COUNTING UNLABELED INTERVAL GRAPHS

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ABSTRACT. We improve the bounds on the number of interval graphs on n vertices. In particular, denoting by I_n the quantity in question, we show that $\log I_n \sim n \log n$ as $n \to \infty$.

A simple undirected graph is an *interval graph* if it is isomorphic to the intersection graph of a family of intervals on the real line. Several characterizations of interval graphs are known; see [4, Chapter 3] for some of them. Linear time algorithms for recognizing interval graphs are given in [1] and [2].

In this paper, we are interested in counting interval graphs. Let I_n denote the number of unlabeled interval graphs on n vertices. (This is the sequence with id A005975 in the On–Line Encyclopedia of Integer Sequences [6].) Initial values of this sequence are given by Hanlon [3]. Answering a question posed by Hanlon [3], Yang and Pippenger [5] proved that the generating function

$$I(x) = \sum_{n \ge 1} I_n x^n$$

diverges for any $x \neq 0$ and they established the bounds

(1)
$$\frac{n\log n}{3} + O(n) \le \log I_n \le n\log n + O(n).$$

The upper bound in (1) follows from $I_n \leq (2n-1)!! = \prod_{j=1}^n (2j-1)$, where the right hand side is the number of matchings on 2n points. For the lower bound, the authors showed

$$I_{3k} \ge k!/3^{3k}$$

by finding an injection from S_k , the set of permutations of length k, to three-colored interval graphs of size 3k.

Using an idea similar to the one in [5], we improve the lower bound in (1) so that the main terms of the lower and upper bounds match. In other words, we find the asymptotic value of $\log I_n$.

For a set S, we denote by $\binom{S}{k}$ the set of k-subsets of S.

Theorem 1. As $n \to \infty$, we have

(2)
$$\log I_n \ge n \log n - 2n \log \log n - O(n).$$

Proof. We consider certain interval graphs on n vertices with colored vertices. Let k be a positive integer smaller than n/2 and ε a positive constant smaller than 1/2. For $1 \leq j \leq k$, let B_j and R_j denote the intervals $[-j - \varepsilon, -j + \varepsilon]$ and $[j - \varepsilon, j + \varepsilon]$, respectively. These 2k pairwise-disjoint intervals will make up 2k vertices in the

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graphs we consider. Now let \mathcal{W} denote the set of k^2 closed intervals with one endpoint in $\{-k, \ldots, -1\}$ and the other in $\{1, \ldots, k\}$. We color B_1, \ldots, B_k with blue, R_1, \ldots, R_k with red, and the k^2 intervals in \mathcal{W} with white.

Together with $S := \{B_1, \ldots, B_k, R_1, \ldots, R_k\}$, each $\{J_1, \ldots, J_{n-2k}\} \in \binom{\mathcal{W}}{n-2k}$ gives an *n*-vertex, three-colored interval graph. For a given $\mathcal{J} = \{J_1, \ldots, J_{n-2k}\}$, let $G_{\mathcal{J}}$ denote the colored interval graph whose vertices correspond to *n* intervals in $S \cup \mathcal{J}$, and let \mathcal{G} denote the set of all $G_{\mathcal{J}}$.

Now let $G \in \mathcal{G}$. For a white vertex $w \in G$, the pair $(d_B(w), d_R(w))$, which represents the numbers of blue and red neighbors of w, uniquely determine the interval corresponding to w; this is the interval $[-d_B(w), d_R(w)]$. In other words, \mathcal{J} can be recovered from $G_{\mathcal{J}}$ uniquely. Thus

$$|\mathcal{G}| = \binom{k^2}{n-2k}.$$

Since there are at most 3^n ways to color the vertices of an interval graph with blue, red, and white, we have

$$I_n \cdot 3^n \ge |\mathcal{G}| = \binom{k^2}{n-2k} \ge \left(\frac{k^2}{n-2k}\right)^{n-2k} \ge \left(\frac{k^2}{n}\right)^n$$

for any k < n/2. Setting $k = \lfloor n/\log n \rfloor$ and taking the logarithms, we get

$$\log I_n \ge n \log(k^2/n) - O(n) = n \log n - 2n \log \log n - O(n).$$

Remark 2. Yang and Pippenger [5] posed the question whether

$$\log I_n = Cn \log n + O(n)$$

for some C or not. According to Theorem 1, this boils down to getting rid of the $2n \log \log n$ term in (2). Such a result would imply that the exponential generating function

$$J(x) = \sum_{n \ge 1} I_n \frac{x^n}{n!}$$

has a finite radius of convergence. (As noted in [5], the bound $I_n \leq (2n-1)!!$ implies that the radius of convergence of J(x) is at least 1/2.) Of course, a strong result would be finding I_n asymptotically.

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