4.7) more concretely. Let S be a finite set of size n. An S-*word* is a finite sequence $a = a_1 a_2 \cdots a_l$ with $a_i \in S$. Given a bijection $\lambda : [n] \to S$, define $(a, \lambda) = [w, \lambda] \in \mathbb{W}_\lambda$ where $w = w_1 w_2 \cdots w_l$ is the word with $\lambda(w) := \lambda(w_1)\lambda(w_2)\cdots\lambda(w_l) = a$. Equation (4.5) is then $\mathscr{W}[\sigma]((a, \lambda)) = (\sigma(a), \sigma \circ \lambda)$ and the formulas in (4.6) become

$$\Delta_\odot((a,\lambda)) = \sum_{i=0}^{l} (a_1 \cdots a_i, \lambda) \otimes (a_{i+1} \dots a_l, \lambda) \quad \text{and} \quad \epsilon_\odot((a,\lambda)) = \begin{cases} 1 & \text{if } a = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

If b is a T-word where $S \cap T = \varnothing$ and $(b, \mu) \in \mathbb{W}_T$, then (4.7) is the continuous linear map with

$$\nabla_{\sqcup\!\sqcup}((a, \lambda) \otimes (b, \mu)) = (a \sqcup\!\sqcup b, \lambda \oplus \mu)$$

where we define $(c_1 w^1 + \cdots + c_k w^k, \lambda) = c_1 (w^1, \lambda) + \cdots + c_k (w^k, \lambda)$. In this way, the product can be defined using the ordinary shuffle operation instead of the shifted shuffle in (2.3).

With slight abuse of notation, we reuse the symbols $\Delta_\odot$ and $\epsilon_\odot$ to denote the families of maps $\mathscr{W}[S] \xrightarrow{\Delta_\odot} \mathscr{W}[S] \hat{\otimes} \mathscr{W}[S]$ and $\mathscr{W}[S] \xrightarrow{\epsilon_\odot} \Bbbk$ for all finite sets S. The following then holds:

Theorem 4.2. $(\mathscr{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ is a species coalgebroid.

Proof. Modify the diagrams in Definition 4.1 by replacing $\mathscr{V}[\emptyset]$, $\mathscr{V}[S]$, $\mathscr{V}[T]$, and $\mathscr{V}[U]$ by $\hat{\mathbf{W}}_0$, $\hat{\mathbf{W}}_{|S|}$, $\hat{\mathbf{W}}_{|T|}$, and $\hat{\mathbf{W}}_{|U|}$. It suffices to show that these modified diagrams each commute. Since all arrows in the diagrams are continuous linear maps, this follows by Theorem 2.4. □

5 Word relations

Here, we characterize the relations on words that generate sub-objects of the bialgebra $\mathbf{W}$, the linearly compact Hopf algebra $\hat{\mathbf{W}}_{\mathsf{P}}$, or the species coalgebroid $\mathscr{W}$. Our starting point is the following:

Definition 5.1. A *word relation* is an equivalence relation $\sim$ on words with the property that $v \sim w$ only if v and w share the same set of letters, not necessarily with the same multiplicities.

5.1 Algebraic relations

Recall that $w \uparrow m$ and $w \downarrow m$ are formed from w by adding and subtracting m to each letter.

Definition 5.2. A word relation $\sim$ is *algebraic* if for all words v and w, the following holds:

(a) If v', w' are words with $v \sim v'$ and $w \sim w'$, then $vw \sim v'w'$.

(b) If $v \sim w$ and $I = \{m+1, m+2, \dots, n\}$ for $m, n \in \mathbb{N}$, then $(v \cap I) \downarrow m \sim (w \cap I) \downarrow m$.

Condition (a) states that $\sim$ is a *congruence* on the free monoid on $\mathbb{P}$, and is equivalent to requiring that $vxw \sim vyw$ whenever v, w, x, y are words with $x \sim y$. A typical example of an algebraic word relation is *K-Knuth equivalence* [8, Definition 5.3], the weakest congruence with $bac \sim bca$, $acb \sim cab$, $aba \sim bab$, and $a \sim aa$ for all integers $a < b < c$. For this relation, Definition 5.2(b) can be checked directly; see also Proposition 5.16.

Fix a word relation $\sim$ and suppose v and w are words. We note two basic facts:

Lemma 5.3. If $\sim$ is algebraic and $v \sim w$, then $v \cap [n] \sim w \cap [n]$ for all $n \in \mathbb{N}$.

Proof. Take $m = 0$ in condition (b) in Definition 5.2. ☐

Lemma 5.4. If $\sim$ is algebraic and $v \uparrow m \sim w \uparrow m$ for some $m \in \mathbb{N}$, then $v \sim w$.

Proof. If $\tilde{v} := v \uparrow m \sim w \uparrow m =: \tilde{w}$, then $v = (\tilde{v} \cap I) \downarrow m \sim (\tilde{w} \cap I) \downarrow m = w$ for $I = m + \mathbb{P}$. ☐

Given a set E of words with letters in $[n]$ and a bijection $\lambda : [n] \to S$, define

$$\kappa_E^\lambda = \sum_{w \in E} [w, \lambda] \in \hat{\mathbf{W}}_\lambda \subset \mathscr{W}[S]. \tag{5.1}$$

For each finite set S of size $n \in \mathbb{N}$, let $\mathbb{K}_S^{(\sim)}$ be the set of elements of the form κ_E^λ where E is a $\sim$-equivalence class of words with letters in $[n]$ and λ is a bijection $[n] \to S$. Let $\mathscr{K}^{(\sim)}[S]$ be the linearly compact $\Bbbk$-vector space with $\mathbb{K}_S^{(\sim)}$ as a pseudobasis. The linearly compact topology on this space is the same as the subspace topology induced by $\mathscr{W}[S]$. Continuous maps to or from $\mathscr{W}[S]$ therefore remain continuous when restricted to $\mathscr{K}^{(\sim)}[S]$. It follows that

$$\mathscr{K}^{(\sim)} : \mathbb{N} \to \widehat{\mathsf{Vec}_\Bbbk} \tag{5.2}$$

defines a subspecies of $\mathscr{W}$.

Theorem 5.5. Suppose $\sim$ is a word relation. Then the species $\mathscr{K}^{(\sim)} : \mathsf{FB} \to \widehat{\mathsf{Vec}_\Bbbk}$ is a sub-coalgebroid of $(\mathscr{W}, \nabla_\sqcup, \iota_\sqcup, \Delta_\odot, \epsilon_\odot)$ if and only if $\sim$ is algebraic.

Proof. Condition (a) in Definition 5.2 holds if and only if $\Delta_\odot(\kappa_E^\lambda) \in \mathscr{K}^{(\sim)}[S] \hat{\otimes} \mathscr{K}^{(\sim)}[S]$ for each bijection $\lambda : [n] \to S$ and basis element $\kappa_E^\lambda \in \mathbb{K}_S^{(\sim)}$. Condition (b) in Definition 5.2 holds if and only if for all words v, w with $v \sim w$ and all integers $n \in \mathbb{N}$, we have both $v \cap [n] \sim w \cap [n]$ and $(v \cap I) \downarrow n \sim (w \cap I) \downarrow n$ for $I = n + \mathbb{P}$. By taking E and F to be the $\sim$-equivalence classes of $v \cap [n]$ and $(v \cap I) \downarrow n$, one checks that this property is necessary and sufficient to have $\nabla_\sqcup(\kappa_E^\lambda \otimes \kappa_F^\mu) \in \mathscr{K}^{(\sim)}[S \sqcup T]$ for all disjoint finite sets S and T and basis elements $\kappa_E^\lambda \in \mathbb{K}_S^{(\sim)}$ and $\kappa_F^\mu \in \mathbb{K}_T^{(\sim)}$. This suffices to show that $\mathscr{K}^{(\sim)}$ is a sub-coalgebroid if and only if $\sim$ is algebraic. ☐

Continue to let $\sim$ be a word relation. For $n \in \mathbb{N}$, write κ_E^n in place κ_E^λ when λ is the identity map $[n] \to [n]$, and let $\mathbb{K}_n^{(\sim)} = \mathbb{K}_{[n]}^{(\sim)} \cap \hat{\mathbf{W}}_n$ be the set of elements κ_E^n where E ranges over all $\sim$-equivalence classes of words with letters in $[n]$. Define

$$\mathbf{K}_n^{(\sim)} = \Bbbk\mathbb{K}_n^{(\sim)} \qquad \text{and} \qquad \mathbf{K}^{(\sim)} = \bigoplus_{n\in\mathbb{N}} \mathbf{K}_n^{(\sim)}. \tag{5.3}$$

The vector space $\mathbf{K}^{(\sim)}$ is a subspace of $\hat{\mathbf{W}}$ but is considered to be a discrete topological space. We say that $\sim$ is of *finite-type* if for each $n \in \mathbb{N}$, the space $\mathbf{K}_n^{(\sim)}$ is finite-dimensional.

Corollary 5.6. If $\sim$ is algebraic and of finite-type then $(\mathbf{K}^{(\sim)}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot) \in \mathsf{Bimon}(\mathsf{Vec}_\Bbbk)$.

Proof. If $\sim$ is algebraic and of finite-type, then the species coalgebroid $(\mathscr{K}^{(\sim)}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$ is finite-dimensional and its image under the functor (4.4) is isomorphic to $(\mathbf{K}^{(\sim)}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$. ☐

The relation $\sim$ is *homogeneous* if $v \sim w$ implies that v and w have the same length. When this holds, each equivalence class in $\mathbb{W}_n$ is finite so $\mathbf{K}^{(\sim)} \subset \mathbf{W}$, and each $\kappa_E^n \in \mathbb{K}_n^{(\sim)}$ is homogeneous.

Theorem 5.7. Suppose $\sim$ is a homogeneous word relation. The vector space $\mathbf{K}^{(\sim)}$ is a graded sub-bialgebra of $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot) \in \mathsf{Bimon}(\mathsf{Vec}_\Bbbk)$ if and only if $\sim$ is algebraic.

Proof. The argument is the same as in the proof of Theorem 5.5, *mutatis mutandis.* ☐

A word of minimal length in its $\sim$-equivalence class is *reduced*. A pair $[w, n] \in \mathbb{W}_n$ is *reduced* with respect to $\sim$ if w is reduced. Let $\mathbb{W}_{\mathsf{R}}^{(\sim)}$ be the set of reduced elements in $\mathbb{W} = \bigsqcup_{n\in\mathbb{N}} \mathbb{W}_n$. Define $\mathbb{K}_{\mathsf{R}}^{(\sim)}$ to be the set of elements of the form $\kappa_E^n \in \mathbf{W}$ where $n \in \mathbb{N}$ and E is the (finite) subset of reduced elements in a single $\sim$-equivalence class of words with letters in $[n]$. Finally, let

$$\mathbf{W}_{\mathsf{R}}^{(\sim)} = \Bbbk\mathbb{W}_{\mathsf{R}}^{(\sim)} \qquad \text{and} \qquad \mathbf{K}_{\mathsf{R}}^{(\sim)} = \Bbbk\mathbb{K}_{\mathsf{R}}^{(\sim)}. \tag{5.4}$$

If $\sim$ is homogeneous then $\mathbf{W}_{\mathsf{R}}^{(\sim)} = \mathbf{W}$ and $\mathbf{K}_{\mathsf{R}}^{(\sim)} = \mathbf{K}^{(\sim)}$.

Proposition 5.8. Suppose $\sim$ is an algebraic word relation. Then $\mathbf{K}_{\mathsf{R}}^{(\sim)}$ and $\mathbf{W}_{\mathsf{R}}^{(\sim)}$ are sub-bialgebras of $(\mathbf{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$.

Proof. Conditions (a) and (b) in Definition 5.2 respectively imply that (1) if v and w are words such that vw is reduced then v and w are reduced, and (2) if v and w are reduced words with $\max(v) \leq m$ then every term in the sum $v \sqcup\!\sqcup (w \uparrow m)$ is reduced. One concludes that $\mathbf{W}_{\mathsf{R}}^{(\sim)}$ is a sub-bialgebra of $\mathbf{W}$. Since $\nabla_{\sqcup\!\sqcup}$ maps $\mathscr{K}^{(\sim)}[m] \hat{\otimes} \mathscr{K}^{(\sim)}[n] \to \mathscr{K}^{(\sim)}[m+n]$ and $\Delta_\odot$ maps $\mathscr{K}^{(\sim)}[n] \to \mathscr{K}^{(\sim)}[n] \hat{\otimes} \mathscr{K}^{(\sim)}[n]$, it follows that $\mathbf{K}_{\mathsf{R}}^{(\sim)}$ is a sub-bialgebra of $\mathbf{W}_{\mathsf{R}}^{(\sim)}$. ☐

5.2 P-algebraic relations

To adapt Theorem 5.5 to packed words, a somewhat technical variation of Definition 5.2 is needed. If $u, v \in \mathbb{W}_{\mathsf{P}}$ are two packed words, then we say that $w \in \mathbb{W}_{\mathsf{P}}$ is a (u, v)-*destandardization* if there are (not necessarily packed) words $\tilde{u}, \tilde{v}$ such that $w = \tilde{u}\tilde{v}$ and $u = \mathrm{fl}(\tilde{u})$ and $v = \mathrm{fl}(\tilde{v})$. For example, 1234, 1324, and 1423 are all $(12, 12)$-destandardizations.

Definition 5.9. A word relation is *P-algebraic* if for all $v, w \in \mathbb{W}_{\mathsf{P}}$, the following holds:

(a) Let $v', w' \in \mathbb{W}_{\mathsf{P}}$ with $v \sim v'$ and $w \sim w'$. In any $\sim$-equivalence class, the numbers of (v,w)- and (v',w')-destandardizations are equal if $\operatorname{char} \Bbbk = 0$ or congruent modulo $p = \operatorname{char} \Bbbk > 0$.

(b) If $v \sim w$ and $I = \{m+1, m+2, \dots, n\}$ for $m, n \in \mathbb{N}$, then $(v \cap I) \downarrow m \sim (w \cap I) \downarrow m$.

Note that property (a) depends on the field $\Bbbk$.

The set of packed words $\mathbb{W}_{\mathsf{P}}$ is a union of equivalence classes under any word relation. Let $\mathbb{K}^{(\sim)}_{\mathsf{P}}$ be the set of sums $\kappa_E := \sum_{w \in E} w \in \hat{\mathbf{W}}_{\mathsf{P}}$ where E is a $\sim$-equivalence classes in $\mathbb{W}_{\mathsf{P}}$. Define $\mathbf{K}^{(\sim)}_{\mathsf{P}} = \Bbbk\mathbb{K}^{(\sim)}_{\mathsf{P}}$ and let $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}} \subset \hat{\mathbf{W}}_{\mathsf{P}}$ be the completion of $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ with respect to $\mathbb{K}^{(\sim)}_{\mathsf{P}}$.

Theorem 5.10. Suppose $\sim$ is a word relation. Then $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is a linearly compact Hopf subalgebra of $(\hat{\mathbf{W}}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot})$ if and only if $\sim$ is P-algebraic. If $\sim$ is homogeneous, then $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ is a graded Hopf subalgebra of $(\mathbf{W}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot})$ if and only if $\sim$ is P-algebraic.

The part of the theorem asserting that $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ is a Hopf algebra when $\sim$ is homogeneous and P-algebraic is formally similar to [17, Theorem 31] and [31, Theorem 2.1], though neither of these results is a special case of our statement, or vice versa.

Proof. Suppose v and w are packed words and $E \subset \mathbb{W}_{\mathsf{P}}$ is a $\sim$-equivalence class. The coefficient of $v \otimes w$ in $\Delta_{\odot}(\kappa_E)$ is the number of (v,w)-destandardizations in E, modulo p if $p = \operatorname{char} \Bbbk > 0$. We have $\Delta_{\odot}(\kappa_E) \in \hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}} \hat{\otimes} \hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ if and only if this coefficient is the same as the corresponding coefficient of $v' \otimes w'$ for any packed words v', w' with $v \sim v'$ and $w \sim w'$. It follows that $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is a linearly compact sub-coalgebra of $\hat{\mathbf{W}}_{\mathsf{P}}$ if and only if condition (a) in Definition 5.9 holds.

One has $\nabla_{\sqcup\!\sqcup}(\kappa_E \otimes \kappa_F) \in \hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ for all basis elements $\kappa_E, \kappa_F \in \mathbb{K}^{(\sim)}_{\mathsf{P}}$ if and only if condition (b) in Definition 5.9 holds by the same reasoning as in the proof of Theorem 5.5. We conclude that $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is a linearly compact Hopf sub-algebra of $\hat{\mathbf{W}}_{\mathsf{P}}$ if and only if $\sim$ is P-algebraic. The second part of theorem holds since if $\sim$ is homogeneous then, in view of Example 3.9, $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ is a graded Hopf subalgebra of $\mathbf{W}_{\mathsf{P}}$ if and only if $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is a linearly compact Hopf sub-algebra of $\hat{\mathbf{W}}_{\mathsf{P}}$. ☐

Corollary 5.11. If $\sim$ is P-algebraic and of finite-type then $(\mathbf{K}^{(\sim)}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot}) \in \mathsf{Bimon}(\mathsf{Vec}_{\Bbbk})$.

This bialgebra is not necessarily graded so may fail to be a Hopf algebra; see Example 6.5.

Proof. Assume $\sim$ is P-algebraic and of finite-type. All products and coproducts of basis elements in $\mathbb{K}^{(\sim)}_{\mathsf{P}}$ are finite sums of (tensors of) other basis elements, so belong to $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ or $\mathbf{K}^{(\sim)}_{\mathsf{P}} \otimes \mathbf{K}^{(\sim)}_{\mathsf{P}}$. The unit element $\emptyset \in \hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ is also in $\mathbf{K}^{(\sim)}_{\mathsf{P}}$, so $(\mathbf{K}^{(\sim)}_{\mathsf{P}}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot})$ is a bialgebra. ☐

An algebraic word relation is not necessarily P-algebraic, or vice versa. The following is a natural sufficient condition for this to occur.

Lemma 5.12. Let $\sim$ be an algebraic word relation. Assume that whenever v and w are words with the same set of letters and $\mathrm{fl}(v) \sim \mathrm{fl}(w)$, it holds that $v \sim w$. Then $\sim$ is P-algebraic.

Proof. Suppose $v, w, v', w' \in \mathbb{W}_{\mathsf{P}}$ and $v \sim v'$ and $w \sim w'$. For any word $\tilde{v}$ with $\mathrm{fl}(\tilde{v}) = v$, there exists a unique word $\tilde{v}'$ in the same alphabet with $\mathrm{fl}(\tilde{v}') = v'$, and we have $\tilde{v} \sim \tilde{v}'$. Given a word $\tilde{w}$ with $\mathrm{fl}(\tilde{w}) = w$, define $\tilde{w}'$ analogously. The map $\tilde{v}\tilde{w} \mapsto \tilde{v}'\tilde{w}'$ is then a bijection between the sets of (v,w)- and (v',w')-destandardizations in any $\sim$-equivalence class, so $\sim$ is P-algebraic. ☐

5.3 Uniformly algebraic relations

Problematically, we do not know of any efficient method to check whether an arbitrary word relation satisfies condition (a) in Definition 5.9, or to generate relations that have this property. It is therefore useful in practice to consider the following less general type of relation:

Definition 5.13. A word relation is *uniformly algebraic* if for all words v, w, the following holds:

(a) If v', w' are words with $v \sim v'$ and $w \sim w'$, then $vw \sim v'w'$.

(b) If $v \sim w$ and $I \subset \mathbb{P}$ is an interval (i.e., a set of consecutive integers), then $v \cap I \sim w \cap I$.

(c) If $v \sim w$ then $\phi(v) \sim \phi(w)$ for any order-preserving injection $\phi : [\min(v), \max(v)] \to \mathbb{P}$.

Condition (b) is the property referred to in [13, §3.1.2], [17, §4.3], and [36, Definition 4] as *compatibility with restriction to alphabet intervals*. Condition (c) is a weaker form of the property referred to in [13, 17] as *compatibility with (de)standardization*.

Lemma 5.14. An algebraic word relation is uniformly algebraic if and only if $\phi(v) \sim \phi(w)$ whenever v, w are words with $v \sim w$ and $\phi : [\max(v)] \to \mathbb{P}$ is an order-preserving injection.

Proof. The given property is a special case of condition (c) in Definition 5.13, so is certainly necessary. Let $\sim$ be an algebraic word relation with this property. If $v \sim w$ and $I = \{m+1, m+2, \dots, n\}$ then $(v \cap I) \downarrow m \sim (w \cap I) \downarrow m$, and applying the map $\phi : i \mapsto i + m$ to both sides gives $v \cap I \sim w \cap I$. Condition (c) in Definition 5.13 holds in view of Lemma 5.4. ☐

Corollary 5.15. A uniformly algebraic word relation is both algebraic and P-algebraic.

Proof. Suppose $\sim$ is uniformly algebraic. Conditions (b) and (c) in Definition 5.13 together imply condition (b) in Definition 5.2. Moreover, it follows that if v and w are words with the same set of letters and $\mathrm{fl}(v) \sim \mathrm{fl}(w)$, then $v \sim w$. By Lemma 5.12, $\sim$ is therefore algebraic and P-algebraic. ☐

Finally, we note a simple way of generating (uniformly) algebraic word relations.

Proposition 5.16. Let $\mathcal{G}$ be a set of unordered pairs of words. Assume that v and w have the same letters if $\{v, w\} \in \mathcal{G}$, and if I is an interval then $v \cap I = w \cap I$ or $\{(v \cap I) \downarrow m, (w \cap I) \downarrow m\} \in \mathcal{G}$ for some $0 \leq m < \min(I)$. The reflexive, transitive closure of the relation $\sim$ with

$$a(v \downarrow m)b \sim a(w \downarrow m)b$$

for all words a and b, pairs $\{v, w\} \in \mathcal{G}$, and integers $0 \leq m < \min(v) = \min(w)$ is then an algebraic word relation. If it holds that $\{\phi(v), \phi(w)\} \in \mathcal{G}$ whenever $\{v, w\} \in \mathcal{G}$ and $\phi : \mathbb{P} \to \mathbb{P}$ is an order-preserving injective map, then $\sim$ is uniformly algebraic.

We refer to $\sim$ as the *weakest algebraic word relation* with $v \sim w$ for $\{v, w\} \in \mathcal{G}$.

Proof. Condition (a) in Definition 5.2 holds if and only if one has $avb \sim awb$ whenever a, b, v, w are words with $v \sim w$, which is evidently the case here. To check condition (b) in Definition 5.2, let $I = \{m+1, m+2, \dots, n\}$ be an interval in $\mathbb{P}$, fix a pair $\{v, w\} \in \mathcal{G}$, and let $0 \leq k < \min(v) = \min(w)$. It suffices to show that $\tilde{v} := ((v \downarrow k) \cap I) \downarrow m \sim ((w \downarrow k) \cap I) \downarrow m =: \tilde{w}$. Since $\tilde{v} = (v \cap J) \downarrow (m+k)$ and $\tilde{w} = (w \cap J) \downarrow (m+k)$ for $J = k + I$, and since we know that either $v \cap J = w \cap J$ or $\{(v \cap J) \downarrow l, (w \cap J) \downarrow l\} \in \mathcal{G}$ for an integer $0 \leq l \leq m + k$, the desired conclusion follows.

If $\{\phi(v), \phi(w)\} \in \mathcal{G}$ whenever $\{v, w\} \in \mathcal{G}$ and $\phi : \mathbb{P} \to \mathbb{P}$ is an order-preserving injective map, then it follows that $\phi(v) \sim \phi(w)$ whenever v and w are words with $v \sim w$ and $\phi : [\max(v)] \to \mathbb{P}$ is an order-preserving injection, so $\sim$ is uniformly algebraic by Lemma 5.14. $\square$

The following example is instructive when comparing the definitions in this section. Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$. There exists a unique associative product $\circ : W \times W \to W$ with the property that $s \circ s = s$ for $s \in S$ and $v \circ w = vw$ if $v, w \in W$ have $\ell(vw) = \ell(v) + \ell(w)$ [19, Theorem 7.1]. Suppose $S = \{s_1, s_2, s_3, \dots\}$ is countably infinite, and let $\overset{\circ}{\sim}$ be the (inhomogeneous) equivalence relation on words with

$$i_1 i_2 \cdots i_m \overset{\circ}{\sim} j_1 j_2 \cdots j_n \quad \text{if and only if} \quad s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} = s_{j_1} \circ s_{j_2} \circ \cdots \circ s_{j_n}.$$

Let $m(i, j) \in \mathbb{P} \sqcup \{\infty\}$ denote the order of the group element $s_i s_j \in W$. The function m can be any map $\mathbb{P} \times \mathbb{P} \to \mathbb{P} \sqcup \{\infty\}$ with $m(i, j) = m(j, i)$ for all i, j and $m(i, j) = 1$ if and only if $i = j$. The word property for Coxeter systems [5, Theorem 3.3.1] implies that $\overset{\circ}{\sim}$ is the weakest relation with $vxw \sim vyw$ whenever $x \sim y$, and with $a \overset{\circ}{\sim} aa$ and $ababa \cdots \overset{\circ}{\sim} babab \cdots$ when both sides have length $m(a, b)$ for all $a, b \in \mathbb{P}$. In particular, $\sim$ is a word relation.

Lemma 5.17. *Let $a, b, n \in \mathbb{P}$ and set $v = ababa \cdots$ and $w = babab \cdots$ where both words have length n. Then $v \overset{\circ}{\sim} w$ if and only if $m(a, b) \leq n$.*

Proof. The result is clear if $m(a, b) \leq n$. If $m(a, b) > n$ then by induction $v = aw' \overset{\circ}{\sim} av' \overset{\circ}{\sim} v' \overset{\circ}{\sim} w' \overset{\circ}{\sim} bw' \overset{\circ}{\sim} bv' = w$ for the words $v' = ababa \cdots$ ($n-1$ letters) and $w' = babab \cdots$ ($n-1$ letters). $\square$

Proposition 5.18. *The relation $\overset{\circ}{\sim}$ is algebraic if and only if $m(i, j) \leq m(i+1, j+1)$ for all $i, j \in \mathbb{P}$, and uniformly algebraic if and only if $m(i, j) \leq m(a, b)$ whenever $0 < |a - b| \leq |i - j|$.*

This means that if $\overset{\circ}{\sim}$ is uniformly algebraic then $m(i, j) = m(i+1, j+1)$ for all $i, j \in \mathbb{P}$.

Proof. Combining Lemmas 5.4, 5.14, and 5.17 shows that the given conditions are necessary. Condition (a) in Definition 5.2 holds for $\overset{\circ}{\sim}$ by construction.

Assume $m(i, j) \leq m(i+1, j+1)$ for all $i, j \in \mathbb{P}$ and let $I = [k+1, n]$ for some $k, n \in \mathbb{N}$. To check condition (b) in Definition 5.2, it suffices to show that if $v = ababa \cdots$ and $w = babab \cdots$ for some $a, b \in \mathbb{P}$, where both words have $m(a, b)$ letters, then $(v \cap I) \downarrow k \sim (w \cap I) \downarrow k$. This is clear when $I \cap \{a, b\} \neq \{a, b\}$ and holds when $\{a, b\} \subset I$ by Lemma 5.17. Thus $\overset{\circ}{\sim}$ is algebraic. It follows by Lemmas 5.14 and 5.17 that the condition for $\overset{\circ}{\sim}$ to be uniformly algebraic is also sufficient. $\square$

Proposition 5.19. *The relation $\overset{\circ}{\sim}$ is algebraic and of finite type in exactly the following cases:*

(a) *One has $m(i, j) = 2$ for all $i < j$.*

(b) *One has $m(i, i+1) = 3$ and $m(i, j) = 2$ for all $i < j \neq i+1$.*

(c) *For some $n \geq 2$, one has $m(i, i+n) = 3$ and $m(i, j) = 2$ for all $i < j \neq i+n$.*

In cases (a) and (b), $\overset{\circ}{\sim}$ is uniformly algebraic. In case (c), $\overset{\circ}{\sim}$ is not P-algebraic over any field.

We discuss the bialgebras $\mathbf{K}^{(\overset{\circ}{\sim})}$ and $\mathbf{K}_{\mathsf{P}}^{(\overset{\circ}{\sim})}$ associated to these cases in the next section.

Proof. The proof depends on the classification of finite Coxeter groups. The $\overset{\circ}{\sim}$-equivalence classes in $\mathbb{W}_n$ are in bijection with the elements of the parabolic subgroup $\langle s_1, s_2, \dots, s_n\rangle \subset W$, so $\overset{\circ}{\sim}$ is of finite type if and only if each of these subgroups is finite. In the listed cases, each subgroup of this form is a finite direct product of finite symmetric groups, and is therefore finite.

In all three cases, the first condition in Proposition 5.18 obviously holds; the second condition is likewise apparent in cases (a) and (b). Hence $\overset{\circ}{\sim}$ is uniformly algebraic in cases (a) and (b) and algebraic in case (c). In case (c), the $\overset{\circ}{\sim}$-equivalence class of the 2-letter word $1(n+1)$ consists of all words of the form $11\cdots 1(n+1)(n+1)\cdots(n+1)$, so contains exactly one $(12,\emptyset)$-destandardization and no $(21,\emptyset)$-destandardizations. But in this case $12 \overset{\circ}{\sim} 21$, so condition (a) in Definition 5.9 fails and $\overset{\circ}{\sim}$ is not P-algebraic.

Now assume $\overset{\circ}{\sim}$ is algebraic and of finite type, so that $m(i,j) \le m(i+1,j+1)$ for all $i,j \in \mathbb{P}$. We cannot have $m(i,j) > 3$ for any $i < j$ since then $3 \le m(j, 2j-i)$ and $\langle s_i, s_j, s_{2j-i}\rangle$ would be infinite. Suppose $n \in \mathbb{P}$ is minimal such that $m(1, n+1) = 3$. If no such n exists then we are in case (a). Otherwise, we have $m(i, i+n) = 3$ for all $i \in \mathbb{P}$, and $m(i,j) = 3$ cannot hold for any $i < j \neq i+n$ since then the Coxeter graph of (W,S) would contain a cycle and one of the subgroups $\langle s_1, s_2, \dots, s_n\rangle$ would be infinite. We are therefore in cases (b) or (c). $\square$

6 Examples

This section presents some further examples of word relations and related bialgebras.

Example 6.1. Define the *commutation relation* on words to be the relation with $v \sim w$ if w is formed by rearranging the letters of v. Both $\mathbf{K}^{(\sim)} \subset \mathbf{W}$ and $\mathbf{K}^{(\sim)}_{\mathsf{P}} \subset \mathbf{W}_{\mathsf{P}}$ are graded sub-bialgebras since $\sim$ is homogeneous and uniformly algebraic. Recording multiplicities of the letters in each equivalence class identifies $\mathbb{K}^{(\sim)}_n$ with $\mathbb{N}^n$. Given $\alpha \in \mathbb{N}^n$, let $[[\alpha]] = \sum_w [w,n] \in \mathbb{K}^{(\sim)}_n$ where the sum is over all words w with $\max(w) \le n$ and with exactly α_i letters equal to i. The product and coproduct of $\mathbf{K}^{(\sim)}$ then have the formulas $\nabla_{\sqcup}([[\alpha]] \otimes [[\beta]]) = [[\alpha\beta]]$ where $\alpha\beta$ means concatenation and $\Delta_{\odot}([[\alpha]]) = \sum_{\alpha=\alpha'+\alpha''} [[\alpha']] \otimes [[\alpha'']]$ where the sum is over $\alpha', \alpha'' \in \mathbb{N}^n$.

Let $H_n \in \mathbb{K}^{(\sim)}_{\mathsf{P}}$ denote the n-letter packed word $111\cdots 1$, so that $H_0 = \emptyset$ is the unit element in $\mathbf{K}^{(\sim)}_{\mathsf{P}}$. Each H_n is homogeneous of degree n, and the algebra structure on $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ is just the polynomial algebra $\Bbbk\langle H_1, H_2, \dots\rangle$ where $H_1, H_2, \dots$ are interpreted as non-commuting indeterminates. The coproduct of $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ satisfies $\Delta_{\odot}(H_n) = \sum_{i=0}^{n} H_i \otimes H_{n-i}$. This graded Hopf algebra is commonly known as the algebra of *noncommutative symmetric functions* NSym [11] or *Leibniz-Hopf algebra*.

Example 6.2. Define K*-equivalence* to be the weakest algebraic word relation with $a \sim aa$ for all $a \in \mathbb{P}$. This is the case of the relation $\overset{\circ}{\sim}$ described in the previous section when (W,S) is a universal Coxeter, i.e., when $m(i,j) = \infty$ for all $i < j$. K-equivalence is therefore uniformly algebraic but neither homogeneous nor of finite type. One has $v \sim w$ if and only if v and w coincide after all adjacent repeated letters are combined. Each equivalence class under $\sim$ contains a unique reduced word with no equal adjacent letters, which we call a *partial (small) multi-permutation*. A *(small) multi-permutation* [21, Definition 4.1] is a partial multi-permutation that is also a packed word.

For a partial multi-permutation w with $\max(w) \le n$, define $[[w,n]] = \sum_{u \sim w} [u,n] \in \mathbb{K}^{(\sim)}_n$. Given an arbitrary list $w^1, w^2, \dots$ of distinct partial multi-permutations with letters in $[n]$ and coefficients $c_1, c_2 \dots \in \Bbbk$, let $[[c_1 w^1 + c_2 w^2 + \dots, n]] = c_1[[w^1,n]] + c_2[[w^2,n]] + \dots \in \mathscr{K}^{(\sim)}[n]$. If v and w are partial multi-permutations with letters in $[m]$ and $[n]$, respectively, then

$$\nabla_{\sqcup\!\sqcup}([[v,m]] \otimes [[w,n]]) = [[v \star (w \uparrow m), m+n]] \in \mathscr{K}^{(\sim)}[m+n]$$

where $\star$ is the *multishuffle product* described by [21, Proposition 3.1], while

$$\Delta_{\odot}([[w,n]]) = [[\blacktriangle w, n]] \in \mathscr{K}^{(\sim)}[n]$$

where $\blacktriangle$ is the *cuut coproduct* defined in [21, §3]. From (4.6) and (4.7), these formulas completely determine the (co)product of the species coalgebroid $(\mathscr{K}^{(\sim)}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot})$.

The linearly compact Hopf algebra $(\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_{\odot}, \epsilon_{\odot})$ is what Lam and Pylyavskyy call the *small multi-Malvenuto-Reutenauer bialgebra* $\mathfrak{m}$MR [21, §4]. Theorem 5.10 for the special case of K-equivalence recovers [21, Theorem 4.2], which asserts somewhat imprecisely that "$\mathfrak{m}$MR is a bialgebra" (despite the fact that its coproduct only makes sense as a map $\mathfrak{m}\mathrm{MR} \to \mathfrak{m}\mathrm{MR} \hat{\otimes} \mathfrak{m}\mathrm{MR}$). The linearly compact Hopf algebra $\mathfrak{m}$MR is the algebraic dual of what Lam and Pylyavskyy call the *big multi-Malvenuto-Reutenauer Hopf algebra* $\mathfrak{M}$MR [21, §7].

Example 6.3. Define the *K-commutation relation* to be the transitive closure $\sim$ of K-equivalence and the commutation relation. This relation is the case of $\overset{\circ}{\sim}$ when W is abelian, as described by Proposition 5.19(a). Therefore $\sim$ is uniformly algebraic, inhomogeneous, and of finite type.

The $\sim$-equivalence classes in $\mathbb{W}_n$ are in bijection with subsets $I \subset [n]$. All packed words w with $\max(w) = n$ belong to the same $\sim$-equivalence class. If we let $\kappa_n \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$ denote the sum of these words, then $\kappa_n = \nabla_{\sqcup\!\sqcup}^{(n-1)}(x \otimes x \otimes \cdots \otimes x)$ for $x = \kappa_1 = 1 + 11 + 111 + \dots$. Thus $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ coincides as an algebra with $\Bbbk[x]$, but its coproduct has $\Delta_{\odot}(x) = x \otimes 1 + x \otimes x + 1 \otimes x$.

Example 6.4. Define *Knuth equivalence* to be the weakest algebraic word relation with

$$bac \sim bca, \qquad acb \sim cab, \qquad aba \sim baa, \qquad \text{and} \qquad bab \sim bba$$

for all $a < b < c$. This relation is of ubiquitous significance in combinatorics. Its equivalence classes are the sets of words with the same insertion tableau under the RSK correspondence.

Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$ is an integer partition and $w = w^1 w^2 \cdots w^m$ is the factorization of a word w into maximal weakly increasing subwords. Slightly abusing standard terminology, say that w is a *semistandard tableau* of shape λ if $\ell(w^i) = \lambda_{m+1-i}$ for $i \in [m]$ and $w^i_j > w^{i+1}_j$ whenever both sides are defined. For example, $\emptyset$, 645123, 2211, and 655133 are semistandard tableaux of the respective shapes $\emptyset$, $(3,2,1)$, $(2,2)$, and $(3,2,1)$.

Each Knuth equivalence class contains a unique semistandard tableau T. When $\max(T) \leq n$, write $[[T,n]] = \sum_{w \sim T} [w,n] \in \mathbb{K}_n^{(\sim)}$. Since $T \cap [n]$ is a semistandard tableau whenever T is, it follows that the product and coproduct of $\mathbf{K}^{(\sim)}$ have the formulas

$$\nabla_{\sqcup\!\sqcup}([[U,m]] \otimes [[V,n]]) = \sum_T [[T, m+n]] \quad \text{and} \quad \Delta_{\odot}([[T,n]]) = \sum_{T \sim UV} [[U,n]] \otimes [[V,n]] \tag{6.1}$$

where the first sum is over semistandard tableaux T with $T \cap [m] = U$ and $T \cap [m+1, \infty) \sim V \uparrow m$, and the second sum is over pairs of semistandard tableaux U and V with $T \sim UV$.

Similar formulas for the (co)product of the Hopf algebra $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ are noted in [22, 33]. The subalgebra of $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ spanned by Knuth equivalence classes of permutations is the *Poirier-Reutenaurer Hopf algebra* PR [35]. Theorem 5.10 for Knuth equivalence recovers [35, Theorem 3.1].

Example 6.5. Recall that *K-Knuth equivalence* is the weakest algebraic word relation with

$$bac \sim bca, \qquad acb \sim cab, \qquad aba \sim bab, \qquad \text{and} \qquad a \sim aa \tag{6.2}$$

for all integers $a < b < c$. Proposition 5.16 implies that this relation is uniformly algebraic. Though less well-studied than its homogenous analogue, K-Knuth equivalence appears to be an equally fundamental case of interest. Its relationship with *Hecke insertion* [7] is parallel to that of Knuth equivalence with the RSK correspondence.

Again with minor abuse of standard terminology, define an *increasing tableau* to be a semistandard tableau with no equal adjacent letters, i.e., in which every weakly increasing consecutive subword is strictly increasing. For example, the words $\emptyset$ or 645123 or 5612 or 545234 are all increasing tableaux under our definition.

There are finitely many increasing tableaux with letters in a given alphabet [33, Lemma 3.2] and every K-Knuth equivalence class contains at least one increasing tableau [8, Theorem 6.2]. Thus, K-Knuth equivalence is of finite type and a somewhat improved way of indexing the elements of $\mathbb{K}_n^{(\sim)}$ is to define $[[T,n]] = \sum_{w\sim T}[w,n]$ for each increasing tableau T with $\max(T) \leq n$. The usefulness of this construction is limited, since it is not known how to easily detect when two increasing tableaux are K-Knuth equivalent. There is an algorithm to compute all K-Knuth classes of words with a given set of letters, however [12].

It is an open problem to find an irredundant indexing set for K-Knuth equivalence classes, with respect to which one can describe explicitly the product and coproduct of the bialgebras $\mathbf{K}^{(\sim)}$ and $\mathbf{K}^{(\sim)}_{\mathsf{P}}$. Patrias and Pylyavskyy [33] refer to the latter as the *K-theoretic Poirier-Reutenauer bialgebra* KPR. They note that KPR is not a Hopf algebra [33, §4] and give some (necessarily inexplicit) formulas for its product and coproduct; see [33, Theorems 4.3, 4.5, 4.10, and 4.12]. Theorem 5.10 for K-Knuth equivalence recovers [33, Theorem 4.15.].

As noted in [8, Remark 5.10] and [12, §4], the set of reduced words in a K-Knuth equivalence class may fail to be spanned by the homogeneous relations $bac \sim bca$, $acb \sim cab$, and $aba \sim bab$ for $a < b < c$. The graded sub-bialgebra $\mathbf{K}^{(\sim)}_{\mathsf{R}} \subset \mathbf{K}^{(\sim)}$ is thus in some sense not any easier to study.

We mention one other property of this relation. Define *weak K-Knuth equivalence* to be the word relation $\approx$ with $v \approx w$ if $v \sim w$ or if $v = v_1v_2v_3\cdots v_n$ and $w = v_2v_1v_3\cdots v_n$. Let $w^{\mathtt{r}}$ be the word obtained by reversing w. If $v \approx w$ then $v^{\mathtt{r}}v \sim w^{\mathtt{r}}w$ since $baab \sim bab \sim aba \sim abba$. Buch and Samuel state the converse as [8, Conjecture 7.10], which appears to be still unresolved:

Conjecture 6.6 (Buch and Samuel [8])**.** Two words v and w are weakly K-Knuth equivalent if and only if $v^{\mathtt{r}}v$ and $w^{\mathtt{r}}w$ are K-Knuth equivalent.

Example 6.7. One avoids many pathologies of K-Knuth equivalence by considering the stronger relation of *Hecke equivalence*, which is the weakest algebraic word relation $\sim$ with

$$ac \sim ca, \qquad aba \sim bab, \qquad \text{and} \qquad a \sim aa$$

for all positive integers $a < b < c$, so that $13 \sim 31$ but $12 \not\sim 21$ [8, Definition 6.4]. This is the case of the relation $\overset{\circ}{\sim}$ corresponding to Proposition 5.19(b), so Hecke equivalence is uniformly algebraic and of finite type. Each set $\mathbb{K}_n^{(\sim)}$ is in bijection with the symmetric group S_{n+1}, which we view as the set of words of length $n+1$ containing each $i \in [n+1]$ as a letter exactly once.

Given $\pi \in S_{n+1}$, let $[[\pi]] = \sum_w [w,n] \in \mathbb{K}_n^{(\sim)}$ denote the sum over *Hecke words* for π, i.e., words $w = w_1w_2\cdots w_m$ with $\pi = s_{w_1} \circ s_{w_2} \circ \cdots \circ s_{w_m}$ where $\circ$ is the product defined in Section 5.3 and

$s_a = (a, a+1) \in S_{n+1}$. The coproduct of $\mathbf{K}^{(\sim)}$ satisfies $\Delta_\odot([[\pi]]) = \sum_{\pi=\pi'\circ\pi''}[[\pi']] \otimes [[\pi'']]$ where the sum is over $\pi', \pi'' \in S_{n+1}$. It is an open problem to describe the product $\nabla_\sqcup([[\pi']] \otimes [[\pi'']])$.

We have a better understanding of the graded bialgebra of reduced classes $\mathbf{K}_{\mathsf{R}}^{(\sim)}$ when $\sim$ is Hecke equivalence. This bialgebra is the main topic of our complementary paper [28], which derives a recursive formula for the product of any two basis elements in $\mathbb{K}_{\mathsf{R}}^{(\sim)}$.

For another point of comparison with K-Knuth equivalence, define *weak Hecke equivalence* to be the word relation $\approx$ with $v \approx w$ if $v \sim w$ or if $v = v_1v_2v_3\cdots v_n$ and $w = v_2v_1v_3\cdots v_n$. The analogue of Conjecture 6.6 for Hecke equivalence is known to be true:

Proposition 6.8 ([15, Theorem 6.4])**.** Two words v and w are weakly Hecke equivalent if and only if $v^{\mathtt{r}}v$ and $w^{\mathtt{r}}w$ are Hecke equivalent.

Example 6.9. Fix an integer $n \geq 2$ and define $\approx$ to be the weakest algebraic word relation with

$$a(a+n)a \approx (a+n)a(a+n), \qquad ab \approx ba, \qquad \text{and} \qquad a \approx aa$$

for all $a, b \in \mathbb{P}$ with $|b-a| \neq n$. This is the case of $\overset{\circ}{\sim}$ in Proposition 5.19(c), so $\approx$ is of finite type but not P-algebraic. Thus, although $\mathbf{K}^{(\approx)}$ is a sub-bialgebra of $\mathbf{W}$, the space $\mathbf{K}_{\mathsf{P}}^{(\approx)}$ is not a sub-bialgebra of $\mathbf{W}_{\mathsf{P}}$. The former is closely related to the n-fold tensor product of the bialgebra in the previous example: if $\sim$ is Hecke equivalence, then the sub-bialgebra ${}^{(n)}\mathbf{K}^{(\approx)} := \bigoplus_{m\in\mathbb{N}} \mathbf{K}_{mn}^{(\approx)} \subset \mathbf{K}^{(\approx)}$ has ${}^{(n)}\mathbf{K}^{(\approx)} \cong \mathbf{K}^{(\sim)} \otimes \mathbf{K}^{(\sim)} \otimes \cdots \otimes \mathbf{K}^{(\sim)}$ (n factors).

Many other algebraic word relations appear in the literature; for example, the *hypoplatic relation* (see [20, Definition 4.16] or [30, Definition 4.2]), *Sylvester equivalence* (see [18, Definition 8]), *hyposylvester equivalence* and *metasylvester equivalence* (see [31, §3]), *Baxter equivalence* (see [13, Definition 3.1]), and the *taïga relation* (see [36, Eq. (8)]) are all uniformly algebraic, homogenous word relations. For Sylvester and Baxter equivalence, the associated Hopf algebra $\mathbf{K}_{\mathsf{P}}^{(\sim)}$ recovers the *Loday-Ronco algebra* of planar binary trees [4, 23] and the *Baxter Hopf algebra* of twin binary trees [13], respectively.

We mention one other miscellaneous example which will be of significance in Section 8.

Example 6.10. Define *exotic Knuth equivalence* to be the weakest algebraic word relation with

$$bac \sim bca, \qquad acb \sim cab, \qquad bba \sim bab \sim abb, \qquad \text{and} \qquad xyzy \sim yzyx$$

for all positive integers $a < b < c$ and $x \leq y < z$. Proposition 5.16 implies that $\sim$ is homogeneous and uniformly algebraic. This relation does not seem to have been studied previously. A sensible invariant to consider is the sequence $(d_n)_{n=0,1,2}$ giving the graded dimension of $\mathbf{K}_{\mathsf{P}}^{(\sim)}$, i.e., in which d_n counts the $\sim$-equivalence classes of packed words of length n. This sequence starts as

$$(d_n)_{n=0,1,2,\ldots} = (1, 1, 3, 9, 31, 110, 412, 1597, 6465, 27021\ldots)$$

but does not match any existing entry in [37].

7 Combinatorial bialgebras

A *composition* α of $n \in \mathbb{N}$, written $\alpha \vDash n$, is a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_l = n$. The unique composition of $n = 0$ is the empty word $\emptyset$. Let $\Bbbk[[x_1, x_2, \ldots]]$ be the algebra of formal power series with coefficients in $\Bbbk$ in a countable set of commuting variables. The *monomial quasi-symmetric function* M_α indexed by a composition $\alpha \vDash n$ with l parts is

$$M_\alpha = \sum_{i_1<i_2<\dots<i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l} \in \Bbbk[[x_1, x_2, \dots]].$$

When α is the empty composition, set $M_\emptyset = 1$.

For each $n \in \mathbb{N}$, the set $\{M_\alpha : \alpha \vDash n\}$ is a basis for a subspace $\mathsf{QSym}_n \subset \Bbbk[[x_1, x_2, \dots]]$. The vector space of *quasi-symmetric functions* $\mathsf{QSym} = \bigoplus_{n\in\mathbb{N}} \mathsf{QSym}_n$ is a subalgebra of $\Bbbk[[x_1, x_2, \dots]]$. This algebra is a graded Hopf algebra whose coproduct is the linear map with $\Delta(M_\alpha) = \sum_{\alpha=\beta\gamma} M_\beta \otimes M_\gamma$ and whose counit is the linear map with $\epsilon(M_\emptyset) = 1$ and $\epsilon(M_\alpha) = 0$ for $\alpha \neq \emptyset$ [1, §3].

Each $\alpha \vDash n$ can be rearranged to form a partition of n, denoted $\mathrm{sort}(\alpha)$. The *monomial symmetric function* indexed by a partition λ is $m_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} M_\alpha$. Write $\lambda \vdash n$ when λ is a partition of n and let $\mathsf{Sym}_n = \Bbbk\text{-span}\{m_\lambda : \lambda \vdash n\}$. The subspace $\mathsf{Sym} = \bigoplus_{n\in\mathbb{N}} \mathsf{Sym}_n \subset \mathsf{QSym}$ is the familiar graded Hopf subalgebra of *symmetric functions*.

Let $\mathsf{NSym} = \Bbbk\langle H_1, H_2, \dots\rangle$ be the graded Hopf algebra of *noncommuative symmetric functions* described in Example 6.1, that is, the $\Bbbk$-algebra of polynomials in non-commuting indeterminates $H_1, H_2, H_3, \dots$, where H_n has degree n and the coproduct has $\Delta(H_n) = \sum_{i=0}^n H_i \otimes H_{n-i}$. Given $\alpha \vDash n$ with l parts, let $H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_l}$ and define $H_\emptyset = H_0 = 1$. NSym is the graded dual of QSym via the bilinear form $\mathsf{NSym} \times \mathsf{QSym} \to \Bbbk$ in which $\{H_\alpha\}$ and $\{M_\alpha\}$ are dual bases [1, §3].

If $\zeta : V \to \Bbbk[t]$ is a map and $a \in \Bbbk$, then let $\zeta|_{t=a} : V \to \Bbbk$ be the map $v \mapsto \zeta(v)(a)$.

Definition 7.1. Suppose $(V, \Delta, \epsilon) \in \mathsf{Comon}(\mathsf{GrVec}_\Bbbk)$ is a graded coalgebra. If $\zeta : V \to \Bbbk[t]$ is a graded linear map with $\zeta|_{t=0} = \epsilon$, then $(V, \Delta, \epsilon, \zeta)$ is a *combinatorial coalgebra.*

Definition 7.2. Suppose $(V, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Bimon}(\mathsf{GrVec}_\Bbbk)$ is a graded bialgebra. If $\zeta : V \to \Bbbk[t]$ is a graded algebra morphism with $\zeta|_{t=0} = \epsilon$, then $(V, \nabla, \iota, \Delta, \epsilon, \zeta)$ is a *combinatorial bialgebra.*

A *combinatorial Hopf algebra* is a combinatorial bialgebra $(V, \nabla, \iota, \Delta, \epsilon, \zeta)$ in which $(V, \nabla, \iota, \Delta, \epsilon)$ is a Hopf algebra. These definitions are minor generalizations of the notions of combinatorial coalgebras and Hopf algebras in [1], where it is required that V have finite graded dimension and $\dim V_0 = 1$.

When the structure maps are clear from context, we refer to just the pair (V, ζ) as a combinatorial coalgebra or bialgebra. A morphism $\phi : (V, \zeta) \to (V', \zeta')$ of combinatorial coalgebras or bialgebras is a graded coalgebra or bialgebra morphism $\phi : V \to V'$ satisfying $\zeta' = \zeta \circ \phi$. The map ζ is the *character* of a combinatorial coalgebra or bialgebra (V, ζ).

Remark. Specifying a graded linear map (respectively, algebra morphism) $V \to \Bbbk[t]$ is equivalent to defining a (multiplicative) linear map $V \to \Bbbk$. We define the character ζ to be a map $V \to \Bbbk[t]$ since this extends more naturally to the linearly compact case. This convention differs from [1, 28], where the character of a combinatorial coalgebra is defined to be a linear map $V \to \Bbbk$.

Example 7.3. Let $\zeta_{\mathsf{QSym}} : \Bbbk[x_1, x_2, \dots] \to \Bbbk[t]$ be the graded algebra morphism which sets $x_1 = t$ and $x_n = 0$ for all $n > 1$. One has $\zeta_{\mathsf{QSym}}(M_\emptyset) = 1$, $\zeta_{\mathsf{QSym}}(M_{(n)}) = t^n$ for each $n \geq 1$, and $\zeta_{\mathsf{QSym}}(M_\alpha) = 0$ for all other compositions. The pair $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ is a combinatorial Hopf algebra.

Suppose $(V, \Delta, \epsilon, \zeta)$ is a combinatorial coalgebra. Let $\zeta_\emptyset := \zeta|_{t=0} = \epsilon$. Given $\alpha \vDash n > 0$, let $\zeta_\alpha : V \to \Bbbk$ be the map whose value at $v \in V$ is the coefficient of $t^{\alpha_1} \otimes t^{\alpha_2} \otimes \cdots \otimes t^{\alpha_m}$ in the image of v under the map $V \xrightarrow{\Delta^{(m-1)}} V^{\otimes m} \xrightarrow{\zeta^{\otimes m}} \Bbbk[t]^{\otimes m}$. Define $\psi : V \to \mathsf{QSym}$ by

$$\psi(v) = \sum_{\alpha} \zeta_\alpha(v) M_\alpha \qquad \text{for } v \in V \tag{7.1}$$

where the sum is over all compositions. This *a priori* infinite sum belongs to QSym since if $v \in V_n$ is homogeneous of degree $n \in \mathbb{N}$ then $\psi(v) = \sum_{\alpha \vDash n} \zeta_\alpha(v) M_\alpha$. Thus, ψ is a graded linear map. The pair $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ is the terminal object in the category of combinatorial (co/bi)algebras:

Theorem 7.4 (Aguiar, Bergeron, Sottile [1])**.** Let (V, ζ) be a combinatorial coalgebra. The map (7.1) is the unique morphism of combinatorial coalgebras $\psi : (V, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. If (V, ζ) is a combinatorial bialgebra, then ψ is a morphism of graded bialgebras.

It is possible to remove the requirement of a grading in the definition of a combinatorial coalgebra while retaining an analogue of Theorem 7.4. However, one must then impose a technical finiteness condition to ensure that the sum (7.1) belongs to QSym.

Proof. This result is only slightly more general than [1, Theorem 4.1] and has essentially the same proof. We sketch the argument. Let $\widehat{\mathsf{NSym}}$ denote the completion of NSym with respect to the basis $\{H_\alpha\}$. Since NSym is a graded algebra, $\widehat{\mathsf{NSym}}$ is a linearly compact algebra. Write $\langle \cdot, \cdot \rangle$ for both the tautological form $V \times V^* \to \Bbbk$ and the bilinear form $\mathsf{QSym} \times \widehat{\mathsf{NSym}} \to \Bbbk$, continuous in the second coordinate, relative to which the pseudobasis $\{H_\alpha\} \subset \widehat{\mathsf{NSym}}$ is dual to the basis $\{M_\alpha\} \subset \mathsf{QSym}$. Both forms are nondegenerate. We view the dual space V^* as the linearly compact algebra with unit element ϵ dual to the coalgebra V via the tautological form. The linearly compact algebra structure on $\widehat{\mathsf{NSym}}$ is the one dual to the coalgebra structure on QSym.

Let $[t^n]f$ denote the coefficient of t^n in $f \in \Bbbk[t]$ and define $\zeta_n \in V^*$ by $\zeta_n(v) = [t^n]\zeta(v)$, so that $\zeta_0 = \epsilon$. Observe that $[t^n]\zeta_{\mathsf{QSym}}(q) = \langle q, H_n \rangle$ for all $n \in \mathbb{N}$ and $q \in \mathsf{QSym}$. It follows that there exists a unique coalgebra morphism $\psi : V \to \mathsf{QSym}$ with $\zeta = \zeta_{\mathsf{QSym}} \circ \psi$ if and only if there exists a unique linearly compact algebra morphism $\phi : \widehat{\mathsf{NSym}} \to V^*$ with $\phi(H_n) = \zeta_n$ for all $n \in \mathbb{N}$, and when this occurs, the two maps satisfy $\langle \psi(v), w \rangle = \langle v, \phi(w) \rangle$ for all $v \in V$ and $w \in \widehat{\mathsf{NSym}}$. Since $\zeta_0 = \epsilon$, since V is a graded coalgebra, and since $V_m \subset \ker \zeta_n$ for all $m \neq n$, there does exist a unique linearly compact algebra morphism $\phi : \widehat{\mathsf{NSym}} \to V^*$ with $\phi(H_n) = \zeta_n$ for all $n \in \mathbb{N}$. For this map one has $\phi(H_\alpha) = \zeta_\alpha$ with ζ_α as in (7.1). Hence, there exists a unique coalgebra morphism $\psi : V \to \mathsf{QSym}$ satisfying $\zeta = \zeta_{\mathsf{QSym}} \circ \psi$, and for this map one has $\langle \psi(v), H_\alpha \rangle = \langle v, \phi(H_\alpha) \rangle = \langle v, \zeta_\alpha \rangle = \zeta_\alpha(v)$ for all $v \in V$ and compositions α; in other words, ψ is the graded linear map (7.1).

Assume (V, ζ) is a combinatorial bialgebra. Use the symbol ∇ to also denote the products of $\Bbbk[t]$ and QSym. Define $\xi = \nabla \circ (\zeta \otimes \zeta)$. Then $(V \otimes V, \xi)$ is a combinatorial coalgebra and it is easy to check that $\nabla \circ (\psi \otimes \psi)$ and $\psi \circ \nabla$ are both morphisms $(V \otimes V, \xi) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. The uniqueness proved in the previous paragraph implies that $\nabla \circ (\psi \otimes \psi) = \psi \circ \nabla$. Since ζ is an algebra morphism, we also have $\psi(1) = 1 \in \mathsf{QSym}$, so ψ is a bialgebra morphism. ☐

The results discussed so far have linearly compact analogues. Let $\Bbbk[[t]]$ denote the algebra of formal power series in t, viewed as a linearly compact space as in Example 3.8. If $V \in \widehat{\mathsf{Vec}_\Bbbk}$ has pseudobasis $\{v_i : i \in I\}$, then a linear map $\phi : V \to \Bbbk[[t]]$ is continuous if and only if for each $n \in \mathbb{N}$, the set of indices $i \in I$ with $[t^n]\phi(v_i) \neq 0$ is finite, and $\phi\left(\sum_{i \in I} c_i v_i\right) = \sum_{i \in I} c_i \phi(v_i)$ for any $c_i \in \Bbbk$.

Definition 7.5. Suppose $(V, \Delta, \epsilon) \in \mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})$. If $\zeta : V \to \Bbbk[[t]]$ is a continuous linear map with $\zeta|_{t=0} = \epsilon$, then $(V, \Delta, \epsilon, \zeta)$ is a *linearly compact combinatorial coalgebra.*

Unlike Definition 7.1, this definition does require any grading on the vector space V.

Definition 7.6. Suppose $(V, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Bimon}(\widehat{\mathsf{Vec}_{\Bbbk}})$. If $\zeta : V \to \Bbbk[[t]]$ is a morphism of linearly compact algebras with $\zeta|_{t=0} = \epsilon$, then $(V, \nabla, \iota, \Delta, \epsilon, \zeta)$ is a *linearly compact combinatorial bialgebra.*

We often refer to just the pair (V, ζ) as a linearly compact combinatorial coalgebra or bialgebra. The map ζ is the *character* of (V, ζ). A morphism $\phi : (V, \zeta) \to (V', \zeta')$ of linearly compact combinatorial (co/bi)algebras is a continuous (co/bi)algebra morphism satisfying $\zeta' = \zeta \circ \phi$.

Example 7.7. Define $\widehat{\mathsf{QSym}} = \prod_{n\in\mathbb{N}} \mathsf{QSym}_n \in \widehat{\mathsf{Vec}_{\Bbbk}}$ and $\widehat{\mathsf{Sym}} = \prod_{n\in\mathbb{N}} \mathsf{Sym}_n \in \widehat{\mathsf{Vec}_{\Bbbk}}$ to be the completions of QSym and Sym with respect to the bases $\{M_\alpha\}$ and $\{m_\lambda\}$. The (co)product and (co)unit maps of QSym extend to make $\widehat{\mathsf{QSym}}$ into a linearly compact bialgebra and $\widehat{\mathsf{Sym}} \subset \widehat{\mathsf{QSym}}$ into a linearly compact sub-bialgebra. The map ζ_{QSym} extends to a linearly compact algebra morphism $\Bbbk[[x_1, x_2, \dots]] \to \Bbbk[[t]]$ and $(\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$ is a linearly compact combinatorial bialgebra.

Suppose $(V, \Delta, \epsilon, \zeta)$ is a linearly compact combinatorial coalgebra. Let $\zeta_\emptyset := \zeta|_{t=0} = \epsilon$. Given $\alpha \vDash n > 0$, let $\zeta_\alpha : V \to \Bbbk$ be the map whose value at $v \in V$ is the coefficient of $t^{\alpha_1} \otimes t^{\alpha_2} \otimes \cdots \otimes t^{\alpha_m}$ in the image of v under $V \xrightarrow{\Delta^{(m-1)}} V^{\hat{\otimes} m} \xrightarrow{\zeta^{\hat{\otimes} m}} \Bbbk[[t]]^{\hat{\otimes} m}$. Define $\psi : V \to \widehat{\mathsf{QSym}}$ to be the map

$$\psi(v) = \sum_{\alpha} \zeta_\alpha(v) M_\alpha \qquad \text{for } v \in V \tag{7.2}$$

where the sum is over all compositions α. This is the same formula as (7.1), except now the sum may have infinitely many nonzero terms.

Theorem 7.8. Let (V, ζ) be a linearly compact combinatorial coalgebra. The map (7.2) is the unique morphism of linearly compact combinatorial coalgebras $\psi : (V, \zeta) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$. If (V, ζ) is a linearly compact combinatorial bialgebra, then ψ is a morphism of linearly compact bialgebras.

Proof. The proof is similar to that of Theorem 7.4. Write $\langle \cdot, \cdot \rangle$ for both the tautological form $V^\vee \times V \to \Bbbk$ and the bilinear form $\mathsf{NSym} \times \widehat{\mathsf{QSym}} \to \Bbbk$, continuous in the second coordinate, relative to which the pseudobasis $\{M_\alpha\} \subset \widehat{\mathsf{QSym}}$ is dual to the basis $\{H_\alpha\} \subset \mathsf{NSym}$. Both forms are nondegenerate. We view the vector space $V^\vee$ of continuous linear maps $V \to \Bbbk$ as the algebra with unit element ϵ dual to the linearly compact coalgebra V via the tautological form. The algebra structure on NSym is the one dual to the linearly compact coalgebra structure on $\widehat{\mathsf{QSym}}$.

Define $\zeta_n \in V^\vee$ by $\zeta_n(v) = [t^n]\zeta(v)$, so that $\zeta_0 = \epsilon$. There is obviously a unique algebra morphism $\phi : \mathsf{NSym} \to V^\vee$ with $\phi(H_n) = \zeta_n$ for all $n \in \mathbb{N}$. The unique map $\psi : V \to \widehat{\mathsf{QSym}}$ satisfying $\langle u, \psi(v) \rangle = \langle \phi(u), v \rangle$ for all $u \in \mathsf{NSym}$ and $v \in V$ is therefore a linearly compact coalgebra morphism. Since $\phi(H_\alpha) = \zeta_\alpha$ with ζ_α as in (7.2), the map ψ coincides with (7.2). Since $\zeta = \zeta_{\mathsf{QSym}} \circ \psi$ if and only if $\langle \zeta_n, v \rangle = \langle H_n, \psi(v) \rangle$ for all $n \in \mathbb{N}$, it follows that ψ is the unique morphism of linearly compact combinatorial coalgebras $(V, \zeta) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$

Assume (V, ζ) is a linearly compact combinatorial bialgebra. Use the symbol ∇ for the products of $\Bbbk[[t]]$ and $\widehat{\mathsf{QSym}}$. Define $\xi = \nabla \circ (\zeta \mathbin{\hat{\otimes}} \zeta)$. Then $(V \mathbin{\hat{\otimes}} V, \xi)$ is a linearly compact combinatorial coalgebra and the maps $\nabla \circ (\psi \mathbin{\hat{\otimes}} \psi)$ and $\psi \circ \nabla$ are morphisms $(V \mathbin{\hat{\otimes}} V, \xi) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$, so they must be equal. By definition $\psi(1) = 1 \in \widehat{\mathsf{QSym}}$, so ψ is a linearly compact bialgebra morphism. $\square$

Theorems 7.4 and 7.8 also have a version for species. Recall the definition of $\mathcal{E}$ from (4.3).

Definition 7.9. Suppose $(\mathscr{V}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$ and $\mathrm{Z} : \mathscr{V} \to \mathcal{E}(\Bbbk[[t]])$ is a natural transformation. Assume, for all disjoint finite sets S and T, that the following conditions hold:

(a) The tuple $(\mathscr{V}[S], \Delta_S, \epsilon_S, \mathrm{Z}_S)$ is a linearly compact combinatorial coalgebra.

(b) One has $\mathrm{Z}_\varnothing \circ \iota_\varnothing(1) = 1 \in \Bbbk[[t]]$.

(c) If $u \in \mathscr{V}[S]$ and $v \in \mathscr{V}[T]$ then $\mathrm{Z}_{S \sqcup T} \circ \nabla_{ST}(u \otimes v) = \mathrm{Z}_S(u)\mathrm{Z}_T(v)$.

Then $(\mathscr{V}, \nabla, \iota, \Delta, \epsilon, \mathrm{Z})$ is a *combinatorial coalgebroid*.

This definition is similar to the notion of a *combinatorial Hopf monoid* given in [27, §5.4]. As usual, when the other data is clear from context, we refer to just $(\mathscr{V}, \mathrm{Z})$ as a combinatorial coalgebroid. The natural transformation $\mathrm{Z} : \mathscr{V} \to \mathcal{E}(\Bbbk[[t]])$ is the *character* of $(\mathscr{V}, \mathrm{Z})$. A morphism of combinatorial coalgebroids $(\mathscr{V}, \mathrm{Z}) \to (\mathscr{V}', \mathrm{Z}')$ is a morphism of species coalgebroids $\phi : \mathscr{V} \to \mathscr{V}'$ such that $\mathrm{Z}' = \mathrm{Z} \circ \phi$.

If (V, ζ) is a linearly compact combinatorial bialgebra, then $(\mathcal{E}(V), \mathcal{E}(\zeta))$ is a combinatorial coalgebroid. Let $\mathcal{E}\widehat{\mathsf{QSym}} = \mathcal{E}(\widehat{\mathsf{QSym}})$ and $\mathrm{Z}_{\mathsf{QSym}} = \mathcal{E}(\zeta_{\mathsf{QSym}})$. Suppose $(\mathscr{V}, \mathrm{Z})$ is a combinatorial coalgebroid. Define $\Psi : \mathscr{V} \to \mathcal{E}\widehat{\mathsf{QSym}}$ to be the natural transformation such that, for each set S, the map Ψ_S is the unique morphism $(\mathscr{V}[S], \mathrm{Z}_S) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$. This is well-defined since if $\sigma : S \to T$ is a bijection then $\zeta_{\mathsf{QSym}} \circ \Psi_T \circ \mathscr{V}[\sigma] = \mathrm{Z}_T \circ \mathscr{V}[\sigma] = \mathrm{Z}_S$, so the maps $\Psi_T \circ \mathscr{V}[\sigma]$ and $\Psi_S = \mathcal{E}\widehat{\mathsf{QSym}}[\sigma] \circ \Psi_S$ must be equal as both are morphisms $(\mathscr{V}[S], \mathrm{Z}_S) \to (\mathcal{E}\widehat{\mathsf{QSym}}, \mathrm{Z}_{\mathsf{QSym}})$.

Corollary 7.10. Let $(\mathscr{V}, \mathrm{Z})$ be a combinatorial coalgebroid. Then Ψ is the unique morphism of combinatorial coalgebroids $(\mathscr{V}, \mathrm{Z}) \to (\mathcal{E}\widehat{\mathsf{QSym}}, \mathrm{Z}_{\mathsf{QSym}})$.

Proof. By Theorem 7.8, Ψ is the unique morphism $\mathscr{V} \to \mathcal{E}\widehat{\mathsf{QSym}}$ in the category $\mathsf{Comon}(\widehat{\mathsf{Vec}_\Bbbk})\text{-}\mathsf{Sp}$ satisfying $\mathrm{Z} = \mathrm{Z}_{\mathsf{QSym}} \circ \Psi$. It remains to show that Ψ is a morphism of species coalgebroids. For this, it suffices to check that $\Psi_\varnothing \circ \iota_\varnothing(1) = 1 \in \widehat{\mathsf{QSym}}$ and $\Psi_{S \sqcup T} \circ \nabla_{ST} = \nabla_{ST} \circ (\Psi_S \mathbin{\hat{\otimes}} \Psi_T)$ for all disjoint finite sets S and T. The first property is evident from (7.2) since $\mathrm{Z}_\varnothing \circ \iota_\varnothing(1) = 1 \in \Bbbk[[t]]$ and $\Delta_\varnothing \circ \iota_\varnothing(1) = \iota_\varnothing(1) \otimes \iota_\varnothing(1)$. The second property follows from Theorem 7.8 since if $V = \mathscr{V}[S] \mathbin{\hat{\otimes}} \mathscr{V}[T]$ and $\xi = \nabla_{\Bbbk[[t]]} \circ (\mathrm{Z}_S \mathbin{\hat{\otimes}} \mathrm{Z}_T)$ then (V, ξ) is a linearly compact combinatorial coalgebra, and both $\Psi_{S \sqcup T} \circ \nabla_{ST}$ and $\nabla_{ST} \circ (\Psi_S \mathbin{\hat{\otimes}} \Psi_T)$ are morphisms $(V, \xi) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$. ☐

Suppose $(V, \nabla, \iota, \Delta, \epsilon)$ is a graded $\Bbbk$-bialgebra. Let $\mathbb{X}(V)$ denote the set of graded linear maps $\zeta : V \to \Bbbk[t]$ for which (V, ζ) is a combinatorial bialgebra. This set is a monoid with unit element ϵ and product $\zeta\zeta' := \nabla_{\Bbbk[t]} \circ (\zeta \otimes \zeta') \circ \Delta$ where $\nabla_{\Bbbk[t]}$ is the product of $\Bbbk[t]$. We refer to $\mathbb{X}(V)$ as the *character monoid* of V. If V is a Hopf algebra with antipode $\mathtt{S}$, then $\zeta^{-1} := \zeta \circ \mathtt{S}$ is the left and right inverse of $\zeta \in \mathbb{X}(V)$, and $\mathbb{X}(V)$ is a group with some notable properties [1].

If $(V, \nabla, \iota, \Delta, \epsilon)$ is a linearly compact $\Bbbk$-bialgebra then we let $\mathbb{X}(V)$ denote the set of continuous linear maps $\zeta : V \to \Bbbk[[t]]$ for which (V, ζ) is a linearly compact combinatorial bialgebra. This set is again a monoid with unit element ϵ and product $\zeta\zeta' := \nabla_{\Bbbk[[t]]} \circ (\zeta \otimes \zeta') \circ \Delta$. In turn, if $(\mathscr{V}, \nabla, \iota, \Delta, \epsilon) \in \mathsf{Mon}(\mathsf{FB}^{\mathsf{Comon}})$ is a species coalgebroid then we define $\mathbb{X}(\mathscr{V})$ to be the set of natural transformations $\mathrm{Z} : \mathscr{V} \to \mathcal{E}(\Bbbk[[t]])$ for which $(\mathscr{V}, \mathrm{Z})$ is a combinatorial coalgebroid. This set is yet another monoid with unit element ϵ, in which the product of $\mathrm{Z}, \mathrm{Z}' \in \mathbb{X}(\mathscr{V})$ is the morphism $\mathrm{Z}\mathrm{Z}' : \mathscr{V} \to \mathcal{E}(\Bbbk[[t]])$ with $(\mathrm{Z}\mathrm{Z}')_S := \mathrm{Z}_S \mathrm{Z}'_S$ for each finite set S.

8 Characters and morphisms

In this section, we assume $\Bbbk$ has characteristic zero and view $\mathbf{W}$ as the bialgebra from Theorem 2.4. Our goal here is to illustrate a variety of cases where well-known symmetric and quasi-symmetric functions may be constructed via the morphisms in Theorems 7.4 and 7.8 and Corollary 7.10.

8.1 Fundamental quasi-symmetric functions

We start by examining four natural elements of $\mathbb{X}(\mathbf{W})$. Let $\zeta_\leq : \mathbf{W} \to \Bbbk[t]$ be the linear map

$$\zeta_\leq([w,n]) = \begin{cases} t^{\ell(w)} & \text{if } w \text{ is weakly increasing} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } [w,n] \in \mathbb{W}. \tag{8.1}$$

Define $\zeta_\geq$, $\zeta_<$, $\zeta_>$ to be the linear maps $\mathbf{W} \to \Bbbk[t]$ given by the same formula but with "weakly increasing" replaced by "weakly decreasing," "strictly increasing," and "strictly decreasing."

Proposition 8.1. For each $\bullet \in \{\leq, \geq, <, >\}$, we have $\zeta_\bullet \in \mathbb{X}(\mathbf{W})$.

Proof. This is equivalent to [28, Proposition 5.4] and easily checked directly. □

For each $\bullet \in \{\leq, \geq, <, >\}$, we let $\psi_\bullet$ denote the unique morphism $(\mathbf{W}, \zeta_\bullet) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vDash n$, let $I(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\alpha_2+\cdots+\alpha_{l-1}\}$. The map $\alpha \mapsto I(\alpha)$ is a bijection from compositions of n to subsets of $[n-1]$. Write $\alpha \leq \beta$ if $\alpha, \beta \vDash n$ and $I(\alpha) \subseteq I(\beta)$. The *fundamental quasi-symmetric function* associated to $\alpha \vDash n$ is

$$L_\alpha = \sum_{\alpha \leq \beta} M_\beta = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in I(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathsf{QSym}_n.$$

The set $\{L_\alpha : \alpha \vDash n\}$ is a second basis of QSym_n. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vDash n$, let $\beta \vDash n$ be such that $I(\beta) = [n-1] \setminus I(\alpha)$ and define the *reversal*, *complement*, and *transpose* of α to be

$$\alpha^{\mathtt{r}} = (\alpha_l, \ldots, \alpha_2, \alpha_1), \qquad \alpha^{\mathtt{c}} = \beta, \qquad \text{and} \qquad \alpha^{\mathtt{t}} = (\alpha^{\mathtt{r}})^{\mathtt{c}} = (\alpha^{\mathtt{c}})^{\mathtt{r}}.$$

For a word $w = w_1 w_2 \cdots w_n$, define $w^{\mathtt{r}} = w_n \cdots w_2 w_1$ and $\mathrm{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}$.

Proposition 8.2 ([28, Proposition 5.5])**.** If $[w,n] \in \mathbb{W}$, $\alpha \vDash \ell(w)$, and $\mathrm{Des}(w) = I(\alpha)$, then

$$\psi_\leq([w,n]) = L_\alpha, \quad \psi_>([w,n]) = L_{\alpha^{\mathtt{c}}}, \quad \psi_\geq([w^{\mathtt{r}},n]) = L_{\alpha^{\mathtt{r}}}, \quad \text{and} \quad \psi_<([w^{\mathtt{r}},n]) = L_{\alpha^{\mathtt{t}}}.$$

Suppose (V, ζ_V) is a combinatorial bialgebra. If $\iota : U \to V$ is an injective bialgebra morphism then $\zeta_U := \zeta_V \circ \iota \in \mathbb{X}(U)$ and ι is a morphism $(U, \zeta_U) \to (V, \zeta_V)$. Similarly, if $\pi : V \to W$ is a surjective bialgebra morphism with $\ker \pi \subset \ker \zeta_V$ then there exists a unique character $\zeta_W \in \mathbb{X}(W)$ with $\zeta_V = \zeta_W \circ \pi$, and π is a morphism $(V, \zeta_V) \to (W, \zeta_W)$. In the first case the unique morphism $(U, \zeta_U) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ factors through (V, ζ_V) and in the second case $(V, \zeta_V) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ factors through (W, ζ_W).

Fix a symbol $\bullet \in \{\leq, \geq, <, >\}$. The bi-ideal $\mathbf{I}_{\mathsf{P}} \subset \mathbf{W}$ is contained in $\ker \zeta_\bullet$, so $\zeta_\bullet$ and $\psi_\bullet$ factor through the quotient map $\pi : \mathbf{W} \to \mathbf{W}_{\mathsf{P}}$. Let $\tilde{\zeta}_\bullet : \mathbf{W}_{\mathsf{P}} \to \Bbbk[t]$ and $\tilde{\psi}_\bullet : \mathbf{W}_{\mathsf{P}} \to \mathsf{QSym}$ be the unique maps with $\zeta_\bullet = \tilde{\zeta}_\bullet \circ \pi$ and $\psi_\bullet = \tilde{\psi}_\bullet \circ \pi$. Then $\tilde{\zeta}_\bullet \in \mathbb{X}(\mathbf{W}_{\mathsf{P}})$ and $\tilde{\psi}_\bullet$ is the unique morphism $(\mathbf{W}_{\mathsf{P}}, \tilde{\zeta}_\bullet) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. If $\sim$ is a homogeneous P-algebraic word relation, so that $\mathbf{K}_{\mathsf{P}}^{(\sim)} \subset \mathbf{W}_{\mathsf{P}}$ is a graded Hopf sub-algebra, then $\tilde{\zeta}_\bullet$ restricts to an element of $\mathbb{X}(\mathbf{K}_{\mathsf{P}}^{(\sim)})$ and $\tilde{\psi}_\bullet$ restricts to the unique morphism of combinatorial Hopf algebras $(\mathbf{K}_{\mathsf{P}}^{(\sim)}, \zeta_\bullet) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$.

Example 8.3. Suppose $\sim$ is the commutation relation from Example 6.1. Recall that NSym can realized as the Hopf algebra $\mathbf{K}^{(\sim)}_{\mathsf{P}}$ by identifying H_n with n-letter word $11\cdots 1 \in \mathbb{K}^{(\sim)}_{\mathsf{P}}$. The character $\tilde{\zeta}_{\leq} \in \mathbb{X}(\mathbf{K}^{(\sim)}_{\mathsf{P}})$ corresponds to the algebra morphism $\mathsf{NSym} \to \Bbbk[t]$ with $H_n \mapsto t^n$, and $\tilde{\psi}_{\leq}(H_n) = L_{(n)} = \sum_{\alpha\vDash n} M_\alpha = \sum_{\lambda\vdash n} m_\lambda = h_n$ is the nth homogeneous symmetric function. Thus $\tilde{\psi}_{\leq}$ gives the natural projection $\mathsf{NSym} \to \mathsf{Sym}$ with $H_n \mapsto h_n$ for $n \in \mathbb{N}$.

If $\sim$ is an algebraic word relation so that $\mathbf{K}^{(\sim)}_{\mathsf{R}} \subset \mathbf{W}$ is a graded sub-bialgebra, then $\zeta_\bullet$ restricts to an element of $\mathbb{X}(\mathbf{K}^{(\sim)}_{\mathsf{R}})$ and $\psi_\bullet$ restricts to the unique morphism $(\mathbf{K}^{(\sim)}_{\mathsf{R}}, \zeta_\bullet) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$.

Example 8.4. Suppose $\sim$ is the Knuth equivalence relation from Example 6.4. If $\lambda \vdash n$, then the Schur function $s_\lambda \in \mathsf{Sym}$ has the formula $s_\lambda = \sum_{\alpha\vDash n} d_{\lambda\alpha} L_\alpha$ where $d_{\lambda\alpha}$ is the number of standard tableaux of shape λ with descent set $I(\alpha)$ [24, Eq. (3.18)]. Let T be a semistandard tableau of shape λ with $\max(T) \leq n$, and set $[[T,n]] = \sum_{w\sim T} [[w,n]] \in \mathbb{K}^{(\sim)}$. The RSK correspondence gives a descent-preserving bijection between the Knuth equivalence class of T and all standard tableaux of shape λ. One therefore has $\psi_{\leq}([[T,n]]) = s_\lambda$ and $\psi_{>}([[T,n]]) = s_{\lambda^T}$ where λ^T is the partition sorting λ^{t}, i.e., the transpose of λ. Applying the bialgebra morphism $\psi_{\leq}$ to the formulas (6.1) for the (co)product of $\mathbf{K}^{(\sim)}$ gives two versions of the Littlewood-Richardson rule; see [33, §2].

8.2 Multi-fundamental quasi-symmetric functions

Fix $\bullet \in \{\leq, \geq, <, >\}$. Since $\mathbf{W}_{\mathsf{P}}$ has finite graded dimension, the character $\tilde{\zeta}_\bullet : \mathbf{W}_{\mathsf{P}} \to \Bbbk[t]$ extends to a continuous linear map $\hat{\mathbf{W}}_{\mathsf{P}} \to \Bbbk[[t]]$, which we denote with the same symbol, and it holds that $\tilde{\zeta}_\bullet \in \mathbb{X}(\hat{\mathbf{W}}_{\mathsf{P}})$. The morphism $\tilde{\psi}_\bullet : (\mathbf{W}_{\mathsf{P}}, \tilde{\zeta}_\bullet) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ likewise extends to a continuous linear map $\hat{\mathbf{W}}_{\mathsf{P}} \to \widehat{\mathsf{QSym}}$, which we also denote with the same symbol. This extension is the unique morphism of linearly compact combinatorial bialgebras $(\hat{\mathbf{W}}_{\mathsf{P}}, \tilde{\zeta}_\bullet) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$.

Given finite, nonempty subsets $S, T \subset \mathbb{P}$, write $S \preceq T$ if $\max(S) \leq \min(T)$ and $S \prec T$ if $\max(S) < \min(T)$, and define $x_S = \prod_{i\in S} x_i$. In [21, §5.3], Lam and Pylyavskyy define the *multi-fundamental quasi-symmetric function* of a composition $\alpha \vDash n$ to be the power series

$$\tilde{L}_\alpha = \sum_{\substack{S_1 \preceq S_2 \preceq \cdots \preceq S_n \\ S_j \prec S_{j+1} \text{ if } j \in I(\alpha)}} x_{S_1} x_{S_2} \cdots x_{S_n} \in \widehat{\mathsf{QSym}} \tag{8.2}$$

where the sum is over finite, nonempty subsets $S_1, S_2, \ldots, S_n$ of positive integers.

If $f \in \Bbbk[[x_1, x_2, \ldots]]$, then we use the shorthand $f(\frac{x}{1-x})$ to denote the power series obtained from f by substituting $x_i \mapsto \frac{x_i}{1-x_i} = x_i + x_i^2 + x_i^3 + \ldots$ for each $i \in \mathbb{P}$. It is easy to check that if $f \in \mathsf{QSym}$ then $f(\frac{x}{1-x}) \in \widehat{\mathsf{QSym}}$. Recall that a multi-permutation is a packed word with no adjacent repeated letters. The functions $\tilde{L}_\alpha$ arise naturally as the images of the basis of the Hopf algebra $\mathfrak{m}\mathrm{MR} = \hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ when $\sim$ is K-equivalence, under the morphisms $(\mathfrak{m}\mathrm{MR}, \tilde{\zeta}_\bullet) \to (\widehat{\mathsf{QSym}}, \zeta_{\mathsf{QSym}})$.

Proposition 8.5. Let $\sim$ be the K-equivalence relation from Example 6.2. Suppose w is a multi-permutation and define $[[w]] = \sum_{v\sim w} v \in \mathbb{K}^{(\sim)}_{\mathsf{P}}$. If $\alpha \vDash \ell(w)$ has $\mathrm{Des}(w) = I(\alpha)$, then

$$\tilde{\psi}_{<}([[w]]) = \tilde{L}_\alpha, \quad \tilde{\psi}_{\leq}([[w]]) = \tilde{L}_\alpha(\tfrac{x}{1-x}), \quad \tilde{\psi}_{>}([[w^{\mathsf{r}}]]) = \tilde{L}_{\alpha^{\mathsf{r}}}, \quad \text{and} \quad \tilde{\psi}_{\geq}([[w^{\mathsf{r}}]]) = \tilde{L}_{\alpha^{\mathsf{r}}}(\tfrac{x}{1-x}).$$

The first identity is equivalent to [21, Theorem 5.11].

Proof. Assume $w = w_1w_2\cdots w_n$ has n letters and let $m_1, m_2, \dots, m_n \in \mathbb{P}$. The first identity holds since, by Proposition 8.2, $\tilde\psi_<$ applied to $w_1w_1\cdots w_1w_2w_2\cdots w_2\cdots w_nw_n\cdots w_n \sim w$ with w_i repeated m_i times gives the sum in (8.2) restricted to subsets with $|S_i| = m_i$. It follows in a similar way that $\tilde\psi_\leq([[w]])$ has the same formula (8.2) except with the sum over finite, nonempty multisets $S_1, S_2, \dots, S_n$, which is just $\tilde L_\alpha(\frac{x}{1-x})$. The other identities hold by Proposition 8.2 since the image of the continuous linear map with $L_\alpha \mapsto L_{\alpha^r}$ has $\tilde L_\alpha \mapsto \tilde L_{\alpha^r}$ and $\tilde L_\alpha(\frac{x}{1-x}) \mapsto \tilde L_{\alpha^r}(\frac{x}{1-x})$. ☐

We shift our attention to the species coalgebroid $(\mathscr{W}, \nabla_{\sqcup\!\sqcup}, \iota_{\sqcup\!\sqcup}, \Delta_\odot, \epsilon_\odot)$. For each $\bullet \in \{\leq, \geq, <, >\}$, let $\mathrm{Z}_\bullet : \mathscr{W} \to \mathcal{E}(\Bbbk[[t]])$ be the natural transformation whose S-component for each finite set S is the continuous linear map $\mathscr{W}[S] \to \Bbbk[[t]]$ with $[w,\lambda] \mapsto \zeta_\bullet([w,n])$ for $[w,\lambda] \in \mathbb{W}_S$. Since there are only finitely many words of a given length with all letters at most n, the following holds:

Corollary 8.6. For each symbol $\bullet \in \{\leq, \geq, <, >\}$, it holds that $\mathrm{Z}_\bullet \in \mathbb{X}(\mathscr{W})$.

For each $\bullet \in \{\leq, \geq, <, >\}$, define $\Psi_\bullet : \mathscr{W} \to \mathcal{E}\widehat{\mathsf{QSym}}$ to be the natural transformation whose S-component is the continuous linear map $\mathscr{W}[S] \to \widehat{\mathsf{QSym}}$ with $[w,\lambda] \mapsto \psi_\bullet([w,n])$ for each finite set S of size n and each pair $[w,\lambda] \in \mathbb{W}_S$. The following is apparent from Corollary 7.10:

Corollary 8.7. For each $\bullet \in \{\leq, \geq, <, >\}$, it holds that $\Psi_\bullet$ is the unique morphism of combinatorial coalgebroids $(\mathscr{W}, \mathrm{Z}_\bullet) \to (\mathcal{E}\widehat{\mathsf{QSym}}, \mathrm{Z}_{\mathsf{QSym}})$.

If $\sim$ is an algebraic word relation so that $\mathscr{K}^{(\sim)} \subset \mathscr{W}$ is sub-coalgebroid, then the natural transformation $\mathrm{Z}_\bullet$ restricts to an element of $\mathbb{X}(\mathscr{K}^{(\sim)})$ and $\Psi_\bullet$ restricts to the unique morphism of combinatorial coalgebroids $(\mathscr{K}^{(\sim)}, \mathrm{Z}_\bullet) \to (\mathcal{E}\widehat{\mathsf{QSym}}, \mathrm{Z}_{\mathsf{QSym}})$.

Example 8.8. Suppose $\sim$ is the Hecke equivalence relation from Example 6.7. Given $\pi \in S_{n+1}$, define $[[\pi]] = \sum_w [w,n] \in \mathbb{K}_n^{(\sim)}$ where the sum is over all Hecke words w for π, and let

$$\tilde K_\pi = \Psi_>([[\pi]]), \qquad J_\pi = \Psi_\leq([[\pi]]), \qquad \text{and} \qquad G_\pi = (-1)^{\ell(\pi)}\tilde K_\pi(-x_1, -x_2, \dots). \tag{8.3}$$

The functions G_π are the *stable Grothendieck polynomials* [6, 7]. Following [21, 33], we call J_π and $\tilde K_\pi$ the *weak stable Grothendieck polynomials* and *signless stable Grothendieck polynomials*. Write ω for the continuous linear involution of $\widehat{\mathsf{QSym}}$ with $L_\alpha \mapsto L_{\alpha^t}$. Proposition 8.2 implies that $J_\pi = \omega(\tilde K_\pi)$. By [6, Theorem 6.12], J_π and $\tilde K_\pi$ are Schur positive elements of $\widehat{\mathsf{Sym}}$ and $G_\pi \in \widehat{\mathsf{Sym}}$.

One says that $\pi \in S_n$ is *Grassmannian* if $\pi_1 < \dots < \pi_p > \pi_{p+1} < \dots < \pi_n$ for some $p \in [n]$. In this case let $\lambda(\pi)$ be the partition sorting $(\pi_1 - 1, \pi_2 - 2, \dots, \pi_p - p)$. If π is Grassmannian then the functions (8.3) depend only on $\lambda(\pi)$. Given a partition λ, define $\tilde K_\lambda = \tilde K_\pi$, $J_\lambda = J_\pi$, and $G_\lambda = G_\pi$, where $\pi \in \bigsqcup_{n\in\mathbb{N}} S_n$ is any Grassmannian permutation with $\lambda = \lambda(\pi)$. By [7, Theorem 1], each J_π is a finite $\mathbb{N}$-linear combination of J_λ's, and each $\tilde K_\pi$ is a finite $\mathbb{N}$-linear combination of $\tilde K_\lambda$'s.

Example 8.9. Let $\sim$ be K-Knuth equivalence so that $\mathsf{KPR} = \mathbf{K}_{\mathsf{P}}^{(\sim)}$ is the K-theoretic Poirier-Reutenauer bialgebra of [33]. That $\tilde\psi_\leq$ is a morphism of linearly compact Hopf algebras $\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)} \to \widehat{\mathsf{Sym}}$ is essentially [33, Theorem 6.23]. If w is a packed word and $[[w]] = \sum_{v\sim w} v \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$, then one has

$$\tilde\psi_\leq([[w]]) = J_{\lambda_1} + J_{\lambda_2} + \dots + J_{\lambda_m} \qquad \text{and} \qquad \tilde\psi_>([[w]]) = \tilde K_{\lambda_1} + \tilde K_{\lambda_2} + \dots + \tilde K_{\lambda_m}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are the (not necessarily distinct shapes) of the finite number of increasing tableaux in the K-Knuth equivalence class of w [33, Theorem 6.24].

8.3 Peak quasi-symmetric functions

Recall the monoidal structure on $\mathbb{X}(\mathbf{W})$: if $\zeta, \zeta' \in \mathbb{X}(\mathbf{W})$ then $\zeta\zeta' := \nabla_{\Bbbk} \circ (\zeta \otimes \zeta') \circ \Delta_{\odot} \in \mathbb{X}(\mathbf{W})$. For any symbols $\bullet, \circ \in \{\leq, \geq, <, >\}$, we can therefore define $\zeta_{\bullet|\circ} = \zeta_\bullet \zeta_\circ \in \mathbb{X}(\mathbf{W})$ and let $\psi_{\bullet|\circ}$ be the unique morphism $(\mathbf{W}, \zeta_{\bullet|\circ}) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. For example, if $[w, n] \in \mathbb{W}$ then

$$\zeta_{>|\leq}([w,n]) = \begin{cases} 1 & \text{if } w = \emptyset, \\ 2t^m & \text{if } w_1 > \cdots > w_i \leq w_{i+1} \leq \cdots \leq w_m \text{ where } 1 \leq i \leq m = \ell(w) \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

Similar formulas hold for the other possibilities of $\zeta_{\bullet|\circ}$.

One calls $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l) \vDash n$ a *peak composition* if $\alpha_i \geq 2$ for $1 \leq i < l$, i.e., if $1 \notin I(\alpha)$ and $i \in I(\alpha) \Rightarrow i \pm 1 \notin I(\alpha)$. The number of peak compositions of n is the nth Fibonacci number. The *peak quasi-symmetric function* [38, Proposition 2.2] of a peak composition $\alpha \vDash n$ is

$$K_\alpha = \sum_{\substack{\beta \vDash n \\ I(\alpha) \subset I(\beta) \cup (I(\beta)+1)}} 2^{\ell(\beta)} M_\beta \in \mathsf{QSym}_n.$$

Such functions are a basis for a graded Hopf subalgebra of QSym, called *Stembridge's peak subalgebra* or the *odd subalgebra* [1, Proposition 6.5], which we denote by $\mathcal{O}\mathsf{QSym}$.

Let $\mathrm{Peak}(w) = \{i \in [2, n-1] : w_{i-1} \leq w_i > w_{i+1}\}$ and $\mathrm{Val}(w) = \{i \in [2, n-1] : w_{i-1} \geq w_i < w_{i+1}\}$ for a word $w = w_1 w_2 \cdots w_n$. For each $\alpha \vDash n$, let $\Lambda(\alpha) \vDash n$ be the peak composition such that

$$I(\Lambda(\alpha)) = \{i \geq 2 : i \in I(\alpha),\ i - 1 \notin I(\alpha)\}.$$

If w is a word and $\alpha \vDash \ell(w)$ and $\mathrm{Des}(w) = I(\alpha)$, then $\mathrm{Peak}(w) = I(\Lambda(\alpha))$. Finally, given a peak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$, define $\alpha^\flat = (\alpha_l + 1, \alpha_{l-1}, \dots, \alpha_2, \alpha_1 - 1)$.

Proposition 8.10 ([28, Proposition 5.7])**.** If $[w, n] \in \mathbb{W}$ and $\alpha, \beta \vDash \ell(w)$ are compositions such that $\mathrm{Peak}(w) = I(\alpha)$ and $\mathrm{Val}(w) = I(\beta)$, then

$$\psi_{>|\leq}([w,n]) = K_\alpha, \quad \psi_{<|\geq}([w,n]) = K_\beta, \quad \psi_{\geq|<}([w^{\mathrm{r}},n]) = K_{\alpha^\flat}, \quad \text{and} \quad \psi_{\leq|>}([w^{\mathrm{r}},n]) = K_{\beta^\flat}.$$

For each $\bullet, \circ \in \{\leq, \geq, <, >\}$, write $\tilde{\zeta}_{\bullet|\circ} = \tilde{\zeta}_\bullet \tilde{\zeta}_\circ : \mathbf{W}_{\mathsf{P}} \to \Bbbk[t]$ and $\tilde{\psi}_{\bullet|\circ} = \tilde{\psi}_\bullet \tilde{\psi}_\circ : \mathbf{W}_{\mathsf{P}} \to \mathsf{QSym}$ for the maps such that $\zeta_{\bullet|\circ} = \tilde{\zeta}_{\bullet|\circ} \circ \pi$ and $\psi_{\bullet|\circ} = \tilde{\psi}_{\bullet|\circ} \circ \pi$ where $\pi : \mathbf{W} \to \mathbf{W}_{\mathsf{P}}$ is the quotient map.

Example 8.11. Again suppose $\sim$ is the commutation relation from Example 6.1, so that we can identify $\mathsf{NSym} \cong \mathbf{K}_{\mathsf{P}}^{(\sim)}$ by setting $H_n = 11\cdots 1 \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$. The character $\tilde{\zeta}_{>|\leq}$ corresponds to the algebra morphism $\mathsf{NSym} \to \Bbbk[t]$ with $H_n \mapsto 2t^n$ for $n > 0$, and we have

$$\tilde{\psi}_{>|\leq}(H_n) = K_{(n)} = \sum_{\alpha \vDash n} 2^{\ell(\alpha)} M_\alpha = \sum_{\lambda \vdash n} 2^{\ell(\lambda)} m_\lambda = q_n$$

where $q_n \in \mathsf{Sym}$ is the symmetric function such that $\sum_{n \geq 0} q_n t^n = \prod_{i \geq 1} \frac{1 + x_i t}{1 - x_i t}$ (see [38, §A.1]). Thus, in this case $\tilde{\psi}_{>|\leq}$ is the composition of the natural projection $\mathsf{NSym} \to \mathsf{Sym}$ with the algebra morphism denoted $\theta : \mathsf{Sym} \to \mathsf{Sym}$ in [38, Remark 3.2].

Define $\mathscr{O}\mathsf{Sym} = \Bbbk[q_1, q_2, q_3, \dots]$. By [38, Theorem 3.8], it holds that $\mathscr{O}\mathsf{Sym} = \mathsf{Sym} \cap \mathscr{O}\mathsf{QSym}$ is a graded Hopf subalgebra Sym. This subalgebra has a distinguised basis $\{Q_\lambda\}$ indexed by strict partitions λ, known as the *Schur Q-functions*; see [38, §A.1] for the definition.

Example 8.12. Suppose $\sim$ is Knuth equivalence and T is a semistandard tableau of shape λ with $\max(T) \leq n$. The morphism $\psi_{>|\leq} : (\mathbf{K}^{(\sim)}, \zeta_{>|\leq}) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ then has $\psi_{>|\leq}([[T, n]]) = S_\lambda$ where $S_\lambda \in \mathscr{O}\mathsf{Sym}$ is the *Schur S-function* of shape λ [25, Chapter III, §8, Ex. 7]. Each S_λ is an $\mathbb{N}$-linear combination of Schur Q-functions, i.e., is *Schur Q-positive.*

Example 8.13. If $\sim$ is Hecke equivalence, then applying $\psi_>$ and $\psi_{>|\leq}$ to the elements of the natural basis $\mathbb{K}^{(\sim)}_{\mathsf{R}}$ of the bialgebra of reduced classes $\mathbf{K}^{(\sim)}_{\mathsf{R}}$ gives the *Stanley symmetric functions* F_π and F^C_π of types A and C; see the discussion in [28].

8.4 Symmetric functions

Suppose $\sim$ is a uniformly algebraic word relation. It is natural to ask when the image of $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ under $\tilde\psi_\bullet$ is contained in $\widehat{\mathsf{Sym}}$, or equivalently when the image of $\mathscr{K}^{(\sim)}$ under $\Psi_\bullet$ is a subspecies of $\mathcal{E}(\widehat{\mathsf{Sym}})$. In turn, one can ask when $\tilde\psi_\bullet(\kappa)$ is Schur positive for all elements $\kappa \in \mathbb{K}^{(\sim)}_{\mathsf{P}}$.

Theorem 8.14. Let $\sim$ be a uniformly algebraic word relation. The following are equivalent:

(a) The image of $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ under $\tilde\psi_\leq$ is contained in $\widehat{\mathsf{Sym}}$.

(b) The image of $\hat{\mathbf{K}}^{(\sim)}_{\mathsf{P}}$ under $\tilde\psi_>$ is contained in $\widehat{\mathsf{Sym}}$.

(c) The relation $\sim$ extends Knuth equivalence or K-Knuth equivalence.

Moreover, if these conditions hold and E is any $\sim$-equivalence class of packed words, then the symmetric functions $\tilde\psi_\leq(\kappa_E)$ and $\tilde\psi_>(\kappa_E)$ are both Schur positive.

There is a left-handed version of this result, in which the symbols $\leq$ and $>$ are replaced by $\geq$ and $<$, and Knuth equivalence in part (c) is replaced by *reverse Knuth equivalence*: the relation with $v \sim w$ if and only if $v^{\mathbf{r}}$ and $w^{\mathbf{r}}$ are Knuth equivalent. One can ask similar questions about (P-)algebraic word relations, but such relations do not seem to have a nice classification.

Proof. The continuous linear map $\widehat{\mathsf{QSym}} \to \widehat{\mathsf{QSym}}$ with $L_\alpha \mapsto L_{\alpha^c}$ restricts to the continuous linear involution of $\widehat{\mathsf{Sym}}$ with $s_\lambda \mapsto s_{\lambda^T}$, so parts (a) and (b) are equivalent by Proposition 8.2.

Suppose (a) holds and write $f \equiv g$ when $f, g \in \widehat{\mathsf{QSym}}$ are such that $f - g \in \widehat{\mathsf{Sym}}$. Consider the six words w of length three involving the letters 1, 2, and 3. By Proposition 8.2, we have $\tilde\psi_\leq(w) \equiv 0$ unless $w \in \{132, 213, 231, 312\}$, and $\tilde\psi_\leq(132) \equiv \tilde\psi_\leq(231) \equiv M_{(2,1)}$ and $\tilde\psi_\leq(213) \equiv \tilde\psi_\leq(312) \equiv M_{(1,2)}$. To have $\tilde\psi_\leq\left(\sum_{v\sim w} v\right) \equiv 0$ for each of these words, it must hold that $132 \sim 213$ and $231 \sim 312$, or $132 \sim 312$ and $231 \sim 213$. The former case implies the latter since if $132 \sim 213$, then $12 = 132 \cap \{1,2\} \sim 213 \cap \{1,2\} = 21$ whence $ab \sim ba$ for all $a, b \in \mathbb{P}$ as $\sim$ is uniformly algebraic. We conclude, by uniformity, that $acb \sim cab$ and $bca \sim bac$ for all positive integers $a < b < c$.

Similarly, if w is one of the eight words of length three involving the letters 1 and 2, then $\tilde\psi_\leq(w) \equiv 0$ unless $w \in \{121, 221, 211, 212\}$, and $\tilde\psi_\leq(121) \equiv \tilde\psi_\leq(221) \equiv M_{(2,1)}$ and $\tilde\psi_\leq(211) \equiv \tilde\psi_\leq(212) \equiv M_{(1,2)}$. To have $\tilde\psi_\leq\left(\sum_{v\sim w} v\right) \equiv 0$ for each of these words, it must hold that $121 \sim 211$ and $212 \sim 221$, or $121 \sim 212$ and $221 \sim 211$. In the first case, the relation $\sim$ extends Knuth equivalence. In the second case, we have $a \sim aa$ for all $a \in \mathbb{P}$ since $1 = 212 \cap \{1\} \sim 121 \cap \{1\} = 11$, so $\sim$ extends K-Knuth equivalence. Thus (a) $\Rightarrow$ (c).

Examples 8.4 and 8.9 show that if (c) holds then $\tilde\psi_{\leq}(\kappa_E)$ and $\tilde\psi_{>}(\kappa_E)$ are both Schur positive for any $\sim$-equivalence class E. In particular, (c) $\Rightarrow$ (a). $\square$

Corollary 8.15. Assume $\sim$ is homogeneous and uniformly algebraic. Then the image of $\mathbf{K}^{(\sim)}$ under $\psi_{\leq}$ (equivalently, $\psi_{>}$) is a sub-bialgebra of Sym if and only if $\sim$ extends Knuth equivalence.

Our last result is an attempt to formulate a version of Theorem 8.14 for the morphisms $\tilde\psi_{\bullet|\circ}$. Recall the definition of *exotic Knuth equivalence* from Example 6.10.

Proposition 8.16. Let $\sim$ be a uniformly algebraic word relation. The image of $\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)}$ under $\tilde\psi_{>|\leq}$ is contained in $\widehat{\mathsf{Sym}}$ only if $\sim$ extends Knuth, K-Knuth, or exotic Knuth equivalence.

Proof. The argument is similar to the proof of Theorem 8.14, although the calculations are more cumbersome to carry out by hand. Again write $f \equiv g$ when $f, g \in \widehat{\mathsf{QSym}}$ are such that $f - g \in \widehat{\mathsf{Sym}}$. Suppose $\sim$ is a uniformly algebraic word relation such that $\tilde\psi_{>|\leq}\left(\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)}\right) \subset \widehat{\mathsf{Sym}}$. If $ab \sim ba$ for any integers $a < b$, then $ab \sim ba$ for all $a < b$ and $\sim$ extends Knuth equivalence. Therefore assume $ab \not\sim ba$ for all integers $a < b$.

Among the permutations $w \in S_4$, the eight elements 1324, 1423, 1432, 2314, 2413, 2431, 3412, and 3421 have $\tilde\psi_{>|\leq}(w) \equiv 4M_{(1,3)}$, the eight elements 1243, 1342, 2143, 2341, 3142, 3241, 4132, and 4231 have $\tilde\psi_{>|\leq}(w) \equiv 4M_{(3,1)}$, and the remaining elements have $\tilde\psi_{>|\leq}(w) \equiv 0$. Since $ab \not\sim ba$ if $a < b$, we must have $1423 \sim 1243$ and $3421 \sim 3241$. It follows for $I = \{2,3,4\}$ that $423 = 1423 \cap I \sim 1243 \cap I = 243$ and $342 = 3421 \cap I \sim 3241 \cap I = 324$. By the uniformity of $\sim$, we conclude that $cab \sim acb$ and $bca \sim bac$ for all $a < b < c$.

To proceed, first suppose that $a \sim aa$ for $a \in \mathbb{P}$. We then have $3231 \sim 3213$ and $3123 \sim 1323$ and it holds that $\tilde\psi_{>|\leq}(2321) \equiv \tilde\psi_{>|\leq}(3123 + 1323) \equiv 4M_{(1,3)}$ and $\tilde\psi_{>|\leq}(1232) \equiv \tilde\psi_{>|\leq}(3231 + 3213) \equiv 4M_{(3,1)}$, while all other words of length 4 with letters in $\{1,2,3\}$ belong to $\sim$-equivalence classes E with $\tilde\psi_{>|\leq}(\kappa_E) \equiv 0$. Since $12 \not\sim 21$, it must hold that $2321 \sim 3231 \sim 3213$ and $1232 \sim 3123 \sim 1323$. Intersecting these relations with the interval $I = \{2,3\}$ shows that $232 \sim 323$, which implies that $aba \sim bab$ for all integers $a < b$. Thus, if $a \sim aa$ then $\sim$ extends K-Knuth equivalence.

Instead suppose that $a \not\sim aa$ for all $a \in \mathbb{P}$. Then $3122 \sim 1322$ and $\tilde\psi_{>|\leq}(2321) \equiv \tilde\psi_{>|\leq}(3122 + 1322) \equiv 4M_{(1,3)}$ and $\tilde\psi_{>|\leq}(3221) \equiv \tilde\psi_{>|\leq}(1232) \equiv 4M_{(3,1)}$, while all other permutations of 1223 belong to $\sim$-equivalence classes E with $\tilde\psi_{>|\leq}(\kappa_E) \equiv 0$. One of two cases must then occur:

- Suppose $2321 \sim 1232$ and $3221 \sim 3122 \sim 1322$, so that $abcb \sim bcba$ and $abb \sim bba$ for all $a < b < c$. Then $2133 \sim 2313$ and $2321 \sim 3213$ and $3123 \sim 1323$, and $\tilde\psi_{>|\leq}(2133 + 2313) \equiv \tilde\psi_{>|\leq}(3123 + 1323) \equiv 4M_{(1,3)}$ and $\tilde\psi_{>|\leq}(3231 + 3213) \equiv \tilde\psi_{>|\leq}(1332) \equiv 4M_{(3,1)}$, while all other permutations of 1233 belong to $\sim$-equivalence classes E with $\tilde\psi_{>|\leq}(\kappa_E) \equiv 0$. Since $12 \not\sim 21$, we must have $3231 \sim 3213 \sim 2133 \sim 2313$ and $3123 \sim 1323 \sim 1332$. Intersecting these equivalences with the interval $I = \{2,3\}$ shows that $233 \sim 323 \sim 332$, so $abb \sim bab \sim bba$ for all $a < b$. Finally, we must have $abaa \sim aaba$ for all $a < b$ since $\tilde\psi_{>|\leq}(1211) \equiv 4M_{(1,3)}$ and $\tilde\psi_{>|\leq}(1121) \equiv 4M_{(3,1)}$, while all other words of length 4 with letters in $\{1,2\}$ belong to $\sim$-equivalence classes E with $\tilde\psi_{>|\leq}(\kappa_E) \equiv 0$. Thus $\sim$ extends exotic Knuth equivalence.
- Suppose $2321 \sim 3221$ and $1232 \sim 3122 \sim 1322$, so that $aba \sim baa$ for all $a < b$. Then $2133 \sim 2313$ and $3123 \sim 1323$ and $3231 \sim 3213$, and $\tilde\psi_{>|\leq}(2133{+}2313) \equiv \tilde\psi_{>|\leq}(3123{+}1323) \equiv$

$\tilde{\psi}_{>|\leq}(3321) \equiv 4M_{(1,3)}$ and $\tilde{\psi}_{>|\leq}(3231 + 3213) \equiv \tilde{\psi}_{>|\leq}(2331) \equiv \tilde{\psi}_{>|\leq}(1332) \equiv 4M_{(3,1)}$, while all other permutations of 1233 belong to $\sim$-equivalence classes E with $\tilde{\psi}_{>|\leq}(\kappa_E) \equiv 0$. Since $12 \not\sim 21$, we must have $1332 \sim 3123 \sim 1323$. Intersecting these equivalences with the interval $I = \{2, 3\}$ shows that $332 \sim 323$, so $bba \sim bab$ for all $a < b$ and $\sim$ extends Knuth equivalence.

We conclude that the relation $\sim$ must extend Knuth, K-Knuth, or exotic Knuth equivalence. □

By Proposition 8.10, the image $\tilde{\psi}_{>|\leq}\left(\hat{\mathbf{K}}_{\mathsf{P}}^{(\sim)}\right)$ is contained in the completion of $\mathcal{O}\mathsf{QSym}$ with respect to its basis of peak quasi-symmetric functions $\{K_\alpha\}$. By [38, Theorem 3.8], the intersection of this completion with $\widehat{\mathsf{Sym}}$ is the linearly compact space of formal power series $\Bbbk[[q_1, q_2, q_3, \dots]]$, which is also the completion of $\mathcal{O}\mathsf{Sym}$ with respect to its basis of Schur Q-functions.

It follows from Examples 8.12 and 8.9 that if $\sim$ extends Knuth equivalence or K-Knuth equivalence then $\tilde{\psi}_{>|\leq}(\kappa)$ is Schur Q-positive for all elements $\kappa \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$. If we could prove the following, then we could upgrade the "only if" in Proposition 8.16 to "if and only if."

Conjecture 8.17. If $\sim$ is exotic Knuth equivalence, then $\tilde{\psi}_{>|\leq}(\kappa) \in \mathsf{Sym}$ for $\kappa \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$.

An even stronger property appears to be true:

Conjecture 8.18. If $\sim$ is exotic Knuth equivalence, then $\tilde{\psi}_{>|\leq}(\kappa)$ is Schur positive for $\kappa \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$.

Curiously, $\tilde{\psi}_{>|\leq}(\kappa)$ is not always Schur Q-positive when $\kappa \in \mathbb{K}_{\mathsf{P}}^{(\sim)}$ and $\sim$ is exotic Knuth equivalence. We have the checked the two conjectures when $\kappa = \kappa_E$ where E is any exotic Knuth equivalence class of words of length at most nine. Among the 27,021 classes E of packed words w with $\ell(w) = 9$, only 35 are such that $\tilde{\psi}_{>|\leq}(\kappa_E)$ is not Schur Q-positive.

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