HONEYCOMB TESSELLATIONS AND CANONICAL BASES FOR PERMUTOHEDRAL BLADES

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Abstract. This paper studies two families of piecewise constant functions which are determined by the $(n-2)$-skeleta of collections of honeycomb tessellations of $\mathbb{R}^{n-1}$ with standard permutohedra. The union of the codimension 1 cones obtained by extending the facets which are incident to a vertex of such a tessellation is called a blade.

We prove ring-theoretically that such a honeycomb, with 1-skeleton built from a cyclic sequence of segments in the root directions $e_i - e_{i+1}$, decomposes locally as a Minkowski sum of isometrically embedded components of hexagonal honeycombs: tripods and one-dimensional subspaces. For each triangulation of a cyclically oriented polygon there exists such a factorization. This consequently gives resolution to an issue proposed and developed by A. Ocneanu, to find a structure theory for an object he discovered during his investigations into higher Lie theories: permutohedral blades.

We introduce a certain canonical basis for a vector space spanned by piecewise constant functions of blades which is compatible with various quotient spaces appearing in algebra, topology and scattering amplitudes. Various connections to scattering amplitudes are discussed, giving new geometric interpretations for certain combinatorial identities for one-loop Parke-Taylor factors. We give a closed formula for the graded dimension of the canonical blade basis. We conjecture that the coefficients of the generating function numerators for the diagonals are symmetric and unimodal.

Contents

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1. Introduction

1.1. **Preface.** For any permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$, choose a point v in the interior of one of the fundamental Weyl alcoves

$$\begin{gathered} 0 \leq x_{\sigma_1} - x_{\sigma_n} \leq 1 \\ 0 \leq x_{\sigma_n} - x_{\sigma_{n-1}} \leq 1 \\ \vdots \\ 0 \leq x_{\sigma_2} - x_{\sigma_1} \leq 1, \end{gathered}$$

and take the orbit obtained by reflecting v across all affine hyperplanes $x_i - x_j = k$, where k is any integer. As shown in [37], the orbit of v then consists of the vertices of a *honeycomb tessellation* of space with weight permutohedra. Then v has n neighbors $v_1, \ldots, v_n$, one for each hyperplane $x_{\sigma_{i+1}} - x_{\sigma_i} = 0$. In this paper we study the complete fan obtained from the union of the convex hulls

$$\bigcup_{1 \leq i < j \leq n} \text{convex hull}\,(v, v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_n)$$

by extending each convex hull to a simplicial cone and translating v to the origin. This complete fan has an $(n-2)$-skeleton

$$((\sigma_1, \ldots, \sigma_n)) := \bigcup_{1 \leq i < j \leq n} \left\{ \sum_{k \notin \{i,j\}} t_k (v_k - v) : t_k \geq 0 \right\}$$

in the hyperplane $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$, which is a particular instance of a *blade.* The portion of the blade $((1, 2, 3, 4))$ inside a unit tetrahedron is shown in Figure 2.

1.2. **Context and history.** In order to construct the geometric foundation for his work on higher representation theory, in his Fall, 2017 Harvard lectures [33] A. Ocneanu has introduced permutohedral *blades*, to study the underlying geometry, combinatorics and representation theory of local aspects of a problem with which for him had a physical motivation, concerning what he has called *the "cracks" in spacetime* arising in the boundary of a tessellation of space with permutohedra, as in Figure 2. However, a significant portion of the algebraic and combinatorial foundations and framework remained elusive and incomplete. Ocneanu calls the core geometric objects blades due to their explicit appearance in dimensions 2 and 3 as *widening the cracks* between by a set of cones in a polyhedral fan, see Figures 1 and 2 for the cases $n = 3, 4$.

The first steps towards assembling a general framework were taken by the author in [16], constructing a new *canonical* basis for a vector space spanned by characteristic functions of certain permutohedral cones called (tectonic) *plates.* In this paper, we first establish the analogous basis result for blades, proving that blades are generated in the Minkowski algebra of polyhedral cones by embeddings of tripods[1] of type A_2 roots.

In Fall, 2015, we conjectured privately to Ocneanu that higher codimension blades could be naturally expressed as Minkowski sums of blades in fewer coordinates. In this paper we go further towards a structure theory for blades using convolutions.

Let us comment briefly on forthcoming joint work of Norledge and Ocneanu, where blades are formulated and studied in a somewhat different way involving certain derivatives defined on functions that are piecewise constant on the faces of the (restricted) all-subset hyperplane

[1] We thank Tamas Kalman for spontaneously suggesting the apt terminology, *tripod.*

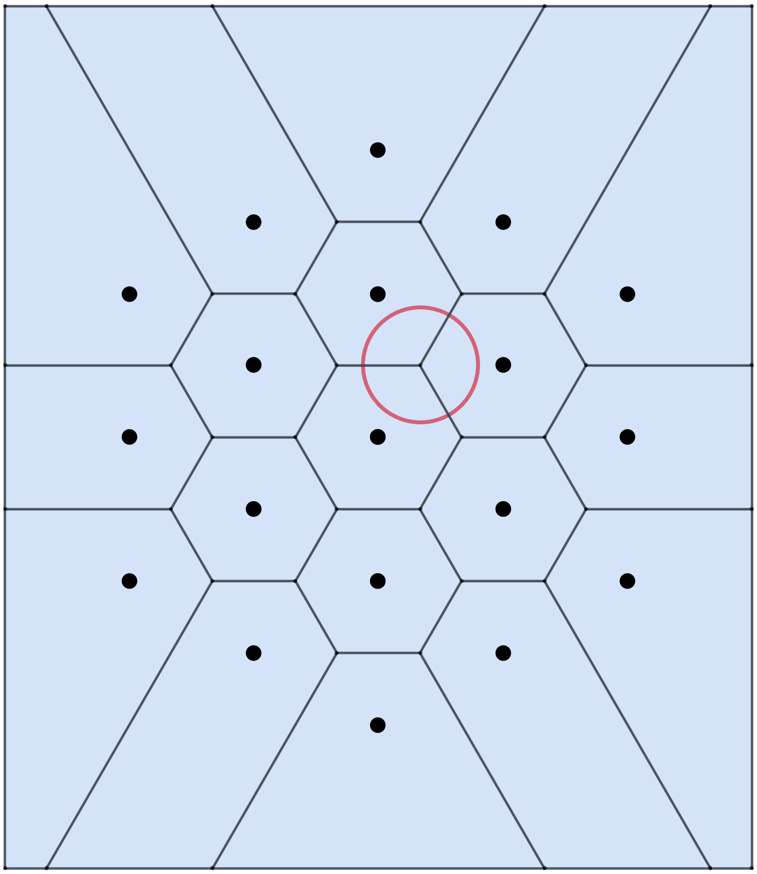

FIGURE 1. Local (uniform) hexagonal honeycomb tessellation: part of the blade $((1,2,3))$ near a vertex of a Voronoi honeycomb. Edges in the tiling are segments parallel to the three root directions $e_i - e_j$. Each $(n=3)$-valent vertex has a neighborhood which intersects the 1-skeleton of the honeycomb in either $((1,2,3))$ or $((1,3,2))$

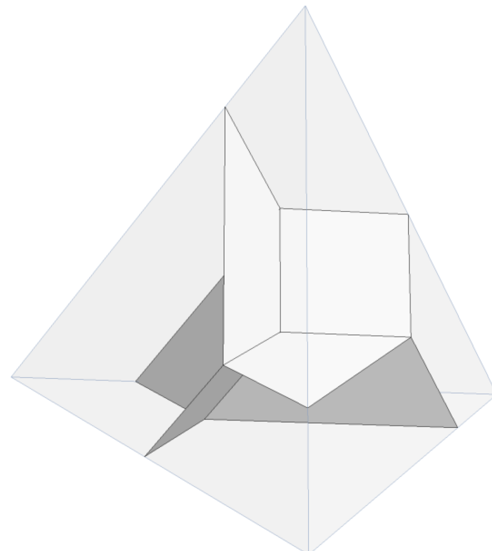

FIGURE 2. Part of the blade $((1,2,3,4))$. It is the 2-skeleton of a complete fan, here viewed inside a tetrahedral frame. Note that here the edges in the 1-skeleton of the blade are not perpendicular to the facets of the tetrahedron, but are rather parallel to the edges of a Hamiltonian cycle on the edges of the tetrahedron.

arrangement. In their setup, blades are shown to be determined by a set of linear equations which follow from a *conservation* principle. It would be very interesting to ask for an analog of the conservation principle, but now for the space spanned by characteristic functions of plates. For a possible starting point, one might try to start with Appendix B.

1.3. **Overview and connections.** In this paper, we prove that the characteristic function of any blade factorizes into characteristic functions of tripods and 1-dimensional subspaces. However, it turns out that the factorization of a blade into tripods is not unique; indeed, we prove that the blade factorizations correspond naturally triangulations of a cyclically-oriented polygon with vertices labeled by an ordered set partition, or equivalently to the blow-ups of an n-valent vertex of a graph into trivalent vertices. We prove that such products are invariant under changes in triangulation and therefore depend only on the cyclic order on the boundary, and we derive their expansions using sums of elementary symmetric functions in characteristic functions of infinite rays in the direction of the collection of n cyclic roots $e_{\sigma_i} - e_{\sigma_{i+1}}$.

We observe that the linear relations which characterize the subspace spanned by characteristic functions of blades in dimension 1 are partially captured by the balancedness condition on weighted directed graphs studied by T. Lam and A. Postnikov in their work on polypositroids [27]. By comparison with the results of [5], blades are also related to zonotopal algebras [2, 5, 24], through Gale duality for the type A reflection arrangement.

In terms of representation theory of the symmetric group, modules spanned by characteristic functions of blades are constructed plethystically using the higher Eulerian representations, which arise as restrictions to $\mathfrak{S}_n$ of representations of $\mathfrak{S}_{n+1}$ of dimensions the Stirling numbers of the first kind, as studied by S. Whitehouse in [39]. For further historical discussion we refer to the introduction of [15], where together with V. Reiner we studied the cohomology ring, denoted there $\mathcal{V}^n$, of the configuration space of n points in $SU(2)$, modulo the diagonal action of $SU(2)$. In fact, it turns out that this cohomology ring is related to (a degeneration of) the Minkowski algebra of blades. The discovery of this geometric connection to blades helped to determine the presentation of $\mathcal{V}^n$ in [15].

The main construction involves characteristic functions of certain embeddings of the type A_2 root system. The characteristic functions of elementary (one-dimensional) tripods, are denoted by $\gamma_{i,j,k}$ and satisfy the cyclic index relation $\gamma_{i,j,k} = \gamma_{k,i,j}$. More generally, lumped tripods are labeled by cyclic classes of 3-block ordered set partitions (S_1, S_2, S_3), of the subset $S_1 \cup S_2 \cup S_3$ of $\{1, \ldots, n\}$. Additionally we have generators 1_S for any proper nonempty subset S of $\{1, \ldots, n\}$. These satisfy the fundamental relation

$$\gamma_{S_1,S_2,S_3} + \gamma_{S_1,S_3,S_2} \;=\; 1_{S_1\cup S_2} + 1_{S_2\cup S_3} + 1_{S_3\cup S_1} - 1_{S_1}1_{S_2}1_{S_3}.$$

A detailed definition of plates and blades will be given later, but let us give an abbreviated version here.

The prototypical *plate* is the cone spanned by a system of simple roots,

$$\langle e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\rangle_+ := \{t_1(e_1 - e_2) + \cdots + t_{n-1}(e_{n-1} - e_n) : t_i \geq 0\}$$

The prototypical *blade* is the complement of the unions of the interiors of the n cones

$$\langle e_1 - e_2, \ldots, e_{n-1} - e_n\rangle_+,\ \langle e_2 - e_3, \ldots, e_n - e_1\rangle_+, \ldots,\ \langle e_n - e_1, \ldots, e_{n-2} - e_{n-1}\rangle_+,$$

where

$$\langle v_1, \ldots, v_k\rangle_+ = \left\{\sum_{i=1}^{k} t_i v_i : t_i \geq 0\right\}.$$

Let us express this directly, as follows (see Definition 14 below). For an ordered set partition $(S_1, \ldots, S_k)$ of $\{1, \ldots, n\}$ with $k \geq 3$, we define the blade $((S_1, \ldots, S_k))$ to be the set-theoretic *union* of the Minkowski sums of cones,

$$\bigcup_{1\leq i<j\leq k} [S_1, S_2] \oplus \cdots \oplus \widehat{[S_i, S_{i+1}]} \oplus \cdots \oplus \widehat{[S_j, S_{j+1}]} \oplus \cdots \oplus [S_k, S_1],$$

where the hat means that the corresponding term has been omitted, and where we adopt the convention $S_{k+1} = S_1$. See [33, 34].

We conclude by summarizing briefly some of the many additional possible directions for future work.

A key observation is that as these n (closed) cones can be seen to cover the whole ambient space (see Corollary 11), coinciding only on common facets, they form what is called a complete fan. In the case of the blade $((1, 2, \ldots, n))$ it can be obtained as the normal fan to the simplex with vertices $e_1 - e_n, e_n - e_{n-1}, \ldots, e_2 - e_1$. Further, one could develop blades using Bergman fans for matroid polytopes; we leave this to future work.

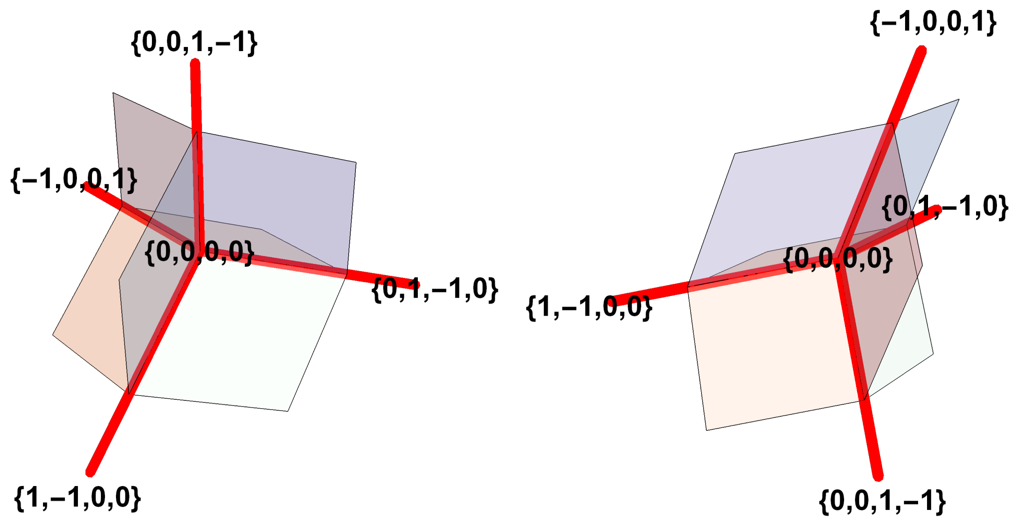

FIGURE 3. The $n = 4$ blade $((1, 2, 3, 4))$: it is the normal fan to the simplex with vertex set $(-1, 1, 0, 0), (0, -1, 1, 0), (0, 0, -1, 1), (1, 0, 0, -1)$; this means that it is the union of the convex hulls of pairs of rays selected from $\{(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1), (-1, 0, 0, 1)\}$

In [16] and here we study modules of plates and blades *modulo* higher codimensions; it would be very interesting to search for a compatible Lie algebra in the direction of [38], see also [8], of certain rational functions modulo products. Such products correspond in our setting to higher codimension faces of the all-subset arrangement.

We point out an interesting apparent connection between blades and certain elliptic functions which appear in string theory [29], where the blade relations are analogous to certain limits of the so-called Fay identities[2]. We would like also to mention another connection, with the related work [28]. Here, modulo the Fay identities, the *canonicalization* relations among pseudoinvariants. These will correspond in our setting to straightening relations into the canonical basis for higher codimension blades which are piecewise constant functions on Minkowski sums of blades.

The utility and beauty of the exponential map used for the cohomology ring of the configuration space of points in $SU(2)$, in Section 9, suggests a connection to some of the combinatorial properties of certain exponential generating functions from [7]. Finally, the examples and discussion in Appendix A suggest a new class of identifications for non-planar on-shell diagrams beyond the well-known square move which deserves further study [9]. Our paper [9] is in preparation.

1.4. **Discussion of main results.** To each blade $((S_1, \ldots, S_k))$, where $S_1, \ldots, S_k$ is an ordered partition of (a subset of) $\{1, \ldots, n\}$ we assign two piecewise-constant surjective functions:

[2]We thank Oliver Schlotterer for this observation.

characteristic functions of blades

$$\Gamma_{S_1,\ldots,S_k} : V_0^n \to \{0,1\}$$

and *graduated* functions of blades

$$[(S_1,\ldots,S_k)] : V_0^n \to \{1,\ldots,k\}.$$

These functions capture essentially different but interdependent aspects of the blades $((S_1,\ldots,S_k))$, as reflected in our results; both will be useful and deserve further study. See Section 11 for a detailed discussion of which properties of these two families remain to be established.

Our main technical results are Theorem 20 and Theorem 27, using the latter to introduce the canonical basis for the graduated blades. In Theorem 20, we prove that the characteristic function $\Gamma_{S_1,S_2,\ldots,S_k}$ of the blade $((S_1,\ldots,S_k))$ with k blocks factors as a convolution product of $k-2$ tripods, using the *flag* factorization:

$$\Gamma_{S_1,S_2,\ldots,S_k} = \Gamma_{S_1,S_2,S_3}\Gamma_{S_1,S_3,S_4}\cdots\Gamma_{S_1,S_{k-1},S_k}.$$

We deduce in Corollary 24 the set-theoretic identity for the Minkowski sum: we prove that any blade can be expressed (not uniquely) as a Minkowski sum of tripods.

We prove that the factorization is independent of the triangulation; this shows that the factorizations of a blade can be represented by with the triangulations of a cyclically oriented n-gon.

We arrive at the punchline of our paper by combining Proposition 13, which identifies the blade $((1,2,\ldots,n))$ with the neighborhood of a vertex of the tessellation of V_0^n with standard permutohedra, with Theorem 20: the honeycomb tessellation by standard permutohedra factorizes locally as Minkowski sums of tripods, where each factorization is encoded by a triangulation of the cyclically oriented n-gon.

In Theorem 27 we derive the expansion of an element in the canonical (candidate) basis as a signed sum of *graduated* functions of blades. In Theorem 35 we remove the word "candidate" by showing that each can be obtained from an upper-unitriangular transformation from a set of graduated blades which we know is linearly independent.

Remark that one could find other possible choices of bases for the space spanned by characteristic functions of plates; for inspiration one could look toward the *forkless* monomial basis from [23] for the so-called subdivision algebra [30]. The case of interest here consists of forkless monomials which would here encode generalized permutohedral cones, labeled by directed trees $\{(i_1,j_1),\ldots,(i_m,j_m)\}$ with $i_a < j_a$, having the property no node of the tree has two outgoing branches: $i_a \neq i_b$ whenever $a \neq b$.

In Section 8, Proposition 38, we enumerate the canonical basis and then ask about a possible interpretation of the coefficients of the generating functions for the diagonals in terms of labeled binary rooted trees, $\frac{(2n)!}{n!}$. Indeed, by way of a simple counting argument we have for the enumeration of the canonical basis of characteristic functions of blades,

$$T_{n,k} = \sum_{i=1}^{n} S(n,i)s(i-1,k-1),$$

where $S(n,i)$ is the Stirling number of the second kind, and $s(i,k)$ is the (unsigned) Stirling number of the first kind.

For $n = 1, 2, \ldots, 6$ we have

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 1 & 4 & 1 \\ 1 & 15 & 9 & 1 \\ 1 & 66 & 66 & 16 & 1 \\ 1 & 365 & 500 & 190 & 25 & 1 \end{array}$$

Note that the rows sum (as they should) to the necklace numbers, which count ordered set partitions up to cyclic block rotation.

The generating function numerators for its diagonals appear to be symmetric and unimodal, and to have coefficients that sum to $(2n)!/n!$, O.E.I.S. A001813, [32]. The coefficients themselves agree with A142459 for the first four rows, and their sums agree, $(2n)!/n!$, but for $n \geq 5$ the coefficients are different. See Section 8 below for more details, as well as a conjecture.

Finally, it turns out that there is an interesting dual grading cyclic ordered set partitions which would be dual to the grading coming from the canonical blade basis. Grouping cyclic ordered set partitions by the number of blocks one obtains the sequence given by O.E.I.S. A028246 [32]:

$$a(n, k) = S(n, k) * (k - 1)!.$$

This gives for $n = 1, 2, \ldots, 6$,

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 1 & 3 & 2 \\ 1 & 7 & 12 & 6 \\ 1 & 15 & 50 & 60 & 24 \\ 1 & 31 & 180 & 390 & 360 & 120 \end{array}$$

Remark that this table appeared has in the enumeration of the basis for a certain quotient word Hopf algebra, see [12].

We shall encounter this enumeration quite naturally in Section 10, where it is observed to count degenerate cyclic orders of n points on the circle.

2. Permutohedral Plates

Let us fix some notation.

We denote $x_S = \sum_{i \in S} x_i$ for any proper nonempty subset S of $\{1, \ldots, n\}$, and abbreviate $x_{S_1 \cup \cdots \cup S_k}$ as $x_{S_1 \cdots S_k}$ for k disjoint subsets $S_1, \ldots, S_k$ of $\{1, \ldots, n\}$. Set $V_0^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$. Let $\bar{e}_J = \sum_{j \in J} e_j - \frac{|J|}{n}(\sum_{i=1}^n e_i)$, if J is a nonempty subset of $\{1, \ldots, n\}$, be the projection of e_J onto the plane V_0^n. Denote by $\pi_1 \oplus \pi_2 = \{u + v : u \in \pi_1,\ v \in \pi_2\}$ the Minkowski sum of the polyhedra $\pi_1, \pi_2 \subseteq V_0^n$. Put

$$[S_i, S_j] = \left\{ \sum_{a \in S_i \cup S_j} t_a \bar{e}_a \in V_0^n : t_{S_i} \geq 0 \right\},$$

that is the upper half of the subspace spanned by $\{\bar{e}_a : a \in S_i \cup S_j\}$.

Recall that a polyhedral cone is a subset of some $\mathbb{R}^n$ which is out by a set of inequalities of the form $\{x \in V_0^n : \sum_{j=1}^n a_{ij} x_j \geq 0,\ i = 1, \ldots, m\}$ for a given coefficient matrix $(a_{ij}) \in \mathbb{R}^{m \times n}$.

An ordered set partition (OSP) is an ordered list $\mathbf{S} = (S_1, \ldots, S_k)$ of disjoint subsets of $\{1, \ldots, n\}$ such that $\bigcup_{i=1}^k S_i$.

Definition 1. A polyhedral cone Π in V_0^n is *permutohedral* if it can expressed as a Minkowski sum of subspaces $[T_i]$ and a sequence of half subspaces $[S_a, S_{a+1}]$,

$$\Pi = [T_1] \oplus \cdots \oplus [T_\ell] \oplus [\mathbf{S}_1] \oplus \cdots \oplus [\mathbf{S}_k],$$

for disjoint subsets $T_1, \ldots, T_\ell$ and ordered set partitions $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ of disjoint subsets $U_1, \ldots, U_k$ of $\{1, \ldots, n\} \setminus \left(\bigcup_{i=1}^k T_i\right)$. In the terminology from below, it is a Minkowski sum of *tangent cones* to the regular permutohedron.

Further, Π is generalized permuhedral if it is a Minkowski sum

$$\Pi = [T_1] \oplus \cdots \oplus [T_\ell] \oplus [S_{i_1}, S_{j_1}] \oplus \cdots \oplus [S_{i_k}, S_{j_k}],$$

where $S_{i_a} \cap S_{j_a} = \emptyset$ for $a = 1, \ldots, k$ and where $\{T_1, \ldots, T_\ell\}$ is a set partition of the complement in $\{1, \ldots, n\}$ of the union of all subsets S_{i_a}, S_{j_a},

$$\{1, \ldots, n\} \setminus \left(\bigcup_{a=1}^k S_{i_a} \cup S_{j_a}\right).$$

In Definition 1, without loss of generality we could further require that the Minkowski sum be *reduced*, that is, we have the property that for each pair $\{S_\ell, S_m\}$, the sets S_ℓ, S_m are either disjoint or equal. For without this assumption, we would have for example the simplification

$$[135, 246] \oplus [127, 345] = [1234567].$$

For detailed examples of generalized permutohedral cones and their functional representations, see Appendix A.2 in [16].

Recall that the tangent cone C of a polyhedron to a face F of a polyhedron P is defined by

$$C = \{x + \lambda(y - x) : x \in F, y \in P, \text{ and } \lambda \geq 0\}.$$

We single out the set of tangent cones to faces of the regular permutohedron, which were studied as *plates* by Ocneanu. See [16] for details of our construction and related results for permutohedral cones, as well as a canonical basis for the space spanned by characteristic functions of such tangent cones. Let $(S_1, \ldots, S_k)$ be an ordered set partition of $\{1, \ldots, n\}$. Namely, we recall the notation $[S_1, \ldots, S_k]$ for the subset in V_0^n which is cut out by the system of inequalities

$$\begin{aligned} x_{S_1} &\geq 0 \\ x_{S_1 S_2} &\geq 0 \\ &\vdots \\ x_{S_1 S_2 \cdots S_{k-1}} &\geq 0 \\ \sum_{i=1}^n x_i &= 0. \end{aligned}$$

This is a permutohedral cone, since it can be expressed as the Minkowski sum of half spaces

$$[S_1, S_2] \oplus [S_2, S_3] \oplus \cdots \oplus [S_{k-1}, S_k].$$

It is a standard result that this is the tangent cone to the face of the usual permutohedron labeled by the ordered set partition $(S_1, \ldots, S_k)$ of $\{1, \ldots, n\}$, see for example [37]. In [16] the same was proved for tree graphs.

Proposition 2. *Let Π be a generalized permutohedral cone which is labeled by a directed tree graph with edge set $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ for some $k \leq n$, with Minkowski sum decomposition*

$$\Pi = [S_{i_1}, S_{j_1}] \oplus \cdots \oplus [S_{i_k}, S_{j_k}].$$

Then it can be expanded as a signed sum of characteristic functions of tangent cones to faces of the usual permutohedron.

In fact the expansion result can be extended to the expansion of generalized permutohedral cones as signed sums of permutohedral cones.

Recall from [16] the notation $\hat{\mathcal{P}}^n$ for the complex linear span of all characteristic functions $[[S_1, \ldots, S_k]]$ of plates $[S_1, \ldots, S_k]$, as $(S_1, \ldots, S_k)$ ranges over all ordered set partitions of $\{1, \ldots, n\}$.

Definition 3. Let C be a polyhedral cone in V_0^n. The dual cone to C, denoted $C^\star$, is defined by the equation

$$C^\star = \{y \in V_0^n : y \cdot x \geq 0 \text{ for all } x \in C\}.$$

In [16] it was an essential property that the plate $[S_1, \ldots, S_k]$ is *dual* (in the sense of convex geometry, see [4]) to the face of the arrangement of type A_{n-1} reflection hyperplanes, given by

$$\{x \in V_0^n : x_{(S_1)} \geq \cdots \geq x_{(S_k)}\},$$

where $x_{(S)}$ is shorthand notation for $x_{i_1} = \cdots = x_{i_{|S|}}$, given that $S = \{i_1, \ldots, i_{|S|}\}$

Let us collect two important results from [4]. It is immediate from Definition 1 that the conical hull of two permutohedral cones is a permutohedral cone; therefore in Theorem 5 the convolution product $\bullet$ in (2) is defined on the subring of characteristic functions of permutohedral cones, $\hat{\mathcal{P}}^n$.

Definition 4. A linear transformation $\hat{\mathcal{P}}^n \to V$, where V is a vector space, is called a *valuation*.

We shall need Theorem 2.5 of [4], which defines the convolution of two polyhedral cones with respect to the Euler characteristic, to prove relations with respect to the convolution product $\bullet$.

Theorem 5 ([4], Theorem 2.5)**.** *There is a unique bilinear operation, denoted here* $\bullet : \hat{\mathcal{P}}^n \times \hat{\mathcal{P}}^n \to \hat{\mathcal{P}}^n$*, called convolution, such that*

$$[\pi_1] \bullet [\pi_2] = [\pi_1 \oplus \pi_2]$$

for any two permutohedral cones π_1, π_2.

The proof given in [4] of Theorem 5 relies a linear map called the *Euler characteristic*, which is proven there to be the unique valuation $\mu : \hat{\mathcal{P}}^n \to \mathbb{Q}$, such that $\mu([\pi]) = 1$ for any nonempty polyhedron π.

Finally, we need a result from [4], which is a statement about the wider class of all polyhedral cones.

Theorem 6 ([4], Theorem 2.7)**.** *There exists a valuation* $\mathcal{D}$ *on the space of characteristic functions of polyhedral cones such that*

$$\mathcal{D}([\pi]) = [\pi^\star].$$

We shall usually abuse notation and use $\star$ for the valuation $\mathcal{D}$, so the relation of Theorem 6 becomes

$$[\pi]^\star = [\pi^\star].$$

The essential property here is that $\star$ interchanges convolution with pointwise product:

$$([C_1] \bullet [C_2] \bullet \cdots \bullet [C_\ell])^\star = ([C_1 \oplus \cdots \oplus C_\ell])^\star = [C_1^\star \cap \cdots \cap C_\ell^\star] = [C_1]^\star \cdot [C_2]^\star \cdots [C_\ell]^\star,$$

where $\bullet$ is convolution and $\cdot$ is the pointwise product of characteristic functions.

We have the following important identity.

Proposition 7. *For any ordered set partition $(S_1, S_2, \ldots, S_k)$ of (any subset of) $\{1, \ldots, n\}$, we have*

$$[[S_1, S_2]] \bullet [[S_2, S_3]] \bullet \cdots \bullet [[S_k, S_1]] = [[S_1 \cup \cdots \cup S_k]].$$

Proof. It is convenient to check this by dualizing. Then, $\star$ gives $[S_i, S_{i+1}]^\star = \{x \in V_0^n : x_{(S_i)} \geq x_{(S_{i+1})}\}$, where $x_{(S)}$ is shorthand for $x_{i_1} = \cdots = x_{i_\ell}$ if $S = \{i_1, \ldots, i_\ell\}$. As $\star$ interchanges convolution and pointwise product of characteristic functions, we have

$$\begin{aligned} ([[S_1, S_2]] \bullet [[S_2, S_3]] \bullet \cdots \bullet [[S_k, S_1]])^\star &= \left\{x \in V_0^n : x_{(S_1)} \geq x_{(S_2)} \geq \cdots \geq x_{(S_k)} \geq x_{(S_1)}\right\} \\ &= [\{(0, \ldots, 0)\}] \\ &= [[S_1 \cup \cdots \cup S_k]]^\star, \end{aligned}$$

and since $\star$ is an involution for closed convex cones, applying $\star$ again completes the proof. □

3. Open plates

For any ordered pair of disjoint subsets (S_1, S_2) of $\{1, \ldots, n\}$, define

$$\mu_{S_1,S_2} = [[S_1, S_2]] - [[S_1]] \bullet [[S_2]],$$

the characteristic function of the set

$$\left\{\sum_{i \in S_1 \cup S_2} t_i e_i \in V_0^n : t_{S_1} > 0\right\},$$

where the inequality is strict. It will be understood that unless indicated otherwise, all products of μ's are convolution, using the Euler characteristic as the measure, see Theorem 5.

Further denote by 1_S the characteristic function of the subspace

$$[S] = \left\{\sum_{i \in S} t_i e_i \in V_0^n : t_i \in \mathbb{R}\right\}.$$

Then for any disjoint nonempty subsets $S_1, \ldots, S_\ell$ of $\{1, \ldots, n\}$, $1_{S_1} 1_{S_2} \cdots 1_{S_k}$ is the characteristic function of the subspace

$$\{x \in V_0^n : x_{S_i} = 0 \text{ and } x_j = 0 \text{ whenever } j \notin S_1 \cup \cdots \cup S_\ell\}.$$

Proposition 8. *We have*

$$\mu^2_{S_1,S_2} = -\mu_{S_1,S_2}, \text{ and } \mu_{S_1,S_2}\mu_{S_2,S_1} = 0.$$

More generally, we have the cycle identities

$$\mu_{S_1,S_2}\mu_{S_2,S_3} \cdots \mu_{S_{k-1},S_k}\mu_{S_k,S_1} = 0$$

for any ordered set partition $(S_1, \ldots, S_k)$ of a subset of $\{1, \ldots, n\}$.

Proof. We supply the algebraic proofs for the identities $\mu^2_{S_1,S_2} = -\mu_{S_1,S_2}$ and $\mu_{S_1,S_2}\mu_{S_2,S_1} = 0$. We have

$$\begin{aligned} \mu^2_{S_1,S_2} &= ([[S_1, S_2]] - [[S_1]] \bullet [[S_2]])^2 = ([[S_1, S_2]]^2 - 2[[S_1, S_2]] + [[S_1]]^2 \bullet [[S_2]]^2) \\ &= ([[S_1, S_2]] - 2[[S_1, S_2]] + [[S_1]] \bullet [[S_2]]) = -([[S_1, S_2]] - [[S_1]] \bullet [[S_2]]) \\ &= -\mu_{S_1,S_2}, \end{aligned}$$

and

$$\begin{aligned}\mu_{S_1,S_2}\mu_{S_2,S_1} &= ([[S_1,S_2]]-[[S_1]]\bullet[[S_2]])([[S_2,S_1]]-[[S_1]]\bullet[[S_2]])\\ &= [[S_1,S_2]]\bullet[[S_2,S_1]]-([[S_1,S_2]]+[[S_2,S_1]])+[[S_1]]\bullet[[S_2]]\\ &= [[S_1\cup S_2]]-([[S_1\cup S_2]]+[[S_1]]\bullet[[S_2]])+[[S_1]]\bullet[[S_2]]\\ &= 0.\end{aligned}$$

The cycle identities can be seen in general through an application of duality for polyhedral cones. Then, as the dual $\mu^\star_{S_i,S_j}$'s are open half-spaces through the origin in V_0^n, and the intersection of the supports of the characteristic functions of the duals $\mu^\star_{S_1,S_2},\ldots,\mu^\star_{S_k,S_1}$'s is empty, their (pointwise) product is identically zero. Further, as $\star$ is an involution on the closed cones $[S_1,S_2]$ and 1_S, we have

$$\begin{aligned}(\mu^\star_{S_1,S_2})^\star &= ([[S_1,S_2]]^\star-1^\star_{S_1}1^\star_{S_2})^\star\\ &= ([[S_1,S_2]]^\star)^\star-(1^\star_{S_1})^\star(1^\star_{S_2})^\star\\ &= [[S_1,S_2]]-1_{S_1}1_{S_2}\\ &= \mu_{S_1,S_2}.\end{aligned}$$

It follows that

$$\begin{aligned}\mu_{S_1,S_2}\cdots\mu_{S_k,S_1} &= (\mu^\star_{S_1,S_2}\cdots\mu^\star_{S_k,S_1})^\star\\ &= 0\end{aligned}$$

□

Denote by $e_j(x_1,\ldots,x_k)$ the jth elementary symmetric function in the variables $x_1,\ldots,x_k$, obtained from the generating function

$$(1+tx_1)(1+tx_2)\cdots(1+tx_k)=\sum_{j=0}^{k}e_j(x_1,\ldots,x_k)t^j$$

Corollary 9. *We have*

$$[(S_1,\ldots,S_k)]=[[S_1\cup\cdots\cup S_k]]+\sum_{j=0}^{k-1}(-1)^{(k-2)-j}e_j([[S_1,S_2]],[[S_2,S_3]],\ldots,[[S_k,S_1]]).$$

Proof. By Proposition 8, we have the cycle identity $\mu_{12}\mu_{23}\cdots\mu_{k1}=0$, hence

$$\begin{aligned}0 &= \mu_{S_1S_2}\mu_{S_2S_3}\cdots\mu_{S_kS_1}\\ &= ([[S_1,S_2]]-1)([[S_2,S_3]]-1)\cdots([[S_k,S_1]]-1),\end{aligned}$$

which expands to an alternating sum of the elementary symmetric functions in the variables

$$[[S_1,S_2]],\ldots,[[S_k,S_1]].$$

Noting that the piecewise constant function $[(S_1,\ldots,S_k)]$ can be expressed as an elementary symmetric function of degree $k-1$, as

$$[(S_1,\ldots,S_k)]=e_{k-1}([[S_1,S_2]],\ldots,[[S_k,S_1]]),$$

we conclude the proof by solving for $e_{k-1}([[S_1,S_2]],\ldots,[[S_k,S_1]])$. □

In particular, for any three disjoint nonempty subsets $S_1,S_2,S_3\subset\{1,\ldots,n\}$ we have the fundamental relation

$$[[S_1,S_2,S_3]]+[[S_2,S_3,S_1]]+[[S_3,S_1,S_2]]=1_{S_1\cup S_2\cup S_3}+\mu_{S_1,S_2}+\mu_{S_2,S_3}+\mu_{S_3,S_1}-1_{S_1}1_{S_2}1_{S_3}.$$

Example 10. In the cases $n = 3, 4$ we have the functional representations of characteristic functions of plates, respectively

$$\begin{aligned}
[[1,2,3]] &\mapsto \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)}\\
[[1,2,3,4]] &\mapsto \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_3+x_4)}
\end{aligned}$$

while for blades we obtain, after partial fraction identities,

$$\begin{aligned}
[(1,2,3)] &= [[1,2,3]]+[[2,3,1]]+[[3,1,2]]\\
&\mapsto \frac{1}{y_1(y_1+y_2)(y_1+y_2+y_3)}+\frac{1}{y_2(y_2+y_3)(y_1+y_2+y_3)}+\frac{1}{y_3(y_1+y_3)(y_1+y_2+y_3)}\\
&= \frac{1}{y_1(y_1+y_2)y_3}+\frac{1}{y_2(y_2+y_3)y_1}+\frac{1}{y_3(y_3+y_1)y_2}-\frac{1}{y_1y_2y_3},
\end{aligned}$$

and

$$\begin{aligned}
&[(1,2,3,4)] = [[1,2,3,4]]+[[2,3,4,2]]+[[3,4,1,2]]+[[4,1,2,3]]\\
\mapsto\ & \frac{1}{y_1(y_1+y_2)(y_1+y_2+y_3)(y_1+y_2+y_3+y_4)}+\frac{1}{y_2(y_2+y_3)(y_2+y_3+y_4)(y_1+y_2+y_3+y_4)}\\
+\ & \frac{1}{y_3(y_3+y_4)(y_1+y_3+y_4)(y_1+y_2+y_3+y_4)}+\frac{1}{y_4(y_1+y_4)(y_1+y_2+y_4)(y_1+y_2+y_3+y_4)}\\
=\ & \frac{1}{y_1(y_1+y_2)(y_1+y_2+y_3)y_4}+\frac{1}{y_2y_3(y_3+y_4)(y_1+y_3+y_4)}+\frac{1}{y_3y_4(y_1+y_4)(y_1+y_2+y_4)}\\
+\ & \frac{1}{y_3y_4(y_1+y_4)(y_1+y_2+y_4)}+\frac{1}{y_1(y_1+y_2)y_3(y_3+y_4)}+\frac{1}{y_2(y_2+y_3)y_4(y_1+y_4)}\\
-\ & \frac{1}{y_1(y_1+y_2)y_3y_4}-\frac{1}{y_1y_2(y_2+y_3)y_4}-\frac{1}{y_1y_2y_3(y_3+y_4)}-\frac{1}{y_2y_3y_4(y_1+y_4)}\\
+\ & \frac{1}{y_1y_2y_3y_4}
\end{aligned}$$

Corollary 11. *Let $(S_1, \ldots, S_k)$ be an ordered set partition. Then, modulo characteristic functions of cones of codimension ≥ 1, we have the cyclic sum relation*

$$[[S_1, S_2, \ldots, S_k]] + [[S_2, S_3, \ldots, S_1]] + \cdots + [[S_k, S_1, \ldots, S_{k-1}]] \equiv [[S_1 \cdots S_k]],$$

that is the union of the cyclic block rotations of the plate $[S_1, \ldots, S_k]$ is the whole ambient space V_0^n.

Proof. We have

$$\begin{aligned}
0 &= \mu_{S_1,S_2}\cdots\mu_{S_k,S_1}\\
&= ([[S_1,S_2]]-1)([[S_2,S_3]]-1)\cdots([[S_k,S_1]]-1)\\
&= [[S_1S_2\cdots S_k]]-([[S_1,S_2,\ldots,S_k]]+[[S_2,S_3,\ldots,S_1]]+[[S_k,S_1,\ldots,S_{k-1}]])+\mathcal{O}(k-2),
\end{aligned}$$

and it follows that

$$[[S_1S_2\cdots S_k]] \equiv [[S_1, S_2, \ldots, S_k]] + [[S_2, S_3, \ldots, S_1]] + \cdots + [[S_k, S_1, \ldots, S_{k-1}]],$$

where we have modded out by $\mathcal{O}(k-2)$, which is an alternating sum of only characteristic functions of cones of dimension $k-2 \leq n-2$, those with codimension at least 1 in V_0^n. $\square$

The identity of Proposition 12 can be recognized as a deformation of the fundamental identity in the so-called subdivision algebra, see [23, 30].

Proposition 12. *We have the triangulation identity for closed cones*

$$([[S_1, S_2]] + [[S_2, S_3]]) \bullet [[S_1, S_3]] = [[S_1, S_2]] \bullet [[S_2, S_3]] + [[S_1, S_3]],$$

while for open cones we have

$$(\mu_{S_1,S_2} + \mu_{S_2,S_3})\mu_{S_1S_3} = \mu_{S_1S_2}\mu_{S_2S_3} - \mu_{S_1S_3}.$$

Proof. It suffices to verify the identity in the plane V_0^3, where the two cones $\langle e_1 - e_3, e_2 - e_3\rangle_+$ and $\langle e_1 - e_2, e_1 - e_3\rangle_+$ intersect on the common line $\langle e_1 - e_3\rangle_+$. Therefore by inclusion/exclusion we have for their characteristic functions the identity

$$[\langle e_1 - e_3, e_2 - e_3\rangle_+] + [\langle e_1 - e_2, e_1 - e_3\rangle_+] = [\langle e_1 - e_2, e_2 - e_3\rangle_+] + [\langle e_1 - e_3\rangle_+],$$

or in the bracket notation,

$$[[1, 3]] \bullet [[2, 3]] + [[1, 2]] \bullet [[1, 3]] = [[1, 2]] \bullet [[2, 3]] + [[1, 3]].$$

For characteristic functions of cones generated by open half lines, $\mu_{i,j} = [[i, j]] - 1$, where 1 is the characteristic function of the point at the origin, the identity can similarly be seen to be

$$(\mu_{ij} + \mu_{jk})\mu_{ik} = \mu_{ij}\mu_{jk} - \mu_{ik}.$$

□

4. Locality for permutohedral honeycombs

Ocneanu was originally partly motivated to study blades by the desire to model moduli spaces of honeycomb tessellations; indeed, this was the guiding principle for the present work as well. With an eye toward future work, we now sketch part of this connection as we see it. See Figure 1.

The vertices of the honeycomb tiling of V_0^n with weight permutohedra, denoted $\mathcal{H}^n$ have a natural construction using the affine Weyl group; we shall use the discussion in Proposition 16.6 in [37] for inspiration. Choose a generic point $x \in \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$ off the affine reflection hyperplanes, that is such that $x_i - x_j \notin \mathbb{Z}$ for all $i \neq j$. Then there exists a unique permutation σ and integers $a_1, \ldots, a_n$ such that

$$\begin{gathered} a_1 < x_{\sigma_1} - x_{\sigma_2} < a_1 + 1 \\ a_2 < x_{\sigma_2} - x_{\sigma_3} < a_2 + 1 \\ \vdots \\ a_n < x_{\sigma_n} - x_{\sigma_1} < a_n + 1. \end{gathered}$$

Then, the n vertices in the honeycomb tiling adjacent to x are obtained by reflecting across hyperplanes parallel to $x_{\sigma_i} = x_{\sigma_{i+1}}$, that is by translation from x in the n directions respectively $e_{\sigma_1} - e_{\sigma_2}, \ldots, e_{\sigma_n} - e_{\sigma_1}$. From Corollary 9 we see that the union of the conical hulls of $n - 1$ at a time gives n cones that intersect only on common boundaries, thereby forming a complete fan; taking the union of conical hulls of $n - 2$ at a time we obtain the $(n - 2)$-skeleton of the honeycomb tiling *near the vertex x*. This part of the $(n-2)$-skeleton coincides up to translation with the blade $((\sigma_1, \ldots, \sigma_n))$, the complement of the unions of the interiors of the n cones

$$\langle e_{\sigma_1} - e_{\sigma_2}, \ldots, e_{\sigma_{n-1}} - e_{\sigma_n}\rangle_+,\ \langle e_{\sigma_2} - e_{\sigma_3}, \ldots, e_{\sigma_n} - e_{\sigma_1}\rangle_+, \ldots,\ \langle e_{\sigma_n} - e_{\sigma_1}, \ldots, e_{\sigma_{n-2}} - e_{\sigma_{n-1}}\rangle_+.$$

We have proved Proposition 13.

Proposition 13. *Let p be a vertex of the honeycomb tiling $\mathcal{H}^n$ of V_0^n. Then, there is an open neighborhood of p which intersects $\mathcal{H}^n$ in a translation of a blade $((\sigma_1, \ldots, \sigma_n))$, for some permutation $\sigma \in \mathfrak{S}_n$.*

Informally, looking forward to Theorem 20, we summarize Proposition 13, as follows: we say that the honeycomb tessellation by standard permutohedra factorizes *locally* in the convolution algebra as a product of characteristic functions of tripods and 1-dimensional subspaces.

5. Blades from symmetric functions on edges of a cyclic Dynkin graph

Recall that for any nonempty subsets $S_1, S_2 \subset \{1, \ldots, n\}$ with $S_1 \cap S_2 = \emptyset$, we denote by

$$\mu_{S_1,S_2} = [[S_1, S_2]] - [[S_1] \cap [S_2]]$$

the characteristic function of the set

$$\{x \in V_0^n : x_{S_1} > 0,\ x_{S_1 \cup S_2} = 0,\ x_i = 0 \text{ for } i \notin S_1 \cup S_2\},$$

where the equality is strict, and we define the convolution product

$$\mu_{S_1,\ldots,S_k} = \mu_{S_1,S_2} \cdots \mu_{S_{k-1},S_k}.$$

If $(S_1, \ldots, S_k)$ is an ordered set partition, the blade $((S_1, S_2, \ldots, S_k))$ is the *complement* in V_0^n of the union of the interiors of the plates

$$[S_1, \ldots, S_k], [S_2, \ldots, S_k, S_1], \ldots, [S_k, S_1 \ldots, S_{k-1}].$$

This union of interiors has characteristic function

$$\mu_{S_1,S_2}\mu_{S_2,S_3} \cdots \mu_{S_{k-1},S_k} + \mu_{S_2,S_3}\mu_{S_3,S_4} \cdots \mu_{S_k,S_1} + \cdots \mu_{S_k,S_1}\mu_{S_1,S_2} \cdots \mu_{S_{k-2},S_{k-1}}.$$

Definition 14. For an ordered set partition $(S_1, \ldots, S_k)$ of $\{1, \ldots, n\}$ with $k \geq 3$, we define the blade $((S_1, \ldots, S_k))$ to be the set theoretic *union* of the Minkowski sums of cones,

$$((S_1, \ldots, S_k)) = \bigcup_{1 \leq i < j \leq k} [S_1, S_2] \oplus \cdots \oplus [\widehat{S_i, S_{i+1}}] \oplus \cdots \oplus [\widehat{S_j, S_{j+1}}] \oplus \cdots \oplus [S_k, S_1],$$

where the hat means that that corresponding term has been omitted, and where we adopt the convention $S_{k+1} = S_1$. When $k = 2$ and $S_1 \sqcup S_2 = \{1, \ldots, n\}$, then we set

$$((S_1, S_2)) = [[S_1]] \oplus [[S_2]].$$

Finally, when $k = 1$ we set

$$((12 \cdots n)) = V_0^n.$$

When the blocks in the ordered set partition are singlets, we have the following interesting interpretation.

Proposition 15. *The (nondegenerate) blade $((\sigma_1, \sigma_2, \ldots, \sigma_n))$ is the union of the codimension 1 facets to the normal fan to the simplex with vertices*

$$e_{\sigma_1} - e_{\sigma_n}, e_{\sigma_2} - e_{\sigma_1}, \ldots, e_{\sigma_n} - e_{\sigma_{n-1}}.$$

Denote by $\Gamma_{S_1,\ldots,S_k}$ the characteristic function of the blade $((S_1, \ldots, S_k))$.
For compactness we further denote

$$\Gamma_S = 1_S \text{ and } \Gamma_{S_1,S_2} = [[S_1, S_2]].$$

We distinguish the case of blades which are labeled by 3-block ordered set partitions.

Definition 16. For a triple of disjoint subsets (S_1, S_2, S_3) of $\{1, \ldots, n\}$, the (characteristic) function

$$\gamma_{S_1,S_2,S_3} = 1_{S_1}1_{S_2}1_{S_3} + \mu_{S_1,S_2} + \mu_{S_2,S_3} + \mu_{S_3,S_1}$$

is called a *tripod.*

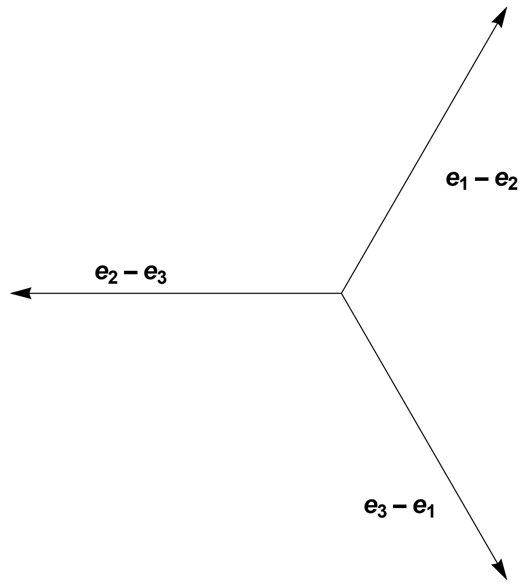

FIGURE 4. Blade in three coordinates, characteristic function $\gamma_{1,2,3} = 1 + \mu_{12} + \mu_{23} + \mu_{31}$. Arrows indicate that the rays extend to infinity. The μ_{ij}'s are characteristic functions of open rays extending from $(0, 0, 0)$, while "1" is the characteristic function of the point $(0, 0, 0)$ itself.

Informally, a blade is the union of the facets of the complete fan built from the k cyclic rotations of the plate $[S_1, \ldots, S_k]$. Note that when the blocks S_i are not singlets, the cones in the fan are not in general pointed.

We show now that lumped blades, labeled by standard ordered set partitions with at least one block of size ≥ 2, reduce naturally.

Proposition 17. *The characteristic function $\Gamma_{S_1,\ldots,S_k}$ of any blade $((S_1, \ldots, S_k))$ with $k \geq 3$ is a convolution product of characteristic functions of (one-dimensional) tripods $\gamma_{i,j,k}$ and sticks 1_{ab} labeled by respectively triples (i, j, k) of integers with $i < j < k$ and $a < b$, where $i, j, k, a, b \in \{1, \ldots, n\}$.*

Proof. Choose elements $i_j \in S_j$ for $j = 1, \ldots, k$. Then, applying the identities $1_{i_1i_2} \cdots 1_{i_{s-1}i_s} = I_S$ and $1_T\gamma_{i,j,k} = \gamma_{T,j,k}$ whenever respectively $S = \{i_1, \ldots, i_s\}$ and $i \in T$, we have

$$\Gamma_{S_1,\ldots,S_k} = 1_{S_1}1_{S_2} \cdots 1_{S_k}\gamma_{i_1,i_2,i_3}\gamma_{i_1,i_3,i_4} \cdots \gamma_{i_1,i_{k-1},i_k}$$

□

Corollary 18. *The characteristic function of the blade $((S_1, \ldots, S_k))$ has the expansion in terms of convolution products, as*

$$\begin{aligned}\Gamma_{S_1,\ldots,S_k} &= 1_{S_1\cup\cdots\cup S_k} - \left(\mu_{S_1,S_2,\ldots,S_k} + \mu_{S_2,S_3,\ldots,S_1} + \cdots + \mu_{S_k,S_1,\ldots,S_{k-1}}\right) \\ &= 1_{S_1\cup\cdots\cup S_k} - \left((\mu_{S_1,S_2} \cdots \mu_{S_{k-1},S_k}) + (\mu_{S_2,S_3} \cdots \mu_{S_k,S_1}) + \cdots + (\mu_{S_k,S_1} \cdots \mu_{S_{k-2},S_{k-1}})\right)\end{aligned}$$

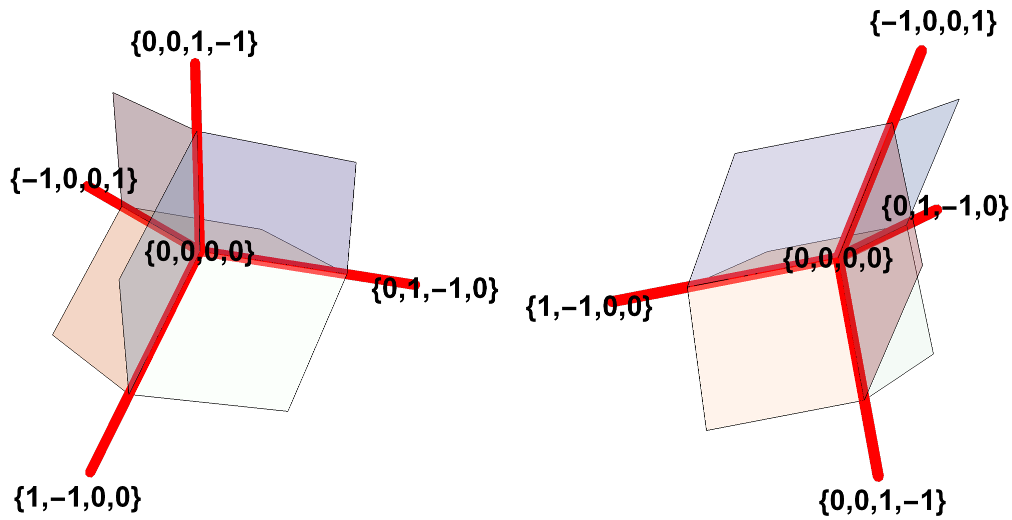

FIGURE 5. The characteristic function of the $n = 4$ blade as a sum of elementary symmetric functions:

$$\Gamma_{1,2,3,4} = 1 + (\mu_{12} + \mu_{23} + \mu_{34} + \mu_{41}) + (\mu_{12}\mu_{23} + \mu_{12}\mu_{34} + \mu_{12}\mu_{41} + \mu_{23}\mu_{34} + \mu_{23}\mu_{41} + \mu_{34}\mu_{41}).$$

Proposition 19. *We have*

$$\Gamma_{S_1,\ldots,S_k} = 1_{S_1}1_{S_2}\cdots 1_{S_k} + \sum_{j=1}^{k-2} \mathbf{e}_j(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1}),$$

where e_j is the jth elementary symmetric function.

Proof. In the following computation, we shall abuse notation and write $\mathbf{1}$ for the product of characteristic functions of any subcollection of the subspaces $[S_i]$ for $i = 1,\ldots,k$. By Proposition 8 we have $\mu_{S_1,S_2}\cdots\mu_{S_k,S_1} = 0$; but $\mathbf{1} + \mu_{S_1,S_2} = [[S_1,S_2]]$, hence

$$\begin{aligned}
[[12\cdots n]] &= [[S_1,S_2]] \bullet [[S_2,S_3]] \bullet \cdots \bullet [[S_k,S_1]] \\
&= (\mathbf{1} + \mu_{S_1,S_2})(\mathbf{1} + \mu_{S_2,S_3})\cdots(\mathbf{1} + \mu_{S_k,S_1}) \\
&= \mathbf{1} + \sum_{j=1}^{k-1} \mathbf{e}_j(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1}) \\
&= \mathbf{1} + \sum_{j=1}^{k-2} \mathbf{e}_j(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1}) + \mathbf{e}_{k-1}(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1}),
\end{aligned}$$

the where $\mathbf{e}_j$ is the jth elementary symmetric function, and $\mathbf{e}_{k-1}(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1})$ is the characteristic function of the complement of the blade $[(S_1,\ldots,S_k)]$. The first equality follows since $[[S_1,S_2]] \bullet \cdots \bullet [[S_k,S_1]] = [[S_1 \cup \cdots \cup S_k]]$.

Consequently we have

$$\Gamma_{S_1,\ldots,S_k} = \mathbf{1} + \sum_{j=1}^{k-2} \mathbf{e}_j(\mu_{S_1,S_2},\ldots,\mu_{S_k,S_1}).$$

$\square$

Theorem 20. *The characteristic function of the blade $\Gamma_{S_1,\ldots,S_k}$ labeled by a standard ordered set partition $(S_1,\ldots,S_k)$ of $\{1,\ldots,n\}$ admits the "flag" factorization*

$$\Gamma_{S_1,\ldots,S_k} = \gamma_{S_1,S_2,S_3}\gamma_{S_1,S_3,S_4}\cdots\gamma_{S_1,S_{k-1},S_k}.$$

Proof. To improve readability, let us temporarily adopt the notation $\mu_{S_i,S_j} = \mu_{i,j}$ and $\gamma_{S_i,S_j,S_k} = \gamma_{i,j,k}$, and

$$\mu_{S_1,S_2}\mu_{S_2,S_3}\cdots\mu_{S_{k-1},S_k} = \mu_{1,2,\ldots,k}.$$

We shall induct on k, using the identities

$$\Gamma_{S_1,\ldots,S_k} = [[S_1\cup\cdots\cup S_k]] - \Big((\mu_{S_1,S_2}\cdots\mu_{S_{k-1},S_k}) + (\mu_{S_2,S_3}\cdots\mu_{S_k,S_1}) + \cdots + (\mu_{S_k,S_1}\cdots\mu_{S_{k-2},S_{k-1}})\Big),$$

that is

$$\Gamma_{S_1,\ldots,S_k} = 1_{S_1\cup\cdots\cup S_k} - \Big(\mu_{12\cdots k} + \mu_{23\cdots k1} + \cdots + \mu_{k1\cdots(k-1)}\Big).$$

For the base case we have

$$(\Gamma_{1,2,3} =)\gamma_{1,2,3} = 1_{123} - (\mu_{1,2}\mu_{2,3} + \mu_{2,3}\mu_{3,1} + \mu_{3,1}\mu_{1,2}) = 1_{1(k-1)(k)} - \mu_{1,k-1,k} - \mu_{k-1,k,1} - \mu_{k,1,k-1}.$$

Computing in the flag triangulation gives

$$\begin{aligned}
& \gamma_{1,2,3}\gamma_{1,3,4}\cdots\gamma_{1,k-2,k-1}\gamma_{1,k-1,k} \\
= \; & (\gamma_{1,2,3}\gamma_{1,3,4}\cdots\gamma_{1,k-2,k-1})\gamma_{1,k-1,k} \\
= \; & (1_{12\cdots(k-1)} - \mu_{1,2,\ldots,k-1} - \mu_{2,3,\ldots,1} - \cdots - \mu_{k-1,1,\ldots,k-2})(1_{1(k-1)(k)} - \mu_{1,k-1,k} - \mu_{k-1,k,1} - \mu_{k,1,k-1}) \\
= \; & 1_{12\cdots k} \\
- \; & 1_{12\cdots(k-1)}(\mu_{1,k-1,k} + \mu_{k-1,k,1} + \mu_{k,1,k-1}) \\
- \; & 1_{1(k-1)(k)}(\mu_{1,2,\ldots,k-1} + \mu_{2,3,\ldots,1} + \cdots + \mu_{k-1,1,\ldots,k-2}) \\
+ \; & (\mu_{1,k-1,k} + \mu_{k-1,k,1} + \mu_{k,1,k-1})(\mu_{1,2,\ldots,k-1} + \mu_{2,3,\ldots,1} + \cdots + \mu_{k-1,1,\ldots,k-2})
\end{aligned}$$

We aim to prove that the lines 2 and 3 are zero.

We first establish some essential (but easily verified) identities. First, we have

$$\mu_{p,q}\mu_{i,\ldots,j} = -\mu_{i,\ldots,j}$$

for any $i \le p < q \le j$, as can be checked geometrically by dualizing with $\star$, in which case the convolution $\bullet$ becomes the pointwise product of characteristic functions. In the dual, $\mu^\star_{p,q}$ takes the value -1 on an open half space containing the support of $\mu^\star_{i,i+1,\ldots,p,\ldots,q,\ldots,i-1}$, (and is zero on the complement), hence the pointwise product acts by switching the sign of $\mu^\star_{i,i+1,\ldots,p,\ldots,q,\ldots,i-1}$.

Whenever $p,q \in \{1,\ldots,j\}$ we have

$$1_{12\cdots j}\mu_{p,q} = 1_{12\cdots j}(1_{pq} - [[p,q]]) = 1_{12\cdots j} - 1_{12\cdots j} = 0.$$

Now on the other hand, if $p \in \{1,2,\ldots,j\}$ but $q \notin \{1,2,\ldots,j\}$ then we have

$$1_{12\cdots j}\mu_{p,q} = 1_{12\cdots j}([[p,q]] - 1) = [[12\cdots j,q]] - 1_{12\cdots j}.$$

Moreover, using

$$0 = \mu_{S_1,S_2}\mu_{S_2,S_1} = ([[S_1,S_2]] - 1_{S_1}1_{S_2})([[S_2,S_1]] - 1_{S_1}1_{S_2})$$

it follows that

$$\begin{aligned}
& 1_{12\cdots(k-1)}(\mu_{1,k-1}\mu_{k-1,k}+\mu_{k-1,k}\mu_{k,1}+\mu_{k,1}\mu_{1,k-1}) \\
= \; & 0\cdot\mu_{k-1,k} \\
+ \; & ([[12\cdots k-1,k]]-1_{12\cdots k-1})([[k,12\cdots k-1]]-1_{12\cdots k-1}) \\
+ \; & \mu_{k,1}\cdot 0 \\
= \; & 0.
\end{aligned}$$

This shows that the second line vanishes. Proving that for the third line we have

$$1_{1(k-1)(k)}(\mu_{1,2,\ldots,k-1}+\mu_{2,3,\ldots,1}+\cdots+\mu_{k-1,1,\ldots,k-2})=0$$

is similar and we omit the computation.

It remains to prove the identity

$$\begin{aligned}
&(\mu_{1,k-1,k}+\mu_{k-1,k,1}+\mu_{k,1,k-1})(\mu_{1,2,\ldots,k-1}+\mu_{2,3,\ldots,1}+\cdots+\mu_{k-1,1,\ldots,k-2}) \\
&= -\left(\mu_{12\cdots k}+\mu_{23\cdots k1}+\cdots+\mu_{k1\cdots(k-1)}\right).
\end{aligned}$$

Indeed, while

$$\begin{aligned}
& \mu_{1,k-1}\mu_{k-1,k}(\mu_{1,2,\ldots,k-1}+\mu_{2,3,\ldots,1}+\cdots+\mu_{k-1,1,\ldots,k-2}) \\
= \; & (-\mu_{1,\ldots,k-1}\mu_{k-1,k})+(-\mu_{2,3,\ldots,k-1,1}\mu_{1,k-1}+\cdots) \\
= \; & (-\mu_{1,\ldots,k-1}\mu_{k-1,k})+(0+\cdots+0) \\
= \; & -\mu_{1,2,\ldots,k}
\end{aligned}$$

and

$$\mu_{k,1}\mu_{1,k-1}(\mu_{1,2,\ldots,k-1}+\mu_{2,3,\ldots,k-1,1}+\cdots+\mu_{k-1,1,\ldots,k-2}) \;=\; -(\mu_{k,1,\ldots,k-1}),$$

we have the remaining $k-2$ nonzero contributions from $\mu_{k-1,k}\mu_{k,1}$:

$$\begin{aligned}
& \mu_{k-1,k}\mu_{k,1}(\mu_{1,2,\ldots,k-1}+\mu_{2,3,\ldots,k-1,1}+\cdots+\mu_{k-1,1,\ldots,k-2}) \\
= \; & -(\mu_{1,2,\ldots,k-1}\mu_{k-1,k}\mu_{k,1})+\mu_{2,3,\ldots,k-2,k-1}(\mu_{k-1,1}\mu_{k-1,k}\mu_{k,1})+\cdots+(\mu_{k-1,1}\mu_{k-1,k}\mu_{k,1})\mu_{1,\ldots,k-2} \\
= \; & -(\mu_{2,3,\ldots,k,1}+\mu_{3,\ldots,k,1,2}+\cdots+\mu_{k-1,1,\ldots,k-2}),
\end{aligned}$$

and we finally obtain

$$\gamma_{1,2,3}\gamma_{1,3,4}\cdots\gamma_{1,k-2,k-1}\gamma_{1,k-1,k}=1_{12\cdots k}-\left(\mu_{1,2,\ldots,k}+\mu_{2,3,\ldots,k,1}+\cdots+\mu_{k,1,\ldots,(k-1)}\right),$$

which, after substituting back in the notation $i\mapsto S_i$, becomes exactly $\Gamma_{S_1,\ldots,S_k}$, as desired. □

6. Blades from triangulations, and factorization independence

Choose a cyclic (counterclockwise, say) order on the vertices of a polygon with vertices labeled by the (standard) ordered set partition $\mathbf{S}=(S_1,\ldots,S_k)$ of $\{1,\ldots,n\}$, where we assume that $1\in S_1$. Let

$$\mathcal{T}=\{((S_{a_1},S_{b_1},S_{c_1})),\ldots,((S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}}))\}$$

be any set of (cyclically oriented) triangles forming a triangulation of the k-gon, labeled such that each (a_i,b_i,c_i) satisfies $a_i<b_i<c_i$ or $b_i<c_i<a_i$ or $c_i<a_i<b_i$.

Definition 21. We shall say that $(S_{a_i},S_{b_i},S_{c_i})$ is a cyclic subword of $(S_1,\ldots,S_k)$ if it satisfies the above condition, $a_i<b_i<c_i$ or $b_i<c_i<a_i$ or $c_i<a_i<b_i$.

Proposition 22. *The product*

$$\Gamma_{\mathcal{T}} = \gamma_{S_{a_1},S_{b_1},S_{c_1}}\gamma_{S_{a_2},S_{b_2},S_{c_2}}\cdots\gamma_{S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}}}$$

is independent of the triangulation $\mathcal{T}$.

Proof. As any two triangulations of an n-gon are related by a sequence of flips, as in Figures 6 and 7, replacing the pair $\{(S_i,S_j,S_k),(S_i,S_k,S_\ell)\}$ with $\{(S_i,S_j,S_\ell),(S_j,S_k,S_\ell)\}$, it suffices to verify directly that $\gamma_{S_i,S_j,S_k}\gamma_{S_i,S_k,S_\ell} = \gamma_{S_i,S_j,S_\ell}\gamma_{S_j,S_k,S_\ell}$. Abbreviating $\mu_{S_iS_j}$ as μ_{ij} and γ_{S_i,S_j,S_k} as γ_{ijk}, and replacing all products of the characteristic functions of subspaces, $[[S_\ell]]$, with 1, we have

$$\begin{aligned}
\gamma_{ijk}\gamma_{ik\ell} &= (1+\mu_{ij}+\mu_{jk}+\mu_{ki})(1+\mu_{ik}+\mu_{k\ell}+\mu_{\ell i})\\
&= 1+(\mu_{ij}+\mu_{jk}+\mu_{k\ell}+\mu_{\ell i})+(\mu_{ki}+\mu_{ik})\\
&+ (\mu_{ij}+\mu_{jk})\mu_{ik}+\mu_{ki}(\mu_{k\ell}+\mu_{\ell i})+(\mu_{ij}\mu_{k\ell}+\mu_{ij}\mu_{\ell i}+\mu_{jk}\mu_{k\ell}+\mu_{jk}\mu_{\ell i})+\mu_{ik}\mu_{ki}
\end{aligned}$$

Using the triangulation identity for a three-block (open) plate, $(\mu_{ij}+\mu_{jk})\mu_{ik} = \mu_{ij}\mu_{jk}-\mu_{ik}$ and $\mu_{ki}(\mu_{k\ell}+\mu_{\ell i}) = \mu_{k\ell}\mu_{\ell i}-\mu_{ki}$, as well as $\mu_{ik}\mu_{ki}=0$, after cancellation we obtain

$$\gamma_{ijk}\gamma_{ik\ell} = 1+(\mu_{ij}+\mu_{jk}+\mu_{k\ell}+\mu_{\ell i})+(\mu_{ij}\mu_{jk}+\mu_{ij}\mu_{k\ell}+\mu_{ij}\mu_{\ell i}+\mu_{jk}\mu_{k\ell}+\mu_{jk}\mu_{\ell i}+\mu_{k\ell}\mu_{\ell i}).$$

Performing an analogous computation for $\gamma_{ij\ell}\gamma_{jk\ell}$ yields the same result. $\square$

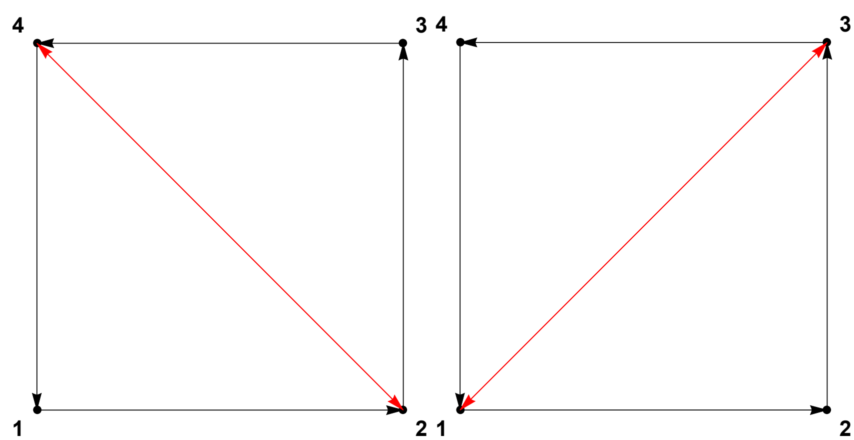

FIGURE 6. Independence of triangulation of the polygon for Proposition 22 $\{(1,2,4),(2,3,4)\} \leftrightarrow \{(1,2,3),(1,3,4)\}$.

Remark 23. Proposition 22 justifies our usual omission of the triangulation, using instead the notation $\Gamma_{S_1,\ldots,S_k}$ for the characteristic function of the blade labeled by the (standard) ordered set partition $(S_1,\ldots,S_k)$.

From the ring-theoretic identities for characteristic functions of blades in Theorem 20 and Proposition 22, we derive the corresponding set-theoretic identity for blades themselves, using Minkowski sums.

Corollary 24. *For a blade $((S_1,\ldots,S_k))$ labeled by a standard ordered set partition $(S_1,\ldots,S_k)$, and any triangulation $\{(S_{a_1},S_{b_1},S_{c_1}),\ldots,(S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}})\}$ of the cyclically-oriented polygon with vertices labeled $S_1,\ldots,S_k$, where $a_i<b_i<c_i$ for all i, we have*

$$((S_1,\ldots,S_k)) = ((S_{a_1},S_{b_1},S_{c_1}))\oplus\cdots\oplus((S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}})).$$

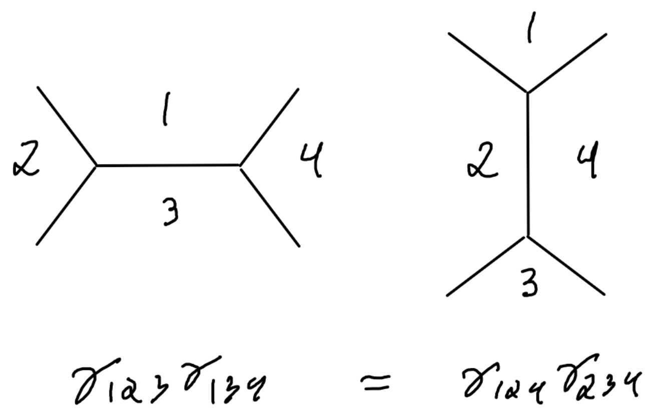

FIGURE 7. Alternate schematic representation for factorization independence, dual to that of Figure 6.

Proof. By Theorem 20 and Proposition 22 we have the identity of piecewise constant functions

$$\Gamma_{S_1,\ldots,S_k} = \Gamma_{S_{a_1},S_{b_1},S_{c_1}} \cdots \Gamma_{S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}}}.$$

Using the homomorphism property of the convolution product from Theorem 5, we express this as

$$[((S_1,\ldots,S_k))] = \Gamma_{S_1,\ldots,S_k} = [((S_{a_1},S_{b_1},S_{c_1})) \oplus \cdots \oplus ((S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}}))],$$

which is a relation of ($\{0,1\}$-valued) characteristic functions which holds identically. In particular the preimages of 1 on both sides are the same, hence

$$((S_1,\ldots,S_k)) = ((S_{a_1},S_{b_1},S_{c_1})) \oplus \cdots \oplus ((S_{a_{k-2}},S_{b_{k-2}},S_{c_{k-2}})),$$

proving that a blade $((S_1,\ldots,S_k))$ can be express (non-uniquely) as a Minkowski sum of tripods $((S_{a_t},S_{b_t},S_{c_t}))$. □

7. GRADUATED BLADES

Definition 25. To each ordered set partition $\mathbf{S} = (S_1,\ldots,S_k)$ we assign a piecewise-constant function

$$[(S_1,\ldots,S_k)] = [[S_1,S_2,\ldots,S_k]] + [[S_2,\ldots,S_k,S_1]] + \cdots + [[S_k,S_1,\ldots,S_{k-1}]].$$

Denote by $\hat{\mathcal{B}}^n$ the complex linear span of all such functions, as $(S_1,\ldots,S_k)$ varies over all standard ordered set partitions of $\{1,\ldots,n\}$.

As the k cyclic block rotations of the standard plate are intersecting, the sum of their characteristic functions surjects onto the set $\{1,2,\ldots,k\}$ and therefore $[(S_1,\ldots,S_k)]$ is *not* itself a characteristic function, see Figure 8. This justifies our new term *graduated.*

Proposition 26. *The set*

$$\{[(S_1,\ldots,S_k)] : \ (S_1,\ldots,S_k) \textit{ is a standard ordered set partition of } \{1,\ldots,n\}\}$$

is linearly independent.

Proof. By Proposition 12 of [16], the set of characteristic functions of plates $[[S_1,\ldots,S_k]]$ labeled by ordered set partitions is linearly independent; it follows that the set of functions $[(S_1,\ldots,S_k)]$, which are cyclic sums over block rotations of plates, is linearly independent as well, since each plate occurs in precisely one of the cyclic rotations of a unique standard ordered set partition. □

$$\varphi(x) = [(1,2,3)](x) = \begin{cases} 1, & \text{plates, dim}=2 \\ 2, & \text{half lines, dim}=1 \\ 3, & \text{dot, dim}=0 \end{cases}$$

Level sets:

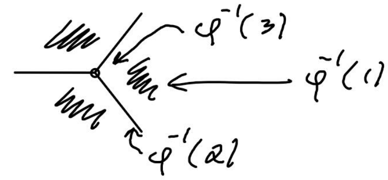

FIGURE 8. Level sets of the function

$$[(1,2,3)] = 3 + 2(\mu_{1,2} + \mu_{2,3} + \mu_{3,1}) + (\mu_{1,2,3} + \mu_{2,3,1} + \mu_{3,1,2}).$$

Recall from [16] that an ordered set partition $(S_1, \ldots, S_k)$ of the set $\{1, \ldots, n\}$ is *standard* if S_1 contains the minimal element in $S_1 \cup \cdots \cup S_k$. A (standard) composite ordered set partition is a set of ordered set partitions $\{\mathbf{S}_1, \ldots, \mathbf{S}_\ell\}$ of a set $\{1, \ldots, n\}$, if each $\mathbf{S}_i$ is a (standard) ordered set partition of a subset T_i with $T_1, \ldots T_\ell$ an (unordered) set partition of $\{1, \ldots, n\}$.

Theorem 27. *Let T be a (possibly empty) subset of $\{1, \ldots, n\}$ and let $\mathbf{S}_1, \ldots, \mathbf{S}_\ell$ be a (standard) composite ordered set partition of the set $\{1, \ldots, n\} \setminus T$. Denote by $m = 1 + |\mathbf{S}_1| + \cdots + |\mathbf{S}_\ell|$ the total number of blocks in the ordered set partitions $\mathbf{S}_1, \ldots, \mathbf{S}_\ell$, together with the set T.*

Then we have

$$[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)] \quad = \quad \sum_{\mathbf{U}} (-1)^{m - |\mathbf{U}|} [(\mathbf{U})],$$

where the sum is over all standard ordered set partitions $\mathbf{U} = (U_1, \ldots, U_k)$, say, such that each of the ordered set partitions

$$(T, \mathbf{S}_1), \ldots, (T, \mathbf{S}_\ell)$$

is a subword of some cyclic rotation of the ordered set partition $(U_1, \ldots, U_k)$.

Proof. We shall apply Theorem 21 of [16] to express

$$[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)]$$

as a signed sum of graduated functions of plates, labeled by ordered set partitions of $\{1, \ldots, n\}$. By definition we have

$$[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)] \quad = \quad \left(\sum_j [[T, \mathbf{S}_1]]^{(j)} \right) \bullet \cdots \bullet \left(\sum_j [[T, \mathbf{S}_\ell]]^{(j)} \right),$$

where the superscript (j) denotes the j^{th} block rotation,

$$[[V_1, \ldots, V_a]]^{(j)} = [[V_j, V_{j+1}, \ldots, V_{j-1}]]$$

for $(V_1, \ldots, V_a)$ any ordered set partition. This expands as

$$[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)] \quad = \quad \sum_{j_1, \ldots, j_\ell} \left([[T, \mathbf{S}_1]]^{(j_1)} \bullet \cdots \bullet [[T, \mathbf{S}_\ell]]^{(j_\ell)} \right),$$

where we remark now that $[[T, \mathbf{S}_i]]^{(j_i)}$ is standard only when $j_i = 1$. Applying Theorem 21 in [16] to expand each summand as a signed sum of characteristic functions of plates, we obtain

$$[[T,\mathbf{S}_1]]^{(j_1)}\bullet\cdots\bullet[[T,\mathbf{S}_\ell]]^{(j_\ell)} = \sum_{\mathbf{U}^{(j_1,\ldots,j_\ell)}}(-1)^{m-|\mathbf{U}^{(j_1,\ldots,j_\ell)}|}[[\mathbf{U}^{(j_1,\ldots,j_\ell)}]]$$

and summing both sides over all $(j_1,\ldots,j_\ell)$, after factorization we recover

$$\begin{aligned}[(T,\mathbf{S}_1)]\bullet\cdots\bullet[(T,\mathbf{S}_\ell)] &= \sum_{(j_1,\ldots,j_\ell)}[[T,\mathbf{S}_1]]^{(j_1)}\bullet\cdots\bullet[[T,\mathbf{S}_\ell]]^{(j_\ell)}\\ &= \sum_{(j_1,\ldots,j_\ell)}\sum_{\mathbf{U}^{(j_1,\ldots,j_\ell)}}(-1)^{m-|\mathbf{U}^{(j_1,\ldots,j_\ell)}|}[[\mathbf{U}^{(j_1,\ldots,j_\ell)}]]\\ &= \sum_{\mathbf{U}}(-1)^{m-|\mathbf{U}|}[(\mathbf{U})].\end{aligned}$$

□

Remark 28. It is interesting to note in Theorem 27 the similarity in the case when T is empty and all blocks $(S_1,\ldots,S_n)$ are singlets, with the so-called *Generalized BDDK relations for the full one-loop Parke-Taylor factors* from [1]. See also Example 29 below.

Example 29. By Lemma 27, the characteristic function $\Gamma_{1,2,3}\Gamma_{1,4,5}$ has the following expansion:

$$\begin{aligned}&[(1,2,3)]\bullet[(1,4,5)]\\ =\ &[(1,2,3,4,5)]+[(1,2,4,3,5)]+[(1,2,4,5,3)]+[(1,4,2,3,5)]+[(1,4,2,5,3)]+[(1,4,5,2,3)]\\ -\ &[(1,2,4,35)]-[(1,2,34,5)]-[(1,4,2,35)]-[(1,4,25,3)]-[(1,24,3,5)]-[(1,24,5,3)]\\ +\ &[(1,24,35)].\end{aligned}$$

In the functional representation $[[i,j]]\mapsto\frac{1}{1-x_i/x_j}$ (See [16] for well-definedness) we have

$$[(1,2,3,4,5)]\mapsto\frac{x_2x_3x_4x_5}{(x_1-x_2)(x_2-x_3)(x_3-x_4)(x_4-x_5)}+\frac{x_3x_4x_5x_1}{(x_2-x_3)(x_3-x_4)(x_4-x_5)(x_5-x_1)}+\cdots$$

$$=\frac{x_3x_4x_5x_1^2+x_2x_3x_4^2x_1+x_2x_3^2x_5x_1+x_2^2x_4x_5x_1-5x_2x_3x_4x_5x_1+x_2x_3x_4x_5^2}{(x_1-x_2)(x_2-x_3)(x_3-x_4)(x_4-x_5)(x_5-x_1)},$$

and one can verify that by partial fraction identities, modulo non-pointed cones (these are labeled by ordered set partitions having a block of size bigger than 1) we have

$$\begin{aligned}&[(1,2,3)]\bullet[(1,4,5)]\\ =\ &[(1,2,3,4,5)]+[(1,2,4,3,5)]+[(1,2,4,5,3)]+[(1,4,2,3,5)]+[(1,4,2,5,3)]+[(1,4,5,2,3)]\\ \mapsto\ &\frac{x_3x_4x_5x_1^2+x_2x_3x_4^2x_1+x_2x_3^2x_5x_1+x_2^2x_4x_5x_1-5x_2x_3x_4x_5x_1+x_2x_3x_4x_5^2}{(x_1-x_2)(x_2-x_3)(x_3-x_4)(x_4-x_5)(x_5-x_1)}+\cdots\\ =\ &\frac{(x_3x_1^2+x_2^2x_1-3x_2x_3x_1+x_2x_3^2)(x_5x_1^2+x_4^2x_1-3x_4x_5x_1+x_4x_5^2)}{(x_1-x_2)(x_2-x_3)(x_3-x_1)(x_1-x_4)(x_4-x_5)(x_5-x_1)}.\end{aligned}$$

Substituting $x_i=e^{-\varepsilon y_i}$ and series expanding in ε, truncating at order $\mathcal{O}(\varepsilon^1)$ we obtain a sum of elementary symmetric functions:

$$[(1,2,3,4,5)]\mapsto\sum_{j=0}^{3}\frac{1}{(4-j)!}e_j\left(\frac{1}{y_1-y_2},\ldots,\frac{1}{y_5-y_1}\right)\varepsilon^{-j}+\mathcal{O}(\varepsilon),$$

where $\mathcal{O}(\varepsilon)$ contains powers of ε^p for $p \geq 1$, and where the right-hand side of the shuffle identity becomes

$$\begin{aligned} & \frac{\left(-y_1^2+y_2y_1+y_3y_1-y_2^2-y_3^2+y_2y_3\right)\left(-y_1^2+y_4y_1+y_5y_1-y_4^2-y_5^2+y_4y_5\right)}{\left(y_1-y_2\right)\left(y_2-y_3\right)\left(y_3-y_1\right)\left(y_1-y_4\right)\left(y_4-y_5\right)\left(y_5-y_1\right)}\varepsilon^{-2} \\ + \quad & \frac{1}{2}\left(\frac{1}{y_1-y_2}+\frac{1}{y_2-y_3}+\frac{1}{y_3-y_1}+\frac{1}{y_1-y_4}+\frac{1}{y_4-y_5}+\frac{1}{y_5-y_1}\right)\varepsilon^{-1} \\ & +\frac{1}{4}+\mathcal{O}(\varepsilon). \end{aligned}$$

Warning: in the expansions of the functional representation built from

$$[[i,j] \mapsto \frac{1}{1-x_i/x_j}$$

above, we have drastically truncated an infinite series; as we have seen that elementary symmetric functions appear in the expansion of the characteristic function of a blade, it is not surprising that it appears for generating functions. However for larger n the structure of the expansion is considerably more complex and interesting, due to the nature of the well-known function

$$\frac{1}{1-e^{-z}},$$

see for example [6]. We leave such questions to future work.

Note that in the functional representation of type $[[i,j]] \mapsto \frac{1}{x_i-x_j}$, since characteristic functions of *both* non-pointed cones *and* higher codimension cones are in the kernel of the Laplace transform valuation, we have trivially

$$[(1,2,3,4,5)] \mapsto \frac{1}{(x_1-x_2)(x_2-x_3)\cdots(x_4-x_5)}+\frac{1}{(x_2-x_3)\cdots(x_4-x_5)(x_5-x_1)}+\cdots=0.$$

Remark 30. Consider now one of the other functional representations mentioned in [16],

$$[[S_1,\ldots,S_k] \quad \mapsto \quad \prod_{i=1}^{k}\frac{1}{1-\prod_{j\in S_i}x_j}, \tag{1}$$

for example

$$[[1,2,3,4,5]] \mapsto \frac{1}{\left(1-x_1\right)\left(1-x_1x_2\right)\left(1-x_1x_2x_3\right)\left(1-x_1x_2x_3x_4\right)\left(1-x_1x_2x_3x_4x_5\right)}.$$

The functional representation of Equation (1) is quite complicated and the numerator for the expansion of $[(1,2,3)] \bullet [(1,4,5)]$ appears not to factor nicely into the product of two irreducible polynomials similar to

$$\frac{\left(x_3x_1^2+x_2^2x_1-3x_2x_3x_1+x_2x_3^2\right)\left(x_5x_1^2+x_4^2x_1-3x_4x_5x_1+x_4x_5^2\right)}{\left(x_1-x_2\right)\left(x_2-x_3\right)\left(x_3-x_1\right)\left(x_1-x_4\right)\left(x_4-x_5\right)\left(x_5-x_1\right)},$$

but rather the numerator is a monstrous polynomial with 14781 monomials which likely doesn't factor at all. The complexity here likely arises because the blades $((1,2,3))$ and $((1,4,5))$ are not in orthogonal subspaces. On the other hand, the expansion using Equation (1) of for example $[(1,2,3)] \bullet (4,5)]$ and $[(1,2,3)] \bullet [(4,5,6)]$ both can be seen to have numerators which factor as products of two irreducible polynomials.

See Examples 47 and 48 in [16] for work with this functional representation for generalized permutohedral cones that are encoded by directed trees.

The leading order term, the coefficient of ε^{-5} in its analogous approximation using $x_i = e^{-\varepsilon y_i}$,

$$[[1,2,3,4,5]] \mapsto \frac{1}{y_1(y_1+y_2)(y_1+y_2+y_3)(y_1+y_2+y_3+y_4)(y_1+y_2+y_3+y_4+y_5)}\varepsilon^{-5}+\mathcal{O}(\varepsilon^{-4}),$$

is also quite interesting, but it also appears to lack a meaningful factorization property for convolutions involving non-orthogonal subspaces.

Example 31.

Recall the definition of the Parke-Taylor factor

$$PT(i_1,\ldots,i_n) = \frac{1}{(x_{i_1}-x_{i_2})(x_{i_2}-x_{i_3})\cdots(x_{i_n}-x_{i_1})},$$

where $x_1,\ldots,x_n$ are complex variables, see [36]. There is a so-called $U(1)$-decoupling identity,

$$PT(i_1,i_2\ldots,i_n)+PT(i_1,i_3\ldots,i_n,i_2)+\cdots+PT(i_1,i_n,i_2,\ldots,i_{n-1}) = 0,$$

which has an analog for blades. Let us illustrate what it looks like in the first nontrivial case.

In the functional representation

$$[[S_1,\ldots,S_k]] \mapsto \prod_{i=1}^{k}\frac{1}{1-\prod_{j\in S_1\cup\cdots\cup S_i}x_j},$$

for example

$$[[1,23,4]] \mapsto \frac{1}{(1-x_1)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},$$

we have

$$\begin{aligned}
[(2,3,4)] \;\mapsto\;& \frac{1}{(1-x_4)(1-x_2x_4)(1-x_2x_3x_4)}+\frac{1}{(1-x_3)(1-x_3x_4)(1-x_2x_3x_4)} \\
+\;& \frac{1}{(1-x_2)(1-x_2x_3)(1-x_2x_3x_4)} \\
=\;& \frac{1}{(1-x_2)(1-x_2x_3)(1-x_4)}+\frac{1}{(1-x_3)(1-x_4)(1-x_2x_4)} \\
+\;& \frac{1}{(1-x_2)(1-x_3)(1-x_3x_4)}+\frac{1}{1-x_2x_3x_4}-\frac{1}{(1-x_2)(1-x_3)(1-x_4)}.
\end{aligned}$$

Recall that

$$[(1,2,3,4)] \;=\; [[1,2,3,4]]+[[2,3,4,1]]+[[3,4,1,2]]+[[4,1,2,3]].$$

Summing over all standard ordered set partitions which contain $(1),(2,3,4)$ as cyclic subwords, after many partial fraction identities we obtain the same, but multiplied by $\frac{1}{1-x_1}$:

$$\begin{aligned}
& [(1,2,3,4)]+[(1,3,4,2)]+[(1,4,2,3)]-[(12,3,4)]-[(13,4,2)]-[(14,2,3)] \\
\mapsto\;& \frac{1}{(1-x_1)(1-x_2)(1-x_2x_3)(1-x_4)}+\frac{1}{(1-x_2)(1-x_3)(1-x_3x_4)(1-x_2)} \\
+\;& \frac{1}{(1-x_1)(1-x_4)(1-x_4x_2)(1-x_3)}+\frac{1}{(1-x_1)(1-x_2x_3x_4)} \\
-\;& \frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}.
\end{aligned}$$

Definition 32. We call an ordered set partition $(S_1, S_2, \ldots, S_k)$ of $\{1, \ldots, n\}$ *2-standard* if both $(S_1, S_2, \ldots, S_k)$ and $(S_2, \ldots, S_k)$ are standard ordered set partitions of respectively $\{1, \ldots, n\}$ and $\{1, \ldots, n\} \setminus S_1$.

The set $\{(T, \mathbf{S}_1), \ldots, (T, \mathbf{S}_\ell)\}$ is a 2-standard composite ordered set partition of $\{1, \ldots, n\}$ if each $(T, \mathbf{S}_i)$ is 2-standard.

In other words, for an ordered set partition of $\{1, \ldots, n\}$ the condition is that S_1 contains 1, and S_2 contains the minimal element in the complement of S_1 in $\{1, \ldots, n\}$. For example, $(1, 2, 4, 3, 5)$ and $(15, 2, 4, 3)$ are 2-standard, but $(125, 4, 3)$ is not.

In what follows, after proving in Theorem 35 that it spans and is linearly independent, we shall call the set

$$\{[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)] : \text{ each } (T, \mathbf{S}_i) \text{ is a 2-standard ordered set partition}\}.$$

the *canonical* basis for $\hat{\mathcal{B}}^n$. If the set $\{(T, \mathbf{S}_1), \ldots, (T, \mathbf{S}_\ell)\}$ contains only 2-standard ordered set partitions, then it shall be called a *blade-canonical* composite ordered set partition.

Following the procedure of [16], we define an endomorphism on $\hat{\mathcal{B}}^n$, taking the given basis of functions $[(S_1, \ldots S_k)]$ labeled by standard ordered set partitions to the canonical one, labeled by 2-standard composite ordered set partitions. This will prove linear independence for the canonical basis.

The bijection $\mathcal{U}_{\mathcal{B}}$, say, between standard ordered set partitions and 2-standard composite ordered set partitions, is closely related to the bijection from [16] between ordered set partitions and standard composite ordered set partitions.

Let $(S_1, \ldots, S_k)$ be a standard ordered set partition of $\{1, \ldots, n\}$. Setting $T = S_1$, we apply the bijection in [16] to the ordered set partition $(S_2, \ldots, S_k)$ of $\{1, \ldots, n\} \setminus S_1$.

We first recall the algorithm from Proposition 25 in [16]. Define $\{i_1, \ldots, i_m\} \subseteq \{2, \ldots, k\}$ with $i_1 > i_2 > \cdots > i_m$, if S_{i_1} contains the smallest label in the set $S_{i_1} \cup S_{i_1+1} \cup \cdots \cup S_k$, and in general S_{i_p} contains the smallest label in the set $S_{i_p} \cup S_{i_p+1} \cup \cdots \cup S_{i_{p-1}-1}$.

Now set

$$\mathcal{U}_{\mathcal{B}}(S_1, \ldots, S_k) = \{(S_1, \mathbf{S}_{i_1}), (S_1, \mathbf{S}_{i_2}) \ldots, (S_1, \mathbf{S}_{i_m})\},$$

where $\mathbf{S}_{i_a} = (S_{i_a}, S_{i_a+1}, \ldots, S_{i_{a-1}-1})$.

Remark 33. If $(S_1, \ldots, S_k)$ is not standard but S_a, say, contains its minimal element, then define $\mathcal{U}_{\mathcal{B}}(S_1, \ldots, S_k) = \mathcal{U}_{\mathcal{B}}(S_a, S_{a+1}, \ldots, S_{a-1})$.

Example 34. We have

$$\begin{aligned}
\mathcal{U}_{\mathcal{B}}(3, 4, 5, 1, 2) &= \mathcal{U}_{\mathcal{B}}(1, 2, 3, 4, 5) = \{(1, 2, 3, 4, 5)\} \\
\mathcal{U}_{\mathcal{B}}(1, 234, 5, 67) &= \{(1, 234, 5, 67)\} \\
\mathcal{U}_{\mathcal{B}}(1, 5, 4, 3, 2) &= \{(1, 2), (1, 3), (1, 4), (1, 5)\} \\
\mathcal{U}_{\mathcal{B}}(12345) &= \{(12345)\} \\
\mathcal{U}_{\mathcal{B}}(1\ 5, 10, 4, 10, 2, 9, 3, 6\ 7) &= \{(1\ 5, 2, 9, 3, 6\ 7), (1\ 5, 4, 10), (1\ 5, 10)\}
\end{aligned}$$

Recall from [16] the lexicographic ordered on ordered set partitions: given two ordered set partitions $(S_1, \ldots, S_k)$ and $(T_1, \ldots, T_\ell)$ of $\{1, \ldots, n\}$ define two sequences $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ whenever $i \in S_{p_i}$ and $j \in T_{q_j}$ we say that in the lexicographic order $(S_1, \ldots, S_k) \prec (T_1, \ldots, T_\ell)$ if the sequence $(p_1, \ldots, p_n)$ is lexicographically smaller than $(q_1, \ldots, q_n)$.

In Theorem 35 finally establishes linear independence for the canonical basis for graduated functions of blades.

Theorem 35. *The linear map induced by the bijection $\mathcal{U}_{\mathcal{B}}$ is invertible. The set*

$$\{[(T, \mathbf{S}_1)] \bullet \cdots \bullet [(T, \mathbf{S}_\ell)] : \text{ each } (T, \mathbf{S}_i) \text{ is a 2-standard ordered set partition}\}.$$

is linearly independent (and hence is a basis).

Proof. The proof that the bijection $\mathcal{U}_{\mathcal{B}}$ is upper-unitriangular with respect to the lexicographic order follows the same procedure which was used in the proof of Theorem 28 in [16].

Indeed, if $(S_1, \ldots, S_k)$ is a standard ordered set partition of $\{1, \ldots, n\}$ and we apply Theorem 27 to expand $\mathcal{U}_{\mathcal{B}}([(S_1, \ldots, S_k)])$, then every blade (except for $[(S_1, \ldots, S_k)]$, which appears) in the expansion is labeled by a standard ordered set partition that is lexicographically strictly smaller than $(S_1, \ldots, S_k)$. It follows that the matrix for $\mathcal{U}_{\mathcal{B}}$ is upper unitriangular, hence invertible. □

8. Canonical blades, graduated and characteristic: enumeration and a conjecture

Recall that

$$\gamma_{i,j,k} = 1 + \mu_{i,j} + \mu_{j,k} + \mu_{k,i} = 1 + ([[i,j]] - 1) + ([[j,k]] - 1) + ([[k,i]] - 1)$$

and that 1_{ab} is the characteristic function of the one-dimensional subspace

$$\{t(e_a - e_b) : t \in \mathbb{R}\}.$$

Note: in Theorem 27 we established the linear independence of the set of canonical *graduated* blades $[(S_1, \ldots, S_k)]$. In what follows we give the set of *canonical* characteristic functions of blades $\Gamma_{S_1,\ldots,S_k}$. We conjecture that these are linearly independent and span the same space as the graduated blades; however the proofs of these two assertions is left to future work.

Example 36. Note that the characteristic function labeled by any set partitions with only singlets is the identity, $[(j)] = 1$.

Then for $n = 3$, the set of canonical characteristic functions of blades consists of the $1+4+1 = 6$ elements

$$\begin{gathered} 1 \\ \Gamma_{1,2,3}, 1_{12}, 1_{23}, 1_{13} \\ 1_{123} \end{gathered}$$

For $n = 4$, the canonical set is as follows:

$$\begin{gathered} 1 \\ \Gamma_{1,2,3}, \Gamma_{1,2,4}, \Gamma_{1,3,4}, 1_{12}, 1_{13}, 1_{14}, 1_{23}, 1_{24}, 1_{34} \\ \Gamma_{1,2,3,4}, \Gamma_{1,2,4,3}, \Gamma_{1,2,34}, \Gamma_{1,23,4}, \Gamma_{12,3,4}, \Gamma_{1,24,3}, \Gamma_{13,2,4}, \Gamma_{14,2,3}, 1_{12}1_{34}, 1_{13}1_{24}, 1_{14}1_{23}, 1_{123}, 1_{124}, 1_{134}, 1_{234} \\ 1_{1234}. \end{gathered}$$

Note that the total count is $1 + 9 + 15 + 1 = 26$, the necklace number for $n = 4$, which counts the number of standard ordered set partitions of $\{1, 2, 3, 4\}$.

For $n = 5$ the first 2-standard composite ordered set partitions appear. The new elements are

$$\Gamma_{1,2,3}\Gamma_{1,4,5},\ \Gamma_{1,2,4}\Gamma_{1,3,5},\ \Gamma_{1,2,5}\Gamma_{1,3,4}.$$

Using the same labeling, but replacing Γ_{S_1,S_2,S_3} with $[(S_1, S_2, S_3)]$ everywhere, we obtain the canonical basis for the space of graduated functions of blades, $\hat{\mathcal{B}}^n$ for $n = 3, 4, 5$.

Remark 37. In Corollary 31 of [16] we found for the count of number $T_{n,k}$ of characteristic functions of plates of dimension $n-k$ in the canonical plate basis, the formula

$$T_{n,k} = \sum_{i=1}^{n} S(n,i)s(i,k-1).$$

Rows $n = 1, \ldots, 6$ are given below.

1					
2	1				
6	6	1			
26	36	12	1		
150	250	120	20	1	
1082	2040	1230	300	30	1

By counting elements in the canonical basis of $\hat{\mathcal{B}}^n$, it is easy to obtain the formula in Proposition 38, by shifting one of the indices from the formula for the canonical plate basis, reflecting the 2-standardness condition.

Proposition 38. *The number of blades of dimension $(n-k)$ in the canonical basis of graduated blades is*

$$T^{B}_{n,k} = \sum_{i=1}^{n} S(n,i)s(i-1,k-1),$$

where $S(n,i)$ is the Stirling number of the second kind, and $s(i,k)$ is the (unsigned) Stirling number of the first kind.

For $n = 2, 3, \ldots, 7$ we have

1					
1	1				
1	4	1			
1	15	9	1		
1	66	66	16	1	
1	365	500	190	25	1

Note that the rows sum (correctly) to the necklace numbers, which count ordered set partitions up to cyclic block rotation.

The numbers that are easy to remember are as follows: the left column counts the occurrence of the whole space $1_{12\cdots n}$. For the rightmost two diagonals, there is always the characteristic function of the unique point at the origin, and there are $(n-1)^2$ linearly-independent characteristic functions of blades each of dimension 1, corresponding to the $\binom{n-1}{2}$ tripod generators γ_{ijk} as in Figure 4 together with the $\binom{n}{2}$ characteristic functions of the one-dimensional subspaces, 1_{ab}.

The generating functions for the first few diagonals of the triangle in Proposition 38 are

$$\frac{1}{1-x}, \frac{x+1}{(1-x)^3}, \frac{x^2+10x+1}{(1-x)^5}, \frac{x^3+59x^2+59x+1}{(1-x)^7}, \frac{x^4+356x^3+966x^2+356x+1}{(1-x)^9},$$

$$\frac{x^5+2517x^4+12602x^3+12602x^2+2517x+1}{(1-x)^{11}},$$

$$\frac{x^6+21246x^5+161967x^4+298852x^3+161967x^2+21246x+1}{(1-x)^{13}},$$

and the coefficients of the numerators sum to the sequence

$$1, 2, 12, 120, 1680, 30240,$$

which appears to be given by O.E.I.S. sequence A001813, $a(m) = \frac{(2m)!}{m!}$, [32]. Clearly, a combinatorial proof and explanation for this formula would be highly desirable.

A vector of real numbers $(b_1, \ldots, b_m)$ is symmetric if $b_i = b_{m-i}$. It is unimodal if it there exists an index i such that $b_1 \leq b_2 \leq \cdots \leq b_i \geq b_{i+1} \geq \cdots \geq b_m$.

Conjecture 39. *The coefficients of the polynomial numerators of the generating functions for the diagonals of the array $T^B_{n,k}$ are symmetric and unimodal, and sum to the sequence $a(m) = \frac{(2m)!}{m!}$.*

The conjecture has been verified numerically in Mathematica: through the 24th diagonal.

The sequence $a(m) = \frac{(2m)!}{m!}$ itself is already suggestive; for example, it can expressed as $a(m) = (m+1)!\left(\frac{(2m)!}{(m+1)!(m!)}\right)$, which is $(m+1)!$ times the Catalan number, suggesting the possibility of a relation to *labeled* (rooted) binary trees (as noted in the comments in sequence A001813), with some additional statistic explaining the numerator coefficients for the generating series. It seems plausible to ask about the possibility of a directly geometric and/or ring-theoretic interpretation of the sequence $a(m)$ and the numerator coefficients in terms of blades or other geometric of physical objects. We leave this question to future research.

9. The graded cohomology ring of a configuration space

In what follows, we connect with joint work with V. Reiner in [15] on the cohomology ring $H^\star(X_n)$, where X_n is the configuration space of n distinct points in $SU(2)$ modulo the diagonal action of $SU(2)$. The presentation of the cohomology ring in Definition 40 suggests that this ring is analogous to a graded analog of the space of characteristic functions of blades labeled by nondegenerate ordered set partitions of $\{1, \ldots, n\}$, having n singleton blocks. We leave the detailed exploration of the connection to future work.

Definition 40. Let $\mathcal{U}^n$ be the commutative algebra on $\binom{n}{2}$ generators $u_{ij} = -u_{ji}$ with $i \neq j$, subject to the relations

$$u_{ij}^2 = 0$$

and

$$u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0.$$

Define $v_{ijk} = u_{ij} + u_{jk} + u_{ki}$, and denote by $\mathcal{V}^n$ the subalgebra of $\mathcal{U}^n$ generated by the v_{ijk}.

Theorem 41 ([15], Theorem 3)**.** *We have an isomorphism of graded rings*

$$\mathcal{V}^n \simeq H^\star(X_n).$$

In the algebra $\mathcal{V}^n$ we have the very useful relation of Proposition 42. It is interesting to compare its relative simplicity to the complexity of the proof of Theorem 20.

Proposition 42. *In $\mathcal{U}^n$ we have the relation*

$$(1 + u_{i_1,i_2}) \cdots (1 + u_{i_k,i_1}) = (1 + v_{i_1,i_2,i_3}) \cdots (1 + v_{i_1,i_{k-1},i_k}),$$

for any sequence $\{i_1, \ldots, i_k\}$ selected from $\{1, \ldots, n\}$ with $k \geq 2$ ($k = 2$ being trivial), and additionally the identities

$$\begin{aligned} v_{ijk} &= v_{jki} = -v_{ikj} \\ v_{ijk}^2 &= 0. \end{aligned}$$

Proof. As $u_{ij}^2 = 0$ and

$$u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0,$$

we have

$$v_{ijk}^2 = u_{ij}^2 + u_{jk}^2 + u_{ki}^2 + 2\left(u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij}\right) = 0.$$

Now let us prove the first assertion by induction.

Since $u_{i,j}^2 = 0$ we have

$$(1 + u_{ij})(1 + u_{ji}) = (1 - u_{ij}^2) = 1.$$

Then,

$$\begin{aligned}
(1+u_{i_1i_2})\cdots(1+u_{i_ki_1}) &= (1+u_{i_1i_2})(1+u_{i_2i_3})\cdots(1+u_{i_{k-1}i_1})(1+u_{i_{k-1}i_k})(1-u_{i_{k-1}i_1})(1+u_{i_ki_1}) \\
&= (1+u_{i_1i_2})(1+u_{i_2i_3})\cdots(1+u_{i_{k-1}i_1})(1+u_{i_1i_{k-1}}+u_{i_{k-1}i_k}+u_{i_ki_1}) \\
&= (1+u_{i_1i_2})(1+u_{i_2i_3})\cdots(1+u_{i_{k-1}i_1})(1+v_{1,k-1,k}) \\
&= (1+v_{i_1i_2i_3})(1+v_{i_1i_3i_4})\cdots(1+v_{i_1i_{k-1}i_k}).
\end{aligned}$$

Here we have used the induction step, $u_{ij}^2 = 0$ and the Jacobi identity to express

$$\begin{aligned}
(1+u_{i_{k-1}i_k})(1-u_{i_{k-1}i_1})(1-u_{i_ki_1}) &= (1+u_{1i_{k-1}})(1+u_{i_{k-1}i_k})(1+u_{i_ki_1}) \\
&= (1+u_{i_1i_{k-1}}+u_{i_{k-1}i_k}+u_{i_ki_1}) \\
&= 1+v_{1i_{k-1}i_k}.
\end{aligned}$$

□

Note that the identity is unchanged if we deform with a formal parameter t to keep track of degree: replace $(1+u_{ij}) \mapsto (1+tu_{ij})$ and accordingly $(1+v_{ijk}) \mapsto (1+tv_{ijk})$.

Then Proposition 42 becomes

$$(1+tu_{i_1i_2})(1+tu_{i_2i_3})\cdots(1+tu_{i_ki_1}) = (1+tv_{i_1i_2i_3})(1+tv_{i_1i_3i_4})\cdots(1+tv_{i_1i_{k-1}i_k}).$$

Corollary 43. *We have*

$$u_{i_1,i_2}\cdots u_{i_{k-1},i_k} + u_{i_2,i_3}\cdots u_{i_{k-1},i_k} + \cdots + u_{i_k,i_1}\cdots u_{i_{k-2},i_3} = 0$$

and

$$u_{i_1,i_2}\cdots u_{i_k,i_1} = 0,$$

for any sequence $\{i_1,\ldots,i_k\}$ *of elements in* $\{1,\ldots,n\}$.

Proof. We need to check that the coefficients of t^k and t^{k-1} in the following expression are zero:

$$(1+tu_{i_1i_2})(1+tu_{i_2i_3})\cdots(1+tu_{i_ki_1}).$$

But by Proposition 42 this equals

$$(1+tv_{i_1i_2i_3})(1+tv_{i_1i_3i_4})\cdots(1+tv_{i_1i_{k-1}i_k})$$

and the highest power of t which appears with nonzero coefficient is t^{k-2}. □

It is interesting to compare the elegance of exponentiation in Example 44 with the relative complexity of the same formally equivalent identity in Proposition 22:

$$(\gamma_{1,2,3}\gamma_{1,3,4} = \gamma_{1,2,4}\gamma_{2,3,4}) \quad \Leftrightarrow (v_{123}v_{134} = v_{124}v_{234}).$$

Example 44. In $\mathcal{U}^n$ we have the relations

$$\exp(v_{123})\exp(v_{134}) = \exp(v_{123}+v_{134}) = \exp(u_{12}+u_{23}+u_{34}+u_{41}),$$

hence the full exponential is an invariant of the boundary 1-skeleton of the polygon! See also Appendix A for a more general construction involving triangulations of polygons. Continuing,

$$\begin{aligned}
& \exp(u_{12}+u_{23}+u_{34}+u_{41}) \\
= & \; 1+(u_{12}+u_{23}+u_{34}+u_{41})+\frac{1}{2}(u_{12}+u_{23}+u_{34}+u_{41})^2+\frac{1}{3!}(u_{12}+u_{23}+u_{34}+u_{41})^3+\cdots \\
= & \; 1+(u_{12}+u_{23}+u_{34}+u_{41})+(u_{12}u_{23}+u_{12}u_{34}+u_{12}u_{41}+u_{23}u_{34}+u_{23}u_{41}+u_{34}u_{41}),
\end{aligned}$$

where the degree 3 term vanishes by direct computation, or by Proposition 42. The sum is in termwise bijection with the expression for $\Gamma_{i,j,k,\ell}$ from Proposition 22 and in particular the expression for $\Gamma_{1,2,3,4}$ as a sum of elementary symmetric functions, as depicted in Figure 5.

By way of a specialization of the canonical basis from Theorem 35 to ordered set partitions having only singleton blocks, one would hope to have the following graded basis for the cohomology ring of the configuration space of points in $SU(2)$ modulo the diagonal action, from [15]. It may be possible to construct an argument using the so-called *nbc* basis, cf [35]; however a self-contained combinatorial proof would be desirable.

Problem 45. *Writing each cycle C of w uniquely as $C = (c_1c_2\cdots c_\ell)$ with convention $c_1 = \min\{c_1,c_2,\ldots,c_\ell\}$, show that $(\mathcal{U}^{n-1})_j$ and $(\mathcal{V}^n)_j$ have the bases respectively*

$$\prod_{\text{cycles } C \text{ of } w} u_{c_1,c_2}\, u_{c_2,c_3}\cdots u_{c_{\ell-1}c_\ell} \qquad \text{and} \qquad \prod_{\text{cycles } C \text{ of } w} v_{c_1,c_2,n}\, v_{c_2,c_3,n}\cdots v_{c_{\ell-1}c_\ell,n},$$

where w runs through all permutations in $\mathfrak{S}_{n-1}$ with $n-1-j$ cycles.

10. Configuration space of points on the circle

Let us denote by

$$\mathcal{O}^n = U(1)^n/U(1) \subset \mathbb{C}^n/U(1)$$

the configuration space of n points on the unit circle $U(1) = \{z \in \mathbb{C} : |z| = 1\}$, modulo simultaneous rotation. In this section we record a convenient labeling and parameterization of the configuration space of n points on the circle modulo simultaneous rotation.

Denote by $\Delta_1^n = \{x \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$ the unit simplex.

Definition 46. Let $\mathbf{S} = (S_1,\ldots,S_k)$ be an ordered set partition of $\{1,\ldots,n\}$. For $x \in \Delta_1^n$, define

$$\begin{aligned}
\varphi_{\mathbf{S}}(x_1,\ldots,x_n) &= e^{2\pi i x_{S_1\cdots S_k}}e_{S_1}+e^{2\pi i x_{S_2\cdots S_k}}e_{S_2}+\cdots+e^{2\pi i x_{S_k}}e_{S_k} \\
&= e_{S_1}+e^{2\pi i x_{S_2\cdots S_k}}e_{S_2}+\cdots+e^{2\pi i x_{S_k}}e_{S_k},
\end{aligned}$$

and denote the equivalence class of $\varphi_{\mathbf{S}}(x_1,\ldots,x_n)$ modulo simultaneous rotation by $U(1)$ by $\overline{\varphi}_{\mathbf{S}}(x_1,\ldots,x_n)$. We further define $[S_1,\ldots,S_k]^{\circledast} = \overline{\varphi}_{\mathbf{S}}(\Delta_1^n)$, and as usual denote by $[[S_1,\ldots,S_k]]^{\circledast}$ the characteristic function of $[S_1,\ldots,S_k]^{\circledast}$.

Then it is easy to see that the image $\varphi_{\mathbf{S}}(\Delta_1^n) \subset \mathcal{O}^n$ fills out the closure of the unique cyclic order of points on the circle[3] which is oriented counterclockwise (i.e. with increasing angle) as

$$x_{(S_1)} \leftarrow x_{(S_2)} \leftarrow \cdots \leftarrow x_{(S_k)} \leftarrow x_{(S_1)},$$

where we recall the shorthand notation $x_{(S)}$ which stands for $(x_{i_1} = \cdots = x_{i_s})$ for $S = \{i_1,\ldots,i_s\}$.

[3] i.e. we include all possible additional collisions

In Proposition 47 we show that the set of composite maps $\overline{\varphi}_{\mathcal{S}} : V_0^n \to \mathcal{O}^n$ identifies all subsets of $\mathcal{O}^n$ which are labeled by cyclic rotations of the same ordered set partition.

Proposition 47. *For any ordered set partition $(S_1,\ldots,S_k)$ of $\{1,\ldots,n\}$ we have invariance under cyclic block rotation:*

$$[S_1,S_2,\ldots,S_k]^{\circledast} = [S_2,S_3,\ldots,S_k,S_1]^{\circledast}.$$

Proof. Let $\mathbf{S} = (S_1,\ldots,S_k)$ be an ordered set partition of $\{1,\ldots,n\}$, and let us denote $\mathbf{S}^{(j)} = (S_j, S_{j+1},\ldots,S_{j-1})$. It suffices to check pointwise that $\varphi_{\mathbf{S}^{(1)}}(x)$ and $\varphi_{\mathbf{S}^{(2)}}(x)$ differ only by a phase.

For each $(x_1,\ldots,x_n) \in \Delta_1^n$ we have

$$\begin{aligned} \varphi_{(S_1,\ldots,S_k)}(x_1,\ldots,x_n) &= e_{S_1} + e^{2\pi i x_{S_2\cdots S_k}} e_{S_2} + e^{2\pi i x_{S_3\cdots S_k}} e_{S_2} + \cdots e^{2\pi i x_{S_k}} e_{S_k} \\ &= e^{2\pi i x_{S_2\cdots S_k}} \left(e^{2\pi i x_{S_1}} e_{S_1} + e_{S_2} + e^{2\pi i x_{S_3\cdots S_k S_1}} e_{S_3} + \cdots + e^{2\pi i x_{S_k S_1}} e_{S_k} \right) \\ &= e^{2\pi i x_{S_2\cdots S_k}} \left(e_{S_2} + e^{2\pi i x_{S_3\cdots S_k S_1}} e_{S_3} + \cdots + e^{2\pi i x_{S_k S_1}} e_{S_k} + e^{2\pi i x_{S_1}} e_{S_1} \right) \\ &= e^{2\pi i x_{S_2\cdots S_k}} \varphi_{(S_2,\ldots,S_k,S_1)}(x_1,\ldots,x_n), \end{aligned}$$

which differs from $\varphi_{(S_2,\ldots,S_k,S_1)}(x_1,\ldots,x_n)$ only by the phase $e^{2\pi i x_{S_2\cdots S_k}}$, hence we have pointwise

$$\overline{\varphi}_{(S_1,\ldots,S_k)}(x_1,\ldots,x_n) = \overline{\varphi}_{(S_2,\ldots,S_k,S_1)}(x_1,\ldots,x_n),$$

and since $(S_1,\ldots,S_k)$ was arbitrary, the identity follows. $\square$

It is natural for $\mathcal{O}^n$ to enumerate the set of degenerate cyclic orders of n particles on the circle,

$$\left\{[S_1,\ldots,S_k]^{\circledast} : (S_1,\ldots,S_k) \text{ is a standard ordered set partition of } \{1,\ldots,n\}\right\},$$

by the number of blocks, that is by the number of distinct uncollided particles in a given configuration. One easily finds O.E.I.S. sequence A028246, see also A053440:

$$T_{n,k} = S(n,k)(k-1)!,$$

where $S(n,k)$ is the Stirling number of the second kind. The number triangle begins with

1					
1	1				
1	3	2			
1	7	12	6		
1	15	50	60	24	
1	31	180	390	360	120

Here the rows total to the necklace numbers, as in Proposition 38.

11. Concluding remarks

In this paper, we have studied a new factorization property for permutohedral honeycomb tessellations: using ring theoretic calculations with characteristic functions of blades, we proved that honeycomb tessellations are locally Minkowski sums of 2-dimensional permutohedral honeycombs. We have also established a certain *canonical* basis for graduated functions of blades.

(1) For graduated functions of blades we prove linear independence for the canonical basis (in Theorem 35), but we do not establish any factorization property. Indeed, it turns out that due to the larger integer multiplicities on shared faces of the cyclic sum the simple factorization property does not hold for graduated blades.

(2) On the other hand, for characteristic functions of blades, which are $\{0,1\}$-valued, we prove the factorization property. We leave all linear independence proofs to future work.
(3) Further, we expect, but it was beyond the scope of the paper to prove, that graduated and characteristic functions of blades and span the same space.

	Basis?	Simple factorization into tripods?
$\{0,1\}$-valued, standard	Expected	✓
$\{0,1\}$-valued canonical	Expected	✓
$\bar{n}$-valued, standard	✓	No
$\bar{n}$-valued, canonical	✓	No

(4) Proving the closed formula for the general straightening relations for both characteristic and graduated functions of blades is beyond the scope of the present work we defer the exposition to future work. Toric posets may be relevant for the general proof, see [13].

However, modulo both characteristic functions of non-pointed and higher codimension cones the solution is very manageable, especially for the top-degree component. In fact in this case it turns out that the straightening relations are combinatorially identical to the *Kleiss-Kuijf* relations for the Parke-Taylor factors, see [25], and have already been formulated graph-theoretically in Section 3.2 of [3], which we can see by way of the homomorphism

$$\Gamma_{i_1,\ldots,i_n} \mapsto PT(i_1,\ldots,i_n),$$

where

$$PT(i_1,\ldots,i_n) = \frac{1}{(x_{i_1}-x_{i_2})(x_{i_2}-x_{i_3})\cdots(x_{i_n}-x_{i_1})},$$

$x_1,\ldots,x_n$ being complex variables. Seeing that this is a homomorphism is not difficult, but it partially involves structures studied in [9] and we leave the proof to future work. See also Example 31 for an analog of the $U(1)$-decoupling identity, for a functional representation of graduated functions of blades. For related straightening relations, see the so-called canonicalization of pseudoinvariants in [28].

(5) As one can derive by simply counting multiplicities obtained by summing the characteristic functions of the blade $((1,2,3))$, see Figure 4, and its mirror image $((1,3,2))$, in dimension ≤ 1, the fundamental blade relations for characteristic functions take the form

$$\Gamma_{1,2,3}+\Gamma_{1,3,2} = 1_{12}+1_{23}+1_{31}-1_1 1_2 1_3.$$

On the other hand, for graduated functions (see Figure 8) on V_0^3 the analogous expression takes the form

$$[(1,2,3)]+[(1,3,2)] = [(1,23)]+[(12,3)]+[(13,2)]-1_{123}+1_1 1_2 1_3.$$

(a) Modulo characteristic functions of cones of *codimension* ≥ 2 in V_0^3, the fundamental blade relations take the form respectively

$$\Gamma_{1,2,3}+\Gamma_{1,3,2} = 1_{12}+1_{23}+1_{31}$$

and

$$[(1,2,3)]+[(1,3,2)] = [(1,23)]+[(12,3)]+[(13,2)]-1_{123}.$$

(b) Modulo characteristic functions of non-pointed cones, the fundamental blade relations take the form respectively

$$\Gamma_{1,2,3}+\Gamma_{1,3,2} = -1_1 1_2 1_3.$$

and

$$[(1,2,3)] + [(1,3,2)] = 1_1 1_2 1_3.$$

(c) Modulo both we have antisymmetry, respectively

$$\Gamma_{1,2,3} + \Gamma_{1,3,2} = 0$$

and

$$[(1,2,3)] + [(1,3,2)] = 0.$$

One could compare these with the antisymmetry of the generator v_{ijk} in the cohomology ring of Section 9.

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Appendix A. Symmetries of a leading singularity

In Theorem 35 we saw that factorizations of characteristic functions of blades correspond to the triangulations of a cyclically-oriented polygon. It is natural to ask about products which do *not* correspond to triangulations, but rather to graphs embedded in higher dimensional objects; indeed, it turns out that this case is similar to something in the scattering amplitudes literature known as a *leading singularity*. Further, the examples and discussion which follows suggests a new class of identifications for non-planar on-shell diagrams beyond the well-known square move which deserves further study [9]. Our paper [9] is in preparation.

While our computations rely on relations which hold in the cohomology ring $\mathcal{U}^n$, namely $v_{ijk} = -v_{ikj}$, $v_{ijk}^2 = 0$ and $v_{ijk}v_{ik\ell} + v_{ik\ell}v_{i\ell j} + v_{i\ell j}v_{ijk} = 0$, it makes sense to ask whether the leading singularities deform manageably when the formal generator v_{ijk} is "replaced" with the characteristic function $\gamma_{i,j,k}$. The practical difficulty is that the above relations on the v_{ijk}'s are degenerations of the relations on the $\gamma_{i,j,k}$, so there are many more moving parts. For example, the relation $v_{123} + v_{132} = 0$ now deforms to the fundamental identity

$$\gamma_{1,2,3} + \gamma_{1,3,2} = 1_{12} + 1_{23} + 1_{31} - 1_{12}1_{23}1_{31}.$$

However, in the algebra $\mathcal{D}^n$ the relations are very simple, and due to the nilpotence of the generators $\Delta_{ijk}^2 = 0$ one has the exponential map.

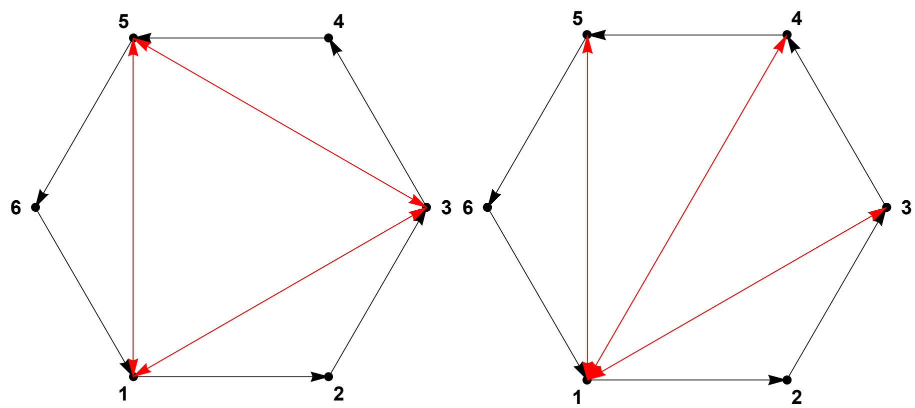

Figure 9. Triangulation change:

$$\{(1,2,3),(3,4,5),(5,6,1),(1,3,5)\} \Leftrightarrow \{(1,2,3),(1,3,4),(1,4,5),(1,5,6)\}.$$

Internal arrows overlap in opposite directions and cancel (triangulation succeeds), see Example 48

Example 48. The simplest example of a triangulation of a polygon that is not a flag occurs for a hexagon at $n = 6$, see Figure 9. This central triangulation corresponds to the coefficient of t^4 in the following expression:

$$\begin{aligned} v_{123}v_{345}v_{561}v_{135} &= \mathrm{coeff}_{t^4}\left((1+tv_{123})(1+tv_{345})(1+tv_{561})(1+tv_{135})\right) \\ &= \mathrm{coeff}_{t^4}((1+tv_{123})(1+tu_{13})(1+tv_{345})(1+tu_{35})(1+tv_{561})(1+tu_{15})) \\ &= \mathrm{coeff}_{t^4}\left((1+tu_{12})(1+tu_{23})(1+tu_{34})(1+tu_{45})(1+tu_{56})(1+tu_{61})\right) \\ &= \mathrm{coeff}_{t^4}\left((1+tv_{123})(1+tv_{134})(1+tv_{145})(1+tv_{156})\right) \\ &= v_{123}v_{134}v_{145}v_{156}, \end{aligned}$$

having used

$$(1+tv_{abc}) = (1+tu_{ab})(1+tu_{bc})(1+tu_{ca}),$$

or, directly, the flip move $v_{135}v_{345} = v_{134}v_{145}$, as also holds for characteristic functions of blades, $\gamma_{135}\gamma_{345} = \gamma_{134}\gamma_{145}$. On the other hand, we have the interesting product which does *not* correspond to a triangulation, but which *does* have meaning as a certain *leading singularity*, as seen in [3]. In the derivation we shall make use of the identities $\exp(tu_{ij}) = 1 + tu_{ij}$ and $\exp(tv_{ijk}) = 1 + tv_{ijk}$, as $u_{ij}^2 = 0$ and $v_{ijk}^2 = 0$.

Then, for the set of triples $\{(1,2,3),(3,4,5),(5,6,1),(2,6,4)\}$ we have

$$\begin{aligned}
& (1+v_{123})(1+v_{345})(1+v_{561})(1+v_{264}) = \exp\left((v_{123}+v_{345}+v_{561}+v_{264})\right) \\
= \;& \exp\left(u_{12}+u_{23}+u_{31}+u_{34}+u_{45}+u_{53}+u_{56}+u_{61}+u_{15}+u_{26}+u_{64}+u_{42}\right) \\
= \;& \exp\left(u_{12}+u_{23}+u_{34}+u_{56}+u_{61}\right)\exp\left(v_{153}+v_{264}\right).
\end{aligned}$$

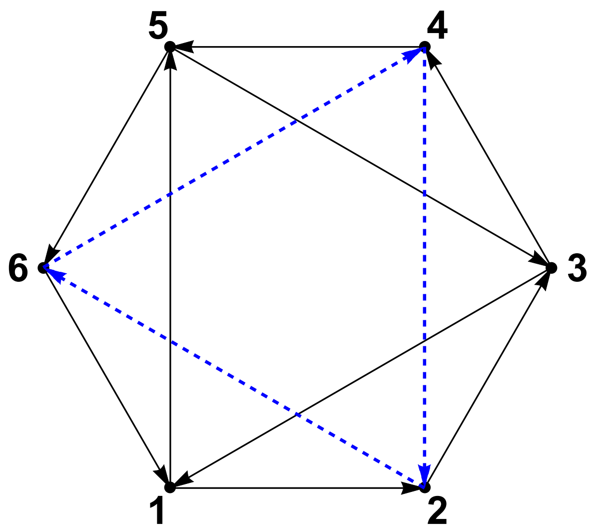

FIGURE 10. No internal edge cancellation occurs (triangulation fails) for the leading singularity of Example 48: $\{(1,2,3),(3,4,5),(5,6,1),(2,6,4)\}$. For further discussion see [9].

Here each u_{ij} corresponds to an oriented edge, as in Figure 10.

It is easy to check that the expansion here is invariant under rotation by the cycle (123456), which would not be obvious from the factorization into triples.

Appendix B. Combinatorial scattering equations and balanced graphs

In this somewhat speculative section, let us look toward scattering amplitudes for inspiration and future work, specifically toward the one-loop worldsheet functions from [20]. The story began in [10], where the *scattering equations* were introduced. These are a system of n highly nonlinear equations in the n complex variables $\sigma_1, \ldots, \sigma_n$:

$$\sum_{b\neq a} G_{ab} = 0$$

for each $a = 1, \ldots, n$, where we define $G_{ab} = \frac{s_{ab}}{\sigma_a - \sigma_b}$.

Above the numerators $s_{ab} = s_{ba}$ for $1 \leq a < b \leq n$ are known as the generalized Mandelstam invariants; for the present purposes, we may assume that they are complex numbers. Remark that we regard these only as combinatorial objects. We shall not here think about the usual linear fractional action of the gauge group SL_2 on the variables σ_a.

Remark that the functions G_{ab} are obtained as limits of Kronecker-Eisenstein series, see [26] and [40]. In the context of scattering amplitudes, a good starting point would be [29] and the references therein. The same functions arise in elliptic solutions to the Classical Dynamical Yang-Baxter equation, see [18] and [17] and the references therein.

The purely combinatorial approach which motivates this section is inspired in part by Appendix A of [20], where the *one-loop* analog of the scattering equations were in effect considered to generate an ideal in a certain ring of one-loop worldsheet functions. That is, one considers the ring generated formally by the functions G_{ab} and mods out by the ideal generated by the one-loop scattering equations, see [21]).

Our aim here is to study a combinatorial interpretation of the quotient ring from [20]: it turns out that this quotient ring is combinatorially temptingly close to (a filtered analog of) $\mathcal{V}^n$, and thus to blades. It is interesting to note that a combinatorially similar quotient ring was studied in [31], where it was conjectured to be isomorphic to the cohomology ring of the configuration space X_n of n distinct points in $SU(2)$, modulo the simultaneous action of $SU(2)$. See Section 9 above.

As we saw in Example 44, the exponential map can be used to efficiently encode the combinatorial structure of the k-skeleta of a blade uniquely from its 1-skeleton. But is there a criterion to determine when a given polynomial is in the subalgebra $\mathcal{V}^n$ of $\mathcal{U}^n$? We give a partial answer which consists of a set of linear relations on the coefficients of the argument of the exponential map $\exp : (\mathcal{U}^n)_1 \to \mathcal{U}^n$, which restricts nicely as $\exp : (\mathcal{V}^n)_1 \to \mathcal{V}^n$. Clearly any monomial in the v_{ijk} is the leading order coefficient of an exponential map, suggesting the possibility that the result could be extended.

Proposition 49. *Let constants $m_{ij} \in \mathbb{C}$, $1 \leq (i \neq j) \leq n$ be given; define $\alpha_{ij} = m_{ij} - m_{ji}$. Then, the product*

$$\prod_{1\leq(i\neq j)\leq n} (1 + u_{ij})^{m_{ij}} \in \mathcal{U}^n$$

is in the subalgebra $\mathcal{V}^n$ if and only if the constants α_{ij} satisfy what we call the combinatorial scattering equations*,*

$$\begin{aligned} \alpha_{12} + \alpha_{13} + \cdots + \alpha_{1n} &= 0 \\ \alpha_{21} + \alpha_{23} + \cdots + \alpha_{2n} &= 0 \\ \alpha_{31} + \alpha_{23} + \cdots + \alpha_{3n} &= 0 \\ &\vdots \\ \alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{n\ n-1} &= 0. \end{aligned}$$

Proof. First note that $u_{ij}^2 = 0$ implies the identity

$$\begin{aligned}\prod_{1\le(i\neq j)\le n}(1+u_{ij})^{m_{ij}} &= \prod_{1\le i<j\le n}\exp(\alpha_{ij}u_{ij})\\ &= \exp\left(\sum_{1\le i<j\le n}\alpha_{ij}u_{ij}\right).\end{aligned}$$

Now, the linear span of all the u_{ij}'s decomposes into a direct sum of two irreducible symmetric group representations:

$$(\mathcal{U}^n)_{(1)} \simeq V_{(n-1,1)} \oplus V_{(n-2,1,1)},$$

with spanning sets respectively

$$\{z_i : i = 1,\ldots,n\} \quad \text{and} \quad \{v_{ijk} = u_{ij} + u_{jk} + u_{ki} : 1 \le i < j < k \le n\},$$

and bases

$$\{z_i : i = 1,\ldots,n-1\} \quad \text{and} \quad \{v_{1jk} = u_{1i} + u_{jk} + u_{k1} : 2 \le j < k \le n\},$$

say, where we define $z_i = \sum_{j\neq i} u_{ij}$.

It is easy to see that the combinatorial scattering equations express the condition on the coefficients α_{ij} for a linear combination

$$\sum_{1\le i<j\le n}\alpha_{ij}u_{ij}$$

to be in $V_{(n-2,1,1)}$. Indeed, in light of the decomposition

$$u_{ij} = \frac{1}{n}(z_i - z_j) + \frac{1}{n}\left(\sum_{k\neq i,j} v_{i,j,k}\right) \in V_{(n,1,1)} \oplus V_{(n-2,1,1)},$$

we have

$$\begin{aligned}\sum_{1\le i<j\le n}\alpha_{ij}u_{ij} &= \frac{1}{n}\sum_{1\le i<j\le n}\alpha_{ij}\left(z_i - z_j + \sum_{k\neq i,j} v_{ijk}\right)\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k\neq i}\alpha_{i,k}\right)z_i + \frac{1}{n}\sum_{1\le i<j\le n}\alpha_{ij}\left(\sum_{k\neq i,j} v_{ijk}\right),\end{aligned}$$

where we have used $\alpha_{ij} = -\alpha_{ji}$. Evidently, this is in $\mathcal{V}^n$ if and only if all coefficients in the first term,

$$\frac{1}{n}\left(\sum_{k\neq i}\alpha_{ik}\right),$$

vanish for each i. This set coincides with exactly the combinatorial scattering equations for the α_{ij}. ◻

Given any element

$$C \in \mathcal{L}_{\mathcal{U}^n} = \{-1,0,1\}^{\binom{n}{2}},$$

we have

$$\exp\left(\sum_{1\le i<j\le n} C_{ij}u_{ij}\right) \in \mathcal{U}^n.$$

In particular, any canonical basis element for $\mathcal{U}^n$, (or $\mathcal{V}^n$), is obtained as the coefficient of the highest power of t of $\exp\left(t\sum_{(i,j)\in C}u_{ij}\right) \in \mathcal{U}^n$ for some $C \in \{-1,0,1\}^{\binom{n}{2}}$. It seems tempting to interpret the combinatorial scattering equations as describing the restriction (of exp) to an $\binom{n-1}{2}$-dimensional subspace of $(\mathcal{U}^n)_1$. In particular, for integer-valued weights, we obtain inside the subspace an $\binom{n-1}{2}$-dimensional sublattice $\mathcal{L}_{\mathcal{V}^n} \subset \mathcal{L}_{\mathcal{U}^n}$. Thus, of particular interest for scattering amplitudes are elements $C \in \mathcal{L}_{\mathcal{V}^n}$ such that $\sum_{(i,j)\in C} u_{ij} = \sum_{a=1}^{n-2} v_{r_a s_a t_a}$ for some list of triples $\{(r_1, s_1, t_1), \ldots, (r_{n-2}, s_{n-2}, t_{n-2})\}$.

There appears to be some graph-theoretic machinery at hand. We now prove that to each balanced weighted graph (in the sense of [27]) there exists an element of the algebra $\mathcal{V}^n$.

Let $\mathcal{G}^n$ be an unoriented graph on n vertices $1, \ldots, n$. Let us choose edge orientations $(i_1 \to j_1), \ldots, (i_\ell \to j_\ell)$ such that $i_a < j_a$ for all $a = 1, \ldots, \ell$.

Definition 50. A graph $\mathcal{G}^n$, with each edge $i \to j$ equipped with a flow m_{ij} (which could be in $\mathbb{C}$) from i to j, for $1 \leq i \neq j \leq n$, is said to be *balanced* provided that the net flux at every vertex is zero. That is, for each $a \in \{1, \ldots, n\}$, the total flow entering a is the same as the total flow leaving a:

$$\sum_{b:\ b\neq a} m_{ab} = \sum_{b:\ b\neq a} m_{ba},$$

or

$$\sum_{b:\ b\neq a} (m_{ab} - m_{ba}) = 0.$$

We recover immediately at a graph theoretic form of the combinatorial scattering equations.

Corollary 51. *Let $\mathcal{G}^n$ be a graph on vertices $\{1, \ldots, n\}$, with flow m_{ij} on the edge $i \to j$ for all distinct $i, j \in \{1, \ldots, n\}$.*

If $\mathcal{G}$ is balanced then the product

$$\prod_{1\leq (i\neq j)\leq n} (1 + u_{ij})^{m_{ij}}$$

is in the subalgebra $\mathcal{V}^n$.

Proof. Suppose that $\mathcal{G}^n$ is balanced. Then for the product

$$\prod_{1\leq (i\neq j)\leq n} (1 + u_{ij})^{m_{ij}} = \prod_{1\leq i<j\leq n} (1 + (m_{ij} - m_{ji})u_{ij}) = \exp\left(\sum_{1\leq i<j\leq n} (m_{ij} - m_{ji})u_{ij}\right),$$

at each vertex $a = 1, \ldots, n$ we have the total flux

$$\sum_{b:\ b\neq a} (m_{ab} - m_{ba}) = \sum_{b:\ b\neq a} \alpha_{ab} = 0.$$

These are the combinatorial scattering equations with coefficients $\alpha_{ab} = m_{ab} - m_{ba}$. $\square$

Of course, one can easily see that the same will be true for the graph of any leading singularity.

Example 52. Assigning the weight $m_{ij} = +1$ to each directed edge (i, j), then the graphs in Figures 9 and 10 are all balanced!

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