

Record values in appending and prepending bitstrings to runs of binary digits

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Abstract

In this short note, we show a simple characterization of integers that reach records for a sequence described by adding binary strings to runs of 1's and 0's in a binary representation. In particular, we show that this set does not depend on the added strings as long as they are nonempty and of the same length.

1 Introduction

The sequence A175046 in the Online Encyclopedia of Integer Sequences (OEIS) [OEIS] is described as follows. For each integer n , take the runs of 1's in a binary representation of n and append 1 to them and take runs of 0's and append a 0 to them, and convert the resulting binary string back into an integer $a(n)$. For example, for $n = 89$, the binary representation is 1011001, and appending each runs of 0's and 1's with 0 and 1 respectively resulted in 110011100011 which implies $a(89) = 3299$. Neil Sloane conjectured and Maximilian Hasler proved that $a(n) \leq \frac{9n^2+12n}{5}$, with equality if and only if $n = \frac{2}{3}(4^k - 1)$, i.e. n is 1010...10 in binary. Neil Sloane also conjectured that the record values of $a(n)$ ¹ (OEIS sequences A319422, A319424) are described as having a binary representation that is either a alternating sequence of 11 and 00 or having a single 00 replaced by 000. In this short note we show that this conjecture is true. In fact, we show that the set of the indices n of the record values is independent of the binary strings that are appended (or prepended) to the runs.

¹i.e. values $a(n)$ such that $a(m) < a(n)$ for all $m < n$.

2 Notation

For an integer n , let $b(n)$ and $c(n)$ be the number of runs of 1's and 0's and the number of bits in the binary representation of n respectively. For an integer n , let $B(n)$ denote the bitstring of its binary representation.

Definition 1. Given two binary strings d_0 and d_1 , let $f(n, d_0, d_1)$ be defined as the number whose binary representation is constructed by appending d_1 to each run of 1's and d_0 to each run of 0's. Similarly $g(n, d_0, d_1)$ is defined by prepending d_i rather than appending. By abuse of notation, we also apply this to the binary representation of n , i.e. $f(B(n), d_0, d_1) = B(f(n, d_0, d_1))$.

We will omit the arguments d_0 and d_1 in $f(n, d_0, d_1)$ and $g(n, d_0, d_1)$ when they are clear from context. Consider the special case of f when $d_0 = 0$ and $d_1 = 1$ (OEIS sequence A156064). A left inverse of f (OEIS sequence A318921) in this case is described in [Lenormand, 2003]. It is easy to see that $b(f(n, 0, 1)) = b(n)$, $c(f(n, 0, 1)) = b(n) + c(n)$, $g(n, 0, 1) = f(n, 0, 1)$ and $g(n, 1, 0) = \lfloor f(n, 0, 1)/2 \rfloor$.

If d_0 and d_1 are both length k bitstrings, then

$$c(f(n, d_0, d_1)) = b(n)k + c(n) \tag{1}$$

$$c(g(n, d_0, d_1)) = b(n)k + c(n) - l \tag{2}$$

$$B(g(n, d_1, d_0)) = d_0 B(\lfloor f(n, d_0, d_1)/2^k \rfloor) \tag{3}$$

where l is the number of leading zeros of d_1 . A consequence of Eq. (3) is that if d_0 does not contain 1's, then $g(n, d_1, d_0) = \lfloor f(n, d_0, d_1)/2^k \rfloor$.

For a binary bitstring x , we denote \bar{x} as the bitstring where the 0's are change to 1's and vice verse. In other words \bar{x} is the 1's complement of x .

3 A characterization of record values of f and g

Theorem 1. Let S_f be the set $\{n : \forall m < n \quad f(m, d_0, d_1) < f(n, d_0, d_1)\}$ where d_1 and d_0 are nonempty bitstrings of the same length. Then $n \in S_f$ if and only if the binary representation of n is either an alternating sequence of 0's and 1's or an alternating sequence of 0's and 1's where exactly one of the 0 is replaced with 00.

Proof. Let T be the set of binary sequences of alternating 0's and 1's plus sequences of alternating 0's and 1's where exactly one of the 0 is replaced with 00. First we show that if both d_0 and d_1 are nonempty bitstrings, then $S_f \subset T$. Consider $n \in S_f$ such that $B(n)$ contains 2 consecutive 1's, i.e. $B(n) = x11y$. Note that x could be the empty bitstring. Then m with $B(m) = x10\bar{y}$ clearly satisfy $m < n$. Both m and n has the same number of bits, but $b(m) = b(n) + 1$, so $c(f(m)) > c(f(n))$ and thus $f(m) > f(n)$. This implies that sequences in S_f are *Fibbinary* numbers (OEIS sequence A003714).

Next suppose that $B(n)$ contains 4 or more consecutive 0's, i.e. $B(n) = x10000y$. Consider $m < n$ with $B(m) = x01010\bar{y}$. Note that because of the above, $n \in S_f$ implies that x

is either the empty string or ends in 0. Then $b(m) = b(n) + 2$ and $c(m) \geq c(n) - 1$ and by Eq. (1), this implies that $f(m) > f(n)$. Now suppose that $B(n)$ contains 3 consecutive 0's, i.e. $B(n) = x1000y$ where y does not start with 0, i.e. y is empty or starts with 1. Consider $m < n$ with $B(m) = x0101\bar{y}$. It is easy to see that $b(m) = b(n) + 1$. If x is not the empty bitstring, then $c(m) = c(n)$ so again $c(f(m)) > c(f(n))$ and $f(m) > f(n)$. If x is the empty bitstring, then $c(m) = c(n) - 1$ and $B(f(n)) = 1d_1000d_0f(y)$, $B(f(m)) = 1d_10d_01d_1f(\bar{y})$. If d_0 and d_1 have length $k > 1$, then $c(f(m)) > c(f(n))$. If d_0 and d_1 are both single-bit strings, then $c(f(m)) = c(f(n))$ and comparing their initial bits shows that $f(m) > f(n)$. Thus elements of S_f cannot contain 3 or more 0's in its binary representation, i.e. S_f is a subset of the terms of OEIS sequence A003796, in particular it is a subset of the terms in OEIS sequence A086638.

Now suppose n is such that $B(n)$ has two occurrences of 00's, i.e. $B(n) = x100y00z$. By the discussion above, y must start and end with a 1. Consider $m < n$ with $B(m) = x010y01\bar{z}$. It can easily be shown that $b(m) = b(n) + 1$, so again $f(m) > f(n)$ if x is not the empty string. If x is the empty string then $B(f(n)) = 1d_100d_0f(y)f(00z) = 1d_100d_0\dots$ and $B(f(m)) = 1d_10d_0f(y)0d_0f(1\bar{z}) = 1d_10d_01\dots$. Again, $f(m) > f(n)$ if d_0 and d_1 are of length $k > 1$. If d_0 and d_1 are single bits, $c(f(n)) = c(f(m))$ and comparing the initial bits shows that $f(m) > f(n)$. This shows that $S_f \subset T$.

Next we show that $T \subset S_f$. Consider an integer n such that $B(n)$ is an alternating sequence of 0's and 1's. This implies that $b(n) = c(n)$. Consider an integer $m < n$. Clearly, $c(m) \leq c(n)$. Since the alternating sequence of 0's and 1's is the only sequence such that $b(n) = c(n)$, this means that $b(m) < b(n)$ and thus $f(m) < f(n)$. Thus $n \in S_f$. Next suppose that n is an integer such that $B(n) = x100y$, where x is either empty or is an alternating sequence of 1's and 0's ending in 0 and y is either empty or an alternating sequence of 1's and 0's. Note that $b(n) = c(n) - 1$ and $B(f(n)) = f(x)1d_100d_0f(y)$. Consider $m < n$. It is clear that $B(m)$ cannot be the alternating string of 1's and 0's of length $c(n)$. Thus $b(m) < c(n)$, i.e. $b(m) \leq b(n)$. Suppose $c(m) < c(n)$, then by Eq. (1) $c(f(m)) < c(f(n))$, i.e. $f(m) < f(n)$.

Suppose $c(m) = c(n)$. If $b(m) < b(n)$, then again $f(m) < f(n)$ by Eq. (1), so we can assume that $b(m) = b(n)$. This implies that $B(m)$ is also of the form $x'100y'$, where x' is either empty or is an alternating sequence of 1's and 0's ending in 0 and y' is either empty or an alternating sequence of 1's and 0's. Since $m < n$, the only possibility is that the 00 of $B(m)$ is to the left of the 00 in $B(n)$. This $B(m) = z1001r$ and $B(n) = z1010r'$. Thus $f(m) = f(z)1d_100d_0f(1r)$ and $f(n) = f(z)1d_10d_01d_1f(0r')$. This implies that $f(n)$ and $f(m)$ has the same initial bits followed by the bits $0d_0$ for $f(m)$ and d_01 for $f(n)$ and this combined with the fact that $c(f(n)) = c(f(m))$ implies that $f(n) > f(m)$. This shows that $T \subset S_f$ and concludes the proof. \square

As a result of Theorem 1 for $d_0 = 0$ and $d_1 = 1$, the values of $f(n)$ for $n \in S_f$ is exactly the numbers whose binary representation is an alternating sequences of 11 and 00 with at most one of the 00 replaced with 000, proving the (second) conjecture stated in Section. 1.

Theorem 1 is also valid for the function g that prepends d_i rather than appends d_i to runs of 0's and 1's.

Theorem 2. Let S_g be the set $\{n : \forall m < n \quad g(m, d_0, d_1) < f(n, d_0, d_1)\}$ where d_1 and d_0 are nonempty bitstrings of the same length. Then $n \in S_g$ if and only if the binary representation of n is either an alternating sequence of 0's and 1's or an alternating sequence of 0's and 1's where exactly one of the 0 is replaced with 00.

Proof. The proof is virtually identical to the proof of Theorem 1 and uses Eq. (2) instead of Eq. (1). The main difference is in some of the cases considered. First, for the case where $B(n) = 1000y$ and $B(m) = 101\bar{y}$ where y is the empty string or starts with 1. In this case, $B(g(n)) = d_11d_0000g(y)$ and $B(g(m)) = d_11d_00d_11g(\bar{y})$. If d_0 and d_1 are of length $k > 1$, then $c(g(m)) < c(g(n))$ and $g(m) > g(n)$. For d_0 and d_1 both a single bit, comparing the initial bits of $g(n)$ and $g(m)$ shows that $g(m) > g(n)$. Second, for $B(n) = 100y00z$ and $B(m) = 10y01\bar{z}$, $B(g(n)) = d_11d_000g(y)d_000\dots = d_11d_000d_1\dots$ and $B(g(m)) = d_11d_00d_11\dots$. For d_0 and d_1 a single-bit string, $c(g(n)) = c(g(m))$ and comparing the initial bits of $g(n)$ and $g(m)$ shows that $g(m) > g(n)$. Third, for the case where $B(m) = z1001r$ and $B(n) = z1010r'$. In this case, $g(m) = g(z)d_11d_000g(1r) = g(z)d_11d_000d_1\dots$ and $g(n) = g(z)d_11d_00d_11g(0r') = g(z)d_11d_00d_11d_0\dots$. This implies that $g(n)$ and $g(m)$ has the same initial bits followed by $0d_1$ for $g(m)$ and d_11 for $g(n)$ which implies that $g(n) > g(m)$. \square

Theorem 1 and 2 imply that when d_0 and d_1 are nonempty and of the same length, $S_g = S_f$ and is equal to the terms in OEIS sequence A319423.

References

- [OEIS] The on-line encyclopedia of integer sequences (<http://oeis.org/>), founded in 1964 by N. J. A. Sloane.
- [Lenormand, 2003] C. Lenormand, “Deux transformations sur les mots”, preprint, 5 pages, Nov. 17, 2003. Available online at: <https://oeis.org/A318921/a318921.pdf>