An Upper Bound for Palindromic and Factor Complexity of Rich Words

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Abstract

A finite word w of length n contains at most $n+1$ distinct palindromic factors. If the bound $n+1$ is reached, the word w is called rich. An infinite word w is called rich if every finite factor of w is rich. Let w be a rich word (finite or infinite) over an alphabet with $q>1$ letters, let $F(w,n)$ be the set of factors of length n of the word w and let $F_p(w,n)\subseteq F(w,n)$ be the set of palindromic factors of length n of the word w. We show that $|F_p(w,n)|\leq (q+1)n(4q^{10}n)^{\log_2 n}$ and $|F(w,n)|\leq (q+1)^2n^4(4q^{10}n)^{2\log_2 n}$.

It is known that $|F_p(w,n)|+|F_p(w,n+1)|\leq |F(w,n+1)|-|F(w,n)|+2$, where w is an infinite word closed under reversal [Baláži, Masáková, Pelantová, Theor. Comput. Sci., 380 (2007)]. We generalize this inequality for finite words and consequently we derive that $|F(w,n)|\leq 2(n-1)\hat{F}_p(w,n)-2(n-1)+q$ and $|F(w,n)|\leq 2(n-1)(q+1)n(4q^{10}n)^{\log_2 n}-2(n-1)+q$, where $\hat{F}_p(w,k)=\max\{|F_p(w,j)| \mid 0\leq j\leq k\}$ and w is a rich word (finite or infinite) such that $F(w,n+1)$ is closed under reversal. Moreover we prove that $|F(w,n)|\leq 2(2n-1)(q+1)2n(8q^{10}n)^{\log_2 2n}-2(2n-1)+q$, where w is a finite rich word.

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1 Introduction

The field of combinatorics on words includes the study of palindromes and rich words. In recent years there have appeared several articles concerning this topic, [8, 5, 3, 17]. Recall that a palindrome is a word that yields the same when being read backward and forward, for example "noon" and "level". Rich words (or also words having *palindromic defect* 0) are words containing maximal number of palindromic factors (it is known that a word of length n can contain at most $n+1$ palindromic factors, including the empty word, [8]). An infinite word is called rich if its every finite factor is rich.

Rich words possess various properties, see for instance [9, 7, 4]. In this article, we will use two of them. First one uses the notion of a *complete return*: Given a word w and a factor r or w. We call the factor r a complete return to u in w if r contains exactly two occurences of u, one as a prefix and one as a suffix. A property of rich words is that all complete returns to any palindromic factor u in w are palindromes, [9].
The second property of rich words, that we use, says that a factor r of a rich word w is uniquely determined by its longest palindromic prefix and its longest palindromic suffix, [7]. Some generalizations of this property may be found in [13].

In this article we present upper bounds for the palindromic and factor complexity of rich words, it means the number of palindromes and factors of given length in a rich word w. There are already some related results:

Let us define $F(w,n)$ to be the set of factors of length n of w and let $F_p(w,n) = |\{v \in F(w,n) \text{ and } v \text{ is a palindrome}\}|$, where w is finite or infinite word. It is clear that $|F_p(w,n)| \leq |F(w,n)|$. Some less obvious inequalities are known; one of the interesting inequalities is the following one, [2], [4]: Given an infinite word w with $F(w,n)$ closed under reversal, then $|F_p(w,n)|+|F_p(w,n+1)| \leq |F(w,n+1)|-|F(w,n)|+2$. In order to prove the presented inequality the authors used the notion of Rauzy graphs; a Rauzy graph is a subgraph of the de Bruijn graph, [16]. In section 3 we generalize this result for finite words, what allows us to improve our upper bound for the factor complexity of finite rich words.

In [1], another inequality has been proven for infinite non-ultimately periodic words:$F_p(w,n) < \frac{16}{n}F(w, n + \lfloor\frac{n}{4}\rfloor)$.

In [14], the authors show that a random word of length n contains, on expectation, $\Theta(\sqrt{n})$ distinct palindromic factors.

Related to the palindromic and factor complexity of rich words is the number of rich words of length n, denoted $\Pi(n)$, since obviously $|F(w,n)| \leq \Pi(n)$, where w is a rich word (finite or infinite). The number of rich words was investigated in [18], where the author gives a recursive lower bound on the number of rich words of length n, and an upper bound on the number of binary rich words. Both these estimates seem to be very rough. In [11], the authors construct for each n a large set of rich words of length n. Their construction gives, currently, the best lower bound on the number of binary rich words, namely $\Pi(n) \geq \frac{C^{\sqrt{n}}}{p(n)}$, where $p(n)$ is a polynomial and the constant $C \approx 37$.
Any factor of a rich word is rich too, see [9]. In other words, the language of rich words is factorial. In particular it means that $\Pi(n)\Pi(m) \leq \Pi(n+m)$ for any $m, n, q \in \mathbb{N}$. Therefore, the Fekete's lemma implies existence of the limit of $\sqrt[n]{\Pi(n)}$ and moreover

$$\lim_{n\to\infty} \sqrt[n]{\Pi(n)} = \inf\left\{ \sqrt[n]{\Pi(n)} \colon n \in \mathbb{N} \right\}.$$

For a fixed n_0, one can find the number of all rich words of length n_0 and obtain an upper bound on the limit. Using a computer Rubinchik counted $\Pi(n)$ for $n \leq 60$, (see the sequence A216264 in OEIS). As $\sqrt[60]{\Pi(60)} < 1.605$, he obtained the upper bound for the binary alphabet: $\Pi(n) < c1.605^n$ for some constant c, [11].
In [15], the author shows that $\Pi(n)$ has a subexponential growth on any alphabet. Formally $\lim\limits_{n\to\infty} \sqrt[n]{\Pi(n)} = 1$. This result is an argument in favor of a conjecture formulated in [11] saying that for some infinitely growing function $g(n)$ the following holds true for a binary alphabet:

$$\Pi(n) = \mathcal{O}\Big(\frac{n}{g(n)}\Big)^{\sqrt{n}}.$$

In this article we construct upper bounds for palindromic and factor complexity. The proof uses the following idea: Let u be a palindromic factor

of a rich word w on the alphabet A, such that aub is factor of w, where $a, b \in A$ and $a \neq b$. Then $lpp(aub)$ and $lps(aub)$ (the longest palindromic prefix and suffix) determine uniquely the factor aub in w. We show that a, b and $lpps(u)$ (the longest proper palindromic suffix) determine uniquely aub too. In addition we observe that either $|lpps(u)| \leq \frac{1}{2}|u|$ or u contains a palindromic factor $\bar{u}$ which determines uniquely u and such that $|\bar{u}| \leq \frac{1}{2}|u|$. Anyway we obtain a "short" palindrome and letters a, b which uniquely determine the "long" palindrome u in case that aub is a factor of w. In these "short" palindromes there are again another "shorter" palindromes and so on. As a consequence we present an upper bound for the number of factors of the form aub with $|aub| = n$.
The property of rich words that all complete returns to any palindromic factor u in w are palindromes, [9], allows us to prove that if w contains factors xux and yuy, where $x, y \in A$ and $x \neq y$, then w must contain a factor of the form aub (recall that $a, b \in A$ and $a \neq b$). This property brings the relation between the factors aub and palindromic factors xux. Due to this we derive an upper bound for the palindromic complexity of rich words. Knowing the upper bound for palindromic complexity and applying again the property from [7] (each factor is uniquely determined by its longest palindromic prefix and its longest palindromic suffix) and the relation $|F_p(w, n)| + |F_p(w, n+1)| \leq |F(w, n+1)| - |F(w, n)| + 2$ from [2] we obtain several upper bounds for the factor complexity.

2 Palindromic complexity of rich words

Consider an alphabet A with q letters, where $q > 1$. A^+ denotes the set of all non empty words over A.
Let ϵ denote the empty word and let $A^* = A^+ \cup \{\epsilon\}$.
Let R_n be the set of rich words of length $n \geq 0$ over A. Let $R^+ = \bigcup_{j>0} R_j$ and $R^* = R^+ \cup \{\epsilon\}$. In addition we define R^∞ to be the set of infinite rich words.
Let $lps(w)$ be the longest palindromic suffix of a word $w \in A^+$ and $lpp(w)$ the longest palindromic prefix. In addition we introduce $lpps(w)$ to be the longest proper palindromic suffix and $lppp(w)$ to be the longest proper palindromic prefix, where $|w| > 1$ (proper means that $lpps(w) \neq w$ and $lppp(w) \neq w$). For a word w with $|w| \leq 1$ we define $lppp(w) = lpps(w) = \epsilon$.
Given a word w of length n, we can write $w = w_1 w_2 \dots w_n$, where $w_i \in A$;

then we define $w[i] = w_i$ and $w[i,j] = w_i w_{i+1} \dots w_j$, where $0 < i \leq j \leq n$.

Moreover we define:
P_n : the set of palindromes of length n
$P^+ = \bigcup_{j>0} P_j$, the set of all palindromes of length > 0
$F(w)$: the set of factors of the word w.
$F(w,n) = \{u \mid u \in F(w) \text{ and } |u| = n\}$ (the set of factors of length n)
$F_p(w) = F(w) \cap \bigcup_{j\geq 0} P_j$ (the set of palindromic factors)
$F_p(w,n) = F(w,n) \cap P_n$ (the set of palindromic factors of length n)

Definition 2.1. *Let $w \in A^*$, we define:*
$Strip(w) = w[2, |w|-1]$, where $|w| > 2$. For $|w| \leq 2$ we define $Strip(w) = \epsilon$. (the function $Strip(w)$ takes off the first and last letter from w). For a set of words S we define $Strip(S) = \{Strip(v) \mid v \in S\}$.

Example 2.2. $w = 01123501$
$Strip(w) = 112350$
$Strip(\{12213, 112, 2, 344\}) = \{221, 1, \epsilon, 4\}$

Definition 2.3. *Let $\gamma(w,n) = \{aub \mid aub \in F(w,n) \text{ and } u \in F_p(w, n-2) \text{ and } a \neq b \text{ and } a,b \in A\}$, where $w \in R^*$ and $n > 2$. For $n \leq 2$ we define $\gamma(w,0) = \gamma(w,1) = \gamma(w,2) = \emptyset$.*
Let $\bar{\gamma}(w,n) = \bigcup_{aub\in\gamma(w,n)}\{(u,a),(u,b)\}$, where $a,b \in A$ (a couple $(u,a) \in \bar{\gamma}(w,n)$ if and only if there is $b \in A$ such that $aub \in \gamma(w,n)$ or $bua \in \gamma(w,n)$).
Let $aub \in \gamma(w,n)$, where $a,b \in A$. We call the word aub a u-switch of w. Alternatively we say that w contains a u-switch.

Example 2.4. $A = \{0,1,2,3,4,5,6\}$
$w = 5112211311001131133114111146$
$\gamma(w,8) = \{51122113, 31133114, 14111146\}$
$Strip(\gamma(w,8)) = \{112211, 113311, 411114\}$
$\bar{\gamma}(w,8) = \{(112211,3),(112211,5),(113311,3),(113311,4),$
$(411114,1),(411114,6)\}$
w does not contain 110011-switch, formally $110011 \notin Strip(\gamma(w,8))$

Remark 2.5. The idea of a u-switch follows from the next lemma. If w contains two different palindromic extensions aba, bub of u, where $a,b \in A$, $a \neq b$ and $|aua| = n$, then w contains a u-switch of length n. The u-switch "switches" from a to b. Note that $aua, bub \in F(w)$ does not imply that $aub \in F(w)$ or $bua \in F(w)$. It may be, for example, that $auc, cub \in F(w)$. Nonetheless $(u, a), (u, b) \in \bar{\gamma}(w, n)$. In addition the next lemma shows that if $aua, xuy \in F(w, n)$ then $(u, a) \in \bar{\gamma}(w, n)$, where $x, y \in A$ and $a \neq x$ or $a \neq y$.

Lemma 2.6. *Given $u \in F_p(w, n-2)$, where $w \in R^* \cup R^\infty$ and $n > 2$. If $aua, b_1ub_2 \in F_p(w, n)$, where $a, b_1, b_2 \in A$ and $|\{a, b_1, b_2\}| > 1$ (it means that at least one letter is different from others), then $(u, a) \in \bar{\gamma}(w, n)$.*

Proof. Recall the definition of a complete return, [9]: Given a word w and a factor r or w. We call the factor r a complete return to u in w if r contains exactly two occurrences of u, one as a prefix and one as a suffix. A characteristic property of rich words is that all complete returns to any palindromic factor u in w are palindromes, [9]. The lemma is a simple consequence of this characteristic property. Obviously there exist factors $r, xuy \in F(w)$ such that ($xuy = b_1ub_2$ if no other factors satisfy the conditions):

- $x, y \in A$
- $|\{x, y, a\}| > 1$
- r has exactly one occurrence of xuy
- r has exactly one occurrence of aua
- aua, xuy are prefix and suffix of r, without loss of generality let aua be a prefix of r and xuy be a suffix of r
- r has exactly two occurrences of u

Then $Strip(r) = ua \dots xu$ is a complete return to the palindrome u, which has to be a palindrome, hence $a = x$ and $x \neq y$ (recall $|\{x, y, a\}| > 1$), in consequence $auy \in \gamma(w, n)$ and $(u, a) \in \bar{\gamma}(w, n)$. □

To clarify the previous proof, let us see the two following examples:

Example 2.7. $A = \{1, 2, 3, 4, 5, 6\}$
$w = 321234321252126$
Let $aua = 32123$, $b_1ub_2 = 52126$, $xuy = 32125$, $xuy \neq b_1ub_2$
$r = 32123432125$, and $Strip(r) = 212343212$ is a complete return to 212.
Then $(212, 3) \in \bar{\gamma}(w, 5)$.

Example 2.8. $A = \{1, 2, 3, 4, 5, 6\}$
$w = 321234321252$
Let $aua = 32123$, $xuy = b_1ub_2 = 32125$
$r = 32123432125$, and $Strip(r) = 212343212$ is a complete return to 212.
Then $(212, 3) \in \bar{\gamma}(w, 5)$.

We show that the number of palindromic factors and the number of u-switches are related:

Proposition 2.9. *For any rich word $w \in R^+ \cup R^\infty$ and $n \geq 2$ it holds:* $2|\gamma(w, n)| + |F_p(w, n-2)| \geq |F_p(w, n)|$

Proof. We define $\omega(w, n) = \{aua | (u, a) \in \bar{\gamma}(w, n)\}$, ($\omega(w, n)$ is a set of palindromes of length n such that if w contains a u-switch aub then $aua, bub \in \omega(w, n)$). Obviously $|\omega(w, n)| \leq 2|\gamma(w, n)|$.
Let us consider the following partition of F_p: $F_p(w, n) = \dot{F}_p(w, n) \cup \ddot{F}_p(w, n)$. Given a palindrome $v \in F_p(w, n)$ with $u = Strip(v)$, then there are two cases:

- $v \in \dot{F}_p(w, n)$ if w contains u-switch xuy, formally $u \in Strip(\gamma(w, n))$. Lemma 2.6 implies that $v \in \omega(w, n)$ (consider $v = aua$ and the u-switch xuy, then $(u, a) \in \bar{\gamma}(w, n)$). It follows that $|\dot{F}_p(w, n)| \leq |\omega(w, n)| \leq 2|\gamma(w, n)|$

- $v \in \ddot{F}_p(w, n)$ if w does not contain u-switch, formally $u \notin Strip(\gamma(w, n))$. Then $u \in F_p(w, n-2) \setminus Strip(\gamma(w, n))$. Given a palindrome $u \in F_p(w, n-2) \setminus Strip(\gamma(w, n))$, then if w has palindromic factors aua and bub, then $a = b$ since w does not contain a u-switch. It follows that $|\ddot{F}_p(w, n)| \leq |F_p(w, n-2)|$

Then $|\dot{F}_p(w, n)| + |\ddot{F}_p(w, n)| = |F_p(w, n)|$ implies the proposition:
$2|\gamma(w, n)| + |F_p(w, n-2)| \geq |F_p(w, n)|$ □

To clarify the previous proof, let us see the following example:

Example 2.10. $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$w = 2110112333211011454110116110116778776$
$\gamma(w, 7) = \{2110114, 4110116\}$
$F_p(w, 7) = \{1233321, 2110112, 1145411, 6110116, 6778776\}$
$\dot{F}_p(w, 7) = \{2110112, 6110116\}$
$\ddot{F}_p(w, 7) = \{1233321, 1145411, 6778776\}$
$F_p(w, 5) = \{23332, 11011, 14541, 77877\}$
$2|\gamma(w, 7)| + |F_p(w, 5)| \geq |F_p(w, 7)|$
$4 + 4 > 5$

In the next proposition we show that the longest proper palindromic suffix r and two different letters $a, b \in A$ determine uniquely a palindromic factor $u \in F_p(w)$ such that $lpps(u) = r$ and $aub \in \gamma(w, |u| + 2)$:

Proposition 2.11. *Let $w \in R^+ \cup R^\infty$, $u, v \in F_p(w)$, $lpps(u) = lpps(v)$, $a, b \in A$ and $a \neq b$. Then $aub, avb \in F(w)$ implies that $u = v$.*

Proof. It is known that if r, t are two factors of a rich word w and $lps(r) = lps(t)$ and $lpp(r) = lpp(t)$, then $r = t$, [7]. We will identify a u-switch by the longest proper palindromic suffix of u and two distinct letters a, b instead of the functions lps and lpp:

Given a u-switch aub where $a \neq b$, $a, b \in A$, we know that $lps(aub)$ and $lpp(aub)$ determine uniquely the factor aub in w. We will prove that for given $a, b \in A$, $a \neq b$, $n \geq 0$ and a palindrome r there is at most one palindrome $u \in F_p(w)$ such that $lpps(u) = r$ and $aub \in \gamma(w, |aub|)$.

Suppose a contradiction: there are $u, v \in F_p(w)$, $u \neq v$, $a, b \in A$, $a \neq b$ such that $lps(aub) = bpb$, $lps(avb) = bsb$, $lpp(aub) = axa$, $lpp(avb) = aya$, $lpps(u) = lpps(v) = r$ and $aub, avb \in \bigcup_{j>0} \gamma(w, j)$. It implies that p, s, x, y are prefixes of r. Thus if $x \neq y$, then $|x| \neq |y|$. Without loss of generality, let $|x| < |y|$. Since y is a prefix of r, then either ya is a prefix of r or $r = y$, consequently aya is a prefix of both aub and avb; and it contradicts the supposition that $lpp(aub) = axa$ (aya is a prefix of aub and $|aya| > |axa|$). Analogously if $p \neq s$. It follows that $x = y$ and $p = s$, in consequence $lpp(aub) = lpp(avb)$ and $lps(aub) = lps(avb)$, which would imply that $u = v$, which is a contradiction.

Hence we conclude that $a, b \in A$, $a \neq b$, and a palindrome r determine uniquely at most one palindrome $u \in F_p(w)$ such that $lpps(u) = r$ and $u \in Strip(\gamma(w, |u| + 2))$. □

In the following we derive an upper bound for the number of u-switches. Before we need one more definition in order to be able to partition the set $Strip(\gamma(w, n+2))$ into subsets based on the longest proper palindromic suffix:

Definition 2.12. *Let $w \in R^+ \cup R^\infty$, $r \in R^+$ and $n \geq 0$, then we define:*
$\Upsilon(w, n, r) = \{u \mid u \in Strip(\gamma(w, n + 2))$ and $lpps(u) = r\}$. ($\Upsilon(w, n, r)$ is the set of palindromic factors u of length n of the word w having the longest

proper palindromic suffix equal to r *and such that* w *contains* u*-switch.)* *Obviously* $\bigcup_{r\in F_p(w)} \Upsilon(w,n,r) = Strip(\gamma(w,n))$ *and* $\Upsilon(w,n,r)\cap\Upsilon(w,n,\bar{r}) = \emptyset$ *if* $r \neq \bar{r}$*.*

Example 2.13. $A = \{0,1,2,3,4,5\}$
$w = 5112211311001131133114$
$\gamma(w,6) = \{51122113, 31133114\}$
$\Upsilon(w,6,11) = \{112211, 113311\}$
$110011 \notin \Upsilon(w,6,11)$, because w does not contain 110011-switch

A simple corollary of the previous proposition is that the size of the set $\Upsilon(w,n,r)$ is limited by the constant $q(q-1)$ (recall that q is the size of the alphabet A).

Corollary 2.14. *For any rich words* $w \in R^+ \cup R^\infty$*,* $r \in R^+$ *and* $n \geq 0$ *it holds:* $|\Upsilon(w,n,r)| \leq q(q-1)$*.*

Proof. From Proposition 2.11 follows that $|\Upsilon(w,n,r)| \leq |\{(a,b) \mid a,b \in A \text{ and } a \neq b\}| = q(q-1)$ (the number of couples (a,b)). □

We define $\bar{\Gamma}(w,n) = \max\{|\gamma(w,i)| \mid 0 \leq i \leq n\}$, where $w \in R^+ \cup R^\infty$ and $n > 0$. Next we define $\Gamma(w,n) = \max\{1, \bar{\Gamma}(w,n)\}$.

Remark 2.15. We defined $\Gamma(w,n)$ as the maximum from the set of sizes of $\gamma(w,i)$, where $0 < i \leq n$. In addition we defined that $\Gamma(w,n) \geq 1$ (hence it cannot be zero); this is just for practical reason: in this way we can find a constant c such that $\Gamma(w,n_1) = c\Gamma(w,n_2)$ for any $n_1,n_2 \geq 0$. The function $\Gamma(w,n)$ will allow us to present another relation between the number of palindromic factors of length n and the number of u-switches, this time without using $F_p(w,n-2)$:

Lemma 2.16. $(q+1)n\Gamma(w,n) \geq |F_p(w,n)|$*, where* $w \in R^+ \cup R^\infty$ *and* $n > 0$*.*

Proof. Let $\bar{\phi}(n) = 2$ if n is even and $\bar{\phi}(n) = 1$ if n is odd and let $\phi(n) = \{2+\bar{\phi}(n), 4+\bar{\phi}(n), \ldots, n\}$; for example $\phi(8) = \{4,6,8\}$ and $\phi(9) = \{3,5,7,9\}$.

Proposition 2.9 states that $2|\gamma(w,n)| + |F_p(w,n-2)| \geq |F_p(w,n)|$, it follows $2|\gamma(w,n-2)| + |F_p(w,n-4)| \geq |F_p(w,n-2)|$ and consequently $2|\gamma(w,n)| + 2|\gamma(w,n-2)| + F_p(w,n-4)| \geq |F_p(w,n)|$. (we replaced $|F_p(w,n-2)|$ by $2|\gamma(w,n-2)| + |F_p(w,n-4)|$).

By iterating the process of replacing $|F_p(w,n-i)|$ by $2|\gamma(w,n-i)|+|F_p(w,n-2i)|$ we achieve:

$$\sum_{j\in\phi(n)} 2|\gamma(w,j)| + |F_p(w,\bar{\phi}(n))| \geq |F_p(w,n)| \tag{1}$$

Note that $|F_p(w,\bar{\phi}(n))| \leq q$ (the number of palindromes of length 1 or 2). Recall that $\Gamma(w,n) \geq |\gamma(w,j)|$ for $2 < j < n$ and note that $|\phi(n)| \leq \lfloor \frac{n}{2} \rfloor$, then it follows $2\lfloor \frac{n}{2} \rfloor \Gamma(w,n) + q \geq |F_p(w,n)|$ and since $2\lfloor \frac{n}{2} \rfloor \leq n$ we obtain from (1) that $n\Gamma(w,n) + q \geq |F_p(w,n)|$.

It is easy to see that $(q+1)n\Gamma(w,n) \geq n\Gamma(w,n) + q$ for $n > 0$, then the lemma follows. We prefer to use $(q+1)n\Gamma(w,n)$ instead of $n\Gamma(w,n) + q$, because it will be easier to handle in Corollary 2.22, even if it makes the upper bound "a little bit worse". □

We need to cope with the longest proper palindromic suffixes that are "too long". We show that if the longest proper palindromic suffix $lpps(v)$ is longer the half of the length of v, then v contains a "short" palindromic factor, that uniquely determines v. Some similar results can be found in [12].

Lemma 2.17. *Let $u, v \in P^+$ be palindromes, where u is a prefix of v and $\frac{1}{2}|v| \leq |u| < |v|$.*
Let $n = \lceil \frac{|v|}{2} \rceil$ if $|v|$ is odd or $n = \frac{|v|}{2}$ if $|v|$ is even.
Let $k = \lceil \frac{|u|}{2} \rceil$ if $|u|$ is odd or $k = \frac{|u|}{2} + 1$ if $|u|$ is even.
We define $\bar{\rho}(u,v) = v[k,n] = v_k v_{k+1} \ldots v_{n-1} v_n$ and we define $\rho(u,v)$ as follows:

- *if $|u|$ is even, then $\rho(u,v) = v_n v_{n-1} \ldots v_{k+1} v_k v_k v_{k+1} \ldots v_{n-1} v_n$*
- *if $|u|$ is odd, then $\rho(u,v) = v_n v_{n-1} \ldots v_{k+1} v_k v_{k+1} \ldots v_{n-1} v_n$*

The palindrome $\rho(u,v)$ and the length $|v|$ determine uniquely v.

Proof. Given n, j such that $1 \leq j \leq n$, we define $mirror(n,j) = n - j + 1$.
Example: $mirror(10,3) = 8$, $mirror(10,8) = 3$, $mirror(9,5) = 5$.
It is easy to see that $mirror(n, mirror(n,j)) = j$.
Given a palindrome w with $|w| = t$, then clearly $w[i] = w[mirror(t,i)]$ for $1 \leq i \leq t$.

Given u, v, k, n as described in the lemma. We will show that for any $1 \leq j < k$, there is $\bar{j}$ such that $j < \bar{j} \leq n$ and $v[j] = v[\bar{j}]$. Let $i = mirror(|u|, j)$, then $v[j] = v[i]$, since the prefix u is a palindrome. Clearly $i \geq k$; if $i \leq n$, then we are done: $i = \bar{j}$. If $i > n$, then let $\bar{j} = mirror(|v|, i)$. Clearly $j < \bar{j} \leq n$, since $|u| < |v|$. Thus we showed that for any $1 \leq j < k$ there is a position $\bar{j}$ which is "closer to the center of v" and such that $v[j] = [\bar{j}]$. Repeating the process, we can show that for any index j there is an index $\tilde{j}$ such that $k \leq \tilde{j} \leq n$ and $v[j] = v[\tilde{j}]$. Then it is a simple exercise to reconstruct v from $\rho(u, v)$ and the length $|v|$ (note that from $|\rho(u, v)|$ you can determine if $|u|$ is odd or even). □

Let us have a look on the next examples that illuminate the proof:

Example 2.18. Let $v = 12321232123212321$, $|v| = 17$, $u = 1232123212321$, $|u| = 13$. Then $n = 9$, $k = 7$, $\bar{\rho}(u, v) = 321$, and $\rho = 12321$.
Let $j = 2$, then $v[2] = 2$, $mirror(|u|, 2) = mirror(13, 2) = 12$, $v[12] = 2$, $\bar{j} = mirror(|v|, 12) = mirror(17, 12) = 6$, $v[6] = 2$.
Let $j = 6$, then $v[6] = 2$, $\bar{j} = mirror(|u|, 6) = mirror(13, 6) = 8$, $v[8] = 2$.

Example 2.19. Let $v = 211221122112$, $|v| = 12$, $u = 21122112$, $|v| = 8$. Then $n = 6$, $k = 5$, $\bar{\rho}(u, v) = 21$, and $\rho = 1221$.

We derive an upper bound for the number of u-switches:

Proposition 2.20. $\Gamma(w, n) \leq q^5(\lceil \frac{n}{2} \rceil)^2 \Gamma(w, \lceil \frac{n}{2} \rceil)$, *where* $w \in R^+ \cup R^\infty$ *and* $n > 0$.

Proof. For a word w with a palindromic factor v, where v has a palindromic suffix u with $\frac{1}{2}|v| \leq |u| < |v|$, the longest proper palindromic suffix $lpps(v)$ would be longer than the half of v, formally $|lpps(u)| \geq \frac{1}{2}|v|$. In such a case the $\rho(u, v)$ is defined and $|\rho(u, v)| \leq \frac{1}{2}|v|$. Anyway, we have a "short" palindrome ($\rho(u, v)$ or the longest proper palindromic suffix $lpps(v)$) that uniquely identifies at most $q(q-1)$ distinct palindromes v of length n. It means that we need only to take into account palindromic factors of length $\leq \lceil \frac{n}{2} \rceil$. Let us express this idea formally:

It is clear that the proposition holds for $n \in \{1, 2\}$. Thus in the proof we consider $n > 2$. We partition $Strip(\gamma(w, n))$ into sets $\Delta_\rho(w, n), \Delta_{lpps}(w, n)$ as follows: given $v \in Strip(\gamma(w, n))$, then $v \in \Delta_\rho(w, n)$ if $\frac{1}{2}|v| \leq |lpps(v)|$, otherwise $v \in \Delta_{lpps}(w, n)$. Obviously $Strip(\gamma(w, n)) = \Delta_\rho(w, n) \cup \Delta_{lpps}(w, n)$ and $\Delta_\rho(w, n) \cap \Delta_{lpps}(w, n) = \emptyset$. Let us investigate the size of $\Delta_\rho(w, n)$ and $\Delta_{lpps}(w, n)$.

- $\rho(u,v)$ and $|v|$ determine uniquely the palindrome v, see Lemma 2.17; in addition note that $|\rho(u,v)| \leq \lceil \frac{|v|}{2} \rceil$. Hence the sum over the number of all palindromic factors of w of length $\leq \lceil \frac{n}{2} \rceil$ must be bigger or equal to the size of $\Delta_\rho(w,n)$.

$$|\Delta_\rho(w,n)| = \sum_{j=1}^{\lceil \frac{n}{2} \rceil} |F_p(w,j)| \tag{2}$$

- the longest proper palindromic suffix $lpps(v)$ identifies at most $q(q-1)$ distinct palindromic factors of w, see Corollary 2.14; by definition $|lpps(v)| < \frac{1}{2}|v|$. Hence the sum over the number of all palindromic factors of w of length $\leq \lceil \frac{n}{2} \rceil$ multiplied by $q(q-1)$ must be bigger or equal to the size of $\Delta_{lpps}(w,n)$

$$|\Delta_{lpps}| = q(q-1) \sum_{j=1}^{\lceil \frac{n}{2} \rceil} |F_p(w,j)| \tag{3}$$

Actually the sets $\Delta_\rho(w,n)$ and $\Delta_{lpps}(w,n)$ contain palindromes of length $n-2$, thus it would be sufficient to sum up to the length $\lceil \frac{n-2}{2} \rceil$ instead of $\lceil \frac{n}{2} \rceil$, but again in Corollary 2.22, it will be more comfortable to handle $\lceil \frac{n}{2} \rceil$.

It is easy to see that $|\gamma(w,n)| \leq q(q-1)|Strip(\gamma(w,n))|$ (for every $u \in Strip(\gamma(w,n))$ and $a,b \in A$, where $a \neq b$ there can be $aub \in \gamma(w,n)$). It follows:

$$|\gamma(w,n)| \leq q(q-1)|Strip(\gamma(w,n))| = q(q-1)(|\Delta_\rho(w,n)| + |\Delta_{lpps}(w,n)|) \tag{4}$$

Then it follows from (2), (3) and (4) that

$$|\gamma(w,n)| \leq q(q-1)(q(q-1)+1) \sum_{j=1}^{\lceil \frac{n}{2} \rceil} |F_p(w,j)| \tag{5}$$

From Lemma 2.16 we know that $|F_p(w,j)| \leq (q+1)j\Gamma(w,j)$, thus we have:

$$\sum_{j=1}^{\lceil \frac{n}{2} \rceil} |F_p(w,j)| \leq \sum_{j=1}^{\lceil \frac{n}{2} \rceil} (q+1)j\Gamma(w,j) \leq \lceil \frac{n}{2} \rceil (q+1) \lceil \frac{n}{2} \rceil \Gamma(w, \lceil \frac{n}{2} \rceil) \tag{6}$$

From (5) and (6):
$|\gamma(w,n)| \leq q(q-1)(q(q-1)+1)(q+1)(\lceil \frac{n}{2} \rceil)^2 \Gamma(w, \lceil \frac{n}{2} \rceil)$.
To simplify the formula we apply $q(q-1)(q(q-1)+1)(q+1) = q(q^2-1)(q^2-q+1) < q^5$, as a result we have $|\gamma(w,n)| \leq q^5(\lceil \frac{n}{2} \rceil)^2 \Gamma(w, \lceil \frac{n}{2} \rceil)$.

Recall the definition of Γ, which is in this case lower of equal to the maximal upper bound from the set $\{q^5(\lceil \frac{j}{2} \rceil)^2 \Gamma(w, \lceil \frac{j}{2} \rceil) \mid 0 \leq j \leq n\}$:
$\Gamma(w,n) \leq \max\{q^5(\lceil \frac{j}{2} \rceil)^2 \Gamma(w, \lceil \frac{j}{2} \rceil) \mid 0 \leq j \leq n\} = q^5(\lceil \frac{n}{2} \rceil)^2 \Gamma(w, \lceil \frac{n}{2} \rceil)$. □

For the next corollary we will need the following lemma.

Lemma 2.21. $\prod_{j\geq 1}^{k} \lceil \frac{n}{2^j} \rceil \leq (2\sqrt{n})^{\log_2 n}$*, where* $k = \lfloor \log_2 n \rfloor$

Proof. $\prod_{j\geq 1}^{k} \lceil \frac{n}{2^j} \rceil = \lceil \frac{n}{2} \rceil \lceil \frac{n}{4} \rceil \lceil \frac{n}{8} \rceil \dots \lceil \frac{n}{2^{k-1}} \rceil \lceil \frac{n}{2^k} \rceil \leq$
$(\frac{n}{2}+1)(\frac{n}{4}+1)(\frac{n}{8}+1)\dots(\frac{n}{2^{k-1}}+1)(\frac{n}{2^k}+1) =$
$(\frac{n+2}{2})(\frac{n+4}{4})(\frac{n+8}{8})\dots(\frac{n+2^{k-1}}{2^{k-1}})(\frac{n+2^k}{2^k}) = \frac{\prod_{j=1}^{k}(n+2^j)}{\prod_{j=1}^{k} 2^j}$
hence we have:

$$\prod_{j\geq 1}^{k} \lceil \frac{n}{2^j} \rceil \leq \frac{\prod_{j=1}^{k}(n+2^j)}{\prod_{j=1}^{k} 2^j} \tag{7}$$

Next we investigate the both products on the right side from (7)

$$\prod_{j=1}^{k}(n+2^j) = (n+2)(n+4)(n+8)\dots(n+2^{k-1})(n+2^k) \leq (2n)^k \tag{8}$$

(note that $n+2^j \leq 2n$, where $j \leq k$)

$$\prod_{j=1}^{k} 2^j = 22^2 2^3 \dots 2^{k-1} 2^k = 2^{\sum_{j=1}^{k} j} = 2^{\frac{k(k+1)}{2}} \tag{9}$$

Then from (7), (8) and (9):
$\prod_{j\geq 1}^{k} \lceil \frac{n}{2^j} \rceil \leq \frac{\prod_{j=1}^{k}(n+2^j)}{\prod_{j=1}^{k} 2^j} \leq \frac{(2n)^k}{2^{\frac{k(k+1)}{2}}} = \left(\frac{2n}{2^{\frac{(k+1)}{2}}}\right)^k$

Since $2^{k+1} \geq n$:

$$\left(\frac{2n}{2^{\frac{(k+1)}{2}}}\right)^k \leq \left(\frac{2n}{n^{\frac{1}{2}}}\right)^k = (2n^{\frac{1}{2}})^k \leq (2\sqrt{n})^{\log_2 n}$$

□

Corollary 2.22. $\Gamma(w,n) \le (4q^{10}n)^{\log_2 n}$*, where* $w \in R^+ \cup R^\infty$ *and* $n > 0$..

Proof. Proposition 2.20 states that

$$\Gamma(w,n) \le q^5(\lceil \frac{n}{2} \rceil)^2 \Gamma(w, \lceil \frac{n}{2} \rceil).$$

By replacing $\Gamma(w, \lceil \frac{n}{j} \rceil)$ by $q^5(\lceil \frac{n}{2j} \rceil)^2\Gamma(w, \lceil \frac{n}{2j} \rceil)$, we obtain
$\Gamma(w,n) \le q^5(\lceil \frac{n}{2} \rceil)^2\Gamma(w, \lceil \frac{n}{2} \rceil) \le q^5(\lceil \frac{n}{2} \rceil)^2 q^5(\lceil \frac{n}{4} \rceil)^2 \Gamma(w, \lceil \frac{n}{4} \rceil) \le$
$q^5(\lceil \frac{n}{2} \rceil)^2 q^5(\lceil \frac{n}{4} \rceil)^2 q^5(\lceil \frac{n}{8} \rceil)^2 \Gamma(w, \lceil \frac{n}{8} \rceil) \le \dots$.
Recall that $\lceil \frac{\lceil m \rceil}{2} \rceil = \lceil \frac{m}{2} \rceil$, where $m \ge 0$ is a real constant (see [10] in chapter 3.2 Floor/ceiling applications).
Finally after $\lfloor \log_2 n \rfloor$ steps:

$$\Gamma(w,n) \le \left(\prod_{j \ge 1}^{\lfloor \log_2 n \rfloor} q^5 \lceil \frac{n}{2^j} \rceil \right)^2 \Gamma(w, h(n)),$$

where $h(n) \in \{1,2\}$ depending on n. Knowing that $\Gamma(w,1) = \Gamma(w,2) = 1$ and using Lemma 2.21 we obtain

$$\Gamma(w,n) \le \left((q^5 2\sqrt{n})^{\log_2 n} \right)^2 \Gamma(w,1) = (4q^{10}n)^{\log_2 n}.$$

□

From Lemma 2.16 and Corollary 2.22 it follows easily:

Corollary 2.23. $|F_p(w,n)| \le (q+1)n(4q^{10}n)^{\log_2 n}$ *where* $w \in R^+ \cup R^\infty$ *and* $n > 0$.

We can simply apply the upper bound for the palindromic complexity to construct an upper bound for factor complexity:

Corollary 2.24. $|F(w,n)| \le (q+1)^2 n^4 (4q^{10}n)^{2\log_2 n}$ *where* $w \in R^+ \cup R^\infty$ *and* $n > 0$.

Proof. We apply again the property of rich words that every factor is determined by its longest palindromic prefix and its longest palindromic suffix, [7]. Hence if there are at most t palindromic factors in w of length $\le n$, than clearly there can be at most t^2 different factors of length n. Let $\hat{F}_p(w,k) = \max\{|F_p(w,j)| \mid 0 \le j \le k\}$. From Lemma 2.23 we can deduce that $t \le \sum_{i=1}^{n} |F_p(w,i)| \le n\hat{F}_p(w,n) \le n(q+1)n(4q^{10}n)^{\log_2 n}$. The lemma follows. □

3 Rich words closed under reversal

Given a word $w = w_1w_2 \dots w_{n-1}w_n \in A^*$, where $w_i \in A$, let w^R denote the reversal of w, formally $w^R = w_nw_{n-1} \dots w_2w_1$. We say that the set $S \in A^*$ is closed under reversal if $w \in S$ implies that $w^R \in S$.

We can achieve an another improvement for the factor complexity if we use the inequality $|F_p(w,n)|+|F_p(w,n+1)| \leq |F(w,n+1)|-|F(w,n)|+2$ from [2], [4]. This inequality was proven for infinite words closed under reversal. The next proposition generalizes the existing proof to allow us to use the result for any word w with $F(w,n+1)$ closed under reversal (including finite words).

Proposition 3.1. *Let $w \in R^+ \cup R^\infty$ be a rich word such that $F(w,n+1)$ is closed under reversal and $|w| \geq n+1$. Then $|F_p(w,n)| + |F_p(w,n+1)| \leq |F(w,n+1)| - |F(w,n)| + 2$.*

Proof. The inequality $|F_p(w,n)|+|F_p(w,n+1)| \leq |F(w,n+1)|-|F(w,n)|+2$ is shown in [2] for infinite words closed under reversal. However, inspecting the proof of Theorem 1.2 (ii) we conclude that the inequality is satisfied if the Rauzy graph Γ_n is strongly connected and if $L_{n+1}(w)$ (or $F(w,n+1)$ with our notation) is closed under reversal, since the map ρ is then defined for all vertices and edges of the Rauzy graph Γ_n. (See in [2] for the details of a construction of the Rauzy graph). Because we require $F(w,n+1)$ to be closed under reversal, we need only to prove that the Rauzy graph Γ_n is strongly connected. For an infinite word w the set L_{n+1} closed under reversal implies that w is recurrent (any factor has at least two occurrences) and in consequence that the Rauzy graph Γ_n is strongly connected.

For a finite word w with $L_{n+1}(w)$ closed under reversal, the Rauzy graph Γ_n is not necessarily strongly connected. Therefore we have to show that we can still apply the existing proof: Let B denote the alphabet $B = A \cup \{x\}$ (without loss of generality suppose that $x \notin A$). Consider the word $\tilde{w} = wxw^R$ on the alphabet B, then $\tilde{w}$ is closed under reversal ($\tilde{w}$ is a palindrome) and it is easy to see that $F(\tilde{w},k) = F(w,k) \cup \bar{F}(w,k)$, where $k \in \{n,n+1\}$ and $\bar{F}(w,k) = \{uxv \mid u \text{ is a suffix of } w \text{ and } v \text{ is a prefix of } w^R \text{ and } |uxv| = k\}$ (u,v may be the empty words). Obviously $|\bar{F}(w,k)| = k$. It follows that $F(w,n) \subset F(\tilde{w},n)$, $F(w,n+1) \subset F(\tilde{w},n+1)$ and

$$|F(\tilde{w},n)| = |F(w,n)| + n \tag{10}$$

$$|F(\tilde{w},n+1)| = |F(w,n+1)| + n + 1 \qquad (11)$$

There is just one palindrome in $\bar{F}(w,n) \cup \bar{F}(w,n+1)$, because every word in $\bar{F}(w,n) \cup \bar{F}(w,n+1)$ has exactly one occurrence of x, consequently only one word $z \in \bar{F}(w,n) \cup \bar{F}(w,n+1)$ has the form uxu^R (uxu^R is of odd length). Therefore it follows:

$$|F_p(w,n)| + |F_p(w,n+1)| + 1 = |F_p(\tilde{w},n)| + |F_p(\tilde{w},n+1)| \qquad (12)$$

The Rauzy graph $\tilde{\Gamma}_n$ of $\tilde{w} = wxw^R$ is strongly connected: realize that $F(w,n+1)$ closed under reversal implies that $F(w,n+1) = F(w^R,n+1)$ and $F(w,n) = F(w^R,n)$. Hence for $\tilde{w}$ it holds $|F_p(\tilde{w},n)| + |F_p(\tilde{w},n+1)| \leq |F(\tilde{w},n+1)| - |F(\tilde{w},n)| + 2$. It follows then from (10), (11) and (12) that $|F_p(w,n)| + |F_p(w,n+1)| \leq |F(w,n+1)| - |F(w,n)| + 2$. □

To clarify the previous proof, let us have a look on the example below:

Example 3.2. Consider the rich word $w = 1100100010011001010$.
Then $F(w,3) = \{110, 100, 001, 010, 000, 011, 101\}, |F(w,3)| = 7$
$F(w,4) = \{1100, 1001, 0010, 0100, 1000, 0001, 0011, 0110, 0101, 1010\}$,
$|F(w,4)| = 10$
$F_p(w,3) = \{010, 000, 101\}, |F_p(w,3)| = 3$
$F_p(w,4) = \{1001, 0110\}, |F_p(w,4)| = 2$
It follows that $|F_p(w,3)| + |F_p(w,4)| = |F(w,4)| - |F(w,3)| + 2$
$3 + 2 = 10 - 7 + 2$
$B = \{0, 1, x\}$
$\tilde{w} = wxw^R = 1100100010011001010x0101001100100010011$
$\bar{F}(\tilde{w},3) = \{10x, 0x0, x01\}$
$\bar{F}(\tilde{w},4) = \{010x, 10x0, 0x01, x010\}$
$(\bar{F}(\tilde{w},3) \cup \bar{F}(\tilde{w},4)) \cap F_p(\tilde{w}) = \{0x0\}$
$|F_p(\tilde{w},3)| + |F_p(\tilde{w},4)| = |F(\tilde{w},4)| - |F(\tilde{w},3)| + 2$
Thus $4 + 2 = 14 - 10 + 2$.

For rich words the inequality may be replaced with equality:

Lemma 3.3. *Let $w \in R^+ \cup R^\infty$ be a rich word such that $F(w,n+1)$ is closed under reversal, $|w| \geq n+1$ and $n > 0$. Then $|F_p(w,n)| + |F_p(w,n+1)| = |F(w,n+1)| - |F(w,n)| + 2$.*

Proof. Note in the proof of Proposition 3.1 that $\tilde{w} = wxw^R$ is rich if w is rich. To see this, note that wx is rich, because $lps(wx) = x$ which is a unioccurrent palindrome in wx and wxw^R is a palindromic closure of wx, which preserves richness, [9]. Then the equality follows from Proposition 3 in [6] (the proposition uses the palindromic defect $D(w)$ of a word, which is, by definition, equal to zero for a rich word). □

Based on Lemma 3.3 we can present a new relation for palindromic and factor complexity:

Proposition 3.4. *Let* $\hat{F}_p(w,k) = \max\{|F_p(w,j)| \mid 0 \leq j \leq k\}$. *Let* $w \in R^+ \cup R^\infty$ *be a rich word such that* $F(w, n+1)$ *is closed under reversal,* $|w| \geq n+1$ *and* $n > 0$. *Then* $|F(w,n)| \leq 2(n-1)\hat{F}_p(w,n) - 2(n-1) + q$.

Proof. Using Lemma 3.3:
$|F_p(w,n)| + |F_p(w,n+1)| = |F(w,n+1)| - |F(w,n)| + 2$
$|F_p(w,n)| + |F_p(w,n+1)| - 2 = |F(w,n+1)| - |F(w,n)|$
Since $F(w,n+1)$ closed under reversal implies that $F(w,i)$ is closed under reversal for $i \leq n+1$, we can sum over all lengths $i \leq n$:
$\sum_{i=1}^{n-1}(|F_p(w,i)| + |F_p(w,i+1)| - 2) = \sum_{i=1}^{n-1}(|F(w,i+1)| - |F(w,i)|)$,
where the sums may be expressed as follows:

$\sum_{i=1}^{n-1}(|F(w,i+1)| - |F(w,i)|) = F(w,2) - F(w,1) + F(w,3) - F(w,2) + F(w,4) - F(w,3) + \cdots + F(w,n-1) - F(w,n-2) + F(w,n) - F(w,n-1) = F(w,n) - F(w,1)$

$\sum_{i=1}^{n-1}(|F_p(w,i)| + |F_p(w,i+1)| - 2) \leq (n-1)(\hat{F}_p(w,n-1) + \hat{F}_p(w,n) - 2)$.

It follows: $F(w,n) - F(w,1) \leq (n-1)(\hat{F}_p(w,n-1) + \hat{F}_p(w,n) - 2)$
$F(w,n) \leq (n-1)(2\hat{F}_p(w,n) - 2) + F(w,1)$
$F(w,n) \leq 2(n-1)\hat{F}_p(w,n) - 2(n-1) + F(w,1)$
obviously $F(w,1) \leq q$, then:
$F(w,n) \leq 2(n-1)\hat{F}_p(w,n) - 2(n-1) + q$

□

The next proposition improves our upper bound for the factor complexity for rich words with $F(w,n+1)$ closed under reversal:

Corollary 3.5. $|F(w,n)| \leq 2(n-1)(q+1)n(4q^{10}n)^{\log_2 n} - 2(n-1) + q$, *where* $w \in R^+ \cup R^\infty$, $F(w,n+1)$ *is closed under reversal and* $|w| \geq n+1$, $n > 0$.

Proof. From Proposition 3.4 and Lemma 2.23 we achieve the result:
$|F(w,n)| \le 2(n-1)(q+1)n(4q^{10}n)^{\log_2 n} - 2(n-1) + q$

□

Since the palindromic closure of finite rich words is closed under reversal, we can improve the upper bound for factor complexity for finite rich words.

Corollary 3.6. *$|F(w,n)| \le 2(2n-1)(q+1)2n(8q^{10}n)^{\log_2 2n} - 2(2n-1) + q$, where $w \in R^+$.*

Proof. Palindromic closure $\hat{w} \in R^+$ of a word $w \in R^+$ preserves richness, $\hat{w}$ is closed under reversal, $F(w) \subseteq F(\hat{w})$ and $|\tilde{w}| \le 2|w|$, [9]. Hence we can apply Corollary 3.5 where we replace n with $2n$. □

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