# FAT AND THIN EMERGENT GEOMETRIES OF HERMITIAN ONE-MATRIX MODELS 

JIAN ZHOU


#### Abstract

We use genus zero free energy functions of Hermitian matrix models to define spectral curves and their special deformations. They are special plane curves defined by formal power series with integral coefficients generalizing the Catalan numbers. This is done in two different versions, depending on two different genus expansions, and these two versions are in some sense dual to each other.


## 1. Introduction

Studies of Hermitian one-matrix models, based on Gaussian type integrals on the spaces of Hermitian $N \times N$-matrices, are often focused on their limiting behaviors as $N \rightarrow \infty$ in the literature. In the early days of string theory, with the applications of fat graphs introduced by 't Hooft [23], a more sophisticated limit called the double scaling limit played an important role in relating matrix models to a wide range of objects interesting to mathematicians and mathematical physicists, including orthogonal polynomials, integrable hierarchies, Virasoro constraints, etc. See e.g. [5] for a survey.

Since its introduction by Euler, graphs have been widely studied by mathematicians, especially in combinatorics, and their studies form an important part of discrete mathematics. They have also been widely used in quantum field theories to formulate Feynman rules for Feynman sums. Of course graph theory are developed by mathematicians and physicists for different purposes and by different techniques. So it is reasonable to expect that techniques developed in quantum field theories should be useful to combinatorial problems related to the enumerations of graphs, and vice versa. See e.g. [2] for an exposition of quantum field techniques in graph enumerations.

By contrast, fat-graphs, even though they are also of combinatorial and physical origins, have more direct geometrical connections. They have also long been studied by combinatorists [24, even before their appearance in the physics literature. In the combinatorics literature they have been called maps, or graphs on Riemann surfaces. However the significance of such geometric connections seemed to be only realized after the advent of string theory in connection with 2D quantum gravity. They were used by Harer-Zagier [13] and Penner [20] to calculate the orbifold Euler characteristics of the moduli spaces of Riemann surfaces $\mathcal{M}_{g, n}$, making a surprising connection to special values of Riemann zeta-function. (Of course, another connection between random matrices and Riemann zeta-function was made by Montgomery and Dyson even earlier [17]. See [3] for an exposition.) Through the work of Witten [26, 27, matrix models are related to intersection theory on the Deligne-Mumford compactifications $\overline{\mathcal{M}}_{g, n}$ and its generalizations to include the
spin structures, and connections to the KdV hierarchies and Virasoro constraints, first discovered in the setting of Hermitian matrix models, was made by Witten for the geometric theories related to $\overline{\mathcal{M}}_{g, n}$ and its generalizations. See the references in Witten [26, 27] for a guide to the literature on this line of developments. Kontsevich [14] introduced a new kind of matrix models to prove Witten's Conjecture.

The generalizations of Witten Conjcture/Kontsevich Theorem to relate other Gromov-Witten type geometric theories to integrable hierarchies and to establish their infinite-dimensional algebraic constraints have been one of the driving forces in the mathematical theory of string theory. Another driving force is the study of mirror symmetry, among which the local mirror symmetry of toric Calabi-Yau 3 -folds has been studied via matrix models. The mirror geometry of a toric CalabiYau 3-fold can be encoded in a plane curve. Dijkgraaf and Vafa [6, 7] identify this curve with the spectral curve of a suitable matrix model.

It is interesting to note that the notion of a spectral curve also appear in at least two other settings. One is in the setting of integrable hierarchies (see e.g. [18), the other is in the setting of Eynard-Orantin topological recursions [10] which itself grows out of the theory of random matrices. So far, both Witten Conjecture/Kontsevich Theorem and local mirror symmetry have been studied from the following three points of view: matrix models, integrable hierarchies, and EO topological recursions. Spectral curves have played an important role in all these different approaches.

Borrowing terminology from statistical physics, we say that the appearance of spectral curves in matrix models are emergent. By emergent geometry we mean the geometric structures that appear when one has an infinite degree of freedom. The spectral curve of matrix models reflects the collective behavior of the spectrum of the $N \times N$-Hermitian matrix as $N$ goes to infinity. In the literature, one introduces a discrete resolvent $\omega_{N}$ for $N \times N$-Hermitian matrix and assume it tends to a differentiable function $\omega(z)$ as $n \rightarrow \infty$. This function satisfies a quadratic equation, hence defining a hyperelliptic plane curves. This is how spectral curves arise in matrix models. See e.g. [4, p. 47].

A natural question one can ask is whether the emergence of spectral curves can only occur when one takes the limit $N \rightarrow \infty$, or it can also occur for finite $N$. This is a legitimate question because for any theory with infinite degree of freedom one can ask this type of questions, and for each finite $N$, the Hermitian one-matrix model based on $N \times N$-matrices can be regarded as a formal quantum field theory whose operator algebra is the algebra of symmetric functions [35] §2], and so it has an infinite degree of freedom.

Another motivation for asking this question comes from our study of the Witten Conjecture/Kontseivich Theorem from the point of view of emergent geometry. In an earlier work [29] we have shown that Virasoro constraints of 8 satisfied by the Witten-Kontsevich $\tau$-function is equivalent to EO toplogical recursion on the Airy curve. This result was later understood from two different perspectives. First in 30, this was interpreted as a geometric reformulation of the Virasoro constraints by introducing the notion of special deformations and quantization of special deformations of the Airy curve. Next, after the author became aware of the notion of emergence through the contacts with some physicists working on condense matter physics, he started to use the terminology of emergent geometry in 32. In that work, the emergent geometry of Witten-Kontsevich partition function was
cast in a more general setting of emergent geometry of KP hierarchy based on boson-fermion correspondence, and in particular a formula for the bosonic $n$-point function associated to $\tau$-function of the KP hierarchy was derived. See 33 for further developments.

We start to address this question in 31. Instead of letting $N \rightarrow \infty$ as in the literature, we go to the other extreme and let $N=1$. The resulting theory is called the theory of topological 1D gravity or the polymer model. The partition function of this theory is just a universal generating series that enumerates all possible graphs, but we study it by quantum field theory techniques, including Feynman sums, Feynman rules, Virasoro constraints and KP hierarchy. Furthermore, we introduce an analogue of Wilson's theory of renormalization in its study. An interesting consequence is that we can reduce the problem of enumeration of graphs with fixed number of loops to the problem of determining finitely many correlators in this theory. Another result obtained in this work is that even for $N=1$, one can talk about the emergent geometry of spectral curves. In this case, we introduce a spectral curve which we call the signed Catalan curve since it is related to Catalan numbers up to signs. We also introduce its special deformation and its quantization. In a more recent development, we unify this theory with the theory of holomorphic anomaly equations [1] via the diagrammatics of Deligne-Mumford compactification 25.

Next, we move on the case of finite $N \geq 1$. Since it is known that each of the partition function $Z_{N}$ of the Hermitian $N \times N$-matrix model is a $\tau$-function of the KP hierarchy [21, we can apply all the results on emergent geometry we have developed for KP hierarchy. We can also try to apply the insight we gain from 31. This is exactly what we do in [35], 36] and this paper. First of all, we identify the element in the Sato grassmannian corresponding to $Z_{N}$ and obtain a formula for the associated bosonic $n$-point functions in [35]. This was achieved by the following techniques well-known in the literature: First, the correlators are expressed as an enumeration problem of fat graphs; next, this is converted to a problem in 2D Yang-Mills theory with finite gauge groups [9, and solved with the help of the representation theory of symmetric groups; finally, with the help of the theory of symmetric functions, one can reformulate the result in terms of dimension formula for irreducible representations of $U(N)$. This version of Schur-Weyl duality goes in the reverse direction to the duality discovered by Gross and Taylor [12], which goes from large $N 2 \mathrm{D} U(N)$-Yang-Mills theory to Hurwitz numbers of branched coverings which corresponds to 2D Yang-Mills theories with symmetric groups as gauge groups.

Recall we are focusing on matrix models with finite $N$. Once we identify the element in the Sato grassmannian that corresponds to $Z_{N}$, we can treat $N$ as a parameter that can take any real or complex values to get a family of tau-functions of the KP hierarchy. This point of view is particularly powerful when we combine it with the introduction of 't Hooft coupling constant which we do in the end of [35]. This leads to a different genus expansion in Hermitian matrix models. In [36] these two different expansions are called the fat expansion and the thin expansion respectively and studied using the results from [31. In particular, we apply the idea of renormalization of coupling constants combined with Virasoro constraints to study the structures of thin and fat free energy functions.

As noted in [36], the distinction of fat and thin genus expansions leads to the fat and thin versions of Virasoro constraints. These are the point of departure for this paper. Starting from [29] we have seen that Virasoro constraints in genus zero is equivalent to the emergence of spectral curve. This idea was applied to the Hermitian matrix model with $N=1$ in 31. So we have already known that the emergence of spectral curve can occur also for finite $N$. However, the spectral curve discovered in 31 is not the spectral curve that leads to the Wigner semicircle law discovered in the limit of $N \rightarrow \infty$. So for a long time our idea of using emergent geometry in the case of finite $N$ to study the emergent geometry for $N \rightarrow \infty$ has seemed to be only partially successful. Even though we were able to develop emergent geometry for finite $N$ as in [31, the result was definitely different from the emergent geometry for $N \rightarrow \infty$ in the literature. So a puzzle arises as to unify these two different kinds of emergent geometry. The solution to this problem naturally emerges after we distinguish for matrix models fat and thin correlators in 35], and consequently distinguish fat and thin versions of free energy functions and their corresponding versions of Virasoro constraints in [36]. We will show that fat and thin free energy functions in genus zero lead to different emergent geometries, to be referred to as the fat and thin emergent geometries respectively. Surprisingly, they are even in some sense dual to each other. But the point is that one of them (the fat one) matches with the emergent geometry in the $N \rightarrow \infty$ case in the literature, the other (the thin one) generalizes the emergent geometry of the case of $N=1$. The thin spectral curve and the thin special deformation turn out to be a deformation of the spectral curve and its special deformation studied in [31], with $N$ as a parameter. It is then natural that we can apply almost all results in that paper to this situation. For the fat spectral curve and the fat special deformation, even though we work with finite $N$, all the major techniques developed in the $N \rightarrow \infty$ setting turn out work as well. So we have a perfect scenario here: the thin emergent geometry is compatible with the $N=1$ case and the fat emergent geometry is compatible with the $N \rightarrow \infty$ case.

Combinatorist may have some interests in this work. The thin special deformation is given by a formal power series with integral coefficients, and it is defined using enumerations of (thin) graphs. The fat special deformation is also given by a formal power series with integral coefficients, but it is defined using enumerations of fat graphs. They both lead to sophisticated generalizations of Catalan numbers. It is a very remarkable fact discovered in this paper that the enumeration problems of two different types of graphs can be unified through the theory of matrix models. In the concrete examples to be presented below, numerous integer sequences on Sloane's Encyclopedia of Integer Sequences [22] appear. They include: A001764, A002293, A002294, A002295, A002296, A007556, A062994, A062744, A230388, A104978, A002005, A001791, A002006-A002010, A000168, A085614, A000309. Some of them even appear in more than one place. They have various enumerative meanings. Our work is in the spirit of [2] in the sense that we want to apply string theoretical idea to the combinatorial problem of enumerations of graphs, with the difference being that we emphasize the possibility to treat both fat and thin graphs simultaneously with the same techniques.

We have divided the rest of the paper into four Sections. In Section 2 and Section 3 we discuss the thin and fat versions of emergent geometries of Hermitian one-matrix models respectively. And in Section 4 we prove a very special property
of fat special deformation as plane curves. In these three Sections we have included numerous concrete examples, they are not only used to illustrate the ideas, but also to present some results that have their own interests. For example, we show that the emergence of spectral curves actually provide a highly nontrivial trick to calculate some special correlators indirectly using the Virasoro constraints. The examples correspond to various combinatorial problems considered in the literature separately. It is interesting to see how they form special cases of a unified picture. We make some concluding remarks in Section 5 .

## 2. Thin Emergent Geometry of Hermitian One-Matrix Models

In this Section we introduce the thin spectral curve and the thin special deformation. We also discuss various concrete examples.
2.1. Virasoro constraints for thin genus expansion. The thin Virasoro constraints are given by the following operators (cf. [36, §3.4]):

$$
\begin{aligned}
L_{-1, N}= & -\frac{\partial}{\partial g_{1}}+\sum_{n \geq 1} n g_{n+1} \frac{\partial}{\partial g_{n}}+N g_{1} g_{s}^{-1}, \\
L_{0, N}= & -2 \frac{\partial}{\partial g_{2}}+\sum_{n \geq 1} n g_{n} \frac{\partial}{\partial g_{n}}+N^{2}, \\
L_{1, N}= & -3 \frac{\partial}{\partial g_{3}}+\sum_{n \geq 1}(n+1) g_{n} \frac{\partial}{\partial g_{n+1}}+2 N g_{s} \frac{\partial}{\partial g_{1}}, \\
L_{m, N}= & \sum_{k \geq 1}(k+m)\left(g_{k}-\delta_{k, 2}\right) \frac{\partial}{\partial g_{k+m}}+g_{s}^{2} \sum_{k=1}^{m-1} k(m-k) \frac{\partial}{\partial g_{k}} \frac{\partial}{\partial g_{m-k}} \\
& +2 N m g_{s} \frac{\partial}{\partial g_{m}}, \quad m \geq 2
\end{aligned}
$$

In genus zero one has following equations for $F_{0, N}$ :

$$
\begin{aligned}
& \frac{\partial F_{0, N}}{\partial g_{1}}=\sum_{n \geq 1} g_{n+1} \cdot n \frac{\partial F_{0, N}}{\partial g_{n}}+N g_{1} \\
& (m+2) \frac{\partial F_{0, N}}{\partial g_{m+2}}=\sum_{k \geq 1} g_{k} \cdot(k+m) \frac{\partial F_{0, N}}{\partial g_{k+m}}, \quad m \geq 0
\end{aligned}
$$

Therefore, if we set

$$
\begin{equation*}
f_{n}=\left.n \frac{\partial F_{0, N}}{\partial g_{n}}\right|_{g_{j}=0, j \geq 2} \tag{1}
\end{equation*}
$$

then we have the following recursion relations and initial value

$$
\begin{equation*}
f_{1}=N g_{1}, \quad f_{n+1}=g_{1} f_{n}, \quad n \geq 1 \tag{2}
\end{equation*}
$$

So we have

$$
\begin{equation*}
\left.n \frac{\partial F_{0, N}}{\partial g_{n}}\right|_{g_{j}=0, j \geq 2}=f_{n}=N g_{1}^{n} \tag{3}
\end{equation*}
$$

2.2. Emergence of the signed Catalan curve. If one introduce the following field $y$ as a generating series of the $f_{n}$ 's:

$$
\begin{equation*}
y=-\frac{z}{\sqrt{2}}+\frac{g_{1}}{\sqrt{2}}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n \geq 1} \frac{f_{n}}{z^{n+1}} \tag{4}
\end{equation*}
$$

then from the above formula for $f_{n}$ we have

$$
\begin{equation*}
y=-\frac{z-g_{1}}{\sqrt{2}}+\sqrt{2} N \sum_{n \geq 0} \frac{g_{1}^{n}}{z^{n+1}}=-\frac{z-g_{1}}{\sqrt{2}}+\frac{\sqrt{2} N}{z-g_{1}} \tag{5}
\end{equation*}
$$

This is a deformation of the following curve:

$$
\begin{equation*}
y=-\frac{1}{\sqrt{2}} z+\frac{\sqrt{2} N}{z} \tag{6}
\end{equation*}
$$

which we call the signed Catalan curve in 31 in the case of $N=1$. Note

$$
\begin{equation*}
\frac{z}{\sqrt{2}}=\frac{-y+\sqrt{y^{2}+4 N}}{2}=\sum_{n=0}^{\infty}(-1)^{n} N^{n+1} \cdot \frac{1}{n+1}\binom{2 n}{n} y^{-2 n-1} \tag{7}
\end{equation*}
$$

The coefficients of the series on the right-hand side are Catalan numbers $\frac{1}{n+1}\binom{2 n}{n}$ with signs $(-1)^{n}$, and an extra factor $N^{n+1}$.

One can rewrite (6) and (5) in the form of algebraic curves as follows:

$$
\begin{align*}
& z^{2}+\sqrt{2} y \cdot z+2 N=0  \tag{8}\\
& \left(z-g_{1}\right)^{2}+\sqrt{2} y \cdot\left(z-g_{1}\right)+2 N=0 \tag{9}
\end{align*}
$$

2.3. Special deformations of the thin spectral curves. We will refer to the curve in (6) the thin spectral curve of the Hermitian matrix models.

The deformation of (6) in (5) defined by (4) is just an example of special deformation deformation defined by the field that appears in the derivation of Virasoro constraints. See e.g. [36, §3.2]. Consider the field

$$
\begin{align*}
\Phi_{N}(z) & =\frac{1}{\sqrt{2}} \sum_{n \geq 1} \tilde{T}_{n} z^{n}-\sqrt{2} \operatorname{tr} \log \left(\frac{1}{z-M}\right)  \tag{10}\\
& =\frac{1}{\sqrt{2}} \sum_{n \geq 1} \tilde{T}_{n} z^{n}+\sqrt{2} N \log z-\sqrt{2} \sum_{n \geq 1} \frac{z^{-n}}{n} \frac{\partial}{\partial T_{n}} \tag{11}
\end{align*}
$$

where $T_{n}=\frac{g_{n}}{n}=\frac{t_{n-1}}{n!}, \tilde{T}_{n}=\frac{g_{n}-\delta_{n, 2}}{n}$, and its derivative:

$$
\begin{equation*}
y:=\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} n T_{n} z^{n-1}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n=1}^{\infty} \frac{1}{z^{n+1}} \frac{\partial F_{0, N}}{\partial T_{n}} . \tag{12}
\end{equation*}
$$

In [36, §] we have shown that

$$
\begin{equation*}
F_{0, N}=N \cdot F_{0}^{1 D} \tag{13}
\end{equation*}
$$

where $F_{0}^{1 D}$ is the genus zero free energy function of the topological 1D gravity studied in 31. The free energy $F^{1 D}$ has extremely nice properties: It satisfies the flow equation [31, (262)] which in genus zero gives:

$$
\begin{equation*}
\frac{\partial F_{0}^{1 D}}{\partial t_{m}}=\frac{1}{(m+1)!}\left(\frac{\partial F_{0}^{1 D}}{\partial t_{0}}\right)^{m+1} \tag{14}
\end{equation*}
$$

It also satisfies the polymer equation [31, (153)] which in genus zero gives [31, (160)]:

$$
\begin{equation*}
\frac{\partial F_{0}^{1 D}}{\partial t_{0}}=\sum_{n \geq 0} \frac{t_{n}}{n!}\left(\frac{\partial F_{0}^{1 D}}{\partial t_{0}}\right)^{n} \tag{15}
\end{equation*}
$$

The solution of this equation is given in 31, Theorem 5.3] by:

$$
\begin{equation*}
\frac{\partial F_{0}^{1 D}}{\partial t_{0}}=I_{0} \tag{16}
\end{equation*}
$$

where $I_{0}$ is defined by:

$$
\begin{equation*}
I_{0}=\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_{1}+\cdots+p_{k}=k-1} \frac{t_{p_{1}}}{p_{1}!} \cdots \frac{t_{p_{k}}}{p_{k}!} . \tag{17}
\end{equation*}
$$

By combining the above results we then get the following
Theorem 2.1. If the field $y$ is defined by (12), then it is given by:

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_{n}-\delta_{n, 1}}{n!} z^{n}+\frac{\sqrt{2} N}{z-I_{0}}, \tag{18}
\end{equation*}
$$

or equivalently, it satisfies the following equation:

$$
\begin{equation*}
\sqrt{2} y\left(z-I_{0}\right)=\left(z-I_{0}\right) \sum_{n=0}^{\infty} \frac{t_{n}-\delta_{n, 1}}{n!} z^{n}+2 N \tag{19}
\end{equation*}
$$

We will refer to either of the two equations above as the thin special deformation of (6) or (8).
2.4. Characterization of the thin special deformation. The following results are the generalizations of Theorem 10.1, Theorem 10.2 and Theorem 10.3 in [31] respectively, which are the corresponding $N=1$ cases. The proofs are exactly the same and so omitted.

Theorem 2.2. Consider the following series:

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_{n}-\delta_{n, 1}}{n!} z^{n}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_{0, N}}{\partial t_{n-1}} . \tag{20}
\end{equation*}
$$

Then one has:

$$
\begin{equation*}
\frac{1}{2}\left(y^{2}\right)_{-}=\left(\sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_{0}}{\partial t_{n-1}}\right)^{2} \tag{21}
\end{equation*}
$$

Here for a formal series $\sum_{n \in \mathbb{Z}} a_{n} f^{n}$,

$$
\begin{equation*}
\left(\sum_{n \in \mathbb{Z}} a_{n} f^{n}\right)_{+}=\sum_{n \geq 0} a_{n} f^{n}, \quad\left(\sum_{n \in \mathbb{Z}} a_{n} f^{n}\right)_{-}=\sum_{n<0} a_{n} f^{n} \tag{22}
\end{equation*}
$$

Theorem 2.3. There exists a unique series

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n \geq 0}\left(v_{n}-\delta_{n, 1}\right) z^{n}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n \geq 0} w_{n} z^{-n-2} \tag{23}
\end{equation*}
$$

such that each $w_{n} \in \mathbb{C}\left[\left[v_{0}, v_{1}, \ldots\right]\right]$ and

$$
\begin{equation*}
\frac{1}{2}\left(y^{2}\right)_{-}=\left(\frac{N}{z}+\sum_{n \geq 0} w_{n} z^{-n-2}\right)^{2} \tag{24}
\end{equation*}
$$

Theorem 2.4. For a series of the form

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \frac{\partial S(z, \mathbf{t})}{\partial z}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n \geq 0} w_{n} z^{-n-2} \tag{25}
\end{equation*}
$$

where $S$ is the universal action defined by:

$$
\begin{equation*}
S(x)=-\frac{1}{2} x^{2}+\sum_{n \geq 1} t_{n-1} \frac{x^{n}}{n!} \tag{26}
\end{equation*}
$$

and each $w_{n} \in \mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right]$, the equation

$$
\begin{equation*}
\frac{1}{2}\left(y^{2}\right)_{-}=\left(\frac{N}{z}+\sum_{n \geq 0} w_{n} z^{-n-2}\right)^{2} \tag{27}
\end{equation*}
$$

has a unique solution given by:

$$
y=\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_{n}-\delta_{n, 1}}{n!} z^{n}+\frac{\sqrt{2}}{z}+\sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_{0, N}}{\partial t_{n-1}},
$$

where $F_{0, N}$ is the free energy of the Hermitian $N \times N$-matrix model.
To gain some better understanding of the thin special deformations, we will first examine some concrete examples in the next few Subsections.
2.5. Thin special deformation on the $\left(g_{1}, g_{2}\right)$-plane. By (17),

$$
\left.I_{0}\right|_{t_{k}=0, k \geq 2}=\sum_{k \geq 1} t_{0} t_{1}^{k-1}=\frac{t_{0}}{1-t_{1}}=\frac{g_{1}}{1-g_{2}}
$$

and so the special deformation of thin spectral curve on the $\left(g_{1}, g_{2}\right)$-space is given by the following plane algebraic curve:

$$
\begin{equation*}
\sqrt{2} y\left(z-\frac{t_{0}}{1-t_{1}}\right)=\left(z-\frac{t_{0}}{1-t_{1}}\right)\left(t_{0}+\left(t_{1}-1\right) z\right)+2 N \tag{28}
\end{equation*}
$$

It can be rewritten in the following form:

$$
\begin{equation*}
\left(z-\frac{g_{1}}{1-g_{2}}\right)^{2}+\sqrt{2} \frac{y}{1-g_{2}}\left(z-\frac{g_{1}}{1-g_{2}}\right)-\frac{2 N}{1-g_{2}}=0 \tag{29}
\end{equation*}
$$

This is just (9) with the following changes of variables:

$$
g_{1} \mapsto \frac{g_{1}}{1-g_{2}}, \quad y \mapsto \frac{y}{1-g_{2}}, \quad \quad N \mapsto \frac{2 N}{1-g_{2}}
$$

One can rewrite (29) as follows:

$$
\begin{equation*}
\left(\left(1-g_{2}\right) z-g_{1}\right)^{2}+\sqrt{2} y\left(\left(1-g_{2}\right) z-g_{1}\right)-2 N\left(1-g_{2}\right)=0 \tag{30}
\end{equation*}
$$

When $g_{2}=1$, it becomes:

$$
\begin{equation*}
y=\frac{g_{1}}{\sqrt{2}} . \tag{31}
\end{equation*}
$$

2.6. Thin special deformation along the $g_{3}$-line. It is easy to see that

$$
\left.I_{0}\right|_{t_{k}=0, k \neq 2}=0
$$

and so the thin special deformation along the $g_{3}$-line is given by:

$$
\sqrt{2} y \cdot z=z \cdot\left(g_{3} z^{2}-z\right)+2 N
$$

It can be rewritten as:

$$
\begin{equation*}
v=\frac{z}{2 N-z^{2}\left(1-g_{3} z\right)}, \tag{32}
\end{equation*}
$$

where $v=1 /(\sqrt{2} y)$. From this one can use Lagrange inversion to express $z$ as a Taylor series in $v$

$$
z=\sum_{n \geq 1} a_{n} v^{n}
$$

as follows:

$$
\begin{aligned}
a_{n} & =\frac{1}{2 \pi i} \oint \frac{z}{v^{n+1}} d v \\
& =\frac{1}{2 \pi i} \oint \frac{z}{\left(\frac{z}{2 N-z^{2}\left(1-g_{3} z\right)}\right)^{n+1}} d \frac{z}{2 N-z^{2}\left(1-g_{3} z\right)} \\
& =\frac{1}{2 \pi i} \oint \frac{\left(2 N-z^{2}\left(1-g_{3} z\right)\right)^{n-1}\left(2 N+z^{2}-2 g_{3} z^{4}\right)}{z^{n}} d z \\
& \left.=\left[2 N-z^{2}\left(1-g_{3} z\right)\right)^{n-1}\left(2 N+z^{2}-2 g_{3} z^{3}\right)\right]_{z^{n-1}},
\end{aligned}
$$

where $[\cdot]_{z^{n-1}}$ means the coefficient of $z^{n-1}$. Using the binomial theorem, it is possible to derive a closed formula for $a_{n}$ as follows. For reason which will become clear later, we will introduce an extra variable $h$, and proceed as follows:

$$
\begin{aligned}
& \left(2 N-w^{2}\left(h-g_{3} w\right)\right)^{n-1} \\
= & \sum_{a=0}^{n-1}(-1)^{a}\binom{n-1}{a}(2 N)^{n-1-a} w^{2 a}\left(h-g_{4} w\right)^{a} \\
= & \sum_{a=0}^{n-1}(-1)^{a}\binom{n-1}{a}(2 N)^{n-1-a} w^{2 a} \sum_{b=0}^{a}(-1)^{b}\binom{a}{b} h^{a-b} g_{4}^{b} w^{b} \\
= & \sum_{0 \leq b \leq a \leq n-1}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} h^{a-b} g_{4}^{b} w^{2 a+b},
\end{aligned}
$$

and then it follows that:

$$
\begin{aligned}
& {\left[\left(2 N-w^{2}\left(h-g_{3} w\right)\right)^{n-1}\left(2 N+h w^{2}-2 g_{3} w^{3}\right)\right]_{w^{n-1}} } \\
= & 2 N\left[\left(2 N-w^{2}\left(h-g_{3} w\right)\right)^{2 m}\right]_{w^{n-1}}+\left[\left(2 N-w^{2}\left(h-g_{3} w\right)\right)^{2 m}\right]_{w^{n-3}} \\
- & 2 g_{3}\left[\left(2 N-w^{2}\left(h-g_{3} w\right)\right)^{2 m}\right]_{w^{n-4}} \\
= & 2 N \sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+b=n-1}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} h^{a-b} g_{4}^{b} \\
+ & h \sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+b=n-3}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} h^{a-b} g_{4}^{b} \\
- & 2 g_{4} \sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+b=n-4}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} h^{a-b} g_{4}^{b} \\
= & 2 N \sum_{(n-1) / 3 \leq a \leq(n-1) / 2}(-1)^{n-1-a} \frac{(n-1)!(2 N)^{n-1-a} h^{3 a-n+1} g_{4}^{n-1-2 a}}{(n-1-a)!(n-1-2 a)!(3 a-n+1)!} \\
+ & h \sum_{(n-3) / 3 \leq a \leq(n-3) / 2}(-1)^{n-3-a} \frac{(n-1)!(2 N)^{n-1-a} h^{3 a-n+3} g_{4}^{n-3-2 a}}{(n-1-a)!(n-3-2 a)!(3 a-n+3)!} \\
- & 2 g_{3} \sum_{(n-4) / 3 \leq a \leq(n-4) / 2}(-1)^{n-4-a} \frac{(n-1)!(2 N)^{n-1-a} h^{3 a-n+4} g_{4}^{n-4-2 a}}{(n-1-a)!(n-4-2 a)!(3 a-n+4)!} \\
= & \sum_{(n-1) / 3 \leq a \leq(n-1) / 2}(-1)^{n-1-a} \frac{(n-1)!(2 N)^{n-1-a} h^{3 a-n+1} g_{4}^{n-1-2 a}}{(n-a)!(n-1-2 a)!(3 a-n+1)!}
\end{aligned}
$$

and so after taking $h=1$ we have:

$$
z=\sum_{m \geq 0} \sum_{m / 3 \leq a \leq m / 2}(-1)^{m-a} \frac{m!(2 N)^{m-a} g_{4}^{m-2 a}}{(m+1-a)!(m-2 a)!(3 a-m)!} v^{m+1}
$$

The first few terms of $z$ are:

$$
\begin{align*}
z & =(2 N) v-(2 N)^{2} v^{3}+(2 N)^{3} g_{3} v^{4}+2(2 N)^{3} v^{5}-5(2 N)^{4} g_{3} v^{6} \\
& +\left(3 g_{3}^{2}(2 N)^{5}-5(2 N)^{4}\right) v^{7}+21 g_{3}(2 N)^{5} v^{8} \\
& +\left(-28 g_{3}^{2}(2 N)^{6}+14(2 N)^{5}\right) v^{9}+\left(12 g_{3}^{3}(2 N)^{7}-84 g_{3}(2 N)^{6}\right) v^{10} \\
& +\left(180 g_{3}^{2}(2 N)^{7}-42(2 N)^{6}\right) v^{11}  \tag{33}\\
& +\left(-165 g_{3}^{3}(2 N)^{8}+330 g_{3}(2 N)^{7}\right) v^{12} \\
& +\left(55 g_{3}^{4}(2 N)^{9}-990 g_{3}^{2}(2 N)^{8}+132(2 N)^{7}\right) v^{13} \\
& +\left(1430 g_{3}^{3}(2 N)^{9}-1287 g_{3}(2 N)^{8}\right) v^{14}+\cdots
\end{align*}
$$

The coefficients $1,0,-1,1,2,-5,(3,-5), 21,(-28,14),(12,-84),(180,42), \cdots$ are up to some signs and orders the sequence A104978 on [22] given by the following formula:

$$
\begin{equation*}
T(n, k)=\frac{(2 n+k)!}{k!(n-k)!(n+k+1)!}, \quad 1 \leq k \leq n \tag{34}
\end{equation*}
$$

FAT AND THIN EMERGENT GEOMETRIES OF HERMITIAN ONE-MATRIX MODELS 11
2.7. Thin special deformation on the $\left(g_{1}, g_{3}\right)$-plane. Next we turn on besides the coupling constant $g_{3}$ also $g_{1}$. Note

$$
\begin{aligned}
\left.I_{0}\right|_{g_{k}=0, k \neq 1,3} & =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{p_{1}+\cdots+p_{k}=k-1 \\
p_{1}, \ldots, p_{k} \in\{0,2\}}} g_{p_{1}+1} \cdots g_{p_{k}+1} \\
& =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{m_{0}+m_{2}=k \\
2 m_{2}=k-1}} \frac{k!}{m_{0}!m_{2}!} g_{1}^{m_{0}} g_{3}^{m_{2}} \\
& =\sum_{m=0}^{\infty} \frac{1}{2 m+1} \cdot \frac{(2 m+1)!}{(m+1)!m!} g_{1}^{m+1} g_{3}^{m} \\
& =g_{1} \sum_{m=0}^{\infty} \frac{1}{m+1}\binom{2 m}{m} g_{1}^{m} g_{3}^{m}
\end{aligned}
$$

the coefficients are Catalan numbers, and so

$$
\begin{equation*}
\left.I_{0}\right|_{g_{k}=0, k \neq 1,3}=g_{1} \cdot \frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{1} g_{3}} \tag{35}
\end{equation*}
$$

Therefore, the thin special deformation on $\left(g_{1}, g_{3}\right)$-plane is given by the equation:

$$
\begin{equation*}
\sqrt{2} y\left(z-\frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{3}}\right)=\left(z-\frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{3}}\right)\left(g_{1}-z+g_{3} z^{2}\right)+2 N \tag{36}
\end{equation*}
$$

If we make the following change of coordinates:

$$
w=z-\frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{3}}, \quad v=\frac{1}{\sqrt{2} y}
$$

then we get the following equation:

$$
w=2 N v+v w\left[g_{1}-\frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{3}}-w+g_{3}\left(w+\frac{1-\sqrt{1-4 g_{1} g_{3}}}{2 g_{3}}\right)^{2}\right]
$$

After simplification,

$$
\begin{equation*}
w=2 N v-v w^{2}\left(\sqrt{1-4 g_{1} g_{3}}-g_{3} w\right) \tag{37}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
v=\frac{w}{2 N-w^{2}\left(\sqrt{1-4 g_{1} g_{3}}-g_{3} w\right)} \tag{38}
\end{equation*}
$$

From this one can use Lagrange inversion to solve for $w$ as in last Subsection to get:

$$
w=\sum_{m \geq 0} \sum_{m / 3 \leq a \leq m / 2}(-1)^{m-a} \frac{m!(2 N)^{m-a} h^{3 a-m} g_{4}^{m-2 a}}{(m+1-a)!(m-2 a)!(3 a-m)!} v^{m+1}
$$

where $h=\sqrt{1-4 g_{1} g_{3}}$. By writing down the first few terms of $w$ explicitly we get:

$$
\begin{aligned}
z & =\frac{1-h}{2 g_{3}}+(2 N) v-(2 N)^{2} h v^{3}+(2 N)^{3} g_{3} v^{4}+2(2 N)^{3} h^{2} v^{5}-5(2 N)^{4} g_{3} h v^{6} \\
& +\left(3 g_{3}^{2}(2 N)^{5}-5 h^{3}(2 N)^{4}\right) v^{7}+21 g_{3} h^{2}(2 N)^{5} v^{8} \\
& +\left(-28 g_{3}^{2} h(2 N)^{6}+14 h^{4}(2 N)^{5}\right) v^{9}+\left(12 g_{3}^{3}(2 N)^{7}-84 g_{3} h^{3}(2 N)^{6}\right) v^{10} \\
& +\left(180 g_{3}^{2} h^{2}(2 N)^{7}-42 h^{5}(2 N)^{6}\right) v^{11} \\
& +\left(-165 g_{3}^{3} h(2 N)^{8}+330 g_{3} h^{4}(2 N)^{7}\right) v^{12} \\
& +\left(55 g_{3}^{4}(2 N)^{9}-990 g_{3}^{2} h^{3}(2 N)^{8}+132 h^{6}(2 N)^{7}\right) v^{13} \\
& +\left(1430 g_{3}^{3} h^{2}(2 N)^{9}-1287 g_{3} h^{5}(2 N)^{8}\right) v^{14}+\cdots
\end{aligned}
$$

from this one sees explicitly how one can first deform along the $g_{3}$-line, then deform along the $g_{1}$-direction.
2.8. Thin special deformation on the $\left(g_{1}, g_{2}, g_{3}\right)$-space. Note

$$
\begin{aligned}
\left.I_{0}\right|_{t_{k}=0, k \geq 3} & =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{p_{1}+\cdots+p_{k}=k-1 \\
0 \leq p_{1}, \ldots, p_{k} \leq 2}} g_{p_{1}+1} \cdots g_{p_{k}+1} \\
& =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{m_{0}+m_{1}+m_{2}=k \\
m_{1}+2 m_{2}=k-1}} \frac{k!}{m_{0}!m_{1}!m_{2}!} g_{1}^{m_{0}} g_{2}^{m_{1}} g_{3}^{m_{2}} .
\end{aligned}
$$

From the system

$$
\begin{aligned}
& m_{0}+m_{1}+m_{2}=k \\
& m_{1}+2 m_{2}=k-1
\end{aligned}
$$

we get $m_{0}=m_{2}+1$ and so $k=m_{1}+2 m_{2}+1, m_{1}$ can be arbitrary, it follows that

$$
\begin{aligned}
\left.I_{0}\right|_{t k}=0, k \geq 3 & =\sum_{m_{2}=0}^{\infty} \sum_{m_{1}=0}^{\infty} \frac{1}{m_{1}+2 m_{2}+1} \frac{\left(m_{1}+2 m_{2}+1\right)!}{\left(m_{2}+1\right)!m_{1}!m_{2}!} g_{1}^{m_{2}+1} g_{2}^{m_{1}} g_{3}^{m_{2}} \\
& =\sum_{m_{2} \geq 0} \frac{\left(2 m_{2}\right)!}{m_{2}!\left(m_{2}+1\right)!} \frac{g_{1}^{m_{2}+1} g_{3}^{m_{2}}}{\left(1-g_{2}\right)^{2 m_{2}+1}} \\
& =\frac{1-\sqrt{1-4 \frac{g_{1}}{1-g_{2}} \frac{g_{3}}{1-g_{2}}}}{2 \frac{g_{3}}{1-g_{2}}}
\end{aligned}
$$

and so the special deformation on the $\left(g_{1}, g_{2}, g_{3}\right)$-space is given by:

$$
\begin{align*}
& \sqrt{2} \frac{y}{1-g_{2}}\left(z-\frac{1-\sqrt{1-4 \frac{g_{1}}{1-g_{2}} \frac{g_{3}}{1-g_{2}}}}{2 \frac{g_{3}}{1-g_{2}}}\right)  \tag{39}\\
= & \left(z-\frac{1-\sqrt{1-4 \frac{g_{1}}{1-g_{2}} \frac{g_{3}}{1-g_{2}}}}{2 \frac{g_{3}}{1-g_{2}}}\right)\left(\frac{g_{1}}{1-g_{2}}-z+\frac{g_{3}}{1-g_{2}} z^{2}\right)+\frac{2 N}{1-g_{2}},
\end{align*}
$$

This is just (36) with the following changes of variables:

$$
g_{1} \mapsto \frac{g_{1}}{1-g_{2}}, \quad g_{3} \mapsto \frac{g_{3}}{1-g_{2}}, \quad y \mapsto \frac{y}{1-g_{2}}, \quad N \mapsto \frac{2 N}{1-g_{2}}
$$

One can rewrite (39) as follows:

$$
\begin{align*}
& \sqrt{2} y\left(z-\frac{1-g_{2}-\sqrt{\left(1-g_{2}\right)^{2}-4 g_{1} g_{3}}}{2 g_{3}}\right) \\
= & \left(z-\frac{1-g_{2}-\sqrt{\left(1-g_{2}\right)^{2}-4 g_{1} g_{3}}}{2 g_{3}}\right)\left(g_{1}-\left(1-g_{2}\right) z+g_{3} z^{2}\right)+2 N, \tag{40}
\end{align*}
$$

and so when $g_{2}=1$,

$$
\begin{equation*}
\sqrt{2} y\left(z+\frac{\sqrt{-g_{1} g_{3}}}{g_{3}}\right)=\left(z+\frac{\sqrt{-g_{1} g_{3}}}{g_{3}}\right)\left(g_{1}+g_{3} z^{2}\right)+2 N . \tag{41}
\end{equation*}
$$

The appearance of fractional powers of the coupling constants indicates the occurrence of some phase transition. If we furthermore take $g_{1}=0$, then we get:

$$
\begin{equation*}
\sqrt{2} y \cdot z=g_{3} z^{3}+2 N \tag{42}
\end{equation*}
$$

2.9. Thin special deformation on the $g_{4}$-line. In this case the thin special deformation is given by

$$
\begin{equation*}
\sqrt{2} y z=-z^{2}\left(1-g_{4} z^{2}\right)+2 N \tag{43}
\end{equation*}
$$

To express $z$ as a Taylor series in $1 / y$, we need to solve:

$$
\begin{equation*}
v=\frac{z}{2 N-z^{2}\left(1-g_{4} z^{2}\right)} . \tag{44}
\end{equation*}
$$

We use Lagrange inversion to solve for $z$ as follows. Write

$$
z=\sum_{n \geq 1} a_{n} v^{n}
$$

then we have

$$
\begin{aligned}
a_{n} & =\frac{1}{2 \pi i} \oint \frac{z}{v^{n+1}} d v \\
& =\frac{1}{2 \pi i} \oint \frac{z}{\left(\frac{z}{2 N-z^{2}\left(1-g_{4} z^{2}\right)}\right)^{n+1}} d \frac{z}{2 N-z^{2}\left(1-g_{4} z^{2}\right)} \\
& =\frac{1}{2 \pi i} \oint \frac{\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{n-1}\left(2 N+z^{2}-3 g_{4} z^{4}\right)}{z^{n}} d z \\
& =\left[\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{n-1}\left(2 N+z^{2}-3 g_{4} z^{4}\right)\right]_{z^{n-1}}
\end{aligned}
$$

It is clear that $a_{2 m}$ all vanish. Using the binomial theorem, it is possible to derive a closed formula for $a_{2 m+1}$ as follows. We first get:

$$
\begin{aligned}
& \left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{2 m} \\
= & \sum_{a=0}^{2 m}(-1)^{a}\binom{2 m}{a}(2 N)^{2 m-a} z^{2 a}\left(1-g_{4} z^{2}\right)^{a} \\
= & \sum_{a=0}^{2 m}(-1)^{a}\binom{2 m}{a}(2 N)^{2 m-a} z^{2 a} \sum_{b=0}^{a}(-1)^{b}\binom{a}{b} g_{4}^{b} z^{2 b} \\
= & \sum_{0 \leq b \leq a \leq 2 m}(-1)^{a+b} \frac{(2 m)!}{(2 m-a)!(a-b)!b!}(2 N)^{2 m-a} g_{4}^{b} z^{2(a+b)},
\end{aligned}
$$

and then it follows that:

$$
\begin{aligned}
& {\left[\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{2 m}\left(2 N+z^{2}-3 g_{4} z^{4}\right)\right]_{z^{2 m}} } \\
= & 2 N\left[\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{2 m}\right]_{z^{2 m}}+\left[\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{2 m}\right]_{z^{2 m-2}} \\
- & 3 g_{4}\left[\left(2 N-z^{2}\left(1-g_{4} z^{2}\right)\right)^{2 m}\right]_{z^{2 m-4}} \\
= & 2 N \sum_{\substack{0 \leq b \leq a \leq 2 m \\
a+b=m}}(-1)^{a+b} \frac{(2 m)!}{(2 m-a)!(a-b)!b!}(2 N)^{2 m-a} g_{4}^{b} \\
+ & \sum_{\substack{0 \leq b \leq a \leq 2 m \\
a+b=m-1}}(-1)^{a+b} \frac{(2 m)!}{(2 m-a)!(a-b)!b!}(2 N)^{2 m-a} g_{4}^{b} \\
- & 3 g_{4} \sum_{\substack{0 \leq b \leq a \leq 2 m \\
a+b=m-2}}(-1)^{a+b} \frac{(2 m)!}{(2 m-a)!(a-b)!b!}(2 N)^{2 m-a} g_{4}^{b} \\
= & \sum_{b=0}^{[m / 2]}(-1)^{m} \frac{(2 m)!}{b!(m-2 b)!(m+b)!}(2 N)^{m+b+1} g_{4}^{b} \\
+ & \sum_{b=0}^{[(m-1) / 2]}(-1)^{m-1} \frac{(2 m)!}{b!(m-1-2 b)!(m+1+b)!}(2 N)^{m+b+1} g_{4}^{b} \\
- & 3 g_{4} \sum_{b=0}^{[(m-2) / 2]}(-1)^{m-2} \frac{(2 m)!}{b!(m-2-2 b)!(m+2+b)!}(2 N)^{m+b+2} g_{4}^{b} .
\end{aligned}
$$

After simplification we get:

$$
\begin{equation*}
a_{2 m+1}=(-1)^{m} \sum_{b=0}^{[m / 2]} \frac{(2 m)!}{b!(m-2 b)!(m+1+b)!}(2 N)^{m+b+1} g_{4}^{b} \tag{45}
\end{equation*}
$$

and so we have:

$$
z=\sum_{m \geq 0}(-1)^{m} \sum_{b=0}^{[m / 2]} \frac{(2 m)!}{b!(m-2 b)!(m+1+b)!}(2 N)^{m+b+1} g_{4}^{b} v^{2 m+1}
$$

The following are the first few terms:

$$
\begin{aligned}
z & =(2 N) v-(2 N)^{2} v^{3}+\left((2 N)^{4} g_{4}+2(2 N)^{3}\right) v^{5} \\
& +\left(6 g_{4}(2 N)^{5}+5(2 N)^{4}\right) v^{7}+\left(4 g_{4}^{2}(2 N)^{7}+28 g_{4}(2 N)^{6}+14(2 N)^{5}\right) v^{9} \\
& -\left(45 g_{4}^{2}(2 N)^{8}+120 g_{4}(2 N)^{7}+42(2 N)^{6}\right) v^{11} \\
& +\left(22 g_{4}^{3}(2 N)^{10}+330 g_{4}^{2}(2 N)^{9}+495 g_{4}(2 N)^{8}+132(2 N)^{7}\right) v^{13} \\
& -\left(364 g_{4}^{3}(2 N)^{11}+2002 g_{4}^{2}(2 N)^{10}+2002 g_{4}(2 N)^{9}+429(2 N)^{8}\right) v^{15}+\cdots,
\end{aligned}
$$

the coefficients (up to signs) $1,1,(1,2),(6,5),(4,28,14),(45,120,42),(22,330,495$, 132), $\cdots$ are generalizations of the Catalan numbers $1,1,2,5,14,42,132, \cdots$, but they have not yet appeared on [22].
2.10. Thin special deformation on the $\left(g_{1}, g_{4}\right)$-plane. In this case

$$
\begin{aligned}
\left.I_{0}\right|_{t_{k}=0, k \neq 0,3} & =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{m_{0}+m_{3}=k \\
3 m_{3}=k-1}} \frac{k!}{m_{0}!m_{3}!} g_{1}^{m_{0}} g_{4}^{m_{3}} \\
& =\sum_{m=0}^{\infty} \frac{1}{3 m+1} \cdot \frac{(3 m+1)!}{(2 m+1)!m!} g_{1}^{2 m+1} g_{4}^{m} \\
& =g_{1} \sum_{m=0}^{\infty} \frac{1}{2 m+1}\binom{3 m}{m} g_{1}^{2 m} g_{4}^{m},
\end{aligned}
$$

the coefficients are the sequence A001764 on [22], they are the numbers of complete ternary trees with $n$ internal nodes. Write

$$
\begin{equation*}
A(x)=\sum_{m=0}^{\infty} \frac{1}{2 m+1}\binom{3 m}{m} x^{m} \tag{46}
\end{equation*}
$$

then one has [22, A001764]:

$$
\begin{equation*}
A(x)=1+x A(x)^{3}=\frac{1}{1-x A(x)^{2}} \tag{47}
\end{equation*}
$$

The thin special deformation is given by the plane algebraic curve:

$$
\begin{equation*}
\sqrt{2} y\left(z-g_{1} A\left(g_{1}^{2} g_{4}\right)\right)=\left(z-g_{1} A\left(g_{1}^{2} g_{4}\right)\right)\left(g_{1}-z+g_{4} z^{3}\right)+2 N \tag{48}
\end{equation*}
$$

Make the following change of coordinates:

$$
w=z-g_{1} A\left(g_{1}^{2} g_{4}\right), \quad v=\frac{1}{\sqrt{2} y}
$$

Then we get the following equation:

$$
w=2 N v+v w\left[g_{1}-g_{1} A\left(g_{1}^{2} g_{4}\right)-w+g_{4}\left(w+g_{1} A\left(g_{1}^{2} g_{4}\right)\right)^{3}\right]
$$

After simplification using (47), we get

$$
\begin{equation*}
w=2 N v-v w^{2}\left(1-3 g_{1}^{2} g_{4} A^{2}\left(g_{1}^{2} g_{4}\right)-3 g_{1} g_{4} A\left(g_{1}^{2} g_{4}\right) w-g_{4} w^{2}\right) \tag{49}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
v=\frac{w}{2 N-w^{2}\left(1-3 g_{1}^{2} g_{4} A^{2}\left(g_{1}^{2} g_{4}\right)-3 g_{1} g_{4} A\left(g_{1}^{2} g_{4}\right) w-g_{4} w^{2}\right)} . \tag{50}
\end{equation*}
$$

From this one can use Lagrange inversion to solve for $w$.
2.11. Thin special deformation on the $g_{k+2}$-line $(k \geq 3)$. It is straightforward to generalize the results in $\$ 2.6$ and $\$ 2.9$. The thin special deformation along the $g_{k+2}$-line is given by

$$
\begin{equation*}
\sqrt{2} y z=-z^{2}+g_{k+2} z^{k+2}+2 N \tag{51}
\end{equation*}
$$

To express $z$ as a Taylor series in $1 / y$, we need to solve:

$$
\begin{equation*}
v=\frac{z}{2 N-z^{2}+g_{k+2} z^{k+2}} . \tag{52}
\end{equation*}
$$

We use Lagrange inversion to solve for $z$ as follows. Write

$$
z=\sum_{n \geq 1} a_{n} v^{n}
$$

then we have

$$
\begin{aligned}
a_{n} & =\frac{1}{2 \pi i} \oint \frac{z}{v^{n+1}} d v \\
& =\frac{1}{2 \pi i} \oint \frac{z}{\left(\frac{z}{2 N-z^{2}\left(1-g_{k+2} z^{k}\right)}\right)^{n+1}} d \frac{z}{2 N-z^{2}\left(1-g_{k+2} z^{k}\right)} \\
& =\frac{1}{2 \pi i} \oint \frac{\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\left(2 N+z^{2}-(k+1) g_{k+2} z^{k+2}\right)}{z^{n}} d z \\
& =\left[\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\left(2 N+z^{2}-(k+1) g_{k+2} z^{k+2}\right)\right]_{z^{n-1}} .
\end{aligned}
$$

Using the binomial theorem, it is possible to derive a closed formula for $a_{2 m+1}$ as follows. We first get:

$$
\begin{aligned}
& \left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1} \\
= & \sum_{a=0}^{n-1}(-1)^{a}\binom{n-1}{a}(2 N)^{n-1-a} z^{2 a}\left(1-g_{k+2} z^{k}\right)^{a} \\
= & \sum_{a=0}^{n-1}(-1)^{a}\binom{n-1}{a}(2 N)^{n-1-a} z^{2 a} \sum_{b=0}^{a}(-1)^{b}\binom{a}{b} g_{k+2}^{b} z^{k b} \\
= & \sum_{0 \leq b \leq a \leq n-1}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} z^{2 a+k b},
\end{aligned}
$$

and then it follows that:

$$
\begin{aligned}
& {\left[\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\left(2 N+z^{2}-(k+1) g_{k+2} z^{k+2}\right)\right]_{z^{n-1}}} \\
& =2 N\left[\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\right]_{z^{n-1}}+\left[\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\right]_{z^{n-3}} \\
& -(k+1) g_{4}\left[\left(2 N-z^{2}\left(1-g_{k+2} z^{k}\right)\right)^{n-1}\right]_{z^{n-k-3}} \\
& =2 N \sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+k b=n-1}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} \\
& +\sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+k b=n-3}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} \\
& -(k+1) g_{k+2} \sum_{\substack{0 \leq b \leq a \leq n-1 \\
2 a+k b=n-k-3}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} \\
& =\sum_{\substack{(n-1) /(k+2) \leq a \leq(n-1) / 2 \\
0 \leq b=(n-1-2 a) / k}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-a} g_{k+2}^{b} \\
& +\sum_{\substack{(n-1) /(k+2) \leq a \leq(n-1) / 2 \\
0 \leq b=(n-3-2 a) / k}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} \\
& -(k+1) g_{k+2} \sum_{\substack{(n-1) /(k+2) \leq a \leq(n-1) / 2 \\
0 \leq b=(n-3-2 a) / k-1}}(-1)^{a+b} \frac{(n-1)!}{(n-1-a)!(a-b)!b!}(2 N)^{n-1-a} g_{k+2}^{b} \\
& =\sum_{\substack{(n-1) /(k+2) \leq a \leq(n-1) / 2 \\
0 \leq b=(n-1-2 a) / k}}(-1)^{a+b} \frac{(n-1)!}{(n-a)!(a-b)!b!}(2 N)^{n-a} g_{k+2}^{b},
\end{aligned}
$$

and so we have:

$$
z=\sum_{m \geq 0} v^{m+1} \sum_{\substack{m /(k+2) \leq a \leq m / 2 \\ 0 \leq b=(m-2 a) / k}}(-1)^{a+b} \frac{m!}{(m+1-a)!(a-b)!b!}(2 N)^{m+1-a} g_{k+2}^{b}
$$

The coefficients (up to signs) are all generalizations of the Catalan numbers.
2.12. Thin special deformation on the $\left(g_{1}, g_{k+2}\right)$-plane. In this case

$$
\begin{aligned}
\left.I_{0}\right|_{g_{j}=0, j \neq 1, k+2} & =\sum_{l=1}^{\infty} \frac{1}{l} \sum_{\substack{m_{1}+m_{k+2}=l \\
(k+1) m_{k+2}=l-1}} \frac{l!}{m_{0}!m_{3}!} g_{1}^{m_{0}} g_{4}^{m_{3}} \\
& =\sum_{m=0}^{\infty} \frac{1}{(k+1) m+1} \cdot \frac{((k+1) m+1)!}{(k m+1)!m!} g_{1}^{k m+1} g_{k+2}^{m} \\
& =g_{1} \sum_{m=0}^{\infty} \frac{1}{k m+1}\binom{(k+1) m}{m} g_{1}^{k m} g_{k+2}^{m}
\end{aligned}
$$

the coefficients are various sequences on [22], they enumerate $(k+1)$-ary rooted trees with $m$ internal nodes. Write

$$
\begin{equation*}
A_{k}(x)=\sum_{m=0}^{\infty} \frac{1}{k m+1}\binom{(k+1) m}{m} x^{m} \tag{53}
\end{equation*}
$$

then one has:

$$
\begin{equation*}
A_{k}(x)=1+x A(x)^{k+1} \tag{54}
\end{equation*}
$$

The thin special deformation is given by the plane algebraic curve:

$$
\begin{equation*}
\sqrt{2} y\left(z-g_{1} A_{k}\left(g_{1}^{k} g_{k+2}\right)\right)=\left(z-g_{1} A_{k}\left(g_{1}^{k} g_{k+2}\right)\right)\left(g_{1}-z+g_{k+2} z^{k+1}\right)+2 N \tag{55}
\end{equation*}
$$

As before, after we make the following change of coordinates:

$$
w=z-g_{1} A_{k}\left(g_{1}^{k} g_{4}\right), \quad v=\frac{1}{\sqrt{2} y}
$$

we get the following equation:

$$
w=2 N v+v w\left[g_{1}-g_{1} A_{k}\left(g_{1}^{k} g_{k+2}\right)-w+g_{k+2} \cdot\left(w+g_{1} A_{k}\left(g_{1}^{k} g_{k+2}\right)\right)^{k+1}\right]
$$

After simplification using (54), we get

$$
\begin{equation*}
w=2 N v+v w^{2}\left(g_{k+2} \sum_{j}\binom{k+1}{j}\left(g_{1} A_{k}\left(g_{a}^{k} g_{k+2}\right)\right)^{j} w^{k-j}-1\right) \tag{56}
\end{equation*}
$$

After rewriting it in the following form:

$$
\begin{equation*}
v=\frac{w}{2 N+w^{2}\left(g_{k+2} \sum_{j}\binom{k+1}{j}\left(g_{1} A_{k}\left(g_{a}^{k} g_{k+2}\right)\right)^{j} w^{k-j}-1\right)} \tag{57}
\end{equation*}
$$

one can use Lagrange inversion to solve for $w$ as a poser series in $v$.
2.13. Thin special deformation in renormalized coupling constants. To understand the examples discussed in the above several Subsections, it is important to turn on all the coupling constants and to consider the problem of expressing $z$ as a Taylor series in $v=\frac{1}{\sqrt{2} y}$. It turns out to be convenient to change to the renormalized coupling constants $I_{k}$ :

$$
\begin{align*}
& I_{0}=\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_{1}+\cdots+p_{k}=k-1} \frac{t_{p_{1}}}{p_{1}!} \cdots \frac{t_{p_{k}}}{p_{k}!},  \tag{58}\\
& I_{k}=\sum_{n \geq 0} t_{n+k} \frac{I_{0}^{n}}{n!}, \quad k \geq 1 . \tag{59}
\end{align*}
$$

These were used in [31] to rewrite the action function:

$$
\begin{align*}
S(z) & =\sum_{n \geq 1}\left(t_{n-1}-\delta_{n, 2}\right) \frac{x^{n}}{n!} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k+1)!}\left(I_{k}+\delta_{k, 1}\right) I_{0}^{k+1}+\sum_{n \geq 2}\left(I_{n-1}-\delta_{n, 2}\right) \frac{\left(z-I_{0}\right)^{n}}{n!} \tag{60}
\end{align*}
$$

Because the thin special deformation is given by:

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} S^{\prime}(z)+\frac{\sqrt{2} N}{z-I_{0}} \tag{61}
\end{equation*}
$$

so in the renormalized coupling constant it can be written as

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n \geq 1}\left(I_{n}-\delta_{n, 1}\right) \frac{\left(z-I_{0}\right)^{n}}{n!}+\frac{\sqrt{2} N}{z-I_{0}} \tag{62}
\end{equation*}
$$

Make the following changes of variables:

$$
\begin{equation*}
w=z-I_{0}, \quad v=\frac{1}{\sqrt{2} y} \tag{63}
\end{equation*}
$$

then we get:

$$
\begin{equation*}
v=\frac{w}{2 N+\sum_{n \geq 1}\left(I_{n}-\delta_{n, 1}\right) \frac{w^{n+1}}{n!}} \tag{64}
\end{equation*}
$$

From this one can use Lagrange inversion to express $w$ as a Taylor series of $v$, with coefficients polynomials in $2 N$ and $I_{n}-\delta_{n, 1}$.

Let us now explain some of the combinatorics involved here. They are all based on an earlier work 31 to which we refer for more details and proofs. First of all, the renormalized coupling constant $I_{0}$, defined by the explicit formula (58) above, with the first few terms explicitly given by

$$
\begin{aligned}
I_{0}= & t_{0}+t_{1} t_{0}+\left(t_{2} \frac{t_{0}^{2}}{2!}+2 \frac{t_{1}^{2}}{2!} t_{0}\right)+\left(t_{3} \frac{t_{0}^{3}}{3!}+3 t_{1} t_{2} \frac{t_{0}^{2}}{2!}+6 \frac{t_{1}^{3}}{3!} t_{0}\right) \\
+ & {\left[t_{4} \frac{t_{0}^{4}}{4!}+\left(6 \frac{t_{2}^{2}}{2!}+4 t_{1} t_{3}\right) \frac{t_{0}^{3}}{3!}+12 t_{2} \frac{t_{1}^{2}}{2!} \frac{t_{0}^{2}}{2!}+24 \frac{t_{1}^{4}}{4!} t_{0}\right] } \\
+ & {\left[t_{5} \frac{t_{0}^{5}}{5!}+\left(5 t_{1} t_{4}+10 t_{2} t_{3}\right) \frac{t_{0}^{4}}{4!}+\left(30 t_{1} \frac{t_{2}^{2}}{2!}+20 t_{3} \frac{t_{1}^{2}}{2!}\right) \frac{t_{0}^{3}}{3!}\right.} \\
& \left.+60 t_{2} \frac{t_{1}^{3}}{3!} \frac{t_{0}^{2}}{2!}+120 \frac{t_{1}^{5}}{5!} t_{0}\right] \\
+ & {\left[t_{6} \frac{t_{0}^{6}}{6!}+\left(20 \frac{t_{3}^{2}}{2!}+6 t_{1} t_{5}+15 t_{2} t_{4}\right) \frac{t_{0}^{5}}{5!}\right.} \\
& +\left(90 \frac{t_{2}^{3}}{3!}+30 t_{4} \frac{t_{1}^{2}}{2!}+60 t_{1} t_{2} t_{3}\right) \frac{t_{0}^{4}}{4!} \\
& \left.+\left(120 t_{3} \frac{t_{1}^{3}}{3!}+180 \frac{t_{1}^{2}}{2!} \frac{t_{2}^{2}}{2!}\right) \frac{t_{0}^{3}}{3!}+360 t_{2} \frac{t_{1}^{4}}{4!} \frac{t_{0}^{2}}{2!}+720 \frac{t_{1}^{6}}{6!} t_{0}\right)+\cdots,
\end{aligned}
$$

has a combinatorial interpretation of an enumeration of rooted trees given by Feynman rules to be specified below. By a rooted tree we mean a tree whose vertices are all marked by $\bullet$, except for a valence-one vertex marked by o. This exceptional vertex was referred to as the root vertex $\circ$ in 31]. This is slightly different from the standard notion of a rooted tree in the literature, which means just a tree with a specified vertex, called the root of the tree. Our version of the rooted tree is obtained from the standard version by attaching an edge to the root, with the other vertex marked by o. With this understood, we can recall the Feynman rules for $I_{0}$
[31, Theorem 3.1]:

$$
\begin{equation*}
I_{0}=\sum_{\Gamma \text { is a rooted tree }} \frac{1}{|\operatorname{Aut} \Gamma|} w_{\Gamma} \tag{65}
\end{equation*}
$$

where the weight of $\Gamma$ is given by

$$
\begin{equation*}
w_{\Gamma}=\prod_{v \in V(\Gamma)} w_{v} \cdot \prod_{e \in E(\Gamma)} w_{e} \tag{66}
\end{equation*}
$$

with $w_{e}$ and $w_{v}$ given by the following Feynman rule:

$$
\begin{align*}
& w(e)=1, \\
& w(v)= \begin{cases}t_{\operatorname{val}(v)-1}, & \text { if } v \text { is not the root vertex } \circ, \\
1, & \text { if } v \text { is the root vertex } \circ .\end{cases} \tag{67}
\end{align*}
$$

For example,

give the first few terms of $I_{0}$. Secondly, all the other renormalized coupling constants $I_{k}$ have similar combinatorial interpretations. We need to extend our version of the rooted trees to a notion of rooted tree of type $k$. This means a tree that is obtained from a standard rooted tree by attaching $k+1$ edges to the root vertex, and mark each of the extra $k+1$ vertices by o. The renormalized coupling constant $I_{k}$ is given by a sum over rooted trees of type $k$ :

$$
\begin{equation*}
\frac{1}{k!} I_{k}=\sum_{\Gamma \text { is a rooted tree of type } k} \frac{1}{\mid \text { Aut } \Gamma \mid} w_{\Gamma} \tag{68}
\end{equation*}
$$

where the weight of $\Gamma$ is given by

$$
\begin{equation*}
w_{\Gamma}=\prod_{v \in V(\Gamma)} w_{v} \cdot \prod_{e \in E(\Gamma)} w_{e} \tag{69}
\end{equation*}
$$

with $w_{e}$ and $w_{v}$ given by the following Feynman rule:

$$
\begin{align*}
& w(e)=1,  \tag{70}\\
& w(v)= \begin{cases}t_{\operatorname{val}(v)-1}, & \text { if } v \text { is not a vertex marked by } \circ, \\
1, & \text { if } v \text { is a vertex marked by } \circ .\end{cases} \tag{71}
\end{align*}
$$

For example,

give the first few terms of $\frac{1}{2!} I_{1}$.
Thirdly, taking the Lagrange inversion also has both a closed formula and a combinatorial interpretation exactly as in the case of $I_{0}$, in fact by the results in [31], the Lagrange inversion of

$$
\begin{equation*}
z=\frac{w}{J_{0}+\sum_{n \geq 1} J_{n} \frac{w^{n}}{n!}} \tag{72}
\end{equation*}
$$

is solved by the following explicit formula:

$$
\begin{equation*}
w=\sum_{k=1}^{\infty} \frac{z^{k}}{k} \sum_{p_{1}+\cdots+p_{k}=k-1} \frac{J_{p_{1}}}{p_{1}!} \cdots \frac{J_{p_{k}}}{p_{k}!} \tag{73}
\end{equation*}
$$

This is proved in [31, Proposition 2.2] with the only change $t_{n} \rightarrow J_{n}$. In particular, the first few terms for $I_{0}$ given above also gives the first few terms of $w$ with this change. Similarly, the Feynman rules that expresses $I_{0}$ as an enumeration of the rooted trees also work for $w$ with this change. So after we apply these ideas to (64), we prove the following:

Theorem 2.5. The thin deformation of the thin spectral curve of Hermitian $N \times N$ matrix model can alternatively be given by a formal power series in $v=y / \sqrt{2}$ defined explicitly by:

$$
\begin{equation*}
z=I_{0}+\sum_{k=1}^{\infty} \frac{v^{k}}{k} \sum_{p_{1}+\cdots+p_{k}=k-1} \frac{J_{p_{1}}}{p_{1}!} \cdots \frac{J_{p_{k}}}{p_{k}!} \tag{74}
\end{equation*}
$$

where $J_{n}$ are given by

$$
\begin{equation*}
J_{0}=2 N, \quad J_{1}=0, \quad J_{n+1}=(n+1)\left(I_{n}-\delta_{n, 1}\right), \quad n \geq 1 \tag{75}
\end{equation*}
$$

Furthermore, this series is also given by enumeration over rooted trees as in (65) (67) but with $t_{n}$ changed to $J_{n}$.

The following are the first few terms of $z$ :

$$
\begin{aligned}
z= & I_{0}+(2 N) v+\frac{1}{2}(2 N)^{2}\left(2 \tilde{I}_{1}\right) v^{3}+\frac{1}{6}(2 N)^{3}\left(3 I_{2}\right) v^{4} \\
+ & \left(\frac{1}{24}\left(4 I_{3}\right)(2 N)^{4}+\frac{1}{2}(2 N)^{3}\left(2 \tilde{I}_{1}\right)^{2}\right) v^{5} \\
+ & \left(\frac{1}{120}(2 N)^{5}\left(5 I_{4}\right)+\frac{5}{12}(2 N)^{4}\left(2 \tilde{I}_{1}\right)\left(3 I_{2}\right)\right) v^{6} \\
+ & \left(\frac{1}{720}(2 N)^{6}\left(6 I_{5}\right)+\frac{1}{12}(2 N)^{5}\left(3 I_{2}\right)^{2}+\frac{1}{8}(2 N)^{5}\left(2 \tilde{I}_{1}\right)\left(4 I_{3}\right)+\frac{5}{8}(2 N)^{4}\left(2 \tilde{I}_{1}\right)^{3}\right) v^{7} \\
+ & \left(\frac{1}{5040}(2 N)^{7}\left(7 I_{6}\right)+\frac{7}{240}(2 N)^{6}\left(2 \tilde{I}_{1}\right)\left(5 I_{4}\right)+\frac{7}{144}\left(4 I_{3}\right)(2 N)^{6}\left(3 I_{2}\right)\right. \\
& \left.+\frac{7}{8}(2 N)^{5}\left(2 \tilde{I}_{1}\right)^{2}\left(3 I_{2}\right)\right) v^{8} \\
+ & \left(\frac{1}{40320}(2 N)^{8}\left(8 I_{7}\right)+\frac{1}{144}(2 N)^{7}\left(4 I_{3}\right)^{2}+\frac{1}{180}(2 N)^{7}\left(2 \tilde{I}_{1}\right)\left(6 I_{5}\right)\right. \\
& +\frac{1}{90}(2 N)^{7}\left(3 I_{2}\right)\left(5 I_{4}\right)+\frac{7}{8}(2 N)^{5}\left(2 \tilde{I}_{1}\right)^{4}+\frac{7}{24}(2 N)^{6}\left(2 \tilde{I}_{1}\right)^{2}\left(4 I_{3}\right) \\
& \left.+\frac{7}{18}(2 N)^{6}\left(3 I_{2}\right)^{2}\left(2 \tilde{I}_{1}\right)\right) v^{9}+\cdots
\end{aligned}
$$

So far in this Subsection we have used the coordinates $t_{n}$ or $I_{n}$, as a result, the coefficients of various expression are often fractional numbers. To obtain integral coefficients, now we change to the coordinates $g_{n}$.
Proposition 2.1. When expressed as formal power series in the coordinates $\left\{g_{n}\right\}_{n \geq 1}$, the renormalized coupling constants $\frac{1}{k!} I_{k}$ have integral coefficients.

Proof. By (58),

$$
\begin{aligned}
I_{0} & =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_{1}+\cdots+p_{k}=k-1} g_{p_{1}+1} \cdots g_{p_{k}+1} \\
& =\sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{\sum_{i=1}^{k} m_{i}=k \\
\sum_{i=1}^{k}(i-1) m_{i}=k-1}} \frac{\left(\sum_{i=1}^{k} m_{i}\right)!}{m_{1}!\cdots m_{k}!} g_{1}^{m_{1}} \cdots g_{k}^{m_{k}} \\
= & \sum_{\substack{\sum_{i=1}^{k}(i-1) m_{i}=k-1}} \frac{1}{\sum_{i=1}^{k} m_{i}}\binom{\sum_{i=1}^{k} m_{i}}{m_{1}, \cdots, m_{k}} g_{1}^{m_{1}} \cdots g_{k}^{m_{k}}
\end{aligned}
$$

Now since $(k, k-1)=1$, we know the greatest common divisor of $m_{1}, \ldots, m_{k}$ is equal to 1, i.e.,

$$
\begin{equation*}
\left(m_{1}, \ldots, m_{k}\right)=1 \tag{76}
\end{equation*}
$$

therefore, by [28, Propospition 3.1],

$$
\begin{equation*}
\frac{1}{\sum_{i=1}^{k} m_{i}}\binom{\sum_{i=1}^{k} m_{i}}{m_{1}, \cdots, m_{k}} \in \mathbb{Z} \tag{77}
\end{equation*}
$$

This shows that $I_{0}$ has integral coefficients. Secondly, by (59), for $k \geq 1$,

$$
\frac{I_{k}}{k!}=\sum_{n \geq 0}\binom{n+k}{k} g_{n+k} I_{0}^{n}
$$

and so $\frac{I_{k}}{k!}$ also has integral coefficients.
Remark 2.1. As the examples in earlier Subsections show, the integral coefficients of $I_{0}$ as formal series in $\left\{g_{n}\right\}_{n \geq 1}$ can be thought of as various generalizations of the Catalan numbers.

As a corollary of Proposition 2.1 and Theorem 2.5, we then have:
Theorem 2.6. The thin deformation of the thin spectral curve of Hermitian $N \times N$ matrix model can be given by a formal power series in $v=y / \sqrt{2}, 2 N$ and $\left\{g_{n}\right\}_{n \geq 1}$ with integral coefficients.

It is clear that the thin special deformation written in the form of (74) is a generalization of (7), a generating series of (signed) Catalan numbers. Therefore, when it is written as formal power series in $1 /(\sqrt{2} y),(2 N)$ and $\left\{g_{n}\right\}_{n \geq 0}$, the coefficients can again be regarded as generalizations of Catalan numbers. So we have reached the following surprising byproduct as the thin emergent geometry of Hermitian matrix models: The thin special deformations of the thin spectral curves of Hermitian matrix models can be used to define sophisticated generalizations of Catalan numbers.

## 3. Fat Emergent Geometry of Hermitian One-Matrix Models

In this Section we discuss the fat special deformation of the fat spectral curve of Hermitian one-matrix models. In the literature (see e.g. [4, §3.2] or [16, §2.2]), the discussions in this Section is usually carried out in the setting of taking $N \rightarrow \infty$. It is crucial for us to note that it also works for finite $N$.
3.1. Fat special deformation. Recall the fat Virasoro constraints are given by the following differential operators (cf. [36, §3.6]):

$$
\begin{aligned}
& L_{-1, t}=-\frac{\partial}{\partial g_{1}}+\sum_{n \geq 1} n g_{n+1} \frac{\partial}{\partial g_{n}}+t g_{1} g_{s}^{-2} \\
& L_{0, t}=-2 \frac{\partial}{\partial g_{2}}+\sum_{n \geq 1} n g_{n} \frac{\partial}{\partial g_{n}}+t^{2} g_{s}^{-2} \\
& L_{1, t}=-3 \frac{\partial}{\partial g_{3}}+\sum_{n \geq 1}(n+1) g_{n} \frac{\partial}{\partial g_{n+1}}+2 t \frac{\partial}{\partial g_{1}} \\
& L_{m, t}=\sum_{k \geq 1}(k+m)\left(g_{k}-\delta_{k, 2}\right) \frac{\partial}{\partial g_{k+m}}+g_{s}^{2} \sum_{k=1}^{m-1} k(m-k) \frac{\partial}{\partial g_{k}} \frac{\partial}{\partial g_{m-k}}+2 t m \frac{\partial}{\partial g_{m}}
\end{aligned}
$$

where $m \geq 2$. As in the thin case,

Theorem 3.1. Consider the following series:

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty}\left(g_{n}-\delta_{n, 2}\right) z^{n-1}+\frac{\sqrt{2} t}{z}+\sqrt{2} \sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}} . \tag{78}
\end{equation*}
$$

Then one has:

$$
\begin{equation*}
\left(y^{2}\right)_{-}=0 . \tag{79}
\end{equation*}
$$

Proof. This is actually equivalent to the Virasoro constraints for $F_{0}(t)$. Indeed,

$$
\begin{aligned}
\frac{y^{2}}{2} & =\left(\frac{1}{2} \sum_{n=1}^{\infty}\left(g_{n}-\delta_{n, 2}\right) z^{n-1}+\frac{1}{z}+\sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}}\right)^{2} \\
& =\frac{1}{4}\left(\sum_{n=1}^{\infty}\left(g_{n}-\delta_{n, 2}\right) z^{n-1}\right)^{2}+\sum_{n=1}^{\infty}\left(g_{n}-\delta_{n, 2}\right) z^{n-1}\left(\frac{t}{z}+\sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}}\right) \\
& +\left(\frac{t}{z}+\sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}}\right)^{2}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{1}{2}\left(y^{2}\right)_{-} & =\frac{1}{z}\left(t g_{1}+\sum_{n=1}^{\infty} n\left(g_{n+1}-\delta_{n, 1}\right) \frac{\partial F_{0}(t)}{\partial g_{n}}\right) \\
& +\sum_{m \geq 0} \sum_{k=1}^{\infty} \frac{(k+m)}{z^{m+2}}\left(g_{k}-\delta_{k, 2}\right) \frac{\partial F_{0}(t)}{\partial g_{k+m}} \\
& +\left(\frac{t}{z}+\sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}}\right)^{2}
\end{aligned}
$$

The fat Virasoro constraints in genus zero take the following form:

$$
\begin{aligned}
& \frac{\partial F_{0}(t)}{\partial g_{1}}=\sum_{n \geq 1} n g_{n+1} \frac{\partial F_{0}(t)}{\partial g_{n}}+t g_{1} \\
& 2 \frac{\partial F_{0}(t)}{\partial g_{2}}=\sum_{n \geq 1} n g_{n} \frac{\partial F_{0}(t)}{\partial g_{n}}+t^{2} \\
& 3 \frac{\partial F_{0}(t)}{\partial g_{3}}=\sum_{n \geq 1}(n+1) g_{n} \frac{\partial F_{0}(t)}{\partial g_{n+1}}+2 t \frac{\partial F_{0}(t)}{\partial g_{1}} \\
& (m+2) \frac{\partial F_{0}(t)}{\partial g_{m+2}}=\sum_{k \geq 1}(k+m) g_{k} \frac{\partial F_{0}(t)}{\partial g_{k+m}}+\sum_{k=1}^{m-1} k(m-k) \frac{\partial F_{0}(t)}{\partial g_{k}} \frac{\partial F_{0}(t)}{\partial g_{m-k}}+2 t m \frac{\partial F_{0}(t)}{\partial g_{m}}
\end{aligned}
$$

where $m \geq 2$. By these, the proof is completed.
We will refer to (79) as the fat special deformation of the fat spectral curve to be discussed below. Similar to the thin case, we can characterize the fat special deformation as follows.

Theorem 3.2. There exists a unique series

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \sum_{n \geq 0}\left(v_{n}-\delta_{n, 1}\right) z^{n}+\frac{\sqrt{2} t}{z}+\sqrt{2} \sum_{n \geq 0} w_{n} z^{-n-2} \tag{80}
\end{equation*}
$$

such that each $w_{n} \in \mathbb{C}\left[\left[v_{0}, v_{1}, \ldots\right]\right]$ and

$$
\begin{equation*}
\frac{1}{2}\left(y^{2}\right)_{-}=0 . \tag{81}
\end{equation*}
$$

Theorem 3.3. For a series of the form

$$
\begin{equation*}
y=\frac{1}{\sqrt{2}} \frac{\partial S(z, \mathbf{t})}{\partial z}+\frac{\sqrt{2} N}{z}+\sqrt{2} \sum_{n \geq 0} w_{n} z^{-n-2} \tag{82}
\end{equation*}
$$

where $S$ is the universal action defined by:

$$
\begin{equation*}
S(x)=-\frac{1}{2} x^{2}+\sum_{n \geq 1} t_{n-1} \frac{x^{n}}{n!} \tag{83}
\end{equation*}
$$

and each $w_{n} \in \mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right]$, the equation

$$
\begin{equation*}
\frac{1}{2}\left(y^{2}\right)_{-}=0 \tag{84}
\end{equation*}
$$

has a unique solution given by:

$$
y=\frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{t_{n}-\delta_{n, 1}}{n!} z^{n}+\frac{\sqrt{2}}{z}+\sqrt{2} \sum_{n=1}^{\infty} \frac{n!}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial t_{n-1}} .
$$

3.2. Fat special deformation in terms of resolvent. The resolvent $\omega$ is defined by:

$$
\begin{equation*}
\omega(z)=\frac{t}{z}+\sum_{n=1}^{\infty} \frac{n}{z^{n+1}} \frac{\partial F_{0}(t)}{\partial g_{n}} . \tag{85}
\end{equation*}
$$

For simplicity of notations, write

$$
\begin{equation*}
f_{n}:=n \frac{\partial F_{0}(t)}{\partial g_{n}} \tag{86}
\end{equation*}
$$

Then by the fat Virasoro constraints in genus zero we have:

$$
\begin{aligned}
\omega(z) & =\frac{t}{z}+\frac{1}{z^{2}}\left(\sum_{n \geq 1} g_{n+1} f_{n}+t g_{1}\right) \\
& +\frac{1}{z^{3}}\left(\sum_{n \geq 1} g_{n} f_{n}+t^{2}\right)+\frac{1}{z^{4}}\left(\sum_{n \geq 1} g_{n} f_{n+1}+2 t f_{1}\right) \\
& +\sum_{m \geq 2} \frac{1}{z^{m+2}}\left(\sum_{k \geq 1} g_{k} f_{k+m}+\sum_{k=1}^{m-1} f_{k} f_{m-k}+2 t m f_{m}\right) \\
& =\frac{1}{z} \omega^{2}(z)+\frac{t}{z}+\frac{g_{1}}{z} \omega(z)+g_{2}\left(\omega(z)-\frac{t}{z}\right) \\
& +g_{3} z\left(\omega(z)-\frac{t}{z}-\frac{f_{1}}{z^{2}}\right)+\cdots \\
& =\frac{1}{z} \omega^{2}(z)+\frac{1}{z}\left(g_{1}+g_{2} z+g_{3} z^{2}+\cdots\right) \omega(z) \\
& +\left(\frac{t}{z}-g_{2} \frac{t}{z}-g_{3} z\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}\right)-\cdots\right) .
\end{aligned}
$$

It can be rewritten in the following form:

$$
\begin{aligned}
& \omega^{2}(z)+\left(g_{1}+\left(g_{2}-1\right) z+g_{3} z^{2}+\cdots\right) \cdot \omega(z) \\
-\quad & \left(g_{2}-1\right) t-g_{3} z^{2}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}\right)-\cdots=0
\end{aligned}
$$

In other words, the fat special deformation in term of the resolvent is given by:

$$
\begin{equation*}
\omega^{2}(z)+S^{\prime}(z) \cdot \omega(z)+P(z)=0 \tag{87}
\end{equation*}
$$

where $S$ is the action

$$
\begin{equation*}
S(x):=\sum_{n=1}^{\infty} \frac{1}{n}\left(g_{n}-\delta_{n, 2}\right) x^{n} \tag{88}
\end{equation*}
$$

and $P(z)$ is determined by $S(z)$ as follows:

$$
\begin{equation*}
P(z)=-\left(g_{2}-1\right) t-g_{3} z^{2}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}\right)-\cdots . \tag{89}
\end{equation*}
$$

It is clear from the above formulas when one takes $g_{k}=0$ for $k \geq n$, then to find the fat spectral curve it suffices to find only $f_{1}, f_{2}, \ldots, f_{n-2}$. From (87) we get:

$$
\begin{equation*}
\omega(z)=\frac{-S^{\prime}(z)-\sqrt{S^{\prime}(z)^{2}-4 P(z)}}{2} \tag{90}
\end{equation*}
$$

For the treatment from the large $N$ point of view, see e.g. [4, §3.2].
3.3. The emergence of semi-circle law. Let us first let all $g_{k}=0$ for all $k \geq 0$. Then we have

$$
S(x)=-\frac{1}{2} x^{2}, \quad P(z)=t
$$

and so

$$
\begin{equation*}
\omega(z)=\frac{z-\sqrt{z^{2}-4 t}}{2} \tag{91}
\end{equation*}
$$

It is well-known that

$$
\begin{equation*}
\omega(z)=\sum_{n \geq 0} C_{n} \frac{t^{n+1}}{z^{2 n+1}} \tag{92}
\end{equation*}
$$

where $C_{n}$ are the Catalan numbers:

$$
\begin{equation*}
C_{n}=\frac{1}{n+1}\binom{2 n}{n} \tag{93}
\end{equation*}
$$

This means that

$$
\left.n \frac{\partial F_{0}(t)}{\partial g_{n}}\right|_{g_{k}=0, k \geq 1}= \begin{cases}C_{m}, & n=2 m  \tag{94}\\ 0, & \text { otherwise }\end{cases}
$$

From (91) we get:

$$
\begin{equation*}
\omega^{2}-z \omega+t=0 \tag{95}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
z=\omega+\frac{t}{\omega} . \tag{96}
\end{equation*}
$$

By comparing with (6), we see a strange duality.
On the other hand,

$$
\begin{equation*}
y=\frac{-\sqrt{z^{2}-4 t}}{\sqrt{2}} \tag{97}
\end{equation*}
$$

So the spectral curve in terms of the field $y=y(z)$ is given by the algebraic curve:

$$
\begin{equation*}
z^{2}-2 y^{2}=4 t \tag{98}
\end{equation*}
$$

For the treatment from the large $N$ point of view, see e.g. the Example in 16, §2.2].

We will refer to the plane algebraic curve (98) as the fat spectral curve of Hermitian one-matrix models. It is equivalently given in terms of the resolvent by (91) or (92). These equations are the dual versions of (7), so it is tempting to say that the thin spectral curve and the fat spectral curve are dual to each other, furthermore, the thin special deformation defined by (74) and the fat deformation defined by (85) are dual to each other. Because $F_{0}(t)$ are formal power series in $t$ and $\left\{g_{n}\right\}_{n \geq 1}$ with integral coefficients, we see that the fat special deformation also can be used to define generalizations of Catalan numbers. In the following several Subsections we will examine some examples to gain more insights into the nature of the plane algebraic curves defined by the fat special deformations.
3.4. Fat special deformation along the $g_{1}$-line. Let us first let all $g_{k}=0$ for all $k \geq 1$. Then we have

$$
S(x)=-\frac{1}{2} x^{2}+g_{1} x, \quad P(z)=t
$$

and so

$$
\begin{equation*}
\omega(z)=\frac{z-g_{1}-\sqrt{\left(z-g_{1}\right)^{2}-4 t}}{2} \tag{99}
\end{equation*}
$$

Its expansion is related to the Motzkin polynomials:

$$
\begin{equation*}
R_{n}(x):=\sum_{k=0}^{[n / 2]} T(n, k) x^{k} \tag{100}
\end{equation*}
$$

where $T_{n, k}$ are defined by:

$$
\begin{equation*}
T(n, k)=\frac{n!}{(n-2 k)!k!(k+1)!} \tag{101}
\end{equation*}
$$

More precisely,

$$
\begin{equation*}
\omega(z)=\sum_{n \geq 0} \sum_{k=0}^{[n / 2]} T(n, k) \frac{t^{k+1} g_{1}^{n-2 k}}{z^{n+1}} \tag{102}
\end{equation*}
$$

This mean

$$
\begin{equation*}
\left.n \frac{\partial F_{0}(t)}{\partial g_{n}}\right|_{g_{k}=0, k \geq 2}=\sum_{k=0}^{[n / 2]} T(n, k) t^{k+1} g_{1}^{n-2 k} \tag{103}
\end{equation*}
$$

Alternatively, write

$$
f_{n}:=\left.n \frac{\partial F_{0}(t)}{\partial g_{n}}\right|_{g_{k}=0, k \geq 2}
$$

Then by the fat Virasoro constraints one has the following recursion relations:

$$
\begin{align*}
& f_{1}=t g_{1} \\
& f_{2}=g_{1} f_{1}+t^{2} \\
& f_{3}=g_{1} f_{2}+2 t f_{1}  \tag{104}\\
& f_{m+2}=g_{1} f_{m+1}+\sum_{k=1}^{m-1} f_{k} f_{m-k}+2 t f_{m}, \quad m \geq 1
\end{align*}
$$

It is natural to set

$$
\begin{equation*}
f_{0}:=t \tag{105}
\end{equation*}
$$

Then one has

$$
\omega(z)=\sum_{n \geq 0} \frac{f_{n}}{z^{n+1}}=\frac{t}{z}+\frac{g_{1}}{z} \omega(z)+\frac{1}{z} \omega^{2}(z)
$$

and from this one can again get (99).
The first few of $f_{n}$ are

$$
\begin{aligned}
& f_{0}=t \\
& f_{1}=t g_{1} \\
& f_{2}=t g_{1}^{2}+t^{2} \\
& f_{3}=t g_{1}^{3}+3 t^{2} g_{1} \\
& f_{4}=t g_{1}^{4}+6 t^{2} g_{1}^{2}+2 t^{3}, \\
& f_{5}=t g_{1}^{5}+10 t^{2} g_{1}^{3}+10 t^{3} g_{1} \\
& f_{6}=t g_{1}^{6}+15 t^{2} g_{1}^{4}+30 t^{3} g_{1}^{2}+5 t^{4} \\
& f_{7}=t g_{1}^{7}+21 t^{2} g_{1}^{5}+70 t^{3} g_{1}^{3}+35 t^{4} g_{1}
\end{aligned}
$$

From (99) we get:

$$
\begin{equation*}
\omega^{2}-\left(z-g_{1}\right) \omega+t=0 \tag{106}
\end{equation*}
$$

and

$$
\begin{equation*}
z=\omega+\frac{t}{\omega}+g_{1} \tag{107}
\end{equation*}
$$

These are deformations of (96) and (95) respectively. The field $y$ is deformed to

$$
\begin{equation*}
y=\frac{-\sqrt{\left(z-g_{1}\right)^{2}-4 t}}{\sqrt{2}} \tag{108}
\end{equation*}
$$

and so (98) is deformed to

$$
\begin{equation*}
\left(z-g_{1}\right)^{2}-2 y^{2}=4 t \tag{109}
\end{equation*}
$$

So as in the thin case, the fat special deformation along the $g_{1}$-line is obtained by changing $z$ to $z-g_{1}$.
3.5. Fat special deformation along the $g_{2}$-line. Next we let $g_{n}=0$ for $n \neq 2$, i.e.,

$$
S(x)=-\frac{1}{2}\left(1-g_{2}\right) x^{2}, \quad P(z)=\left(1-g_{2}\right) t
$$

From this we get the resolvent:

$$
\begin{equation*}
\omega(z)=\frac{\left(1-g_{2}\right) z-\sqrt{\left(1-g_{2}\right)^{2} z^{2}-4\left(1-g_{2}\right) t}}{2} \tag{110}
\end{equation*}
$$

and so

$$
\begin{equation*}
\frac{y}{\sqrt{2}}=\frac{1}{2}\left(g_{2}-1\right) z+\omega(z)=-\frac{\sqrt{\left(1-g_{2}\right)^{2} z^{2}-4\left(1-g_{2}\right) t}}{2} \tag{111}
\end{equation*}
$$

Therefore, the fat special deformation along the $g_{2}$-line leads to the plane algebraic curve:

$$
\left(1-g_{2}\right)^{2} z^{2}-2 y^{2}=4\left(1-g_{2}\right) t
$$

this can be obtained from (98) by making the following changes:

$$
y \mapsto \frac{y}{1-g_{2}}, \quad \quad t \mapsto \frac{t}{1-g_{2}}
$$

The expansion of the resolvent in this case is

$$
\omega(z)=\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2 n}{n} \frac{t^{n+1}}{\left(1-g_{2}\right)^{2 n+1}} \frac{1}{z^{2 n+1}}
$$

From this one gets

$$
\begin{equation*}
\left.n \frac{\partial F_{0}}{\partial g_{n}}\right|_{g_{k}=0, k \neq 2}=\frac{1}{n+1}\binom{2 n}{n} \frac{t^{n+1}}{\left(1-g_{2}\right)^{2 n+1}} \tag{112}
\end{equation*}
$$

3.6. Fat special deformations on the $\left(g_{1}, g_{2}\right)$-plane. Next we let $g_{n}=0$ for $n \geq 3$, i.e.,

$$
S(x)=g_{1} x-\frac{1}{2}\left(1-g_{2}\right) x^{2}, \quad P(z)=\left(1-g_{2}\right) t
$$

From this we get the resolvent:

$$
\begin{equation*}
\omega(z)=\frac{\left(1-g_{2}\right) z-g_{1}-\sqrt{\left[\left(1-g_{2}\right) z-g_{1}\right]^{2}-4\left(1-g_{2}\right) t}}{2} \tag{113}
\end{equation*}
$$

and so

$$
\begin{equation*}
\frac{y}{\sqrt{2}}=\frac{1}{2}\left(\left(g_{2}-1\right) z+g_{1}\right)+\omega(z)=-\frac{\sqrt{\left(\left(1-g_{2}\right) z-g_{1}\right)^{2}-4\left(1-g_{2}\right) t}}{2} . \tag{114}
\end{equation*}
$$

This can be obtained from (108) by making the following changes:

$$
y \mapsto \frac{y}{1-g_{2}}, \quad \quad t \mapsto \frac{t}{1-g_{2}}, \quad \quad g_{1} \mapsto \frac{g_{1}}{1-g_{2}} .
$$

We rewrite (113) in the following form

$$
\begin{equation*}
\left(\frac{\sqrt{2} \omega}{1-g_{2}}\right)^{2}-\sqrt{2} \frac{\sqrt{2} \omega}{1-g_{2}}\left(z-\frac{g_{1}}{1-g_{2}}\right)+\frac{2 t}{1-g_{2}}=0 \tag{115}
\end{equation*}
$$

This can be regarded as a dual to (29): $\frac{\sqrt{2} \omega}{1-g_{2}}$ in this equation plays the role of $z-\frac{g_{1}}{1-g_{2}}$ in (29), $z-\frac{g_{1}}{1-g_{2}}$ in this equation plays the file of $\frac{y}{1-g_{2}}$ in (29), and $\frac{2 t}{1-g_{2}}$ in this equation plays the role of $-\frac{2 N}{1-g_{2}}$ in (29).

The expansion of (113) is

$$
\begin{equation*}
\omega(z)=\sum_{n \geq 0} \sum_{k=0}^{[n / 2]} T(n, k) \frac{t^{k+1} g_{1}^{n-2 k}}{\left(1-g_{2}\right)^{n-k} z^{n+1}} \tag{116}
\end{equation*}
$$

from this one can get closed formula for $\left\langle p_{1}^{k}\left(\frac{p_{2}}{2}\right)^{l} p_{n}\right\rangle_{0}(t)$.
3.7. Fat special deformations along the $g_{3}$-line. Next we let $g_{n}=0$ for $n \neq 2$, i.e.,

$$
S(x)=-\frac{1}{2} x^{2}+\frac{g_{3}}{3} x^{3}, \quad P(z)=t-g_{3} z^{2}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}\right)
$$

where

$$
f_{n}=\left.n \frac{\partial F_{0}(t)}{\partial g_{n}}\right|_{g_{k}=0, k \neq 3}
$$

From this we get the resolvent:

$$
\begin{equation*}
\omega(z)=\frac{z-g_{3} z^{2}-\sqrt{\left(z-g_{3} z^{2}\right)^{2}-4\left(t-g_{3} t z-g_{3} f_{1}\right)}}{2} \tag{117}
\end{equation*}
$$

and so

$$
\begin{equation*}
y=\frac{1}{2}\left(g_{3} z^{2}-z\right)+\omega(z)=-\frac{\sqrt{\left(z-g_{3} z^{2}\right)^{2}-4\left(t-g_{3} t z-g_{3} f_{1}\right)}}{2} . \tag{118}
\end{equation*}
$$

The spectral curve is then deformed to the following hyperelliptic curve on the ( $y, z$ )-plane:

$$
\begin{equation*}
4 y^{2}=\left(z-g_{3} z^{2}\right)^{2}-4\left(t-g_{3} t z-g_{3} f_{1}\right) \tag{119}
\end{equation*}
$$

To make this result more explicit, we need to find explicit formula for $f_{1}$. Note in the above expressions,

$$
\begin{equation*}
f_{1}=\sum_{n \geq 0} \frac{1}{n!}\left\langle p_{1}\left(\frac{p_{3}}{3}\right)^{n}\right\rangle_{0}(t) \cdot g_{3}^{n} \tag{120}
\end{equation*}
$$

where $\left\langle p_{1}\left(\frac{p_{3}}{3}\right)^{n}\right\rangle_{0}(t)$ denote the fat genus zero correlators in the sense of 36]. For small $n$, these correlators can be computed using the fat Virasoro constraints,

$$
\begin{aligned}
f_{1} & =\sum_{n \geq 1} \frac{1}{n!} \cdot n \cdot\left\langle p_{2}\left(\frac{p_{3}}{3}\right)^{n-1}\right\rangle_{0}(t) \cdot g_{3}^{n} \\
& =g_{3} t^{2}+\sum_{n \geq 2} \frac{1}{(n-1)!} \cdot 3(n-1) \cdot\left\langle\left(\frac{p_{3}}{3}\right)^{n-1}\right\rangle_{0}(t) \cdot g_{3}^{n} \\
& =g_{3} t^{2}+3 \sum_{m \geq 1} \frac{1}{(2 m-1)!} \cdot\left\langle\left(\frac{p_{3}}{3}\right)^{2 m}\right\rangle_{0}(t) \cdot g_{3}^{2 m+1} .
\end{aligned}
$$

Using the fat graphs, one see that the computation of $\frac{1}{n!}\left\langle p_{1}\left(\frac{p_{3}}{3}\right)^{n}\right\rangle_{0}(t)$ is equivalent to the problem of rooted planar trivalent graphs, hence is given by the sequence A002005 on [22], and so by e.g. [15], §3] we have for $n \geq 1$,

$$
\frac{1}{n!}\left\langle p_{1}\left(\frac{p_{3}}{3}\right)^{n}\right\rangle_{0}(t)= \begin{cases}\frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2}, & n=2 m+1  \tag{121}\\ 0, & n=2 m\end{cases}
$$

One can derive from this

$$
\begin{equation*}
\left\langle p_{1}\left(\frac{p_{3}}{3}\right)^{2 m+1}\right\rangle_{0}(t)=\frac{2^{2 m+1}(2 m+1)!(3 m)!!}{(m+2)!m!!} t^{m+2} \tag{122}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\left(\frac{p_{3}}{3}\right)^{2 m}\right\rangle_{0}(t)=\frac{2^{2 m+1}(2 m-1)!(2 m+1)!(3 m)!!}{3 \cdot(m+2)!m!!} t^{m+2} \tag{123}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
f_{1}=\sum_{m \geq 0} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2} g_{3}^{2 m+1} \tag{124}
\end{equation*}
$$

and so the specially deformed spectral curve (119) along the $g_{3}$-line is explicitly given by

$$
\begin{equation*}
y^{2}=\frac{1}{4}\left(z-g_{3} z^{2}\right)^{2}-t+g_{3} t z+\sum_{m \geq 0} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2} g_{3}^{2 m+2} \tag{125}
\end{equation*}
$$

To understand this hyperelliptic curve, we need to study the discriminant $\Delta$ of the right-hand side. Using Maple, we find:

$$
\begin{equation*}
\Delta=\frac{1}{16} g_{3}^{5}\left(64 g_{3}^{4} f_{1}^{3}+\left(g_{3}-96 g_{3}^{3} t\right) f_{1}^{2}+\left(30 g_{3}^{2} t^{2}-t\right) f_{1}-27 g_{3}^{3} t^{4}+g_{3} t^{3}\right) \tag{126}
\end{equation*}
$$

this vanishes because it is known from Entry A002005 on [22] that if

$$
y(x)=\sum_{m \geq 0} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} x^{m}
$$

then

$$
\begin{equation*}
64 x^{3} y^{3}+x(1-96 x) y^{2}+(30 x-1) y-27 x+1=0 \tag{127}
\end{equation*}
$$

It follows that the specially deformed spectral curve has the following form:

$$
\begin{equation*}
y^{2}=\left(1-g_{3} z+\alpha\right)^{2}\left(z-a_{+}\right)\left(z-a_{-}\right) . \tag{128}
\end{equation*}
$$

In $\$ 4.6$ we will go in the reverse direction and use this as a method to compute computing $f_{1}$ and hence $\left\langle\left(\frac{p_{3}}{3}\right)^{2 m}\right\rangle_{0}(t)$ for all $m$.

The first few of terms of $f_{1}$ are:

$$
\begin{equation*}
f_{1}=g_{3} t^{2}+4 g_{3}^{3} t^{3}+32 g_{3}^{5} t^{4}+336 g_{3}^{7} t^{5}+4096 g_{3}^{9} t^{6}+54912 g_{3}^{11} t^{7}+\cdots \tag{129}
\end{equation*}
$$

By (117) the first few terms of the resolvent are:

$$
\begin{aligned}
\omega & =\frac{t}{z}+\frac{f_{1}}{z^{2}}+\frac{f_{1}}{g_{3} z^{3}}+\frac{f_{1}-g_{3} t^{2}}{g_{3}^{2} z^{4}}+\frac{f_{1}-g_{3} t^{2}-2 g_{3}^{2} t f_{1}}{g_{3}^{3} z^{5}} \\
& +\frac{f_{1}-g_{3} t^{2}-4 g_{3}^{2} t f_{1}-g_{3}^{3} f_{1}^{2}}{g_{3}^{4} z^{6}}+\frac{f_{1}+2 g_{3}^{3} t^{3}-g_{3} t^{2}-6 g_{3}^{2} t f_{1}-3 g_{3}^{3} f_{1}^{2}}{g_{3}^{5} z^{7}}+\cdots \\
& =\frac{t}{z}+\frac{1}{z^{2}}\left(g_{3} t^{2}+4 g_{3}^{3} t^{3}+32 g_{3}^{5} t^{4}+336 g_{3}^{7} t^{5}+4096 g_{3}^{9} t^{6}+\cdots\right) \\
& +\frac{1}{z^{3}}\left(t^{2}+4 g_{3}^{2} t^{3}+32 g_{3}^{4} t^{4}+336 g_{3}^{6} t^{5}+4096 g_{3}^{8} t^{6}+\cdots\right) \\
& +\frac{1}{z^{4}}\left(4 g_{3} t^{3}+32 g_{3}^{3} t^{4}+336 g_{3}^{5} t^{5}+4096 g^{7} t^{6}+54912 g_{3}^{9} t^{7}+\cdots\right) \\
& +\frac{1}{z^{5}}\left(2 t^{3}+24 g_{3}^{2} t^{4}+272 g_{3}^{4} t^{5}+3424 g_{3}^{6} t^{6}+46720 g_{3}^{8} t^{7}+\cdots\right) \\
& +\frac{1}{z^{6}}\left(15 g_{3} t^{4}+200 g_{3}^{3} t^{5}+2672 g_{3}^{5} t^{6}+37600 g_{3}^{7} t^{7}+\cdots\right)+\cdots
\end{aligned}
$$

We regard this as a dual to (33), in both cases the right-hands are formal power series with integral coefficients which generalize the Catalan numbers. The leading terms of the coefficients of $\frac{1}{z^{2 n+1}} t^{n+1}$ are $\frac{1}{n+1}\binom{2 n}{n}$. We have checked that the leading terms of the coefficients of $\frac{1}{z^{2 n}}$ are $g_{3} t^{n+1}$ times $\binom{2 n}{n-1}$ which are the sequence A001791 on [22].

From the above calculation of $\omega$, one also gets a way to compute the first few $f_{n}$ 's and hence the corresponding correlators:

$$
\begin{aligned}
f_{2} & =\frac{f_{1}}{g_{3}}=\sum_{m \geq 0} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2} g_{3}^{2 m} \\
f_{3} & =\frac{f_{1}-g_{3} t^{2}}{g_{3}^{2}}=\sum_{m \geq 1} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2} g_{3}^{2 m-1} \\
f_{4} & =\frac{f_{1}-g_{3} t^{2}-2 g_{3}^{2} t f_{1}}{g_{3}^{3}} \\
& =\sum_{m \geq 1} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+2} g_{3}^{2 m-2}-2 \sum_{m \geq 0} \frac{2^{2 m+1}(3 m)!!}{(m+2)!m!!} t^{m+3} g_{3}^{2 m} \\
& =\sum_{m \geq 0}\left(\frac{2^{2 m+3}(3 m+3)!!}{(m+3)!(m+1)!!}-\frac{2^{2 m+2}(3 m)!!}{(m+2)!m!!}\right) t^{m+3} g_{3}^{2 m} \\
f_{5} & =\frac{f_{1}-g_{3} t^{2}-4 g_{3}^{2} t f_{1}-g_{3}^{3} f_{1}^{2}}{g_{3}^{4}} \\
f_{6} & =\frac{f_{1}+2 g_{3}^{3} t^{3}-g_{3} t^{2}-6 g_{3}^{2} t f_{1}-3 g_{3}^{3} f_{1}^{2}}{g_{3}^{5}}
\end{aligned}
$$

The coefficients of $f_{4}$ are the integer sequence A002006 on [22]: 2, 24, 272, 3424, $46720, \cdots$; the coefficients of $f_{5}$ are the integer sequence A002007: 15, 200, 2672, $37600, \cdots$; the coefficients of $f_{6}$ are the integer sequence A002008 5, 120, 1840, $27552, \cdots$; the coefficients of $f_{7}$ and $f_{8}$ are the sequences A002009 and A002010 respectively. These numbers are the numbers of almost trivalent maps in the sense of [19]. Our discussion in this Subsection shows that the plane algebraic curve (119) encode all the numbers of almost trivalent maps.
3.8. Fat special deformations on the $\left(g_{1}, g_{2}, g_{3}\right)$-space. Next we let $g_{n}=0$ for $n \geq 4$, i.e.,

$$
S(x)=-\frac{1}{2} x^{2}+g_{1} x+\frac{g_{2}}{2} x^{2}+\frac{g_{3}}{3} x^{3}, \quad P(z)=\left(1-g_{2}\right) t-g_{3} z^{2}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}\right)
$$

where

$$
f_{1}=\left.\frac{\partial F_{0}(t)}{\partial g_{1}}\right|_{g_{k}=0, k \geq 4}
$$

From this we get the resolvent:

$$
\begin{align*}
\omega(z) & =\frac{1}{2}\left(-g_{1}+\left(1-g_{2}\right) z-g_{3} z^{2}\right. \\
& \left.-\sqrt{\left(-g_{1}+\left(1-g_{2}\right) z-g_{3} z^{2}\right)^{2}-4\left(\left(1-g_{2}\right) t-g_{3} t z-g_{3} f_{1}\right)}\right) \tag{130}
\end{align*}
$$

and so

$$
\begin{equation*}
y=-\frac{\sqrt{\left(-g_{1}+\left(1-g_{2}\right) z-g_{3} z^{2}\right)^{2}-4\left(\left(1-g_{2}\right) t-g_{3} t z-g_{3} f_{1}\right)}}{2} \tag{131}
\end{equation*}
$$

The spectral curve is then deformed to

$$
\begin{equation*}
4 y^{2}=\left(-g_{1}+\left(1-g_{2}\right) z-g_{3} z^{2}\right)^{2}-4\left(\left(1-g_{2}\right) t-g_{3} t z-g_{3} f_{1}\right) \tag{132}
\end{equation*}
$$

Again, to make this special deformation more explicit, we need to find explicit formula for $f_{1}$. One can apply the fat dilaton equation and the fat string equation to reduce the computation of $f_{1}$ to the computations of $\left\langle\left(\frac{p_{3}}{3}\right)^{2 m}\right\rangle_{0}(t)$. The following are the first few terms of $f_{1}$ :

$$
f_{1}=g_{1} t+g_{3} t^{2}+t g_{2} g_{1}+2 t^{2} g_{3} g_{2}+t g_{3} g_{1}^{2}+t g_{2}^{2} g_{1}+\cdots
$$

On the other hand, using Maple we find the discriminant of the degree four polynomial on the right-hand side of (132) is

$$
\begin{aligned}
\Delta= & \frac{g_{3}^{5}}{16}\left(64 g_{3}^{4} f_{1}^{3}+g_{3}\left[\left(\left(1-g_{2}\right)^{2}-4 g_{1} g_{3}\right)^{2}-96\left(1-g_{2}\right) g_{3}^{2} t\right] f_{1}^{2}\right. \\
+ & {\left[30 t g_{3}^{2}\left(1-g_{2}\right)^{2}-\left(1-g_{2}\right)\left[\left(1-g_{2}\right)^{2}-4 g_{1} g_{3}\right]^{2}+72 t g_{1} g_{3}^{3}\right] f_{1} } \\
+ & {\left[-27 t^{4} g_{3}^{3}+16 t^{2} g_{1}^{3} g_{3}^{2}-t^{3} g_{3}\left(1-g_{2}\right)^{3}-t^{2} g_{1}\left(1-g_{2}\right)^{4}\right.} \\
& \left.+8 t^{2} g_{1}^{2} g_{3}\left(1-g_{2}\right)^{2}+36 t^{3} g_{1} g_{3}^{2}\left(1-g_{2}\right)\right]
\end{aligned}
$$

One can use the vanishing of $\Delta$ to compute $f_{1}$ and $\omega$. This will be justified in 4.7
3.9. Fat special deformations on the $g_{4}$-line. Let $g_{n}=0$ for $n \neq 4$, i.e.,

$$
S(x)=-\frac{1}{2} x^{2}+\frac{g_{4}}{4} x^{4}, \quad P(z)=t-g_{4} z^{3}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}+\frac{f_{2}}{z^{3}}\right)
$$

where

$$
f_{1}=\left.\frac{\partial F_{0}(t)}{\partial g_{1}}\right|_{g_{k}=0, k \neq 4}, \quad f_{2}=\left.2 \frac{\partial F_{0}(t)}{\partial g_{2}}\right|_{g_{k}=0, k \neq 4}
$$

From this we get the resolvent:

$$
\begin{equation*}
\omega(z)=\frac{1}{2}\left(z-g_{4} z^{3}-\sqrt{\left(z-g_{4} z^{3}\right)^{2}-4\left(t-g_{4} t z^{2}-g_{4} f_{1} z-g_{4} f_{2}\right)}\right) \tag{133}
\end{equation*}
$$

and so

$$
\begin{equation*}
y=-\frac{\sqrt{\left(z-g_{4} z^{3}\right)^{2}-4\left(t-g_{4} t z^{2}-g_{4} f_{1} z-g_{4} f_{2}\right)}}{2} \tag{134}
\end{equation*}
$$

The special deformation of the spectral curve in this case is

$$
\begin{equation*}
4 y^{2}=\left(z-g_{4} z^{3}\right)^{2}-4\left(t-g_{4} t z^{2}-g_{4} f_{1} z-g_{4} f_{2}\right) \tag{135}
\end{equation*}
$$

We need to find explicit formulas for $f_{1}$ and $f_{2}$. Note

$$
f_{1}=\sum_{n \geq 0} \frac{g_{4}^{n}}{n!}\left\langle p_{1}\left(\frac{p_{4}}{4}\right)^{n}\right\rangle_{0}^{c}(t)=0
$$

where in the second equality we have used the fat selection rule [36, (21)]. By the fat dilaton equation,

$$
f_{2}=\sum_{n \geq 0} \frac{g_{4}^{n}}{n!}\left\langle p_{2}\left(\frac{p_{4}}{4}\right)^{n}\right\rangle_{0}^{c}(t)=t^{2}+\sum_{n \geq 1} 4 n \frac{g_{4}^{n}}{n!}\left\langle\left(\frac{p_{4}}{4}\right)^{n}\right\rangle_{0}^{c}(t)
$$

By using the fat graphs, up to powers of $t$, the correlators $\frac{1}{n!}\left\langle p_{2}\left(\frac{p_{4}}{4}\right)^{n}\right\rangle_{0}^{c}(t)$ are the number of rooted 4-regular planar maps, i.e., they are A000168 in [22]:

$$
\begin{equation*}
\frac{1}{n!}\left\langle p_{2}\left(\frac{p_{4}}{4}\right)^{n}\right\rangle_{0}^{c}(t)=2 \cdot 3^{n} \cdot \frac{(2 n)!}{n!(n+2)!} t^{n+2} \tag{136}
\end{equation*}
$$

The first few terms of $f_{2}$ are:

$$
f_{2}=t^{2}+2 t^{3} g_{4}+9 t^{4} g_{4}^{2}+54 t^{5} g_{4}^{3}+378 t^{6} g_{4}^{4}+2916 t^{7} g_{4}^{5}+24057 t^{8} g_{4}^{6}+\cdots
$$

The generating series

$$
A(z)=\sum_{n \geq 0} 2 \cdot 3^{n} \cdot \frac{(2 n)!}{n!(n+2)!} z^{n}
$$

can be summed up:

$$
A(z)=\frac{(1-12 z)^{3 / 2}-1+18 z}{54 z^{2}}
$$

hence $A$ satisfies the equation:

$$
1-16 z+(18 z-1) A(z)-27 z^{2} A(z)^{2}=0
$$

The discriminant of the right-hand side of (135) is up to a constant

$$
\begin{equation*}
\left(f_{2} g_{4}-t\right) g^{10}\left(27 f_{2}^{2} g_{4}^{2}+16 g_{4} t^{3}-18 f_{2} g_{4} t-t^{2}+f_{2}\right)^{2} \tag{137}
\end{equation*}
$$

so it vanishes. Conversely, one can use this vanishing to compute $f_{2}$ :

$$
\begin{equation*}
f_{2}=\frac{1}{54 g_{4}^{2}}\left(\left(1-12 g_{4} t\right)^{3 / 2}-1+18 g_{4} t\right) \tag{138}
\end{equation*}
$$

For a different approach, see 4.8 .
The first few terms of the resolvent are

$$
\begin{aligned}
\omega & =t x+\frac{f_{2}}{z^{3}}+\frac{f_{2}-t^{2}}{g_{4} z^{5}}+\frac{f_{2}-t^{2}-2 t g_{4} f_{2}}{g_{4}^{2} z^{7}} \\
& +\frac{f_{2}+2 g_{4} t^{3}-t^{2}-4 t g_{4} f_{2}-g_{4}^{2} f_{2}^{2}}{g_{4}^{3} z^{9}} \\
& +\frac{f_{2}+4 g_{4} t^{3}-t^{2}-6 t g_{4} f_{2}-3 g_{4}^{2} f_{2}^{2}+6 g_{4}^{2} t^{2} f_{2}}{g_{4}^{4} z^{11}}+\cdots,
\end{aligned}
$$

it follows that $f_{2 n-1}=0$ for $n \geq 1$, and

$$
\begin{aligned}
f_{4} & =\frac{f_{2}-t^{2}}{g_{4}}=\frac{1}{54 g_{4}^{3}}\left(\left(1-12 g_{4} t\right)^{3 / 2}-1+18 g_{4} t-54 g_{4}^{2} t^{2}\right) \\
& =2 t^{3}+9 t^{4} g_{4}+54 t^{5} g_{4}^{2}+378 t^{6} g_{4}^{3}+2916 t^{7} g_{4}^{4}+24057 t^{8} g_{4}^{5}+\cdots \\
f_{6} & =\frac{f_{2}-t^{2}-2 t g_{4} f_{2}}{g_{4}^{2}} \\
& =\frac{1}{54 g_{4}^{3}}\left(\left(1-2 g_{4} t\right)\left(1-12 g_{4} t\right)^{3 / 2}-1+20 g_{4} t-90 g_{4}^{2} t^{2}\right) \\
& =5 t^{4}+36 g_{4} t^{5}+270 g_{4}^{2} t^{6}+2160 g_{4}^{3} t^{7}+18225 g_{4}^{4} t^{8}+160380 g_{4}^{5} t^{9}+\cdots \\
f_{8} & =\frac{f_{2}+2 g_{4} t^{3}-t^{2}-4 t g_{4} f_{2}-g_{4}^{2} f_{2}^{2}}{g_{4}^{3}} \\
& =\frac{7}{54 g_{4}^{5}}\left(\left(2-9 g_{4} t\right)\left(1-12 g_{4} t\right)^{3 / 2}-2+45 g_{4} t-270 g_{4}^{2} t^{2}+270 g_{4}^{3} t^{3}\right) \\
& =14 t^{5}+140 g_{4} t^{6}+1260 g_{4}^{2} t^{7}+11340 g_{4}^{3} t^{8}+103950 g_{4}^{4} t^{9}+\cdots \\
f_{10} & =\frac{f_{2}+4 g_{4} t^{3}-t^{2}-6 t g_{4} f_{2}-3 g_{4}^{2} f_{2}^{2}+6 g_{4}^{2} t^{2} f_{2}}{g_{4}^{4}} \\
& =42 t^{6}+540 g_{4} t^{7}+5670 g+4^{2} t^{8}+56700 g_{4}^{3} t^{9}+561330 g_{4}^{4} t^{10}+\cdots
\end{aligned}
$$

The coefficients count fat graphs whose vertices are all 4 -valent except for one vertex. These numbers have not yet appeared on [22].
3.10. Fat special deformation along the $g_{6}$-line. Let $g_{n}=0$ for $n \neq 6$, i.e.,

$$
S(x)=-\frac{1}{2} x^{2}+\frac{g_{6}}{6} x^{6}, \quad P(z)=t-g_{6} z^{5}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}+\frac{f_{2}}{z^{3}}+\frac{f_{3}}{z^{4}}+\frac{f_{4}}{z^{5}}\right)
$$

where

$$
f_{j}=\left.j \frac{\partial F_{0}(t)}{\partial g_{j}}\right|_{g_{k}=0, k \neq 6}
$$

From this we get the resolvent:

$$
\begin{equation*}
\omega(z)=\frac{1}{2}\left(z-g_{6} z^{5}-\sqrt{\left(z-g_{6} z^{5}\right)^{2}-4\left(t-g_{6} t z^{4}-g_{6} \sum_{j=1}^{4} f_{j} z^{4-j}\right)}\right) \tag{139}
\end{equation*}
$$

and so the special deformation of the spectral curve in this case is

$$
\begin{equation*}
4 y^{2}=\left(z-g_{6} z^{5}\right)^{2}-4\left(t-g_{6} t z^{4}-g_{6} f_{1} z^{3}-g_{6} f_{2} z^{2}-g_{6} f_{3} z-g_{6} f_{4}\right) \tag{140}
\end{equation*}
$$

Note by the fat selection rule [36, (21)],

$$
\begin{aligned}
& f_{1}=\sum_{n \geq 0} \frac{g_{6}^{n}}{n!}\left\langle p_{1}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t)=0 \\
& f_{3}=\sum_{n \geq 0} \frac{g_{6}^{n}}{n!}\left\langle\frac{p_{3}}{3}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t)=0
\end{aligned}
$$

and by the fat dilaton equation,

$$
\begin{equation*}
f_{2}=\sum_{n \geq 0} \frac{g_{6}^{n}}{n!}\left\langle p_{2}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t)=t^{2}+\sum_{n \geq 1} 6 n \frac{g_{6}^{n}}{n!}\left\langle\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t) \tag{141}
\end{equation*}
$$

The first few terms of $f_{2}$ can be computed by e.g. 35:

$$
f_{2}=t^{2}+5 t^{4} g_{6}+100 t^{6} g_{6}^{2}+\cdots
$$

Similarly, the first few terms of

$$
\begin{equation*}
f_{4}=\sum_{n \geq 0} \frac{g_{6}^{n}}{n!}\left\langle p_{4}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t) \tag{142}
\end{equation*}
$$

can be computed:

$$
f_{4}=2 t^{2}+24 t^{4} g_{6}+\cdots
$$

One can also consider the discriminant in this case, but unfortunately its vanishing gives an equation that involves both $f_{2}$ and $f_{4}$, so one needs extra conditions. In next Section we will find closed formulas for $f_{2}$ and $f_{4}$ in this case by first establishing a result that is stronger than the vanishing of the discriminant.

## 4. What is Really Special about Fat Special Deformations?

The examples in last Section lead to a remarkable property of the plane algebraic curves defined by fat special deformation. Because of the lack of a terminology in the literature we will say the fat special deformation is formally one-cut. This is inspired by the method of one-cut solutions of Hermitian one-matrix models in the literature. Again this is often carried in the setting of large $N$-limit in the literature (see e.g. [16, §2.2]), but here we are working in the case with finite $N$.
4.1. Fat special deformation is formally one-cut. By this we mean $S^{\prime}(z)^{2}-$ $4 P(z)$ has the following form:

$$
\begin{equation*}
S^{\prime}(z)^{2}-4 P(z)=Q(z)^{2}\left(z-a_{-}\right)\left(z-a_{+}\right) \tag{143}
\end{equation*}
$$

where $Q(z)$ is a formal deformation of $1, a_{ \pm}$are formal deformations of $\pm 2 \sqrt{t}$ respectively:

$$
\begin{equation*}
a_{ \pm}= \pm 2 \sqrt{t}+2 b_{ \pm}\left(g_{1}, g_{2}, \ldots\right) \tag{144}
\end{equation*}
$$

I.e., $b_{ \pm}$are formal power series in $g_{1}, g_{2}, \ldots$, with coefficients in $\mathbb{C}\left[t^{1 / 2}\right]$, such that

$$
\begin{equation*}
\left.b_{ \pm}\left(g_{1}, g_{2}, \ldots\right)\right|_{g_{k}=0, k \geq 1}=0 \tag{145}
\end{equation*}
$$

In other words, we may write $\omega$ as

$$
\begin{equation*}
\omega(z)=\frac{1}{2}\left(-S^{\prime}(z)-Q(z) \sqrt{\left(z-a_{-}(t)\right)\left(z-a_{+}(t)\right)}\right) \tag{146}
\end{equation*}
$$

Theorem 4.1. The fat special deformation is formally one cut, i.e., it has the form:

$$
\begin{equation*}
y^{2}=Q(z)^{2}\left(z-a_{-}\right)\left(z-a_{+}\right) \tag{147}
\end{equation*}
$$

To prove this Theorem, we will present an algorithm that enables us to find $a_{ \pm}$ and $Q$. We adapt from a well-known method in the literature often used in the large $N$-limit. Introduce

$$
\begin{equation*}
H(z)=-\frac{S^{\prime}(z)}{\sqrt{\left(z-a_{-}\right)\left(z-a_{+}\right)}} \tag{148}
\end{equation*}
$$

considered as a series expansion for large $z$. It is an element in $\left.\mathbb{C}\left[z, z^{-1}\right]\right]$

$$
\begin{equation*}
H(z)=\sum_{i \in \mathbb{Z}} H_{i} z^{i} \tag{149}
\end{equation*}
$$

Write $H(z)=H_{+}(z)+H_{-}(z)$, where

$$
H_{+}(z)=\sum_{i \geq 0} H_{i} z^{i}, \quad H_{-}(z)=\sum_{i<0} H_{i} z^{-i}
$$

Then one has $Q(z)=H_{+}(z)$ and

$$
\begin{equation*}
\omega(z)=\frac{1}{2} H_{-}(z) \sqrt{\left(z-a_{-}\right)\left(z-a_{+}\right)} \tag{150}
\end{equation*}
$$

In particular, when we take $g_{n}=0$ for $n \geq N$ for some $N$, i.e., $S(x)$ is just a polynomials, then $H_{i}=0$ for $i \gg 0 . H_{+}(z)$ is the polynomial part of $H(z)$ in $z$. Now

$$
\begin{equation*}
\sqrt{\left(z-a_{-}\right)\left(z-a_{+}\right)}=z-\frac{a_{-}+a_{+}}{2}-\frac{\left(a_{+}-a_{-}\right)^{2}}{8 z}+\cdots \tag{151}
\end{equation*}
$$

So one has

$$
\begin{equation*}
\omega \sim \frac{H_{-1}}{2}+\frac{2 H_{-2}-H_{-1} \cdot\left(a_{+}+a_{-}\right)}{4 z}+\cdots \tag{152}
\end{equation*}
$$

and by (85) we must have

$$
\begin{equation*}
H_{-1}=0, \quad H_{-2}=2 t \tag{153}
\end{equation*}
$$

These coefficients can be easily computed. One first gets:

$$
\begin{aligned}
H(z) & =-\frac{S^{\prime}(z)}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}} \\
& =-\frac{S^{\prime}(z)}{z}\left(1-\frac{a_{+}}{z}\right)^{-1 / 2} \cdot\left(1-\frac{a_{-}}{z}\right)^{-1 / 2} \\
& =-\frac{S^{\prime}(z)}{z} \sum_{i=0}^{\infty}(-1)^{i}\binom{-1 / 2}{i} \frac{a_{+}^{i}}{z^{i}} \cdot \sum_{j=0}^{\infty}(-1)^{j}\binom{-1 / 2}{j} \frac{a_{-}^{j}}{z^{j}} \\
& =\left(z-\sum_{m=1}^{d} g_{m} z^{m-1}\right) \cdot \frac{1}{z} \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{(2 i-1)!!(2 j-1)!!}{2^{n} i!j!} \frac{a_{+}^{i}}{z^{n}},
\end{aligned}
$$

from this we get

$$
\begin{align*}
& H_{-1}=-\sum_{n=0}^{\infty} \tilde{g}_{n+1} c_{n}=0  \tag{154}\\
& H_{-2}=-\sum_{n=1}^{\infty} \tilde{g}_{n} c_{n}=2 t \tag{155}
\end{align*}
$$

where $\tilde{g}_{n}=g_{n}-\delta_{n, 2}$, and $c_{n}$ is defined by:

$$
\begin{equation*}
c_{n}:=\sum_{i+j=n} \frac{(2 i-1)!!(2 j-1)!!}{2^{n} i!j!} a_{+}^{i} a_{-}^{j}=\frac{1}{2^{2 n}} \sum_{i+j=n}\binom{2 i}{i}\binom{2 j}{j} a_{+}^{i} a_{-}^{j} . \tag{156}
\end{equation*}
$$

Change these into expressions in $b_{+}$and $b_{-}$:

$$
\begin{aligned}
c_{n} & =\frac{1}{2^{2 n}} \sum_{i+j=n}\binom{2 i}{i}\binom{2 j}{j}\left(2 \sqrt{t}+2 b_{+}\right)^{i}\left(-2 \sqrt{t}-2 b_{-}\right)^{j} \\
& =\frac{1}{2^{n}} \sum_{i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j} \sum_{k=0}^{i}\binom{i}{k} t^{(i-k) / 2} b_{+}^{k} \sum_{l=0}^{j}\binom{j}{l} t^{(j-l) / 2} b_{-}^{l} \\
& =\frac{1}{2^{n}} \sum_{k, l} b_{+}^{k} b_{-}^{l} \sum_{i \geq k, j \geq l, i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{k}\binom{j}{l} t^{(n-k-l) / 2} .
\end{aligned}
$$

The leading term of $c_{n}$ is

$$
\frac{1}{2^{n}} \sum_{i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{0}\binom{j}{0} t^{n / 2}= \begin{cases}\binom{2 m}{m} t^{m}, & n=2 m \\ 0, & n=2 m-1\end{cases}
$$

This can be proved as follows:

$$
\begin{aligned}
& \sum_{n=0}^{\infty} x^{n} \frac{1}{2^{n}} \sum_{i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}=\sum_{i=0}^{\infty} \frac{1}{2^{i}}\binom{2 i}{i} x^{i} \cdot \sum_{j=0}^{\infty}(-1)^{j} \frac{1}{2^{j}}\binom{2 j}{j} x^{j} \\
= & (1-2 x)^{-1 / 2}(1+2 x)^{-1 / 2}=\left(1-4 x^{2}\right)^{-1 / 2}=\sum_{m=0}^{\infty}\binom{2 m}{m} x^{2 m} .
\end{aligned}
$$

Similarly, the subleading term is

$$
\begin{aligned}
& b_{+} \cdot \frac{1}{2^{n}} \sum_{i \geq 1, j \geq 0, i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{1}\binom{j}{0} t^{(n-1-0) / 2} \\
+ & b_{-} \cdot \frac{1}{2^{n}} \sum_{i \geq 0, j \geq 1, i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{0}\binom{j}{1} t^{(n-0-1) / 2} \\
= & n\binom{n}{[n / 2]}\left(b_{+}+(-1)^{n} b_{-}\right) t^{(n-1) / 2} .
\end{aligned}
$$

We have used the following two identities:

$$
\begin{aligned}
& \frac{1}{2^{n}} \sum_{i \geq 1, j \geq 0, i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{1}\binom{j}{0}=n\binom{n-1}{[(n-1) / 2]}, \\
& \frac{1}{2^{n}} \sum_{i \geq 0, j \geq 1, i+j=n}(-1)^{j}\binom{2 i}{i}\binom{2 j}{j}\binom{i}{0}\binom{j}{1} t^{(n-0-1) / 2}=(-1)^{n} n\binom{n-1}{[(n-1) / 2]} .
\end{aligned}
$$

The can be proved using genrating series as follows. For the first identity, we have

$$
\begin{aligned}
& \sum_{n=0}^{\infty} x^{n} \frac{1}{2^{n}} \sum_{i+j=n}(-1)^{j} i\binom{2 i}{i}\binom{2 j}{j}=\sum_{i=0}^{\infty} \frac{i}{2^{i}}\binom{2 i}{i} x^{i} \cdot \sum_{j=0}^{\infty}(-1)^{j} \frac{1}{2^{j}}\binom{2 j}{j} x^{j} \\
= & x \frac{d}{d x} \sum_{i=0}^{\infty} \frac{1}{2^{i}}\binom{2 i}{i} x^{i} \cdot \sum_{j=0}^{\infty}(-1)^{j} \frac{1}{2^{j}}\binom{2 j}{j} x^{j}=x \frac{d}{d x}(1-2 x)^{-1 / 2} \cdot(1+2 x)^{-1 / 2} \\
= & x(1-2 x)^{-3 / 2} \cdot(1+2 x)^{-1 / 2}=x(1-2 x)^{-1} \cdot\left(1-4 x^{2}\right)^{-1 / 2},
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n\binom{n-1}{[(n-1) / 2]} x^{n}=\sum_{m=1}^{\infty} 2 m\binom{2 m-1}{m-1} x^{2 m}+\sum_{m=0}^{\infty}(2 m+1)\binom{2 m}{m} x^{2 m+1} \\
= & x \frac{d}{d x} \sum_{m=1}^{\infty}\binom{2 m-1}{m-1} x^{2 m}+x \frac{d}{d x} \sum_{m=0}^{\infty}\binom{2 m}{m} x^{2 m+1} \\
= & x \frac{d}{d x}\left(\frac{1-\sqrt{1-4 x^{2}}}{2 \sqrt{1-4 x^{2}}}+\frac{x}{\sqrt{1-4 x^{2}}}\right)=x \frac{d}{d x} \frac{1+2 x}{2\left(1-4 x^{2}\right)^{1 / 2}} \\
= & \frac{x}{(1-2 x)\left(1-4 x^{2}\right)^{3 / 2}}
\end{aligned}
$$

So the first identity is proved. Similarly,

$$
\begin{aligned}
& \sum_{n=0}^{\infty} x^{n} \frac{1}{2^{n}} \sum_{i+j=n}(-1)^{j} j\binom{2 i}{i}\binom{2 j}{j}=\sum_{i=0}^{\infty} \frac{1}{2^{i}}\binom{2 i}{i} x^{i} \cdot \sum_{j=0}^{\infty}(-1)^{j} \frac{j}{2^{j}}\binom{2 j}{j} x^{j} \\
= & \sum_{i=0}^{\infty} \frac{1}{2^{i}}\binom{2 i}{i} x^{i} \cdot x \frac{d}{d x} \sum_{j=0}^{\infty}(-1)^{j} \frac{1}{2^{j}}\binom{2 j}{j} x^{j}=(1-2 x)^{-1 / 2} \cdot x \frac{d}{d x}(1+2 x)^{-1 / 2} \\
= & -(1-2 x)^{-1 / 2} \cdot x(1+2 x)^{-3 / 2}=-x(1+2 x)^{-1} \cdot\left(1-4 x^{2}\right)^{-1 / 2},
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{n=1}^{\infty}(-1)^{n} n\binom{n-1}{[(n-1) / 2]} x^{n}=\sum_{m=1}^{\infty} 2 m\binom{2 m-1}{m-1} x^{2 m}-\sum_{m=0}^{\infty}(2 m+1)\binom{2 m}{m} x^{2 m+1} \\
= & x \frac{d}{d x} \sum_{m=1}^{\infty}\binom{2 m-1}{m-1} x^{2 m}-x \frac{d}{d x} \sum_{m=0}^{\infty}\binom{2 m}{m} x^{2 m+1} \\
= & x \frac{d}{d x}\left(\frac{1-\sqrt{1-4 x^{2}}}{2 \sqrt{1-4 x^{2}}}-\frac{x}{\sqrt{1-4 x^{2}}}\right)=x \frac{d}{d x} \frac{1-2 x}{2\left(1-4 x^{2}\right)^{1 / 2}} \\
= & -\frac{x}{(1+2 x)\left(1-4 x^{2}\right)^{3 / 2}}
\end{aligned}
$$

so the second identity is prove.
For example,

$$
\begin{aligned}
c_{1} & =\frac{1}{2} a_{+}+\frac{1}{2} a_{-}=b_{+}-b_{-}, \\
c_{2} & =\frac{3}{8} a_{+}^{2}+\frac{1}{4} a_{+} a_{-}+\frac{3}{8} a_{-}^{2}=2 t+2 \sqrt{t}\left(b_{+}+b_{-}\right)+\left(\frac{3}{2} b_{+}^{2}-b_{+} b_{-}+\frac{3}{2} b_{-}^{2}\right), \\
c_{3} & =\frac{5}{16} a_{+}^{3}+\frac{3}{16} a_{+}^{2} a_{-}+\frac{3}{16} a_{-} a_{+}^{2}+\frac{5}{16} a_{-}^{3} \\
& =6\left(b_{+}-b_{-}\right) t+6\left(b_{+}^{2}-b_{-}^{2}\right) \sqrt{t}+\left(\frac{5}{2} b_{+}^{3}-\frac{3}{2} b_{+}^{2} b_{-}+\frac{3}{2} b_{+} b_{-}^{2}-\frac{5}{2} a_{-}^{3}\right), \\
c_{4} & =\frac{35}{128} a_{+}^{4}+\frac{5}{32} a_{+}^{3} a_{-}+\frac{9}{64} a_{+}^{2} a_{-}^{2}+\frac{5}{32} a_{+} a_{-}^{3}+\frac{35}{128} a_{-}^{4} \\
& =6 t^{2}+12\left(b_{+}+b_{-}\right) t^{3 / 2}+\left(21 b_{+}^{2}-6 b_{+} b_{-}+21 b_{-}^{2}\right) t \\
& +\left(15 b_{+}^{3}-3 b_{+}^{2} b_{-}-3 b_{+} b_{-}^{2}+15 b_{-}^{3}\right) t^{1 / 2} \\
& +\left(\frac{35}{8} b_{+}^{4}-\frac{5}{2} b_{+}^{3} b_{-}+\frac{9}{4} b_{+}^{2} b_{-}^{2}-\frac{5}{2} b_{+} b_{-}^{3}+\frac{35}{8} b_{-}^{4}\right) .
\end{aligned}
$$

Therefore, after rewriting (154) and (155) in the following form:

$$
\begin{align*}
& H_{-1}=-\sum_{m=0}^{\infty} \tilde{g}_{2 m+1} c_{2 m}-\sum_{m=0}^{\infty} \tilde{g}_{2 m+2} c_{2 m+1}=0,  \tag{157}\\
& H_{-2}=-\sum_{m=1}^{\infty} \tilde{g}_{2 m} c_{2 m}-\sum_{m=0}^{\infty} \tilde{g}_{2 m+1} c_{2 m+1}=2 t, \tag{158}
\end{align*}
$$

we get:

$$
\begin{aligned}
b_{+}-b_{-} & =\sum_{m=0}^{\infty} g_{2 m+1}\left(\binom{2 m}{m} t^{m}+2 m\binom{2 m}{m}\left(b_{+}+b_{-}\right) t^{(2 m-1) / 2}+\cdots\right) \\
& +\sum_{m=0}^{\infty} g_{2 m+2}\left((2 m+1)\binom{2 m+1}{m}\left(b_{+}-b_{-}\right) t^{m}+\cdots\right), \\
2 \sqrt{t}\left(b_{+}+b_{-}\right) & =-\left(\frac{3}{2} b_{+}^{2}-b_{+} b_{-}+\frac{3}{2} b_{-}^{2}\right) \\
& +\sum_{m=1}^{\infty} g_{2 m}\left(\binom{2 m}{m} t^{m}+2 m\binom{2 m}{m}\left(b_{+}+b_{-}\right) t^{(2 m-1) / 2}+\cdots\right) \\
& +\sum_{m=0}^{\infty} g_{2 m+1}\left((2 m+1)\binom{2 m+1}{m}\left(b_{+}-b_{-}\right) t^{m}+\cdots\right) .
\end{aligned}
$$

This system of equations can be recursive solved as follows. Write

$$
\begin{equation*}
b_{ \pm}=\sum_{n=1}^{\infty} b_{ \pm}^{(n)} \tag{159}
\end{equation*}
$$

where each $b_{ \pm}^{(n)}$ is homogeneous of degree $n$ in $g_{1}, g_{2}, \ldots$ For example, the degree 1 part of the system is

$$
\begin{aligned}
& b_{+}^{(1)}-b_{-}^{(1)}=\sum_{m=0}^{\infty} g_{2 m+1}\binom{2 m}{m} t^{m}, \\
& 2 \sqrt{t}\left(b_{+}^{(1)}+b_{-}^{(1)}\right)=\sum_{m=1}^{\infty} g_{2 m}\binom{2 m}{m} t^{m} .
\end{aligned}
$$

And so

$$
\begin{aligned}
& b_{+}^{(1)}=\frac{1}{2} \sum_{m=0}^{\infty} g_{2 m+1}\binom{2 m}{m} t^{m}+\frac{1}{4} \sum_{m=1}^{\infty} g_{2 m}\binom{2 m}{m} t^{m-1 / 2} \\
& b_{-}^{(1)}=-\frac{1}{2} \sum_{m=0}^{\infty} g_{2 m+1}\binom{2 m}{m} t^{m}+\frac{1}{4} \sum_{m=1}^{\infty} g_{2 m}\binom{2 m}{m} t^{m-1 / 2}
\end{aligned}
$$

Next we get the system

$$
\begin{aligned}
b_{+}^{(2)}-b_{-}^{(2)} & =\sum_{m=0}^{\infty} g_{2 m+1} \cdot 2 m\binom{2 m}{m}\left(b_{+}^{(1)}+b_{-}^{(1)}\right) t^{(2 m-1) / 2} \\
& +\sum_{m=0}^{\infty} g_{2 m+2} \cdot(2 m+1)\binom{2 m+1}{m}\left(b_{+}^{(1)}-b_{-}^{(1)}\right) t^{m} \\
2 \sqrt{t}\left(b_{+}^{(2)}+b_{-}^{(2)}\right) & =-\left(\frac{3}{2}\left(b_{+}^{(1)}\right)^{2}-b_{+}^{(1)} b_{-}^{(1)}+\frac{3}{2}\left(b_{-}^{(1)}\right)^{2}\right) \\
& +\sum_{m=1}^{\infty} g_{2 m} \cdot 2 m\binom{2 m}{m}\left(b_{+}^{(1)}+b_{-}^{(1)}\right) t^{(2 m-1) / 2} \\
& +\sum_{m=0}^{\infty} g_{2 m+1} \cdot(2 m+1)\binom{2 m+1}{m}\left(b_{+}^{(1)}-b_{-}^{(1)}\right) t^{m} .
\end{aligned}
$$

From this we can get $b_{ \pm}^{(2)}$, and so on. So the proof of Theorem4.1 is complete.
4.2. The case of even potential function. For even potential function

$$
S(x)=-\frac{1}{2} x^{2}+\sum_{n \geq 1} g_{2 n} x^{2 n}
$$

we have $a_{+}=-a_{-}$,

$$
\begin{aligned}
H(z) & ==-\frac{S^{\prime}(z)}{\sqrt{z^{2}-a_{+}^{2}(N)}}=-\frac{S^{\prime}(z)}{z}\left(1-\frac{a_{+}^{2}}{z^{2}}\right)^{-1 / 2} \\
& =\left(1-\sum_{m \geq 1} g_{2 m} z^{2 m-2}\right) \cdot \sum_{n=0}^{\infty} \frac{1}{2^{2 n}}\binom{2 n}{n} \frac{a_{+}^{2 n}(N)}{z^{2 n}}
\end{aligned}
$$

from this we get

$$
\begin{aligned}
& H_{-1}=0 \\
& H_{-2}=-\sum_{n=1}^{\infty} \tilde{g}_{2 n} \frac{1}{2^{2 n}}\binom{2 n}{n} a_{+}^{2 n}=2 t .
\end{aligned}
$$

From these we get:

$$
\begin{equation*}
\sqrt{t} b_{+}=-\frac{1}{2} b_{+}^{2}+\frac{1}{2} \sum_{n=1}^{\infty} g_{n}\binom{2 n}{n}\left(t^{n}+2 n t^{n-1 / 2} b_{+}+\cdots\right) \tag{160}
\end{equation*}
$$

and hence one can solve $b_{+}$as a formal power series in $g_{k}$ 's.
In the next few Subsections we will work out some concrete examples to illustrate the applications of Theorem 4.1 and the even potential function case.
4.3. Fat special deformation along the $g_{1}$-line again. For the action

$$
S(z)=-\frac{1}{2} z^{2}+g_{1} z
$$

one has

$$
S^{\prime}(z)=-\left(z-g_{1}\right)
$$

From

$$
\begin{aligned}
& \frac{z-g_{1}}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}} \\
= & \left(1-\frac{g_{1}}{z}\right) \cdot\left(1+\frac{a_{+}+a_{-}}{2 z}+\frac{3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}}{8 z^{2}}+\cdots\right) \\
= & 1+\frac{a_{+}+a_{-}-2 g_{1}}{2 z}+\frac{3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}-4 g_{1}\left(a_{+}+a_{-}\right)}{8 z^{2}}+\cdots,
\end{aligned}
$$

one gets two equations

$$
a_{+}+a_{-}-2 g_{1}=0, \quad 3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}-4 g_{1}\left(a_{+}+a_{-}\right)=16
$$

with solutions

$$
\begin{equation*}
a_{+}+a_{-}=2 g_{1}, \quad a_{+} a_{-}=g_{1}^{2}-4 t \tag{161}
\end{equation*}
$$

It follows that the spectral curve in resolvent is

$$
\begin{equation*}
\omega=\frac{z-g_{1}-\sqrt{z^{2}-2 g_{1} z+g_{1}^{2}-4 t}}{2} \tag{162}
\end{equation*}
$$

In this case one has

$$
Q(z)=1
$$

We have recovered (99) in $\$ 3.4$
4.4. Fat special deformation along the $g_{2}$-line again. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{2}}{2} z^{2}
$$

one has

$$
\begin{equation*}
S^{\prime}(z)=-\left(z-g_{2} z\right) \tag{163}
\end{equation*}
$$

From

$$
\begin{aligned}
& \frac{z-g_{2} z}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}} \\
= & \left(1-g_{2}\right) \cdot\left(1+\frac{a_{+}+a_{-}}{2 z}+\frac{3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}}{8 z^{2}}+\cdots\right)
\end{aligned}
$$

we get

$$
a_{+}+a_{-}=0, \quad a_{+} a_{-}=\frac{-4 t}{1-g_{2}}
$$

The spectral curve deforms to

$$
\omega=\frac{\left(1-g_{2}\right) z-\left(1-g_{2}\right) \sqrt{z^{2}-\frac{4 t}{1-g_{2}}}}{2}
$$

with

$$
\begin{equation*}
Q(z)=1-g_{2} . \tag{164}
\end{equation*}
$$

Alternatively, one can proceed from

$$
\frac{z-g_{2} z}{\sqrt{z^{2}-a_{+}^{2}}}=\left(1-g_{2}\right) \cdot\left(1-\frac{a_{+}^{2}}{2 z^{2}}+\cdots\right)
$$

to get $a_{+}^{2}=\frac{4 t}{1-g_{2}}$. We have recovered (110) in $\$ 3.5$
4.5. Fat special deformation on the $\left(g_{1}, g_{2}\right)$-plane again. For the action function

$$
\begin{equation*}
S(z)=-\frac{1}{2} z^{2}+g_{1} z+\frac{g_{2}}{2} z^{2} \tag{165}
\end{equation*}
$$

one has

$$
\begin{aligned}
& -\frac{S^{\prime}(z)}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}}=\frac{-g_{1}+\left(1-g_{2}\right) z}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}} \\
= & \left(\left(1-g_{2}\right)-\frac{g_{1}}{z}\right) \cdot\left(1+\frac{a_{+}+a_{-}}{2 z}+\frac{3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}}{8 z^{2}}+\cdots\right) \\
= & \left(1-g_{2}\right)+\frac{\left(1-g_{2}\right)\left(a_{+}+a_{-}\right)-2 g_{1}}{2 z} \\
+ & \frac{\left(1-g_{2}\right)\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)-4 g_{1}\left(a_{+}+a_{-}\right)}{8 z^{2}}+\cdots,
\end{aligned}
$$

and one gets the following two equations

$$
\left(1-g_{2}\right)\left(a_{+}+a_{-}\right)-2 g_{1}=0, \quad\left(1-g_{2}\right)\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)-4 g_{1}\left(a_{+}+a_{-}\right)=0
$$

with solutions:

$$
a_{+}+a_{-}=\frac{2 g_{1}}{1-g_{2}}, \quad a_{+} a_{-}=\frac{-4 t}{1-g_{2}}
$$

The spectral curve deforms to

$$
\begin{equation*}
\omega=\frac{\left(1-g_{2}\right) z-g_{1}-\left(1-g_{2}\right) \sqrt{z^{2}-\frac{2 g_{1} z}{1-g_{2}}-\frac{4 t}{1-g_{2}}+\frac{g_{1}^{2}}{\left(1-g_{2}\right)^{2}}}}{2} . \tag{166}
\end{equation*}
$$

This recovers (113) in \$3.6.
4.6. Fat special deformation along the $g_{3}$-line again. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{3}}{3} z^{3},
$$

we have

$$
\begin{aligned}
& -\frac{S^{\prime}(z)}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}}=\frac{z-g_{3} z^{2}}{\sqrt{\left(z-a_{+}\right)\left(z-a_{-}\right)}} \\
= & \left(1-g_{3} z\right) \cdot\left(1+\frac{a_{+}+a_{-}}{2 z}+\frac{3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}}{8 z^{2}}\right. \\
+ & \left.\frac{5 a_{+}^{3}+3 a_{+}^{2} a_{-}+3 a_{+} a_{-}^{2}+5 a_{-}^{3}}{16 z^{3}}+\cdots\right) \\
= & -g_{3} z+\left(1-\frac{1}{2} g_{3}\left(a_{+}+a_{-}\right)\right)+\frac{4\left(a_{+}+a_{-}\right)-g_{3}\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)}{8 z} \\
+ & \frac{2\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)-g_{3}\left(5 a_{+}^{3}+3 a_{+}^{2} a_{-}+3 a_{+} a_{-}^{2}+5 a_{-}^{3}\right)}{16 z^{2}}+\cdots,
\end{aligned}
$$

and so the following two equations:

$$
\begin{align*}
& 4\left(a_{+}+a_{-}\right)-g_{3}\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)=0  \tag{167}\\
& 2\left(3 a_{+}^{2}+2 a_{+} a_{-}+3 a_{-}^{2}\right)-g_{3}\left(5 a_{+}^{3}+3 a_{+}^{2} a_{-}+3 a_{+} a_{-}^{2}+5 a_{-}^{3}\right)=32 t \tag{168}
\end{align*}
$$

From the first equation we

$$
\begin{equation*}
g_{3}^{2} a_{+} a_{-}=\frac{3}{4}\left(g_{3}\left(a_{+}+a_{-}\right)\right)^{2}-g_{3}\left(a_{+}+a_{-}\right) \tag{169}
\end{equation*}
$$

and plug this into the second equation we find:

$$
\begin{equation*}
2\left(g_{3}\left(a_{+}+a_{-}\right) / 2\right)^{3}-3\left(g_{3}\left(a_{+}+a_{-}\right) / 2\right)^{2}+\left(g_{3}\left(a_{+}+a_{-}\right) / 2\right)=2 g_{3}^{2} t . \tag{170}
\end{equation*}
$$

By Lagrange inversion,

$$
\begin{equation*}
g_{3}\left(a_{+}+a_{-}\right) / 2=\sum_{n=0}^{\infty} \frac{2^{2 n}}{n+1}\binom{3 n / 2}{n}\left(2 g_{3}^{2} t\right)^{n+1} . \tag{171}
\end{equation*}
$$

The coefficients $1,3,16,105,768,6006,49152, \ldots$ are sequence A085614 of the OnLine Encyclopedia of Integer Sequences [22]. They are the numbers of elementary arches of size $n$. For the explicit expressions in the above equality, see [11. The coefficients of $\left(a_{+}+a_{-}\right)^{2}$ also have combinatorial meanings,

$$
\begin{aligned}
& \left(x+3 x^{2}+16 x^{3}+105 x^{4}+768 x^{5}+6006 x^{6}+49152 x^{7}+\ldots\right)^{2} \\
= & x^{2}+6 x^{3}+41 x^{4}+306 x^{5}+2422 x^{6}+19980 x^{7}+169941 x^{8}+1479786 x^{9}+\cdots
\end{aligned}
$$

the coefficients $1,6,41,306,2422,19980,169941,1479786, \cdots$ are the integer sequence A143023. By (167),

$$
\begin{aligned}
& g_{3}^{2} a_{+} a_{-}=\frac{3}{4}\left(g_{3}\left(a_{+}+a_{-}\right)\right)^{2}-g_{3}\left(a_{+}+a_{-}\right) \\
= & \frac{3}{4}\left(\sum_{n=0}^{\infty} \frac{2^{3 n+2}}{n+1}\binom{3 n / 2}{n}\left(g_{3}^{2} t\right)^{n+1}\right)^{2}-\sum_{n=0}^{\infty} \frac{2^{3 n+2}}{n+1}\binom{3 n / 2}{n}\left(g_{3}^{2} t\right)^{n+1},
\end{aligned}
$$

therefore, one gets

$$
\begin{align*}
g_{3}^{2} a_{+} a_{-} & =-4 g_{3}^{2} t-\sum_{n=1}^{\infty}\left[\frac{2^{3 n+2}}{n+1}\binom{3 n / 2}{n}\right. \\
& \left.-\frac{3}{4} \sum_{i+j=n-1} \frac{2^{3 n+1}}{(i+1)(j+1)}\binom{3 i / 2}{i}\binom{3 j / 2}{j}\right]\left(g_{3}^{2} t\right)^{n+1} . \tag{172}
\end{align*}
$$

The following are the first few terms of $a_{+}+a_{-}$and $a_{+} a_{-}$respectively.

$$
\begin{aligned}
a_{+}+a_{-} & =4 g_{3} t+24 g_{3}^{3} t^{2}+256 g_{3}^{5} t^{3}+3360 g_{3}^{7} t^{4}+49152 g_{3}^{9} t^{5} \\
& +768768 g_{3}^{11} t^{6}+\cdots, \\
a_{+} a_{-} & =-4 t-12 g_{3}^{2} t^{2}-112 g_{3}^{4} t^{3}-1392 g_{3}^{5} t^{4}-19776 g_{3}^{6} t^{5} \\
& -303744 g_{3}^{10} t^{6}-\cdots .
\end{aligned}
$$

From these and $a_{+}-a_{-}=\left(\left(a_{+}+a_{-}\right)^{2}-4 a_{+} a_{-}\right)^{1 / 2}$, one can compute $a_{+}-a_{-}$and get the first few terms of $a_{+}$and $a_{-}$:

$$
\begin{aligned}
a_{+} & =2 t^{1 / 2}+2 g_{3} t+4 g_{3}^{2} t^{3 / 2}+12 g_{3}^{3} t^{2}+36 g_{3}^{4} t^{5 / 2}+128 g_{3}^{5} t^{3} \\
& +440 g_{3}^{6} t^{7 / 2}+1680 g_{3}^{7} t^{4}+6188 g_{3}^{8} t^{9 / 2}+24576 g_{3}^{9} t^{5}+94392 g_{3}^{10} t^{11 /}+\cdots \\
a_{-} & =-2 t^{1 / 2}+2 g_{3} t-4 g_{3}^{2} t^{3 / 2}+12 g_{3}^{3} t^{2}-36 g_{3}^{4} t^{5 / 2}+128 g_{3}^{5} t^{3} \\
& -440 g_{3}^{6} t^{7 / 2}+1680 g_{3}^{7} t^{4}-6188 g_{3}^{8} t^{9 / 2}+24576 g_{3}^{9} t^{5}-94392 g_{3}^{10} t^{11 /}+\cdots
\end{aligned}
$$

The fat spectral curve is deformed to

$$
\begin{equation*}
\omega=\frac{1}{2}\left(\left(z-g_{3} z^{2}\right)-\left(1-g_{3} z-\frac{1}{2} g_{3}\left(a_{+}+a_{-}\right)\right) \sqrt{z^{2}-\left(a_{+}+a_{-}\right) z+a_{+} a_{-}}\right), \tag{173}
\end{equation*}
$$

where closed formula for $a_{+}+a_{-}$and $a_{+} a_{-}$are given by (171) and (172) respectively.
More explicitly,

$$
\begin{gathered}
\omega=\frac{1}{2}\left(\left(z-g_{3} z^{2}\right)-\left(1-g_{3} z-\frac{1}{2} g_{3}\left(4 g_{3} t+24 g_{3}^{3} t^{2}+256 g_{3}^{5} t^{3}+3360 g_{3}^{7} t^{4}+\cdots\right)\right)\right. \\
\left.\cdot \sqrt{z^{2}-\left(4 g_{3} t+24 g_{3}^{3} t^{2}+256 g_{3}^{5} t^{3}+3360 g_{3}^{7} t^{4}+\cdots\right) z-4 t-12 g_{3}^{2} t^{2}-112 g_{3}^{4} t^{3}-\cdots}\right)
\end{gathered}
$$

Alternatively, we can solve for $b=a_{+}+a_{-}$and $c=a_{+} a_{-}$in the following way. Since we know the resolvent has the following two forms:

$$
\begin{aligned}
\omega & =\frac{1}{2}\left(\left(z-g_{3} z^{2}\right)-\left(1-g_{3} z-\frac{1}{2} g_{3} b\right) \sqrt{z^{2}-b z+c}\right) \\
& =\frac{1}{2}\left(z-g_{3} z^{2}-\sqrt{\left(z-g_{3} z^{2}\right)^{2}-4\left(t-g_{3} t z-g_{3} f_{1}\right)}\right)
\end{aligned}
$$

one can rewrite the first line in the following form:

$$
\begin{aligned}
\omega= & \frac{1}{2}\left[\left(z-g_{3} z^{2}\right)-\left(\left(z-g_{3} z^{2}\right)^{2}+\left(g_{3} b+g_{3}^{2} c-\frac{3}{4} g_{3}^{2} b^{2}\right) z^{2}\right.\right. \\
& \left.\left.+\left(-b-2 g_{3} c+g_{3} b^{2}+g_{3}^{2} b c-\frac{1}{4} g_{3}^{2} b^{3}\right) z+\left(c-g_{3} b c+\frac{1}{4} g_{3}^{2} b^{2} c\right)\right)^{1 / 2}\right]
\end{aligned}
$$

and compare it with the second line to get the following system of equations:

$$
\begin{align*}
& g_{3} b+g_{3}^{2} c-\frac{3}{4} g_{3}^{2} b^{2}=0 \\
& -b-2 g_{3} c+g_{3} b^{2}+g_{3}^{2} b c-\frac{1}{4} g_{3}^{2} b^{3}=4 g_{3} t  \tag{174}\\
& c-g_{3} b c+\frac{1}{4} g_{3}^{2} b^{2} c=4 g_{3} f_{1}-4 t
\end{align*}
$$

From the first equation we get

$$
c=\frac{3}{4} b^{2}-\frac{1}{g_{3}} b .
$$

This is just (169). Plug this into the second equation:

$$
2\left(g_{3} b / 2\right)^{3}-3\left(g_{3} b / 2\right)^{2}+\left(g_{3} b / 2\right)=2 g_{3}^{2} t
$$

this is just (170), and so $b$ and $c$ can be solved as above. But now we have the third equation in the above system, from which we get:

$$
\begin{equation*}
f_{1}=\frac{1}{4 g_{3}}\left(4 t+c\left(1-\frac{1}{2} g_{3} b\right)^{2}\right) \tag{175}
\end{equation*}
$$

Now we use this to recover (124). Recall that $b=a_{+}+a_{-}$and $c=a_{+} a_{-}$, so by (171) and (172) we have

$$
\begin{aligned}
f_{1} & =\frac{t}{g_{3}}+\frac{7}{16} \frac{b^{2}}{g_{3}}-\frac{1}{4} \frac{b}{g_{3}^{2}}-\frac{1}{4} b^{3}+\frac{3}{64} g_{3} b^{4} \\
& =\frac{t}{g_{3}}+\frac{11}{32} \frac{b^{2}}{g_{3}}-\frac{1}{4} \frac{b}{g_{3}^{2}}-\frac{7}{64} b^{3}+\frac{3}{8} b t \\
& =\frac{1}{8} \frac{t}{g}-\frac{1}{32} \frac{b}{g_{3}^{2}}+\frac{3}{8} b t+\frac{1}{64} \frac{b^{2}}{g_{3}} \\
& =\frac{1}{8} \frac{t}{g}-\frac{1}{32 g_{3}^{3}} \sum_{n=0}^{\infty} \frac{2^{2 n+1}}{n+1}\binom{3 n / 2}{n}\left(2 g_{3}^{2} t\right)^{n+1}+\frac{3 t}{8 g_{3}} \sum_{n=0}^{\infty} \frac{2^{2 n+1}}{n+1}\binom{3 n / 2}{n}\left(2 g_{3}^{2} t\right)^{n+1} \\
& +\frac{1}{64 g_{3}^{3}}\left(\sum_{n=0}^{\infty} \frac{2^{2 n+1}}{n+1}\binom{3 n / 2}{n}\left(2 g_{3}^{2} t\right)^{n+1}\right)^{2} \\
& =\frac{3}{8} \sum_{n=0}^{\infty} \frac{2^{3 n+2}}{n+1}\binom{3 n / 2}{n} t^{n+2} g_{3}^{2 n+1}-\frac{1}{32} \sum_{n=0}^{\infty} \frac{2^{3 n+5}}{n+2}\binom{(3 n+3) / 2}{n+1} t^{n+2} g_{3}^{2 n+1} \\
& +\frac{1}{64} \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{2^{3 i+2}}{i+1}\binom{3 i / 2}{i} \frac{2^{3 j+2}}{j+1}\binom{3 j / 2}{j} \cdot t^{n+2} g_{3}^{2 n+1} .
\end{aligned}
$$

By comparing with the formula for $f_{1}$ in 3.7 we get a combinatorial identity:

$$
\begin{aligned}
& \frac{3}{8} \sum_{n=0}^{\infty} \frac{2^{3 n+2}}{n+1}\binom{3 n / 2}{n} x^{n}-\frac{1}{32} \sum_{n=0}^{\infty} \frac{2^{3 n+5}}{n+2}\binom{(3 n+3) / 2}{n+1} x^{n} \\
+ & \frac{1}{64} \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{2^{3 i+2}}{i+1}\binom{3 i / 2}{i} \frac{2^{3 j+2}}{j+1}\binom{3 j / 2}{j} \cdot x^{n}=\sum_{n \geq 0} 2^{2 n+1} \cdot \frac{(3 n)!!}{(n+2)!n!!} x^{n} .
\end{aligned}
$$

It does not seem to be easy to directly prove this.
4.7. Fat special deformation on the $\left(g_{1}, g_{2}, g_{3}\right)$-space again. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+g_{1} z+\frac{g_{2}}{2} z^{2}+\frac{g_{3}}{3} z^{3}
$$

we have

$$
\begin{aligned}
& -\frac{S^{\prime}(z)}{\sqrt{z^{2}-b z+c}}=\frac{z-g_{1}-g_{2} z-g_{3} z^{2}}{\sqrt{z^{2}-b z+c}} \\
= & \left(-g_{3} z+\left(1-g_{2}\right)-\frac{g_{1}}{z}\right) \cdot\left(1+\frac{b}{2 z}+\cdots\right) \\
= & -g_{3} z+\left(1-g_{2}-\frac{1}{2} g_{3} b\right)+\cdots,
\end{aligned}
$$

and so we get $Q(z)=1-g_{2}-g_{3} z-\frac{1}{2} g_{3} b$ and the fat spectral curve is deformed to

$$
\begin{equation*}
\omega=\frac{1}{2}\left(\left(z-g_{3} z^{2}\right)-\left(1-g_{2}-g_{3} z-\frac{1}{2} g_{3} b\right) \sqrt{z^{2}-b z+c}\right) . \tag{176}
\end{equation*}
$$

By comparing it with (130) we get the following system of equations:

$$
\begin{aligned}
& g_{3} b+g_{3}^{2} c-\frac{3}{4} g_{3}^{2} b^{2}=2 g_{1} g_{3} \\
& -\left(1-g_{2}\right)^{2} b-2\left(1-g_{2}\right) g_{3} c+\left(1-g_{2}\right) g_{3} b^{2}+g_{3}^{2} b c-\frac{1}{4} g_{3}^{2} b^{3}+2 g_{1}\left(1-g_{2}\right)=4 g_{3} t \\
& \left(1-g_{2}\right)^{2} c-\left(1-g_{2}\right) g_{3} b c+\frac{1}{4} g_{3}^{2} b^{2} c+g_{1}^{2}=4 g_{3} f_{1}-4 t
\end{aligned}
$$

This system deforms the system (174), one can follow the same procedures to solve for $b, c$ and $f_{1}$.
4.8. Fat special deformation along the $g_{4}$-line again. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{4}}{4} z^{3},
$$

one has:

$$
\begin{aligned}
-\frac{S^{\prime}(z)}{\sqrt{z^{2}-a^{2}}} & =\frac{z-g_{4} z^{3}}{\sqrt{z^{2}-a^{2}}} \\
& =\left(\left(1-\frac{g_{4} a^{2}}{2}\right)-g_{4} z^{2}\right)+\left(\frac{a^{2}}{2}-\frac{3}{8} g_{4} a^{4}\right) \frac{1}{z^{2}}+\cdots,
\end{aligned}
$$

hence one has

$$
\frac{a^{2}}{2}-\frac{3}{8} g_{4} a^{4}=2 t
$$

From this one gets

$$
a^{2}=\frac{2\left(1-\sqrt{1-12 g_{4} t}\right)}{3 g_{4}} .
$$

and

$$
Q(z)=\frac{2}{3}+\frac{1}{3} \sqrt{1-12 g_{4} t}-g_{4} z^{2}
$$

and so the resolvent is given by:
(177) $\omega=\frac{1}{2}\left(z-g_{4} z^{3}-\left(\frac{2}{3}+\frac{1}{3} \sqrt{1-12 g_{4} t}-g_{4} z^{2}\right) \sqrt{z^{2}-\frac{2\left(1-\sqrt{1-12 g_{4} t}\right)}{3 g_{4}}}\right)$.

By comparing this with (133) one gets:

$$
f_{2}=\frac{1}{54 g_{4}^{2}}\left(\left(1-12 g_{4} t\right)^{3 / 2}-1+18 g_{4} t\right)=\sum_{n=0}^{\infty} 2 \cdot 3^{n} \frac{(2 n)!}{n!(n+2)!} g_{4}^{n} t^{n+2}
$$

the coefficients are the sequence A000168 in [22]. This recovers the formula for $f_{2}$ in 3.9
4.9. Fat special deformation along the $g_{6}$-line again. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{6}}{6} z^{3},
$$

one has:

$$
\begin{aligned}
-\frac{S^{\prime}(z)}{\sqrt{z^{2}-a^{2}}} & =\frac{z-g_{6} z^{5}}{\sqrt{z^{2}-a^{2}}} \\
& =\left(\left(1-\frac{3}{8} g_{6} a^{4}\right)-\frac{g_{6} a^{2}}{2} z^{2}-g_{6} z^{4}\right)+\left(\frac{a^{2}}{2}-\frac{5}{16} g_{6} a^{6}\right) \frac{1}{z^{2}}+\cdots
\end{aligned}
$$

hence one has

$$
\begin{equation*}
\frac{a^{2}}{2}-\frac{5}{16} g_{6} a^{6}=2 t \tag{178}
\end{equation*}
$$

and

$$
Q(z)=\left(1-\frac{3}{8} g_{6} a^{4}\right)-\frac{g_{6} a^{2}}{2} z^{2}-g_{6} z^{4}
$$

and so the resolvent is given by:

$$
\begin{aligned}
\omega= & \frac{1}{2}\left(z-g_{6} z^{5}-\left(\left(1-\frac{3}{8} g_{6} a^{4}\right)-\frac{g_{6} a^{2}}{2} z^{2}-g_{6} z^{4}\right) \sqrt{z^{2}-a^{2}}\right) \\
= & \frac{1}{2}\left(z-g_{6} z^{5}-\left(\left(z-g_{6} z^{5}\right)^{2}+\left(-\frac{5}{8} a^{6} g_{6}^{2}+g_{6} a^{2}\right) z^{4}\right.\right. \\
& \left.\left.+\left(-\frac{15}{64} g_{6}^{2} a^{8}+\frac{1}{4} g_{6} a^{4}\right) z^{2}+\left(-\frac{9}{64} a^{10} g_{6}^{2}+\frac{3}{4} g_{6} a^{6}-a^{2}\right)\right)^{1 / 2}\right)
\end{aligned}
$$

By comparing this with (139) one gets:

$$
\begin{equation*}
\omega(z)=\frac{1}{2}\left(z-g_{6} z^{5}-\sqrt{\left(z-g_{6} z^{5}\right)^{2}-4\left(t-g_{6} t z^{4}-g_{6} f_{2} z^{2}-g_{6} f_{4}\right)}\right) \tag{179}
\end{equation*}
$$

one gets a system of three equations:

$$
\begin{aligned}
& -\frac{5}{8} a^{6} g_{6}^{2}+g_{6} a^{2}=4 g_{6} t \\
& -\frac{15}{64} g_{6}^{2} a^{8}+\frac{1}{4} g_{6} a^{4}=4 g_{6} f_{2}, \\
& -\frac{9}{64} a^{10} g_{6}^{2}+\frac{3}{4} g_{6} a^{6}-a^{2}=4 g_{6} f_{4}-4 t
\end{aligned}
$$

The first equation is just (178), from which we get by Lagrange inversion:

$$
\begin{equation*}
a^{2}=\sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1} \tag{180}
\end{equation*}
$$

the coefficients $\frac{1}{2 n+1}\binom{3 n}{n}$ are the sequence A001764 in [22], they have various combinatorial origins, e.g., they are the numbers of complete ternary trees with $n$ internal nodes, or $3 n$ edges. From the second equation in the system we get

$$
\begin{aligned}
f_{2} & =-\frac{15}{256} g_{6} a^{8}+\frac{1}{16} a^{4} \\
& =\frac{15}{256} g_{6} a^{2} \cdot \frac{8}{5 g_{6}}\left(4 t-a^{2}\right)+\frac{1}{16} a^{4} \\
& =\frac{3}{8} a^{2} t-\frac{1}{32} a^{4}
\end{aligned}
$$

therefore we get a closed formula for $f_{2}$ :

$$
\begin{aligned}
f_{2} & =\frac{3}{8} t \sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1} \\
& -\frac{1}{32}\left[\sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1}\right]^{2} .
\end{aligned}
$$

After using a combinatorial identity

$$
\frac{3}{2} \sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n} x^{n}-\frac{1}{2}\left[\sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n} x^{n}\right]^{2}=\sum_{n \geq 0} \frac{1}{(n+1)(2 n+1)}\binom{3 n}{n} x^{n}
$$

we get:

$$
\begin{equation*}
f_{2}=\sum_{n \geq 0} \frac{2^{n}}{(n+1)(2 n+1)}\binom{3 n}{n}\left(5 g_{6}\right)^{n} t^{2 n+2} \tag{181}
\end{equation*}
$$

The coefficients $\frac{2^{n}}{(n+1)(2 n+1)}\binom{3 n}{n}=1,1,4,24,176,1456,13056, \cdots$ are the sequence A000309 on [22] with one of the combinatorial meaning being the number of rooted planar bridgeless cubic maps with $2 n$ nodes. By comparing with (141),

$$
\begin{equation*}
\frac{1}{n!}\left\langle p_{2}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t)=\frac{6 n}{n!}\left\langle\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t)=\frac{2^{n}}{(n+1)(2 n+1)}\binom{3 n}{n} \cdot 5^{n} t^{2 n+2} \tag{182}
\end{equation*}
$$

Similarly, by the third equation in the system,

$$
\begin{aligned}
& f_{4}=\frac{t}{g_{6}}-\frac{9}{256} a^{10} g_{6}+\frac{3}{16} a^{6}-\frac{a^{2}}{4 g_{6}} \\
= & \frac{t}{g_{6}}+\frac{9}{256} a^{4} g_{6} \cdot \frac{8}{5 g_{6}}\left(4 t-a^{2}\right)-\frac{3}{16} \cdot \frac{8}{5 g_{6}}\left(4 t-a^{2}\right)-\frac{a^{2}}{4 g_{6}} \\
= & -\frac{9}{160} a^{6}-\frac{9}{40} a^{4} t+\frac{1}{20} \frac{a^{2}}{g_{6}}-\frac{1}{5} \frac{t}{g_{6}} \\
= & \frac{9}{160} \cdot \frac{8}{5 g_{6}}\left(4 t-a^{2}\right)-\frac{9}{40} a^{4} t+\frac{1}{20} \frac{a^{2}}{g_{6}}-\frac{1}{5} \frac{t}{g_{6}} \\
= & \frac{9}{40} a^{4} t-\frac{1}{25} \frac{a^{2}}{g_{6}}+\frac{4}{25} \frac{t}{g_{6}} .
\end{aligned}
$$

So we get a closed formula for $f_{4}$ :

$$
\begin{aligned}
f_{4} & =\frac{9}{40} t\left(\sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1}\right)^{2} \\
& -\frac{1}{25 g_{6}} \sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1}+\frac{4}{25} \frac{t}{g_{6}} .
\end{aligned}
$$

The first few terms of $f_{4}$ are:

$$
\begin{aligned}
f_{4} & =2 t^{3}+24 g_{6} t^{5}+600 g_{6}^{2} t^{7}+20000 g_{6}^{3} t^{9} \\
& +780000 g_{6}^{4} t^{11}+33600000 g_{6}^{5} t^{13}+\cdots
\end{aligned}
$$

Recall the definition of $f_{4}$ in this case is given by (142), so we get

$$
\begin{align*}
& \sum_{n \geq 0} \frac{g_{6}^{n}}{n!}\left\langle p_{4}\left(\frac{p_{6}}{6}\right)^{n}\right\rangle_{0}^{c}(t) \\
= & \frac{9}{40} t\left(\sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1}\right)^{2}  \tag{183}\\
- & \frac{1}{25 g_{6}} \sum_{n \geq 0} \frac{1}{2 n+1}\binom{3 n}{n}\left(\frac{5}{8} g_{6}\right)^{n}(4 t)^{2 n+1}+\frac{4}{25} \frac{t}{g_{6}} .
\end{align*}
$$

So we have solved the problem of finding $f_{2}$ and $f_{4}$ in this case from $\$ 3.10$.
4.10. Fat special deformation along the $g_{5}$-line. For the action function

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{5}}{5} z^{5}
$$

we have

$$
\begin{aligned}
& -\frac{S^{\prime}(z)}{\sqrt{z^{2}-b z+c}}=\frac{z-g_{5} z^{4}}{\sqrt{z^{2}-b z+c}} \\
= & \left(1-g_{5} z^{3}\right) \cdot\left(1+\frac{b}{2 z}+\frac{3 b^{2}-4 c}{8 z^{2}}+\frac{5 b^{3}-12 b c}{16 z^{3}}+\frac{35 b^{4}-120 b^{2} c+48 c^{2}}{128 z^{4}}+\cdots\right) \\
= & 1-g_{5} z^{3}-\frac{1}{2} g_{5} b z^{2}-\frac{g_{5}}{8}\left(3 b^{2}-4 c\right) z-g_{5} \frac{5 b^{3}-12 b c}{16}+\cdots,
\end{aligned}
$$

and so the following two equations:

$$
\begin{equation*}
Q(z)=1-g_{5} z^{3}-\frac{1}{2} g_{5} b z^{2}-\frac{g_{5}}{8}\left(3 b^{2}-4 c\right) z-g_{5} \frac{5 b^{3}-12 b c}{16} \tag{184}
\end{equation*}
$$

From the equation

$$
Q(z)^{2}\left(z^{2}-b z+c\right)=S^{\prime}(z)^{2}-P(z)
$$

for

$$
S(z)=-\frac{1}{2} z^{2}+\frac{g_{5}}{5} z^{5}, \quad P(z)=t-g_{5} z^{4}\left(\frac{t}{z}+\frac{f_{1}}{z^{2}}+\frac{f_{2}}{z^{3}}+\frac{f_{3}}{z^{4}}\right)
$$

we get:

$$
\begin{aligned}
& \left(1-g_{5} z^{3}-\frac{1}{2} g_{5} b z^{2}-\frac{g_{5}}{8}\left(3 b^{2}-4 c\right) z-g_{5} \frac{5 b^{3}-12 b c}{16}\right)^{2}\left(z^{2}-b z+c\right) \\
= & \left(z-g_{5} z^{4}\right)^{2}-4\left(t-g_{5} t z^{3}-g_{5} f_{1} z^{2}-g_{5} f_{2} z-g_{5} f_{3}\right) .
\end{aligned}
$$

By comparing the coefficients of $z^{k}$, we get a system of six equations:

$$
\begin{aligned}
& g_{5} b-\frac{3}{4} g_{5}^{2} c^{2}-\frac{35}{64} g_{5}^{2} b^{4}+\frac{15}{8} g_{5}^{2} b^{2} c=0 \\
& -g_{5} c+\frac{1}{4} g_{5} b^{2}-\frac{7}{32} g_{5}^{2} b^{5}-\frac{3}{2} g_{5}^{2} b c^{2}+\frac{5}{4} g_{5}^{2} b^{3} c=4 g_{5} t \\
& \frac{1}{8} g_{5} b^{3}+\frac{1}{4} g_{5}^{2} c^{3}-\frac{35}{256} g_{5}^{2} b^{6}-\frac{1}{2} g_{5} b c-\frac{21}{16} g_{5}^{2} b^{2} c^{2}+\frac{55}{64} g_{5}^{2} b^{4} c=4 g_{5} f_{1}, \\
& -b+\frac{5}{8} g_{5} b^{4}-\frac{25}{256} g_{5}^{2} b^{7}+g_{5} c^{2}-\frac{9}{4} g_{5} b^{2} c+\frac{45}{64} g_{5}^{2} b^{5} c \\
& \quad-\frac{23}{16} g_{5}^{2} b^{3} c^{2}+\frac{3}{4} g_{5}^{2} b c^{3}=4 g_{5} f_{2} \\
& -\frac{5}{8} g_{5} b^{3} c+c+\frac{25}{256} g_{5}^{2} b^{6} c-\frac{15}{32} g_{5}^{2} b^{4} c^{2}+\frac{9}{16} g_{5}^{2} b^{2} c^{3}+\frac{3}{2} g_{5} b c^{2}=4 g_{5} f_{3}-4 t
\end{aligned}
$$

After rewriting the first two equations in the following form:

$$
\begin{aligned}
& b=\frac{3}{4} g_{5} c^{2}+\frac{35}{64} g_{5} b^{4}-\frac{15}{8} g_{5} b^{2} c \\
& c=-4 t+\frac{1}{4} b^{2}-\frac{7}{32} g_{5} b^{5}-\frac{3}{2} g_{5} b c^{2}+\frac{5}{4} g_{5} b^{3} c
\end{aligned}
$$

one can solve for $b$ and $c$ recursively from the initial values:

$$
b=O\left(t^{2}\right), \quad c=-4 t+O\left(t^{2}\right)
$$

we get:

$$
\begin{aligned}
& b=12 g_{5} t^{2}+2592 g_{5}^{3} t^{5}+1143072 g_{5}^{5} t^{8}+638254080 g_{5}^{7} t^{11} \\
& c=-4 t-252 g_{5}^{2} t^{4}-91584 g_{5}^{4} t^{7}-47262528 g_{5}^{6} t^{10}-\cdots
\end{aligned}
$$

plug these into the next equations in the system one can compute $f_{1}, f_{2}, f_{4}$ :

$$
\begin{aligned}
& f_{1}=2 g_{5} t^{3}+216 g_{5}^{3} t^{6}+63504 g_{5}^{5} t^{9}+26593920 g_{5}^{7} t^{12}+\cdots \\
& f_{2}=t^{2}+36 g_{5}^{2} t^{5}+8640 g_{5}^{4} t^{8}+3312576 g_{5}^{6} t^{11}+\cdots \\
& f_{3}=9 g_{5} t^{4}+1512 g_{5}^{3} t^{7}+509328 g_{5}^{5} t^{10}+\cdots
\end{aligned}
$$

4.11. Fat special deformation in renormalized coupling constants. As in the thin case, we can work with the action function written in the renormalized coupling constants:

$$
\begin{aligned}
S(z) & =\sum_{n \geq 1}\left(t_{n-1}-\delta_{n, 2}\right) \frac{x^{n}}{n!} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k+1)!}\left(I_{k}+\delta_{k, 1}\right) I_{0}^{k+1}+\sum_{n \geq 2}\left(I_{n-1}-\delta_{n, 2}\right) \frac{\left(z-I_{0}\right)^{n}}{n!} \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k+1)!}\left(I_{k}+\delta_{k, 1}\right) I_{0}^{k+1}+\sum_{n \geq 2}\left(I_{n-1}-\delta_{n, 2}\right) \frac{w^{n}}{n!}
\end{aligned}
$$

where $w=z-I_{0}$. From now on we work in the $w$-coordinate. By Theorem 4.1,

$$
\begin{equation*}
\left(S^{\prime}\right)^{2}-4 P=R(w)^{2}\left(w^{2}-4 p w-4 q\right) \tag{185}
\end{equation*}
$$

where $w^{2}-4 p w-4 q=z^{2}-b z+c$. To find $R$, we use the expansion

$$
\begin{aligned}
\frac{1}{\sqrt{w^{2}-4 p w-4 q}} & =\frac{1}{w} \sum_{m=0}^{\infty}\binom{2 m}{m}\left(\frac{p}{w}+\frac{q}{w^{2}}\right)^{m} \\
& =\frac{1}{w} \sum_{m=0}^{\infty}\binom{2 m}{m} \sum_{k=0}^{m}\binom{m}{k} \frac{p^{m-k} q^{k}}{w^{m+k}} \\
& =\sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} \frac{(2 n-2 k)!}{k!(n-k)!(n-2 k)!} \frac{p^{n-2 k} q^{k}}{w^{n+1}}
\end{aligned}
$$

and we have

$$
\begin{aligned}
& -\frac{S^{\prime}}{\sqrt{w^{2}-4 p w-4 q}} \\
= & -\sum_{m \geq 1}\left(I_{m}-\delta_{m, 1}\right) \frac{w^{m-1}}{(m-1)!} \cdot \sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} \frac{(2 n-2 k)!}{k!(n-k)!(n-2 k)!} \frac{p^{n-2 k} q^{k}}{w^{n+1}} \\
= & -\sum_{m \geq 1} \sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} \frac{(2 n-2 k)!}{k!(n-k)!(n-2 k)!} p^{n-2 k} q^{k} \frac{I_{m}-\delta_{m, 1}}{(m-1)!} w^{m-n-2},
\end{aligned}
$$

and so

$$
\begin{equation*}
R(w)=\sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} \frac{(2 n-2 k)!}{k!(n-k)!(n-2 k)!} p^{n-2 k} q^{k} \sum_{m \geq n+2} \frac{I_{m}-\delta_{m, 1}}{(m-1)!} w^{m-n-2} \tag{186}
\end{equation*}
$$

On the other hand, by (89),

$$
\begin{aligned}
P(z) & =-\sum_{n \geq 2}\left(g_{n}-\delta_{n, 2}\right) z^{n-1} \sum_{k=1}^{n-1} \frac{f_{k}}{z^{k}} \\
& =-\sum_{n \geq 2}\left(g_{n}-\delta_{n, 2}\right) \sum_{k=1}^{n-1} f_{k}\left(w+I_{0}\right)^{n-1-k} \\
& =-\sum_{n \geq 2}\left(g_{n}-\delta_{n, 2}\right) \sum_{l=0}^{n-2} f_{n-1-l} \sum_{j=0}^{l}\binom{l}{j} I_{0}^{l-j} w^{j} \\
& =-\sum_{j \geq 0} w^{j} \sum_{l \geq j} \sum_{n \geq l+2}\left(g_{n}-\delta_{n, 2}\right) f_{n-1-l}\binom{l}{j},
\end{aligned}
$$

where we set $f_{0}=t$ as a convention. Here $g_{n}$ can be changed into a formal power series in $\left\{I_{m}\right\}$ (31, Proposition 2.4]):

$$
\begin{equation*}
g_{n}=\frac{t_{n-1}}{(n-1)!}=\frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^{k} I_{0}^{k}}{k!} I_{n-1+k} \tag{187}
\end{equation*}
$$

Now we plug all these expression into (185) to get:

$$
\begin{aligned}
& \left(\sum_{m \geq 1}\left(I_{m}-\delta_{m, 1}\right) \frac{w^{m-1}}{(m-1)!}\right)^{2} \\
- & 4 \sum_{j \geq 0} w^{j} \sum_{l \geq j} \sum_{n \geq l+2}\left(\frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^{k} I_{0}^{k}}{k!} I_{n-1+k}-\delta_{n, 2}\right) f_{n-1-l}\binom{l}{j} \\
= & \left(\sum_{n=0}^{\infty} \sum_{k=0}^{[n / 2]} \frac{(2 n-2 k)!}{k!(n-k)!(n-2 k)!} p^{n-2 k} q^{k} \sum_{m \geq n+2} \frac{I_{m}-\delta_{m, 1}}{(m-1)!} w^{m-n-2}\right)^{2} \\
& \cdot\left(w^{2}-4 p w-4 q\right) .
\end{aligned}
$$

By expanding both sides as formal power series in $w$, one gets a sequence of equations to solve for $p, q$ and to compute the $f_{n}$ 's in the same fashion as we have done in the $g_{n}$-coordinates.

To illustrate the idea, let all $I_{n}$ vanish except for $I_{0}$ and $I_{2}$, i.e.,

$$
S=-\frac{1}{2} w^{2}+I_{2} \frac{w^{3}}{3!}
$$

and so we have

$$
\begin{aligned}
& -\frac{S^{\prime}}{\sqrt{w^{2}-4 p w-4 q}}=\frac{w-\frac{I_{2}}{2} w^{2}}{w \sqrt{1-\frac{4 p}{w}-\frac{4 q}{w^{2}}}} \\
= & \left(1-\frac{I_{2}}{2} w\right) \cdot\left(1+\frac{2 p}{w}+\cdots\right) \\
= & -\frac{I_{2}}{2} w+\left(1-I_{2} p\right)+\cdots .
\end{aligned}
$$

From this we get

$$
R(w)=1-I_{2} p-\frac{I_{2}}{2} w
$$

On the other hand, we have

$$
g_{3}=\frac{I_{2}}{2}, \quad g_{2}=-I_{0} I_{2}, \quad g_{1}=\frac{1}{2} I_{0}^{2} I_{2}
$$

and all other $g_{n}=0$, therefore,

$$
\begin{aligned}
P & =\left(1-g_{2}\right) t-g_{3} t z-g_{3} f_{1} \\
& =\left(1+I_{0} I_{2}\right) t-\frac{I_{2}}{2} t\left(w+I_{0}\right)-\frac{I_{2}}{2} f_{1} \\
& =\left(1+\frac{1}{2} I_{0} I_{2}\right) t-\frac{I_{2}}{2} t w-\frac{I_{2}}{2} f_{1} .
\end{aligned}
$$

Plug all the above into (185):

$$
\begin{aligned}
& \left(w-\frac{I_{2}}{2} w^{2}\right)^{2}-\left(1-I_{2} p-\frac{I_{2}}{2} w\right)^{2}\left(w^{2}-4 p w-4 q\right) \\
= & 4\left(\left(1+\frac{1}{2} I_{0} I_{2}\right) t-\frac{I_{2}}{2} t w-\frac{I_{2}}{2} f_{1}\right) .
\end{aligned}
$$

By comparing the coefficients we get a system of equations:

$$
\begin{aligned}
& I_{2}^{2} p^{2}+I_{2}^{2} q-2 I_{2} p=0 \\
& 4 I_{2}^{2} p^{3}+4 I_{2}^{2} p q-8 I_{2} p^{2}-4 I_{2} q+4 p=-2 I_{2} t \\
& 4 I_{2}^{2} p^{2} q-8 I_{2} p q+4 q=4\left(\left(1+\frac{1}{2} I_{0} I_{2}\right) t-\frac{I_{2}}{2} f_{1}\right)
\end{aligned}
$$

The situation is exactly like in $\$ 4.6$

## 5. Concluding Remarks

Duality is an idea that has been widely used in mathematics and string theory. In this work we have just witnessed more examples of duality in the framework of random matrix models. The original goal for understanding the duality between matrix models with $N \rightarrow \infty$ with matrix models with $N=1$ is almost fulfilled by considering matrix models for finite $N$ for all $N$. It is natural to expect that for matrix models of finite $N$, they should have some features of the theory as $N \rightarrow \infty$, and also some features of the theory as $N=1$. It is fortunate that such ideas can be actually worked out. The remaining next step will be to reexamine the procedure of taking $N \rightarrow \infty$ and in particular the double scaling limit from our new perspectives.

In the process of studying the $N=1 / N \rightarrow \infty$-duality for matrix models, we have found a surprising duality between the enumeration problems for ordinary (thin) graphs and fat graphs. More surprisingly, this fat/thin duality comes by with a unification by working with Hermitian matrix models for all finite $N$.

Another remarkable duality is the discrete mathematics of enumerations of fat or thin graphs leading to various generalizations of the Catalan numbers is dual to the algebraic geometry of some special plane curves defined using formal power series with integral coefficients.

It is fair to say that all these dualities are emergent. They are possible because even though we working with only matrices of finite sizes, we still have an infinite degree of freedom by working with all symmetric functions. We would like to mention that the motivation for the author to work on emergent geometry is the wish to answer the following question: Is mirror symmetry an emergent phenomenon? We
believe the answer is yes. In some sense in this paper we are working in a universal setting by working on the space of all possible coupling constants. It should be possible to get special examples that are dual to the local Gromov-Witten theory of toric Calabi-Yau three-folds. When this is done, one will obtain a way to embed local mirror symmetry into the emergent geometry of matrix models. In this way, local mirror symmetry will be unified with many other examples in the even more framework of emergent geometry of KP hierarchy explained in 33, and 34. We expect that mirror symmetry for compact Calabi-Yau 3-folds also fits in a similar picture of emergent geometry, but more complicated integrable hierarchies are needed for this purpose.

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
E-mail address: jzhou@math.tsinghua.edu.cn

