# COUNTING UNLABELED INTERVAL GRAPHS 

HUSEYIN ACAN


#### Abstract

We improve the bounds on the number of interval graphs on $n$ vertices. In particular, denoting by $I_{n}$ the quantity in question, we show that $\log I_{n} \sim n \log n$ as $n \rightarrow \infty$.


A simple undirected graph is an interval graph if it is isomorphic to the intersection graph of a family of intervals on the real line. Several characterizations of interval graphs are known; see [4, Chapter 3] for some of them. Linear time algorithms for recognizing interval graphs are given in [1] and [2].

In this paper, we are interested in counting interval graphs. Let $I_{n}$ denote the number of unlabeled interval graphs on $n$ vertices. (This is the sequence with id A005975 in the On-Line Encyclopedia of Integer Sequences [6].) Initial values of this sequence are given by Hanlon [3]. Answering a question posed by Hanlon [3], Yang and Pippenger [5] proved that the generating function

$$
I(x)=\sum_{n \geq 1} I_{n} x^{n}
$$

diverges for any $x \neq 0$ and they established the bounds

$$
\begin{equation*}
\frac{n \log n}{3}+O(n) \leq \log I_{n} \leq n \log n+O(n) \tag{1}
\end{equation*}
$$

The upper bound in (1) follows from $I_{n} \leq(2 n-1)!!=\prod_{j=1}^{n}(2 j-1)$, where the right hand side is the number of matchings on $2 n$ points. For the lower bound, the authors showed

$$
I_{3 k} \geq k!/ 3^{3 k}
$$

by finding an injection from $S_{k}$, the set of permutations of length $k$, to three-colored interval graphs of size $3 k$.

Using an idea similar to the one in [5], we improve the lower bound in (1) so that the main terms of the lower and upper bounds match. In other words, we find the asymptotic value of $\log I_{n}$.

For a set $S$, we denote by $\binom{S}{k}$ the set of $k$-subsets of $S$.
Theorem 1. As $n \rightarrow \infty$, we have

$$
\begin{equation*}
\log I_{n} \geq n \log n-2 n \log \log n-O(n) \tag{2}
\end{equation*}
$$

Proof. We consider certain interval graphs on $n$ vertices with colored vertices. Let $k$ be a positive integer smaller than $n / 2$ and $\varepsilon$ a positive constant smaller than $1 / 2$. For $1 \leq j \leq k$, let $B_{j}$ and $R_{j}$ denote the intervals $[-j-\varepsilon,-j+\varepsilon]$ and $[j-\varepsilon, j+\varepsilon]$, respectively. These $2 k$ pairwise-disjoint intervals will make up $2 k$ vertices in the

[^0]graphs we consider. Now let $\mathcal{W}$ denote the set of $k^{2}$ closed intervals with one endpoint in $\{-k, \ldots,-1\}$ and the other in $\{1, \ldots, k\}$. We color $B_{1}, \ldots, B_{k}$ with blue, $R_{1}, \ldots, R_{k}$ with red, and the $k^{2}$ intervals in $\mathcal{W}$ with white.

Together with $\mathcal{S}:=\left\{B_{1}, \ldots, B_{k}, R_{1}, \ldots, R_{k}\right\}$, each $\left\{J_{1}, \ldots, J_{n-2 k}\right\} \in\binom{\mathcal{W}}{n-2 k}$ gives an $n$-vertex, three-colored interval graph. For a given $\mathcal{J}=\left\{J_{1}, \ldots, J_{n-2 k}\right\}$, let $G_{\mathcal{J}}$ denote the colored interval graph whose vertices correspond to $n$ intervals in $\mathcal{S} \cup \mathcal{J}$, and let $\mathcal{G}$ denote the set of all $G_{\mathcal{J}}$.

Now let $G \in \mathcal{G}$. For a white vertex $w \in G$, the pair $\left(d_{B}(w), d_{R}(w)\right)$, which represents the numbers of blue and red neighbors of $w$, uniquely determine the interval corresponding to $w$; this is the interval $\left[-d_{B}(w), d_{R}(w)\right]$. In other words, $\mathcal{J}$ can be recovered from $G_{\mathcal{J}}$ uniquely. Thus

$$
|\mathcal{G}|=\binom{k^{2}}{n-2 k}
$$

Since there are at most $3^{n}$ ways to color the vertices of an interval graph with blue, red, and white, we have

$$
I_{n} \cdot 3^{n} \geq|\mathcal{G}|=\binom{k^{2}}{n-2 k} \geq\left(\frac{k^{2}}{n-2 k}\right)^{n-2 k} \geq\left(\frac{k^{2}}{n}\right)^{n}
$$

for any $k<n / 2$. Setting $k=\lfloor n / \log n\rfloor$ and taking the logarithms, we get

$$
\log I_{n} \geq n \log \left(k^{2} / n\right)-O(n)=n \log n-2 n \log \log n-O(n)
$$

Remark 2. Yang and Pippenger [5] posed the question whether

$$
\log I_{n}=C n \log n+O(n)
$$

for some $C$ or not. According to Theorem 1, this boils down to getting rid of the $2 n \log \log n$ term in (2). Such a result would imply that the exponential generating function

$$
J(x)=\sum_{n \geq 1} I_{n} \frac{x^{n}}{n!}
$$

has a finite radius of convergence. (As noted in [5], the bound $I_{n} \leq(2 n-1)$ !! implies that the radius of convergence of $J(x)$ is at least $1 / 2$.) Of course, a strong result would be finding $I_{n}$ asymptotically.

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Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: huseyin.acan@rutgers.edu


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