arXiv:1810.00993v4 [math.CO] 2 Nov 2018

BALLOT PERMUTATIONS, ODD ORDER PERMUTATIONS, AND A NEW PERMUTATION STATISTIC

SAM SPIRO

Abstract. We say that a permutation π is ballot if, for all i, the word $\pi_1 \cdots \pi_i$ has at least as many ascents as it has descents. We say that π is an odd order permutation if π has odd order in S_n. Let $b(n)$ denote the number of ballot permutations of order n, and let $p(n)$ denote the number of odd order permutations of order n. Callan conjectured that $b(n) = p(n)$ for all n, and this was later proven by Bernardi, Duplantier, and Nadeau.

In this paper we propose a refinement of Callan's original conjecture. Let $b(n, d)$ denote the number of ballot permutations with d descents. Let $p(n, d)$ denote the number of odd order permutations with $M(\pi) = d$, where $M(\pi)$ is a certain statistic related to the cyclic descents of π. We conjecture that $b(n, d) = p(n, d)$ for all n and d. We prove this stronger conjecture for the cases $d = 1$, 2, and $d = \lfloor (n-1)/2 \rfloor$, and in each of these cases we establish formulas for $b(n, d)$ involving second-order Eulerian numbers and Eulerian-Catalan numbers.

1. Introduction

Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation. Define the up-down signature $Q^\pi = (q_1^\pi, q_2^\pi, \ldots, q_{n-1}^\pi)$ by

$$q_i^\pi = \begin{cases} 1 & \pi_i < \pi_{i+1}, \\ -1 & \pi_i > \pi_{i+1}. \end{cases}$$

For example, if $\pi = 31452$, then $Q^\pi = (-1, 1, 1, -1)$. The problem of enumerating the number of permutations in S_n with a given up-down signature started with André [1] who deduced the exponential generating function for the number of permutations π with up-down signature $Q^\pi = (1, -1, 1, -1, \ldots)$. This work was generalized by Niven [6] who provided a formula for the number of $\pi \in S_n$ such that $Q^\pi = Q$ for any fixed up-down signature Q. More recent results related to up-down signatures include work by Brown, Fink, and Willbrand [4], Shevelev [8], and Shevelev and Spilker [9].

In this paper we are interested in permutations whose up-down signatures satisfy a certain property. We will say that a permutation π is ballot if $\sum_1^k q_i^\pi \ge 0$ for all $1 \le k \le n-1$. For example, $\pi = 31452$ is not ballot since $\sum_1^1 q_i^\pi = -1$, but one can verify that $\sigma = 14352$ is ballot. We let $B(n)$ denote the set of all ballot permutations of order n, and we let $b(n) = |B(n)|$.

We will say that a permutation π is an odd order permutation (abbreviated OOP) if the order of π is odd in S_n, which is equivalent to π being the product of only odd cycles. For example, $\pi = (3, 1, 4)(2, 5, 6, 7, 9)$ is an OOP since it has order 15 in S_9. We let $P(n)$ denote the set of OOP's of order n, and we let $p(n) = |P(n)|$.

Callan [7] conjectured that ballot permutations and OOP's are equinumerous, and this was proven by Bernardi, Duplantier, and Nadeau.

Theorem 1.1. [2] *$b(n) = p(n)$ for all n.*

Based on experimental data, we believe that a refined version of Theorem 1.1 is true. Let $B(n, d)$ denote the set of permutations of $B(n)$ with exactly d descents, and let $b(n, d) = |B(n, d)|$. We note that $B(n, d) = \emptyset$ whenever $d > \lfloor (n-1)/2 \rfloor$ since any $\pi \in B(n, d)$ would have $\sum_1^{n-1} q_i^\pi < 0$, and hence π wouldn't be ballot.

We wish to define an analog for the descent statistic in the context of OOP's. Given a cycle $\bar{c} = (c_1, \ldots, c_k)$ of a permutation π, we let $A_{\text{cyc}}(\bar{c})$ denote the number of cyclic ascents of $\bar{c}$. That

Date: November 5, 2018.

is, $A_{\mathrm{cyc}}(\bar{c})$ is the number of ascents in the word $c_1c_2\cdots c_kc_1$. We similarly define $D_{\mathrm{cyc}}(\bar{c})$ to be the number of cyclic descents of $\bar{c}$. We let $M(\bar{c})=\min(A_{\mathrm{cyc}}(\bar{c}),D_{\mathrm{cyc}}(\bar{c}))$. For example, if $\bar{c}=(4,2,8,5,6)$ we have $A_{\mathrm{cyc}}(\bar{c})=2,\ D_{\mathrm{cyc}}(\bar{c})=3$, and hence $M(\bar{c})=2$. For a permutation $\pi=\bar{c}_1\bar{c}_2\cdots\bar{c}_k$ written in cycle notation, we define $M(\pi)=\sum_1^k M(\bar{c}_i)$. For example, if $\pi=(1,3,9)(4,2,8,5,6)(7)$, then $M(\pi)=1+2+0=3$. Let $P(n,d)$ denote the set of permutations of $P(n)$ with $M(\pi)=d$, and let $p(n,d)=|P(n,d)|$.

Conjecture 1.2. $b(n,d)=p(n,d)$ for all n and d.

We provide proofs for this conjecture in several special cases. Before stating these results, we first introduce some notation. We define the Eulerian number $E(n,k)$ to be the number of permutations of order n with exactly k descents. Let $A(n)=2^n-2n$ denote the second-order Eulerian numbers. For more information on (second order) Eulerian numbers, we refer the reader to Graham, Knuth, Patashnik, and Liu [5]. Let $EC(n)=2E(2n,n-1)$ denote the Eulerian-Catalan numbers, which have been studied recently by Bidkhori and Sullivant [3].

Conjecture 1.2 is trivially true for $d=0$. We show that it is also true for $d=1$ and $d=2$.

Theorem 1.3.

$$b(n,1)=p(n,1)=A(n).$$

Moreover, there exists an explicit bijection between $B(n,1)$ and $P(n,1)$.

Theorem 1.4. *For $n\geq 2$,*

$$b(n,2)=p(n,2)=$$

$$\sum_{k=0}^{n}\binom{n}{k}A(k)-n2^{n-1}-2\binom{n}{3}+2n^2-2n-1.$$

Observe that $b(n,d)=p(n,d)=0$ if $d>\lfloor (n-1)/2\rfloor$. Thus the largest value for d such that Conjecture 1.2 is non-trivial is $d=\lfloor (n-1)/2\rfloor$, and in this case the conjecture does indeed hold.

Theorem 1.5.

$$b(2n+1,n)=p(2n+1,n)=EC(n).$$

Moreover, there exists an explicit bijection between $B(2n+1,n)$ and $P(2n+1,n)$.

Theorem 1.6.

$$b(2n,n-1)=p(2n,n-1)=\frac{1}{2}\sum_{k\ odd}\binom{2n}{k}EC\left(\frac{k-1}{2}\right)EC\left(\frac{2n-k-1}{2}\right).$$

Lastly, we provide a formula for $p(2n+1,n-1)$ which we predict also holds for $b(2n+1,n-1)$.

Proposition 1.7. *For $n\geq 2$, $p(2n+1,n-1)=$*

$$2E(2n,n-2)+\frac{1}{6}\sum_{\substack{1\leq k\leq 2n-1,\ k\ odd\\ 1\leq \ell\leq 2n+1-k,\ \ell\ odd}}\binom{2n+1}{k}\binom{2n+1-k}{\ell}EC\left(\frac{k-1}{2}\right)EC\left(\frac{\ell-1}{2}\right)EC\left(\frac{2n-k-\ell}{2}\right)$$

Notation. We collect some notation that will be used in various places throughout the text. If $\bar{c}$ is a cycle, we let $|\bar{c}|$ denote its length. We will say that $\bar{c}$ is mostly increasing if $M(\bar{c})=D_{\mathrm{cyc}}(\bar{c})$, or equivalently if $A_{\mathrm{cyc}}(\bar{c})\geq D_{\mathrm{cyc}}(\bar{c})$. We say that $\bar{c}$ is mostly decreasing if $M(\bar{c})=A_{\mathrm{cyc}}(\bar{c})$. We note that if $|\bar{c}|$ is odd, then $\bar{c}$ is either mostly increasing or mostly decreasing, but not both. We let $C(n,d)$ denote the set of n-cycles of S_n which have $M(\bar{c})=d$, and we let $c(n,d)=|C(n,d)|$.

Outline. Theorem 1.3 is proven in Section 2, Theorem 1.4 is proven in Section 3, and the remaining results are proven in Section 4. In the appendix we've included tables of values for some of the relevant statistics that we consider.

2. Proof of Theorem 1.3

In order to prove Theorem 1.3, we will need a simple combinatorial lemma.

Lemma 2.1. *For $n \geq 1$,*

$$\sum_{k\geq 3,\ k \text{ odd}} \binom{n}{k} = \frac{1}{2}A(n).$$

Proof. We claim that $\sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$. Indeed, consider the involution $\phi : 2^{[n]} \to 2^{[n]}$ defined by

$$\phi(S) = \begin{cases} S \cup \{1\} & 1 \notin S, \\ S \setminus \{1\} & 1 \in S. \end{cases}.$$

Note that ϕ bijectively maps the subsets of $[n]$ of odd cardinality to those of even cardinality. Since $\sum_{k \text{ odd}} \binom{n}{k}$ counts the number of subsets of $[n]$ of odd cardinality, we conclude that

$$\sum_{k \text{ odd}} \binom{n}{k} = \frac{1}{2}2^n = 2^{n-1}.$$

We thus have

$$\sum_{k\geq 3,\ k \text{ odd}} \binom{n}{k} = 2^{n-1} - n = \frac{1}{2}A(n).$$

□

Proof of Theorem 1.3. We first prove that $p(n, 1) = A(n)$. From the definitions we have $\pi \in P(n, 1)$ if and only if π contains exactly one non-trivial cycle which can be written as $(c_1, c_2, \ldots, c_k)$ with $k \geq 3$ odd and either $c_i < c_{i+1}$ for all i or $c_i > c_{i+1}$ for all i. Thus π is uniquely determined by first choosing the set of elements to go into its non-trivial cycle, and then choosing whether this cycle is mostly increasing or mostly decreasing. We conclude that

$$p(n, 1) = 2 \sum_{k\geq 3,\ \text{odd}} \binom{n}{k} = A(n)$$

by Lemma 2.1.

We now construct a bijection ϕ between $B(n, 1)$ and $P(n, 1)$. Any $\pi \in B(n, 1)$ can be written as $\pi = xdy'$, where d is the unique descent of π and x, y' are words containing only ascents. Note that x does not contain any letter $d' > d$, as otherwise we would have $\pi_{i-1} > d$, which would imply that π has at least two descents. Thus we can write $y' = yz$ where $z = (d + 1) \cdots n$ and y contains only ascents. For example, if $\sigma = 125783469$ we have $x = 1257,\ d = 8,\ y = 346,\ z = 9$. Note that x is always non-empty (otherwise π wouldn't be ballot) and y is always non-empty (otherwise d wouldn't be a descent), and this latter statement is equivalent to saying $x \neq 12 \cdots (d - 1)$.

We first define our map ϕ for the permutations π which have 1 appearing in x, such as the permutation σ given above. We will call a word $a = a_1 \cdots a_r$ a consecutive run if $a_{i+1} = a_i + 1$ for all $1 \leq i < r$. We rewrite xd as $x_1 \cdots x_{k+1}$, where each x_i is a maximal consecutive run. For example, if $\sigma = 125783469$ we have $xd = 12578 = x_1x_2x_3$ with $x_1 = 12,\ x_2 = 5,\ x_3 = 78$. Since we assumed that xd contains 1 and is not equal to $12 \cdots d$, we have that xd is not itself a consecutive run, and thus we always have $k \geq 1$.

We now rewrite y as $y_1 \cdots y_k$, where y_i denotes the consecutive run consisting of all the elements that are larger than every element of x_i and smaller than every element of x_{i+1}. For example, if $\sigma = 125783469$ we have $y = 346 = y_1y_2$ with $y_1 = 34,\ y_2 = 6$. Note that each y_i is non-empty, as otherwise x_ix_{i+1} would be a consecutive run, contradicting the maximality of x_i and x_{i+1}.

Let x'_i and y'_i be the largest values of x_i and y_i. Note that $x'_1 < y'_1 < \cdots < x'_k < y'_k < x'_{k+1}$. We define, for all π with 1 in x, $\phi(\pi) = (x'_1, y'_1, \ldots, x'_k, y'_k, x'_{k+1})$. We note that this is an element of $P(n, 1)$ since $\phi(\pi)$ consists of a single non-trivial cycle on $2k+1$ elements which has exactly one cyclic descent. For example, if $\sigma = 125783469$ we have $\phi(\sigma) = (2, 4, 5, 6, 8)$.

It remains to define ϕ for the case that π has 1 in y. If $\pi = xdyz$ as in the notation above, let $\pi' = ydxz$, noting that $\pi' \in B(n,1)$ since x and y are non-empty. If π has 1 in y, we define $\phi(\pi) = \widetilde{\phi(\pi')}$, where $\widetilde{\tau}$ denotes τ with all of its cycles reversed. For example, if $\sigma = 125783469$ we have $\sigma' = 346812579$ and $\phi(\sigma') = \widetilde{\phi(\sigma)} = \widetilde{(2,4,5,6,8)} = (8,6,5,4,2)$. In this case we again have $\phi(\pi) \in P(n,1)$, so ϕ is indeed a map from $B(n,1)$ to $P(n,1)$.

We claim that ϕ is invertible. Namely, let $\pi \in P(n,1)$ be such that its non-trivial cycle $\bar{c}$ is mostly increasing, say $(\bar{c}) = (x'_1, y'_1, \ldots, x'_k, y'_k, x'_{k+1})$ with these values increasing. Let y_i be the consecutive run starting at $x'_i + 1$ and ending with y'_i, let x_i be the consecutive run starting at $y'_{i-1} + 1$ and ending at x'_i (where we let $y'_0 = 0$), and let z consist of the consecutive run from $d+1$ to n. We define $\psi(\pi) = x_1 x_2 \cdots x_{k+1} y_1 y_2 \cdots y_k z$, which is the unique preimage of π under ϕ. For example, if $\sigma = (2,4,5,6,8) \in P(9,1)$ then $\psi(\sigma) = 125783469$. If the cycle of π is mostly decreasing we define $\psi(\pi) = \psi(\widetilde{\pi})'$, with the operation $'$ defined as before, and again one can verify that this sends π to its unique preimage under ϕ. We conclude that ϕ and ψ are inverses of each other, and hence that ϕ is a bijection. ◻

3. Proof of Theorem 1.4

Let $B(n,d,k)$ denote the subset of $B(n,d)$ which has $\pi_n = k$, and let $b(n,d,k) = |B(n,d,k)|$. We prove several formulas related to $b(n,d,k)$, and we will use these results to obtain a formula for $b(n,2)$. In the appendix we've included some tables of values for $b(n,d,k)$, giving some concrete examples of these formulas. We note that $B(n,d,k) = \emptyset$ whenever $d < 0$.

Lemma 3.1. *For $n \geq 2$,*

$$b(n,d,k) = \sum_{k' \geq k} b(n-1,d-1,k') + \sum_{k' < k} b(n-1,d,k').$$

Proof. Given $\pi \in B(n,d,k)$, let $\phi(\pi) \in S_{n-1}$ be defined by

$$\phi(\pi)_i = \begin{cases} \pi_i & \pi_i < k, \\ \pi_i - 1 & \pi_i > k. \end{cases}$$

Assume $\pi_{n-1} = j$. If $j > k$ then $\phi(\pi) \in B(n-1,d-1,j-1)$, and if $j < k$ then $\phi(\pi) \in B(n-1,d,j)$. Every element of $\bigcup_{k' \geq k} B(n-1,d-1,k') \cup \bigcup_{k' < k} B(n-1,d,k')$ is the image of a unique element of $B(n,d,k)$ under ϕ, so ϕ is a bijection between these two sets and we conclude the result. ◻

Lemma 3.2. *For $n \geq 2$,*

$$b(n,d,1) = b(n,d-1,n).$$

Proof. By Lemma 3.1 applied twice,

$$b(n,d,1) = \sum_{k=1}^{n-1} b(n-1,d-1,k) = b(n,d-1,n).$$

◻

Lemma 3.3. *For $n, k \geq 2$,*

$$b(n,d,k) = b(n,d,k-1) + b(n-1,d,k-1) - b(n-1,d-1,k-1).$$

Proof. This follows by considering $b(n,d,k) - b(n,d,k-1)$ and then applying Lemma 3.1 to $b(n,d,k)$ and $b(n,d,k-1)$. ◻

Proposition 3.4. *For $n \geq 2$,*

$$b(n,1,k) = \begin{cases} 2^{k-1} & k \leq n-2, \\ 2^{n-2} - 1 & k = n-1, \\ A(n-1) & k = n. \end{cases}$$

Proof. We will prove these formulas by double induction. Note that all of these formulas hold for $n = 2$ (see the appendix). Observe that $b(n, 0, k) = 0$ if $k < n$ and $b(n, 0, n) = 1$. Thus $b(n, 1, 1) = 1$ for all $n \geq 2$ by Lemma 3.2, which agrees with our proposed formula for $k = 1$. Inductively assume that we've verified the formula for $b(n, 1, k')$ for all $n \geq 2$ and $1 \leq k' < k$, and then that we've inductively verified the formula for $b(n', 1, k)$ for all $2 \leq n' < n$. By Lemma 3.3, $b(n, 1, k)$ is equal to

$$b(n, 1, k-1) + b(n-1, 1, k-1) - b(n-1, 0, k-1). \tag{1}$$

If $k \leq n-2$, inductively we know that (1) is equal to $2^{k-2} + 2^{k-2} - 0 = 2^{k-1}$. If $k = n-1$, then (1) is equal to $2^{n-3} + (2^{n-3} - 1) - 0 = 2^{n-2} - 1$. If $k = n$, then (1) is equal to $(2^{n-2} - 1) + A(n-2) - 1 = 2^{n-1} - 2(n-1) = A(n-1)$. We conclude by induction that these formulas hold. □

We note that the formulas of Proposition 3.4 can also be proven combinatorially. If $\pi \in B(n, 1, n)$, then $\pi_1 \cdots \pi_{n-1} \in B(n-1, 1)$, and the implicit map between these two sets is invertible. Thus $b(n, 1, n) = b(n-1, 1) = A(n-1)$ by Theorem 1.3. For $k \neq n$, $\pi \in B(n, 1, k)$ is uniquely determined by the set of values of π that appear in between n (which is necessarily the descent of π) and k. If $k \leq n-2$ this set can be any subset of $[k-1]$, so $b(n, 1, k) = 2^{k-1}$. If $k = n-1$ then it can be any subset except all of $[n-2]$ (as then the word wouldn't be ballot), so $b(n, 1, n-1) = 2^{n-2} - 1$.

Let $a(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} A(n-1-i) - (k-1)2^{k-2}$.

Lemma 3.5.

$$a(n, k-1) + a(n-1, k-1) - 2^{k-2} = a(n, k).$$

Proof. The left-hand side is equal to

$$\sum_{i=0}^{k-2} \binom{k-2}{i} A(n-1-i) - (k-2)2^{k-3} + \sum_{i=0}^{k-2} \binom{k-2}{i} A(n-2-i) - (k-2)2^{k-3} - 2^{k-2} =$$

$$\sum_{i=0}^{k-1} \left(\binom{k-2}{i} + \binom{k-2}{i-1} \right) A(n-1-i) - (k-1)2^{k-2} =$$

$$\sum_{i=0}^{k-1} \binom{k-1}{i} A(n-1-i) - (k-1)2^{k-2} = a(n, k),$$

with the second to last equality coming from Pascal's rule. □

Proposition 3.6. *For $n \geq 5$,*

$$b(n, 2, k) = \begin{cases} a(n, k) & k \leq n-4, \\ a(n, n-3) - 2 & k = n-3, \\ a(n, n-2) - 2n + 7 & k = n-2, \\ a(n, n-1) - 2\binom{n+1}{2} + 8n - 10 & k = n-1, \\ a(n, n) - 2\binom{n+2}{3} + 10\binom{n+1}{2} - 14n + 5 & k = n. \end{cases}$$

Proof. One can verify that these formulas all hold for $n = 5$ (see the appendix). By Lemma 3.2 and Proposition 3.4, $b(n, 2, 1) = b(n, 1, n) = A(n-1) = a(n, 1)$ for all $n \geq 5$, so the formula is correct for $k = 1$. Inductively assume that we've verified the formula for $b(n, 1, k')$ for all $n \geq 5$ and $1 \leq k' < k$, and then that we've inductively verified the formula for $b(n', 2, k)$ for all $5 \leq n' < n$. By Lemma 3.3 we have $b(n, 2, k)$ equal to

$$b(n, 2, k-1) + b(n-1, 2, k-1) - b(n-1, 1, k-1). \tag{2}$$

If $k \leq n-4$, then by Proposition 3.4 and Lemma 3.5 we know inductively that (2) is equal to

$$a(n, k-1) + a(n-1, k-1) - 2^{k-2} = a(n, k).$$

If $k = n-3$ then (2) is equal to

$$a(n, n-4) + a(n-1, n-4) - 2 - 2^{n-5} = a(n, n-3) - 2.$$

If $k = n-2$ then (2) is equal to

$$a(n,n-3) - 2 + a(n-1,n-3) - 2(n-1) + 7 - 2^{n-4} =$$

$$a(n,n-2) - 2n + 7.$$

If $k = n-1$ then, recalling that $b(n,1,n-1) = 2^{n-1} - 1$, (2) is equal to

$$a(n,n-2) - 2n + 7 + a(n-1,n-2) - 2\binom{n}{2} + 8(n-1) - 10 - 2^{n-3} + 1 =$$

$$a(n,n-1) - 2\binom{n+1}{2} + 8n - 10,$$

where we used the fact that $n + \binom{n}{2} = \binom{n+1}{2}$, a special case of Pascal's rule. Finally, if $k = n$ then, recalling that $b(n,1,n) = A(n-1) = 2^{n-1} - 2(n-1)$, (2) is equal to

$$a(n,n-1) - 2\binom{n+1}{2} + 8n - 10 + a(n-1,n-1) - 2\binom{n+1}{3} + 10\binom{n}{2} - 14(n-1) + 5 - 2^{n-2} - 2(n-2) =$$

$$a(n,n) - 2\binom{n+2}{3} + 10\binom{n+1}{2} - 14n + 5,$$

where we've used the fact that $\binom{n+1}{2} + \binom{n+1}{3} = \binom{n+2}{3}$. □

Corollary 3.7. *For $n \geq 2$,*

$$b(n,2) = \sum_{k=0}^{n} \binom{n}{k} A(k) - n2^{n-1} - 2\binom{n}{3} + 2n^2 - 2n - 1.$$

Proof. One can verify that this formula is equal to $0 = b(n,2)$ for $n = 2,3$. For $n \geq 4$, note that by Lemma 3.1 and Proposition 3.6 we have

$$b(n,2) = \sum_{k=1}^{n} b(n,2,k) = b(n+1,2,n+1) =$$

$$\sum_{i=0}^{n} \binom{n}{i} A(i) - n2^{n-1} - 2\binom{n+3}{3} + 10\binom{n+2}{2} - 14(n+1) + 5.$$

One can verify that $-2\binom{n+3}{3} + 10\binom{n+2}{2} - 14(n+1) + 5 = -2\binom{n}{3} + 2n^2 - 2n - 1$, proving the result. □

We now wish to find formulas related to $p(n,d)$. Recall that $c(n,d)$ denotes the number of n-cycles π of S_n with $M(\pi) = d$.

Lemma 3.8. *For n odd, $c(n,d) = 0$ if $d > (n-1)/2$, and otherwise $c(n,d) = 2E(n-1,d-1)$.*

Proof. $A_{\text{cyc}}(\bar{c}) + D_{\text{cyc}}(\bar{c}) = n$, and since n is odd, one of these values is at most $(n-1)/2$. We conclude that $M(\bar{c}) \leq (n-1)/2$ for all $\bar{c}$, and hence $c(n,d) = 0$ if $d > (n-1)/2$.

Let $S(n,d)$ denote the permutations of S_n with exactly d descents, and let $C^+(n,d)$ denote the cycles of $C(n,d)$ which are mostly increasing. Let n be odd and $d \leq (n-1)/2$. If $\pi \in S(n-1,d-1)$, define $\phi(\pi) = (\pi_1,\ldots,\pi_{n-1},n)$. $\phi(\pi)$ has exactly one more cyclic ascent than π has ascents, and similarly with regards to descents. We conclude that $M(\phi(\pi)) = D_{\text{cyc}}(\phi(\pi)) = d$ since $d \leq (n-1)/2$, so $\phi(\pi) \in C^+(n,d)$. It's not too difficult to see that ϕ is a bijection onto $C^+(n,d)$ and that $|C^+(n,d)| = \frac{1}{2}c(n,d)$. Since $|S(n-1,d-1)| = E(n-1,d-1)$, we conclude the result. □

Lemma 3.9.

$$E(n,0) = 1,$$

$$E(n,1) = \frac{1}{2}A(n+1).$$

Proof. The first equation is straightforward. For the second, let π be a permutation with exactly one descent. Either $\pi \in B(n,1)$ if the descent of π is not in the first position, or π is of the form $d12\cdots n$ with $d \neq 1$ and $12\cdots n$ consists of all the elements of $[n] \setminus \{d\}$ in increasing order. We conclude by Theorem 1.3 that

$$E(n,1) = b(n,1) + n - 1 = 2^n - 2n + n - 1 = 2^n - (n+1) = \frac{1}{2}A(n+1).$$

□

Proposition 3.10.

$$p(n,2) = \sum_{k\geq 5,\ k\ odd} \binom{n}{k} A(k) + \sum_{k,\ell\geq 3,\ k,\ell\ odd} 2\binom{n}{k}\binom{n-k}{\ell}.$$

Proof. For $\pi \in P(n)$, $M(\pi) = 2$ implies that either π contains exactly one non-trivial cycle $\bar{c}$ with $M(\bar{c}) = 2$, or that π contains exactly two non-trivial cycles $\bar{c}_1$ and $\bar{c}_2$ with $M(\bar{c}_1) = M(\bar{c}_2) = 1$. Let P_1 denote the permutations of the first type and P_2 those of the second.

Each $\pi \in P_1$ is uniquely determined by first choosing the k elements to be in its non-trivial odd cycle, and then arranging the elements of this cycle in one of $c(k,2)$ ways. By Lemma 3.8, $c(k,2)$ will be non-zero when $k \geq 5$, in which case it will be equal to $2E(k-1,1) = A(k)$ by Lemma 3.9. We conclude that

$$|P_1| = \sum_{k\geq 5,\ k \text{ odd}} \binom{n}{k} A(k).$$

Similarly, one can construct each $\pi \in P_2$ by choosing k elements to go into its first cycle, ℓ elements to go into its second cycle, and then arranging the elements of each cycle in one of $c(k,0) = c(\ell,0) = 2$ ways for $k,\ell \geq 3$. However, this construction implicitly puts an ordering on the cycles of π, and hence double counts each element of P_2. After taking this double counting into account, we conclude that

$$|P_2| = \sum_{k,\ell\geq 3,\ k,\ell \text{ odd}} 2\binom{n}{k}\binom{n-k}{\ell},$$

and the result follows. □

It remains to show that the formulas of Corollary 3.7 and Proposition 3.10 are equal to each other. In order to do this, we first prove a few more combinatorial lemmas.

Lemma 3.11. *For $n \geq 2$,*

$$\sum_{k\ odd} \binom{n}{k} A(k) = (-1)^{n+1} + \sum_{k\ even} \binom{n}{k} A(k)$$

Proof. We have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k = (1-2)^n = (-1)^n$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k = \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}(-x)^k\Big|_{x=1} =$$
$$\frac{d}{dx}(1-x)^n\Big|_{x=1} = 0,$$

provided $n \geq 2$. Since $A(k) = 2^k - 2k$, we have

$$\sum_{k \text{ even}} \binom{n}{k} A(k) - \sum_{k \text{ odd}} \binom{n}{k} A(k) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k A(k) =$$
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^k + 2\sum_{k=0}^{n} \binom{n}{k} (-1)^k k = (-1)^n,$$

giving the desired result.

∎

Lemma 3.12.

$$\sum_{k,\ell\geq 3,\ k,\ell\ odd} 2\binom{n}{k}\binom{n-k}{\ell} = \sum_{k\ even}\binom{n}{k}A(k) - n2^{n-1} + 2n^2 - 2n - 1.$$

Proof. Note that the $k = n$ terms of $\sum_{k,\ell\geq 3,\ k,\ell \text{ odd}} 2\binom{n}{k}\binom{n-k}{\ell}$ contribute nothing to the sum. Thus ignoring these terms and using Lemma 2.1 we get

$$\sum_{k,\ell\geq 3,\ k,\ell \text{ odd}} 2\binom{n}{k}\binom{n-k}{\ell} = \sum_{3\leq k\leq n-1,\ k \text{ odd}}\binom{n}{k}A(n-k) = \sum_{1\leq k\leq n-3,\ n-k \text{ odd}}\binom{n}{k}A(k).$$

If n is odd this sum is equal to

$$\sum_{1\leq k\leq n-3,\ k \text{ even}}\binom{n}{k}A(k) = \sum_{k \text{ even}}\binom{n}{k}A(k) - nA(n-1) - A(0).$$

If n is even this sum is equal to

$$\sum_{1\leq k\leq n-3,\ k \text{ odd}}\binom{n}{k}A(k) = \sum_{k \text{ odd}}\binom{n}{k}A(k) - nA(n-1) = \sum_{k \text{ even}}\binom{n}{k}A(k) - nA(n-1) - 1,$$

with the last equality coming from Lemma 3.11. We conclude the result by noting that $A(0) = 1$ and $A(n-1) = 2^{n-1} - 2n + 2$. ∎

Proof of Theorem 1.4. We already know that $b(n, 2)$ satisfies this formula by Corollary 3.7. Note that

$$\sum_{k\geq 5,\ k \text{ odd}}\binom{n}{k}A(k) = \sum_{k \text{ odd}}\binom{n}{k}A(k) - \binom{n}{3}A(3) - \binom{n}{1}A(1) = \sum_{k \text{ odd}}\binom{n}{k}A(k) - 2\binom{n}{3}.$$

Combining this observation with Proposition 3.10 and Lemma 3.12 shows that $p(n, 2)$ also satisfies this formula.

∎

In principle one should be able to generalize the methods used in this section to compute formulas for $b(n, d)$ and $p(n, d)$ for any finite d, though the computations would be somewhat tedious.

4. Formulas for large d

Recall that $|\bar{c}|$ denotes the length of the cycle $\bar{c}$.

Lemma 4.1. *If $\pi = \bar{c}_1 \cdots \bar{c}_k \in P(n)$, then*

$$M(\pi) \leq \frac{n-k}{2},$$

with equality if and only if $M(\bar{c}_i) = \frac{|\bar{c}_i|-1}{2}$ for all i.

Proof. As noted in the proof of Lemma 3.8, if $\bar{c}$ is a cycle of odd length then $M(\bar{c}) \leq \frac{|\bar{c}|-1}{2}$. The result follows by applying this inequality to each $\bar{c}_i$ and noting that $\sum \bar{c}_i = n$. ∎

Proof of Theorem 1.5. By Lemma 4.1, we have $\pi \in P(2n+1, n)$ if and only if π is a $(2n+1)$-cycle $\bar{c}$ with $M(\bar{c}) = n$. Thus Lemma 3.8 implies that

$$p(2n+1, n) = c(2n+1, n) = 2E(2n, n-1) = EC(n).$$

It remains to establish a bijection from $B(2n+1, n)$ to $P(2n+1, n)$.

If $\pi = \pi_1 \cdots \pi_{2n+1} \in B(2n+1, n)$, let $\phi(\pi) = (\pi_1, \pi_2, \ldots, \pi_{2n+1})$. Since π contained n descents, $\pi_1\pi_2 \cdots \pi_{2n+1}\pi_1$ contains exactly n or $n+1$ descents, and hence $M(\phi(\pi)) = n$ and the codomain of this map is $P(2n+1, n)$. The fact that this map is invertible is implicitly proven in the second proof of Theorem 1.1 of [3]. Explicitly (using the notation of [3]), it is shown that if $w = (w_1, \ldots, w_{2n+1})$ has n or $n+1$ cyclic descents (i.e. if $w \in P(2n+1, n)$), then there exists $n+1$ choices of i such that $w_i w_{i+1} \cdots w_{2n+1} w_1 \cdots w_{i-1}$ has n descents, and exactly one of these choices for i makes this word have excedence 0 (i.e. makes the word be ballot). We thus define $\psi(w) = w_i w_{i+1} \cdots w_{2n+1} w_1 \cdots w_{i-1}$ with i the unique value such that this word has n descents and is ballot. ψ is the inverse of ϕ, and hence these maps are bijections. □

With Lemma 4.1 we can compute formulas for $p(n, d)$ when d is large.

Proposition 4.2.

$$p(2n, n-1) = \frac{1}{2} \sum_{k\ odd} \binom{2n}{k} EC\left(\frac{k-1}{2}\right) EC\left(\frac{2n-k-1}{2}\right)$$

Proof. Since $2n$ is even, any $\pi \in P(2n)$ is the product of at least two odd cycles. By Lemma 4.1 we have that $\pi \in P(2n, n-1)$ if and only if $\pi = \bar{c}\bar{d}$ with $\bar{c}, \bar{d}$ odd cycles such that $M(\bar{c}) = (|\bar{c}| - 1)/2$ and $M(\bar{d}) = (|\bar{d}| - 1)/2$.

Consider the following procedure for generating an element $\pi \in P(2n, n-1)$. Choose k elements to be in the first cycle of π (which also determines the elements of the second cycle), and then arrange the elements of these two cycles in $c(k, (k-1)/2)$ and $c(2n-k, (2n-k-1)/2)$ ways, respectively. By Lemma 3.8 and the fact that we defined $EC(\ell) = 2E(2\ell, \ell - 1)$, we conclude that $c(k, (k-1)/2) = EC((k-1)/2)$ and $c(2n-k, (2n-k-1)/2) = EC((2n-k-1)/2)$. Putting these results together (and noting that this procedure double counts the elements of $P(2n, n-1)$) gives the desired formula. □

Proof of Proposition 1.7. By Lemma 3.8, there are exactly $c(2n+1, n-1) = 2E(2n, n-2)$ elements $\pi \in P(2n+1, n-1)$ that consist of a single cycle, so it remains to count the elements of $P(2n+1, n-1)$ that are not of this form.

If π is not a single cycle then, since $2n+1$ is odd, π must be the product of at least 3 odd cycles. By Lemma 4.1, we must have $\pi = \bar{c}_1\bar{c}_2\bar{c}_3$ with $M(\bar{c}_i) = \frac{|\bar{c}_i| - 1}{2}$ for $i = 1, 2, 3$. We can construct such a π by choosing k elements (with $k < 2n+1$ odd) to go into the first cycle of π, ℓ of the remaining $2n+1-k$ elements to go into the second cycle (which determines the elements of the third cycle), and then arranging the elements of each cycle. As argued in the proof of Proposition 4.2, there will be $EC((k-1)/2)$ ways to arrange the first cycle, $EC((\ell-1)/2)$ ways to arrange the second, and $EC((2n-k-\ell)/2)$ ways to arrange the third cycle. This argument overcounts the elements of $P(2n+1, n-1)$ by a factor of 6 since we've implicitly placed an order on the cycles. Putting all these results together gives the desired formula. □

In principle one should be able to generalize these methods to compute $p(n, \lfloor (n-1)/2 \rfloor - d)$ for any finite d, though the computations would be somewhat tedious.

We now wish to find a formula for $b(2n, n-1)$, and to do so we introduce an additional statistic. Given any $\pi \in S_n$ and $0 \le k \le n-1$, let $T_k(\pi) = \sum_{i=1}^{k} q_i^\pi$, with the convention that $T_0(\pi) = 0$. Define $T(\pi) = \min_{0 \le k \le n-1}\{T_k(\pi)\}$. We let $S(n, d, t)$ denote the set of permutations of S_n with exactly d descents and with $T(\pi) = t$, and we let $s(n, d, t) = |S(n, d, t)|$. Note that $S(n, d, 0) = B(n, d)$. We further define $A(\pi)$ to denote the number of ascents of π, $D(\pi)$ to denote the number of descents of π, and $\hat{\pi} := \pi_n \pi_{n-1} \cdots \pi_2 \pi_1$.

Lemma 4.3. *If $\pi \in S(n, d, t)$, then $\hat{\pi} \in S(n, n-1-d, t+2d-n+1)$.*

Proof. Observe that $q_i^{\hat{\pi}} = -q_{n-i}^\pi$. In particular this implies that $D(\hat{\pi}) = n - 1 - D(\pi) = n - 1 - d$. Define $R_k(\pi) = \sum_{i=k}^{n-1} -q_i^\pi$ with the convention that $R_n(\pi) = 0$, and let $R(\pi) = \min_{1 \le k \le n}\{R_k(\pi)\}$. Observe that $T_k(\hat{\pi}) = R_{n-k}(\pi)$ for all $0 \le k \le n-1$, and hence $T(\hat{\pi}) = R(\pi)$.

Let k and ℓ be the smallest integers such that $T(\pi) = T_k(\pi)$ and $R(\pi) = R_\ell(\pi)$. We claim that $k = \ell - 1$. Indeed, assume $k > \ell - 1$. By the minimality of k, we must have

$$0 < T_{\ell-1}(\pi) - T_k(\pi) = \sum_{i=\ell}^{k} -q_i^\pi = \sum_{i=\ell}^{n-1} -q_i^\pi + \sum_{i=k+1}^{n-1} q_i^\pi = R_\ell(\pi) - R_{k+1}(\pi),$$

a contradiction to ℓ being such that R_ℓ is minimal. Similarly, if $k < \ell - 1$ we have

$$0 < R_{k+1}(\pi) - R_\ell(\pi) = T_k(\pi) - T_{\ell-1}(\pi),$$

a contradiction, so we conclude that $k = \ell - 1$.

With this we have

$$T(\pi) - R(\pi) = T_{\ell-1}(\pi) - R_\ell(\pi) = \sum_{i=0}^{n-1} q_i^\pi = A(\pi) - D(\pi).$$

Since $R(\pi) = T(\hat{\pi})$, we conclude that

$$T(\hat{\pi}) = T(\pi) + D(\pi) - A(\pi) = t + d - (n - 1 - d) = t + 2d - n + 1$$

as desired. □

Corollary 4.4. *$s(n, d, t) = s(n, n-1-d, t+2d-n+1)$. In particular, $s(2n, n-1, 0) = s(2n, n, -1)$.*

Proof. By the previous lemma, the map $\phi : S_n \to S_n$ defined by $\phi(\pi) = \hat{\pi}$ is an involution sending $S(n, d, t)$ to $S(n, n-1-d, t+2d-n+1)$ and vice versa, so ϕ is a bijection between these two sets. □

Proof of Theorem 1.6. We already know $p(2n, n-1)$ satisfies this formula by Proposition 4.2, so it remains to prove that this is the case for $b(2n, n-1)$. Let w be a word composed of k distinct elements. We define q_i^w analogous to how q_i^π was defined, and we will say that w is a Dyck word if $\sum_1^\ell q_i^w \geq 0$ for all $1 \leq \ell \leq k-1$ and if $\sum_1^{k-1} q_i^w = 0$. Observe that w being a Dyck word implies that k is odd.

We generate a permutation π as follows. Given an odd number k, choose a subset $S \subseteq [2n]$ of size k. Choose an ordering of the elements of S in such a way that the resulting word w_1 is a Dyck word, and similarly choose an ordering of $[2n] \setminus S$ to get a Dyck word w_2. The procedure then outputs $\pi = w_1 w_2$.

Let $\pi = w_1 w_2$ be a permutation generated by this procedure such that w_1 has length k. w_1 has $(k-1)/2$ descents and w_2 has $(2n-k-1)/2$ descents, so $D(\pi) = n-1$ if $q_k^\pi = 1$ and otherwise $D(\pi) = n$. Since w_1 is a Dyck word we have $T_r(\pi) \geq 0$ if $r < k$ and that $T_k(\pi) = q_k^\pi$, and since w_2 is a Dyck word we have $T_r(\pi) = q_k^\pi + \sum_{k+1}^r q_i^\pi \geq q_k^\pi$ for $r > k+1$. Thus $T(\pi) = 0$ if $q_i^k = 1$ and otherwise $T(\pi) = -1$. We conclude that this procedure always generates an element of $S(2n, n-1, 0) \cup S(2n, n, -1)$. We claim that every permutation of $S(2n, n-1, 0) \cup S(2n, n, -1)$ is generated in a unique way by this procedure.

Let $\pi \in S(2n, n-1, 0)$, noting that $\sum_1^\ell q_i^\pi \geq 0$ for all ℓ and that $\sum_1^{n-1} q_i^\pi = 1$. Let k denote the largest value such that $\sum_{i=1}^{k-1} q_i^\pi = 0$, allowing for the case $k = 1$. Then k is odd, and $w_1 := \pi_1 \cdots \pi_k$ and $w_2 := \pi_{k+1} \cdots \pi_n$ are both Dyck words (w_1 is obvious, w_2 is due to the maximality of k), so $\pi = w_1 w_2$ arises from this procedure. Assume that we can also write $\pi = w_1' w_2'$ with w_1', w_2' Dyck words with w_1' having length ℓ. Note that $q_\ell^\pi = 1$, as otherwise we would have $T(\pi) < 0$. If $\ell < k$ then

$$\sum_1^{k-1} q_i^\pi = \sum_1^{\ell-1} q_i^\pi + q_\ell^\pi + \sum_{\ell+1}^{k-1} q_i^\pi = 1 + \sum_{\ell+1}^{k} q_i^\pi \geq 1,$$

a contradiction to the fact that $\sum_1^{k-1} q_i^\pi = 0$. A symmetric argument shows that we must have $k = \ell$, and hence the decomposition $\pi = w_1 w_2$ is unique.

Similarly given $\pi \in S(2n, n, -1)$, let k denote the smallest value such that $\sum_1^k q_i^\pi = -1$. Again we have that k is odd and that $\pi_1 \cdots \pi_k$ and $\pi_{k+1} \cdots \pi_n$ are Dyck word, so π is generated by this procedure, and uniqueness follows a similar argument as before. Thus each element of $S(2n, n-1, 0) \cup S(2n, n, -1)$ is generated uniquely by this procedure.

It's not too difficult to see that the number of Dyck words using the letters $\{a_1, \ldots, a_k\}$ is precisely $b(k, (k-1)/2)$, which is equal to $EC((k-1)/2)$ by Theorem 1.5. Thus the total number of ways to carry out this procedure is

$$\sum_{k \text{ odd}} \binom{2n}{k} EC\left(\frac{k-1}{2}\right) EC\left(\frac{2n-k-1}{2}\right).$$

We conclude that

$$\sum_{k \text{ odd}} \binom{2n}{k} EC\left(\frac{k-1}{2}\right) EC\left(\frac{2n-k-1}{2}\right) = |S(2n, n-1, 0) \cup S(2n, n, -1)| =$$

$$s(2n, n-1, 0) + s(2n, n, -1) = 2s(2n, n-1, 0) = 2b(2n, n-1),$$

with the second to last equality coming from Corollary 4.4. We conclude the result. □

5. Acknowledgments

The author would like to thank Fan Chung for suggesting this research topic, as well as for her many helpful comments and suggestions.

References

[1] Désiré André. Sur les permutations alternées. *Journal de mathématiques pures et appliquées*, 7:167–184, 1881.

[2] Olivier Bernardi, Bertrand Duplantier, and Philippe Nadeau. A bijection between well-labelled positive paths and matchings. *Séminaire Lotharingien de Combinatoire*, 63:B63e, 2010.

[3] Hoda Bidkhori and Seth Sullivant. Eulerian-catalan numbers. *The Electronic Journal of Combinatorics*, 18(1):187, 2011.

[4] Francis CS Brown, Thomas MA Fink, and Karen Willbrand. On arithmetic and asymptotic properties of up–down numbers. *Discrete mathematics*, 307(14):1722–1736, 2007.

[5] Ronald L Graham, Donald E Knuth, Oren Patashnik, and Stanley Liu. Concrete mathematics: a foundation for computer science. *Computers in Physics*, 3(5):106–107, 1989.

[6] Ivan Niven. A combinatorial problem of finite sequences. *Nieuw Arch. Wisk*, 16(3):116–123, 1968.

[7] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A000246, 2018.

[8] Vladimir Shevelev. Number of permutations with prescribed up-down structure as a function of two variables. *Integers*, 12(4):529–569, 2012.

[9] Vladimir Shevelev and Juergen Spilker. Up-down coefficients for permutations. *Elemente der Mathematik*, 68(3):115–127, 2013.

Appendix: Computational Data

Below we've included some computational data for some of the statistics that we've considered. The first table consists of values for $b(n,d)$. We note that all of the values listed agree with the values for $p(n,d)$.

b(n,d)	d=0	d=1	d=2	d=3	d=4	d=5
n=1	1	0	0	0	0	0
n=2	1	0	0	0	0	0
n=3	1	2	0	0	0	0
n=4	1	8	0	0	0	0
n=5	1	22	22	0	0	0
n=6	1	52	172	0	0	0
n=7	1	114	856	604	0	0
n=8	1	240	3488	7296	0	0
n=9	1	494	12746	54746	31238	0
n=10	1	1004	43628	330068	518324	0
n=11	1	2026	143244	1756878	5300418	2620708
n=12	1	4072	457536	8641800	43235304	55717312
n=13	1	8166	1434318	40298572	309074508	728888188
n=14	1	16356	4438540	180969752	2026885824	7589067592
n=15	1	32738	13611136	790697160	12512691028	69028576454
n=16	1	65504	41473216	3385019968	73898171456	573754927712
n=17	1	131038	125797010	14270283414	422060869866	4470473831914
n=18	1	262108	380341580	59457742524	2349012559564	33181419358420
n=19	1	524250	1147318004	245507935018	12811010885886	237191391335758
n=20	1	1048536	3455325600	1006678811272	68751877461032	1645761138814040
n=21	1	2097110	10394291094	4105447763032	364232722279840	11148787030131978
n=22	1	4194260	31242645420	16672235476128	1909625025412472	74065171862108524
n=23	1	8388562	93853769320	67482738851220	9927594128105024	484210423704506108
n=24	1	16777168	281825553760	272439143364672	51256011278005824	3123806527720851840
n=25	1	33554382	846030314842	1097660274098482	263144690491841262	19930831004237505532

~

Below we've included tables for $b(n,k,d)$, along with row and column sums for each table.

b(1,k,d)	d=0	Row Sum:
k=1	1	1
Col Sum:	1	1

~

b(2,k,d)	d=0	Row Sum:
k=1	0	0
k=2	1	1
Col Sum:	1	1

~

b(3,k,d)	d=0	d=1	Row Sum:
k=1	0	1	1
k=2	0	1	1
k=3	1	0	1
Col Sum:	1	2	3

~

b(4,k,d)	d=0	d=1	Row Sum:
k=1	0	1	1
k=2	0	2	2
k=3	0	3	3
k=4	1	2	3
Col Sum:	1	8	9

~

b(5,k,d)	d=0	d=1	d=2	Row Sum:
k=1	0	1	8	9
k=2	0	2	7	9
k=3	0	4	5	9
k=4	0	7	2	9
k=5	1	8	0	9
Col Sum:	1	22	22	45

~

b(6,k,d)	d=0	d=1	d=2	Row Sum:
k=1	0	1	22	23
k=2	0	2	29	31
k=3	0	4	34	38
k=4	0	8	35	43
k=5	0	15	30	45
k=6	1	22	22	45
Col Sum:	1	52	172	225

~

b(7,k,d)	d=0	d=1	d=2	d=3	Row Sum:
k=1	0	1	52	172	225
k=2	0	2	73	150	225
k=3	0	4	100	121	225
k=4	0	8	130	87	225
k=5	0	16	157	52	225
k=6	0	31	172	22	225
k=7	1	52	172	0	225
Col Sum:	1	114	856	604	1575