

Eta quotients and Rademacher sums

Kevin Acres and David Broadhurst

Abstract Eta quotients on $\Gamma_0(6)$ yield evaluations of sunrise integrals at 2, 3, 4 and 6 loops. At 2 and 3 loops, they provide modular parametrizations of inhomogeneous differential equations whose solutions are readily obtained by expanding in the nome q . Atkin-Lehner transformations that permute cusps ensure fast convergence for all external momenta. At 4 and 6 loops, on-shell integrals are periods of modular forms of weights 4 and 6 given by Eichler integrals of eta quotients. Weakly holomorphic eta quotients determine quasi-periods. A Rademacher sum formula is given for Fourier coefficients of an eta quotient that is a Hauptmodul for $\Gamma_0(6)$ and its generalization is found for all levels with genus 0, namely for $N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$. There are elliptic obstructions at $N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$, with genus 1. We surmount these, finding explicit formulas for Fourier coefficients of eta quotients in thousands of cases. We show how to handle the levels $N = 22, 23, 26, 28, 29, 31, 37, 50$, with genus 2, and the levels $N = 30, 33, 34, 35, 39, 40, 41, 43, 45, 48, 64$, with genus 3. We also solve examples with genera 4, 5, 6, 7, 8, 13.

1 Introduction

Elliptic obstructions to the evaluation of massive Feynman diagrams were recognized and surmounted more than 50 years ago by Afaf Sabry [30]. They occur in two-loop two-point integrals when three massive particles appear in an intermediate state [7]. The simplest example is the two-loop sunrise diagram with unit masses in two space-time dimensions, whose study was revolutionized in 2013, when Spencer

Kevin Acres, Monash University, Victoria 3800, Australia,
e-mail: research@research-systems.com,
David Broadhurst, Open University, Milton Keynes MK76AA, UK,
e-mail: David.Broadhurst@open.ac.uk

Bloch and Pierre Vanhove [4] showed how to parametrize and solve its second order differential equation using eta quotients on $\Gamma_0(6)$.

Their solution was particularly bold, since they expand in a nome q that is small near the physical threshold where the external energy w is close to 3. Thus they achieve fast convergence near the branchpoint that frustrates other methods. The price to pay is that convergence is slow near any of the other three cusps of $\Gamma_0(6)$, which occur at $w = 0, 1, \infty$. We shall show how to use Atkin-Lehner transformations of eta quotients to expand about those cusps, achieving optimal efficiency.

Bloch, Kerr and Vanhove [5] conquered the corresponding three-loop problem, also using eta quotients on $\Gamma_0(6)$, thanks to the remarkable circumstance, noted more than 40 years ago by Geoffrey Joyce [20], that a transformation of variables relates solutions of the relevant homogeneous third-order differential equation to products of solutions of the second-order equation at two loops. Joyce's observation was made in the context of the physics of condensed matter. The relevance of his work on the diamond lattice to Feynman integrals was decoded in [2]. We shall use an Atkin-Lehner transformation to achieve optimal efficiency at three loops.

The role of $\Gamma_0(6)$ does not end at three loops. It is of the essence for the on-shell problems at 4 and 6 loops, where the relevant Bessel moments turn out to be Eichler integrals of eta quotients that are cusp forms of level 6 with modular weights 4 and 6, respectively. We shall review key results, which were until recently only conjectures [8, 9, 11, 12], tested to many thousands of digits. For an account of how they were proved [34–37], see the lucid review by Yajun Zhou [38].

It is notable that this connection between number theory and Feynman integrals persists in the real world of four-dimensional space-time. The four-loop radiative corrections to the magnetic moment of the electron in quantum electrodynamics, evaluated with breath-taking skill by Stefano Laporta [23], contain a pair of Bessel moments [37] that are Eichler integrals. We conclude Section 2 with results that indicate that one of these is a quasi-period, in the sense of Francis Brown [13]. Moreover we conjecturally identify quasi-periods at 6 loops.

Section 3 concerns a searching question raised by Johannes Blümlein at a recent conference held at the Hausdorff Centre for Mathematics, in Bonn. Is there a closed formula for the Fourier coefficients of the Hauptmodul of $\Gamma_0(6)$, of the type that Petersson [26] and Rademacher [22, 27, 28] found for Klein's j -invariant? We conjecturally answer in the affirmative, by giving a formula that serves this purpose for all levels with genus 0. Moreover we are able to extend its use to higher genera.

2 Eta quotients in quantum field theory

Broadhurst, Fleischer and Tarasov [10] gave the differential equation for the two-loop unit-mass sunrise integral in an arbitrary number D of space-time dimensions. At $D = 2$, this integral is a Bessel moment [2]

$$I(w^2) = 4 \int_0^\infty I_0(wx) K_0^3(x) x dx, \quad (1)$$

where w is the external energy, which enters the Bessel function $I_0(wx)$ via Fourier transformation. The Bessel function $K_0(x)$ is cubed, since three particles of unit mass connect the two vertices. Bloch and Vanhove [4] found a very neat modular parametrization of the differential equation at $D = 2$, which we here write as

$$-\left(q \frac{d}{dq}\right)^2 \frac{I(w^2)}{6f} = g, \quad w = \frac{3\eta_2^2 \eta_3^4}{\eta_1^4 \eta_6^2}, \quad f = \frac{\eta_1^6 \eta_6}{\eta_2^3 \eta_3^2}, \quad (2)$$

$$g = \frac{\eta_2^5 \eta_3^4 \eta_6}{\eta_1^4} = \frac{\eta_3^9}{\eta_1^3} + \frac{\eta_6^9}{\eta_2^3} = \sum_{n>0} \frac{n^2(q^n - q^{5n})}{1 - q^{6n}}, \quad \eta_n = q^{n/24} \prod_{k>0} (1 - q^{nk}), \quad (3)$$

with eta quotients determining the energy w , the integrating factor f , which is an elliptic integral determining the discontinuity across the cut for $w > 3$, and the inhomogeneous term g . Two easy integrations of the Lambert series for g then yield

$$\frac{I(w^2)}{f} = \frac{\pi \log(-1/q)}{\sqrt{3}} - 3 \sum_{n>0} \frac{\chi_6(n)}{n^2} \frac{1 + q^n}{1 - q^n} \quad (4)$$

with $\chi_6(n) = \pm 1$ for $n = \pm 1 \pmod{6}$ and $\chi_6(n) = 0$, otherwise. This solution is determined by the discontinuity across the cut and the finiteness of $I(1) = \pi^2/4$ [2]. In summary: after dividing $I(w^2)$ by the modular form f , with weight 1 and level 6, we obtain solution (4) by two integrations of the weight 3 modular form g with respect to z , where $q = \exp(2\pi iz)$. Such integrals of modular forms are referred to as Eichler integrals.

We remark that modular parametrizations of differential equations were used in [3], to elucidate proofs of rationality of zeta values, and in [21], for problems in statistical physics.

2.1 Atkin-Lehner transformations of eta quotients

Now set $q = \exp(2\pi iz)$ with $\Im z > 0$ and consider the transformations [15]

$$z \mapsto z_2 = \frac{2z-1}{6z-2}, \quad z \mapsto z_3 = \frac{3z-2}{6z-3}, \quad z \mapsto z_6 = \frac{-1}{6z}, \quad (5)$$

which permute the cusps at $z = 0, \frac{1}{2}, \frac{1}{3}, \infty$. Then, with $q_k = \exp(2\pi iz_k)$,

$$-\left(q_k \frac{d}{dq_k}\right)^2 \frac{I(w^2)}{6f_k(z_k)} = g_k(z_k), \quad (6)$$

$$f_2(z) = \frac{\eta_2^6 \eta_3}{\eta_1^3 \eta_6^2}, \quad f_3(z) = \frac{\eta_2 \eta_3^6}{\eta_1^2 \eta_6^3}, \quad f_6(z) = \frac{\eta_1 \eta_6^6}{\eta_2^2 \eta_3^3}, \quad (7)$$

$$g_2(z) = \frac{\eta_1^5 \eta_3 \eta_6^4}{\eta_2^4}, \quad g_3(z) = \frac{\eta_1^4 \eta_2 \eta_6^5}{\eta_3^4}, \quad g_6(z) = \frac{\eta_1 \eta_2^4 \eta_3^5}{\eta_6^4}. \quad (8)$$

From the alternative differential equations (6), we obtain alternative expansions:

$$\frac{I(w^2)}{f_2(z_2)} = \sum_{n=1}^{\infty} \frac{3\chi_3(n)}{n^2} \frac{(1-q_2^n)^2}{1+q_2^{2n}} = I(0) - \sum_{n=1}^{\infty} \frac{6\chi_3(n)}{n^2} \frac{q_2^n}{1+q_2^{2n}}, \quad (9)$$

$$\frac{I(w^2)}{f_3(z_3)} = \sum_{n=1}^{\infty} \frac{2\chi_2(n)}{n^2} \frac{(1-q_3^n)^3}{1-q_3^{3n}} = I(1) - \sum_{n=1}^{\infty} \frac{6\chi_2(n)}{n^2} \frac{q_3^n}{1+q_3^n+q_3^{2n}}, \quad (10)$$

$$\frac{I(w^2)}{f_6(z_6)} = -3\log^2(-q_6) + \sum_{n=1}^{\infty} \frac{6}{n^2} \frac{q_6^n}{1-q_6^n+q_6^{2n}}, \quad (11)$$

with $\chi_2(n) = 0, 1$, for $n = 0, 1 \pmod{2}$, and $\chi_3(n) = -1, 0, 1$, for $n = -1, 0, 1 \pmod{3}$.

Then for any real value of w^2 there is an optimal choice of nome in which to expand, which may be determined as follows. Let

$$w_1^2 = w^2, \quad w_2^2 = \frac{w^2 - 9}{w^2 - 1}, \quad w_3^2 = \frac{9}{w^2}, \quad w_6^2 = \frac{9}{w_2^2}. \quad (12)$$

For $w^2 \in [-3, 9 - 6\sqrt{2}]$ set $k = 2$, else for $w^2 \in [9 - 6\sqrt{2}, 3]$ set $k = 3$, else for $w^2 \in [3, 9 + 6\sqrt{2}]$ set $k = 1$, else set $k = 6$. Then compute $w_k \in [\sqrt{3}, \sqrt{3} + \sqrt{6}]$ and obtain the optimal nome $q_k = Q(w_k)$ from

$$Q(x) = \exp\left(\frac{-\pi \operatorname{agm}(1, \sqrt{r})}{\operatorname{agm}(1, \sqrt{1-r})}\right), \quad r = \frac{16x}{(x+3)(x-1)^3}, \quad (13)$$

by the lightning-fast process of the arithmetic-geometric mean. This results in a small real nome $q_k \in [-\exp(-\pi/\sqrt{3}), \exp(-\pi\sqrt{2/3})]$ and hence $|q_k| < 0.16304$. If $k = 2$, use (9); if $k = 3$, use (10); if $k = 6$, use (11); if $k = 1$ use $q = q_1$ in (4) and extract a Clausen value from

$$\sum_{n>0} \frac{\chi_6(n)}{n^2} \frac{1+q^n}{1-q^n} = C_2 + \sum_{n>0} \frac{\chi_6(n)}{n^2} \frac{2q^n}{1-q^n}, \quad C_2 = \frac{5\operatorname{Cl}_2(\pi/3)}{\sqrt{27}} = \frac{5I(0)}{12}. \quad (14)$$

The authors of [6] expand in $q_M = -q_2$, thereby encountering η_4 and η_{12} in

$$f_M(z) = f_2\left(z + \frac{1}{2}\right) = \frac{\eta_1^3 \eta_4^3 \eta_6}{\eta_2^3 \eta_3 \eta_{12}}, \quad g_M(z) = g_2\left(z + \frac{1}{2}\right) = -\frac{\eta_2^{11} \eta_6^7}{\eta_1^5 \eta_3 \eta_4^5 \eta_{12}}. \quad (15)$$

Since they expand about the cusp at $w = 0$, they inevitably face issues of slow convergence near the cusps at $w = 1, 3, \infty$. Moreover they had to address delicate questions of analytic continuation at the on-shell point $w = 1$. Our procedure of invariably expanding about the *nearest* cusp avoids all such problems.

Such use of Atkin-Lehner transformations to achieve efficient expansions in small nomes is well known to mathematicians who compute with modular forms [16]. We recommend exploitation of the transformations (5) to physicists who encounter problems that involve the congruence subgroup $\Gamma_0(6)$. For example, the authors of [1] encountered, at 3 loops, a second-order equation with complicated coefficients

and powers of $\log(x)$ in the inhomogeneous term. Their homogeneous equation has a hypergeometric solution

$$H(x) = \frac{(x^2 - 1)^2}{9(x^2 + 3)} \sum_{n=0}^{\infty} \frac{(4/3)_n (5/3)_n}{n!(n+1)!} \left(\frac{x^2(x^2 - 9)^2}{(x^2 + 3)^3} \right)^{n+1} \quad (16)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. We obtained

$$H \left(3 \frac{\eta_1^2 \eta_6^4}{\eta_2^4 \eta_3^2} \right) = \frac{1}{2} \left(\frac{\eta_1^{14} \eta_6^{10}}{\eta_2^{22} \eta_3^2} + \frac{\eta_1^6 \eta_6^4}{\eta_2^{12} \eta_3^2} \left(\frac{\eta_1^4 \eta_6^8}{\eta_2^8 \eta_3^4} + \frac{1}{3} \right) q \frac{d}{dq} \right) \frac{\eta_2^6 \eta_3}{\eta_1^3 \eta_6^2} \quad (17)$$

as a modular parametrization of the homogeneous solution, where the derivative with respect to q results from a complete integral of the second kind. It would be interesting to investigate whether an Atkin-Lehner transformation may be used to avoid a singularity that was encountered at $x = \frac{1}{3}$ at intermediate stages of the work in [1].

For our next advertisement of the virtue of Atkin-Lehner transformation, we turn to the three-loop equal-mass sunrise integral. Bailey, Borwein, Broadhurst and Glasser [2] developed the expansion in t of

$$J(t) = 8 \int_0^{\infty} I_0(\sqrt{t}x) K_0^4(x) x dx = 7\zeta(3) + O(t). \quad (18)$$

A neat and novel q -expansion comes from exploiting a transformation [2, 20] from w^2 , at two loops, to $t = 10 - w^2 - 9/w^2$, at three loops. Then we obtain the modular parametrization

$$t = 10 - w^2 - \frac{9}{w^2} = -64 \left(\frac{\eta_2 \eta_6}{\eta_1 \eta_3} \right)^6, \quad (19)$$

$$\left(q \frac{d}{dq} \right)^3 \frac{J(t)}{(wf/3)^2} = 24h, \quad h = \frac{\eta_2^{16}}{\eta_1^8} - 9 \frac{\eta_6^{16}}{\eta_3^8} = \sum_{n>0} \frac{n^3 (q^n - 8q^{3n} + q^{5n})}{1 - q^{6n}}, \quad (20)$$

$$\frac{J(t)}{(wf/3)^2} = J(0) + 24 \sum_{n>0} \frac{\phi(n)}{n^3} \frac{q^n}{1 - q^n}, \quad (21)$$

with $\phi(n) = 0, 1, 0, -8, 0, 1$, for $n = 0, 1, 2, 3, 4, 5 \pmod 6$. The Pari-GP procedure

```
z(t) = { local (x=2/(sqrt(4-t)+sqrt(16-t)),
a=sqrt((1-x)^3*(1+3*x)));
I/2*agm(a, 4*x*sqrt(x))/agm(a, sqrt((1+x)^3*(1-3*x))) ; }
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returns the correct value of z for the nome $q = \exp(2\pi iz)$, for all real t . Expansion (21) is highly efficient for $t \in [-8, 8]$, where $|q| \leq \exp(-\sqrt{2}\pi/3) < 0.22742$. For the rest of the real t -axis, we exploit the involution $z \mapsto z_6 = -1/(6z)$, which gives $t \mapsto t_6 = 64/t$, with fixed points at $t = \pm 8$. For $t_6 \in [-8, 8]$ we use

$$\left(q_6 \frac{d}{dq_6}\right)^3 \frac{J(t)}{(wf_6(z_6))^2} = -24h_6(z_6), \quad (22)$$

$$h_6(z) = -t_6 h = 1 + 2h - 30 \sum_{n>0} \frac{n^3(q^{2n} + q^{4n} - 8q^{6n})}{1 - q^{6n}}, \quad (23)$$

$$\frac{J(t)}{(wf_6(z_6))^2} = -4 \log^3(q_6) + 24 \sum_{n>0} \frac{15\phi(n+3) - \phi(n)}{n^3} \frac{1 + q_6^n}{1 - q_6^n}, \quad (24)$$

in agreement with the result proved by Bloch, Kerr and Vanhove [5]. Extracting

$$C_3 = \sum_{n>0} \frac{15\phi(n+3) - \phi(n)}{n^3} = \frac{2\zeta(3)}{3} \quad (25)$$

we achieve a highly efficient expansion in $q_6 = \exp(-\pi i/(3z))$ for $64/t \in [-8, 8]$, with a strong check of consistency with (21) in the neighbourhoods of $t = \pm 8$, where both expansions work well.

2.2 Eichler integrals of eta quotients for on-shell sunrise integrals

On-shell sunrise integrals lead us to consider Bessel moments of the form

$$M(a, b, c) = \int_0^\infty I_0^a(x) K_0^b(x) x^c dx. \quad (26)$$

For $L > 3$, the off-shell L -loop integral

$$S_L(t) = \int_0^\infty \frac{dx_1}{x_1} \dots \int_0^\infty \frac{dx_L}{x_L} \frac{1}{(1 + \sum_{j=1}^L x_j)(1 + \sum_{k=1}^L 1/x_k) - t} \quad (27)$$

has not yielded to the methods given above for $L = 2, 3$. By contrast, the on-shell values $S_L(1) = 2^L M(1, L+1, 1)$ with $L+2$ Bessel functions yield Eichler integrals of cusp forms of weights 4 and 6 on $\Gamma_0(6)$ at $L = 4$ and $L = 6$ loops, namely integrals of the form $\int_0^\infty f(iy)y^{s-1} dy$, where $f(z)$ is a cusp form with weight L and s is a integer with $L > s > 0$. First we consider the situation at $L = 3$ loops, where a modular form of weight 3 occurs.

At 3 loops, with 5 Bessel functions, the on-shell problem is solved by the weight 3 level 15 cusp form

$$f_{3,15}(z) = (\eta_3 \eta_5)^3 + (\eta_1 \eta_{15})^3 = \sum_{n>0} A_5(n) q^n = -\frac{f_{3,15}(-1/(15z))}{(-15)^{3/2} z^3} \quad (28)$$

with complex multiplication in $\mathbf{Q}(\sqrt{-15})$. If the Kronecker symbol $(\frac{p}{15}) = (\frac{p}{3})(\frac{p}{5})$ is negative, for prime p , then $A_5(p) = 0$. For $\Re s > 2$, there is a convergent L-series

$$L_5(s) = \sum_{n>0} \frac{A_5(n)}{n^s} = \prod_p \frac{1}{1 - A_5(p)p^{-s} + \left(\frac{p}{15}\right) p^{2-2s}} \quad (29)$$

whose analytic continuation is provided by the Eichler integral

$$L_5(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{3,15}(iy) y^{s-1} dy \quad (30)$$

with critical values

$$L_5(1) = \frac{5}{\pi^2} M(1, 4, 1), \quad L_5(2) = \frac{4}{3} M(2, 3, 1). \quad (31)$$

At 4 loops, with 6 Bessel functions, the on-shell problem is solved by the weight 4 level 6 cusp form

$$f_{4,6}(z) = (\eta_1 \eta_2 \eta_3 \eta_6)^2 = \sum_{n>0} A_6(n) q^n = \frac{f_{4,6}(-1/(6z))}{6^2 z^4}. \quad (32)$$

For $\Re s > 5/2$, there is a convergent L-series

$$L_6(s) = \sum_{n>0} \frac{A_6(n)}{n^s} = \frac{1}{1+2^{1-s}} \frac{1}{1+3^{1-s}} \prod_{p>3} \frac{1}{1 - A_6(p)p^{-s} + p^{3-2s}} \quad (33)$$

whose analytic continuation is provided by the Eichler integral

$$L_6(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{4,6}(iy) y^{s-1} dy \quad (34)$$

with critical values

$$L_6(2) = \frac{2}{\pi^2} M(1, 5, 1) = \frac{2}{3} M(3, 3, 1), \quad (35)$$

$$L_6(1) = \frac{2}{\pi^2} M(2, 4, 1) = \frac{3}{\pi^2} L_6(3). \quad (36)$$

At 6 loops, with 8 Bessel functions, the on-shell problem is solved by the weight 6 level 6 cusp form

$$f_{6,6}(z) = \frac{\eta_2^9 \eta_3^9}{\eta_1^3 \eta_6^3} + \frac{\eta_1^9 \eta_6^9}{\eta_2^3 \eta_3^3} = \sum_{n>0} A_8(n) q^n = -\frac{f_{6,6}(-1/(6z))}{6^3 z^6}. \quad (37)$$

For $\Re s > 7/2$, there is a convergent L-series

$$L_8(s) = \sum_{n>0} \frac{A_8(n)}{n^s} = \frac{1}{1-2^{2-s}} \frac{1}{1+3^{2-s}} \prod_{p>3} \frac{1}{1 - A_8(p)p^{-s} + p^{5-2s}} \quad (38)$$

whose analytic continuation is provided by the Eichler integral

$$L_8(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{6,6}(iy) y^{s-1} dy \quad (39)$$

with critical values

$$L_8(4) = \frac{4}{9\pi^2} M(1, 7, 1) = \frac{4}{9} M(3, 5, 1) = \frac{\pi^2}{9} L_8(2), \quad (40)$$

$$L_8(5) = \frac{4}{27} M(2, 6, 1) = \frac{2\pi^2}{21} M(4, 4, 1) = \frac{2\pi^2}{21} L_8(3) = \frac{\pi^4}{54} L_8(1). \quad (41)$$

2.3 Eichler integrals for quasi-periods at level 6

In [13] Francis Brown associated a pair of periods and a pair of quasi-periods to the weight 12 level 1 modular form $\Delta(z) = \eta_1^{24}$. The periods are a pair of Eichler integrals that determine critical values of the L-series at odd and even integers. No concrete integrals were given for the quasi-periods. Rather it was asserted that numerical values may be obtained by an undeclared regularization of integrals of a weakly holomorphic modular form $\Delta'(z) = 1/q + O(q^2)$.

In the case of the level 6 modular forms that yield 4-loop and 6-loop Feynman integrals the situation is cleaner, since the periods are Eichler integrals of eta quotients, $f_{4,6}$ and $f_{6,6}$, with 4 cusps. Thus we may hope to find weakly holomorphic modular forms, $g_{4,6}$ and $g_{6,6}$, that yield quasi-periods as well defined Eichler integrals with a base-point at a cusp free of singularities. A test is provided by the condition that a 2×2 determinant formed from a pair of periods and a pair of quasi-periods should be an algebraic multiple of a power of π , as is trivially ensured for modular forms of weight 2 by Legendre's relation between pairs of complete elliptic integrals of first and second kind.

At 4 loops, we achieved this with Eichler integrals

$$\frac{D_2}{2} = \frac{M(1, 5, 1)}{\pi^4} = \frac{4M(1, 5, 3)}{\pi^4} + \frac{5E_2}{18} \quad (42)$$

$$\frac{3D_1}{5} = \frac{M(2, 4, 1)}{\pi^3} = \frac{4M(2, 4, 3)}{\pi^3} + \frac{E_1}{3} \quad (43)$$

$$\begin{bmatrix} D_s \\ E_s \end{bmatrix} = - \int_{1/\sqrt{3}}^\infty \begin{bmatrix} f_{4,6} \left(\frac{1+iy}{2} \right) \\ g_{4,6} \left(\frac{1+iy}{2} \right) \end{bmatrix} y^{s-1} dy, \quad (44)$$

$$g_{4,6}(z) = \frac{(w^2 - 3)^2(w^4 + 9)}{8w^4} f_{4,6}(z) = 5q + 102q^2 + 945q^3 + O(q^4), \quad (45)$$

$$D_1 E_2 - D_2 E_1 = \frac{1}{24\pi^3}. \quad (46)$$

At 6 loops, it is conjecturally achieved by

$$\det \begin{bmatrix} M(1, 7, 1) & 32M(1, 7, 3) - 64M(1, 7, 5) \\ M(2, 6, 1) & 32M(2, 6, 3) - 64M(2, 6, 5) \end{bmatrix} = \frac{5\pi^6}{192}, \quad (47)$$

$$\frac{F_2}{4} = \frac{M(1, 7, 1)}{\pi^6} \stackrel{?}{=} \frac{32M(1, 7, 3) - 64M(1, 7, 5)}{\pi^6} + \frac{35G_2}{108}, \quad (48)$$

$$\frac{9F_1}{28} = \frac{M(2, 6, 1)}{\pi^5} \stackrel{?}{=} \frac{32M(2, 6, 3) - 64M(2, 6, 5)}{\pi^5} + \frac{5G_1}{12}, \quad (49)$$

$$\begin{bmatrix} F_s \\ G_s \end{bmatrix} = - \int_{1/\sqrt{3}}^{\infty} \begin{bmatrix} f_{6,6} \left(\frac{1+iy}{2} \right) \\ g_{6,6} \left(\frac{1+iy}{2} \right) \end{bmatrix} (3y^2 - 1)y^{s-1} dy, \quad (50)$$

$$g_{6,6}(z) = \frac{(w^2 - 3)^4}{16w^4} f_{6,6}(z) = q + 36q^2 + 567q^3 + 5264q^4 + O(q^5), \quad (51)$$

$$F_1 G_2 - F_2 G_1 \stackrel{?}{=} \frac{1}{4\pi^5}, \quad (52)$$

where the question marks indicate unproven discoveries, checked to thousands of digits of numerical precision.

3 Rademacher sums for Fourier coefficients of eta quotients

For positive integers N , M and n , we define the Rademacher sums

$$R_{N,M}(n) = \sum_{c>0, \gcd(c,N)=1} \frac{2\pi I_1(4\pi\sqrt{nM/N}/c)}{\sqrt{nN/Mc}} K(c, N, M, n) \quad (53)$$

as sums of Bessel functions multiplied by Kloosterman sums

$$K(c, N, M, n) = \sum_{r \in [1, c], \gcd(r, c)=1} \exp\left(\frac{2\pi i(Mr - ns)}{c}\right) \Big|_{Nr s = 1 \pmod c}. \quad (54)$$

In (53) the sum is over all positive integers c coprime to N . In (54) the sum is over the integers $r \in [1, c]$ coprime to c and $s \in [1, c]$ is the inverse of Nr modulo c . It follows from these definitions that $R_{N,M}(n)/M = R_{N,n}(M)/n$ and that $R_{N,M}(n) = R_{dN,dM}(n)$ for every positive integer d that divides N .

3.1 Genus 0

We found that $R_{N,1}(n)/R_{N,1}(1)$ is the coefficient of q^n in an eta quotient T_N/B_N defining an OEIS sequence in the genus 0 cases of Table 1, where the eta quotients agree with the canonical Hauptmoduln in [24, Table 8].

For $n > 0$, Rademacher [27] obtained $R_{1,1}(n)$ as the coefficient of q^n of

N	$R_{N,1}(1)$	T_N	B_N	OEIS
2	4096	η_2^{24}	η_1^{24}	A014103
3	729	η_3^{12}	η_1^{12}	A121590
4	256	η_4^8	η_1^8	A092877
5	125	η_5^6	η_1^6	A121591
6	72	$\eta_2\eta_6^5$	$\eta_1^5\eta_3$	A128638
7	49	η_7^4	η_1^4	A121593
8	32	$\eta_2^2\eta_8^4$	$\eta_1^4\eta_4^2$	A107035
9	27	η_9^3	η_1^3	A121589
10	20	$\eta_2\eta_{10}^3$	$\eta_1^3\eta_5$	A095846
12	12	$\eta_2^2\eta_3\eta_{12}^3$	$\eta_1^3\eta_4\eta_6^2$	A187100
13	13	η_{13}^2	η_1^2	A121597
16	8	$\eta_2\eta_{16}^2$	$\eta_1^2\eta_8$	A123655
18	6	$\eta_2\eta_3\eta_{18}^2$	$\eta_1^2\eta_6\eta_9$	A128129
25	5	η_{25}	η_1	A092885

Table 1 Eta quotients T_N/B_N of genus 0 with Fourier coefficients $R_{N,1}(n)/R_{N,1}(1)$

$$j(z) = \frac{1}{\eta_1^{24}} \left(1 + 240 \sum_{n>0} \frac{n^3 q^n}{1 - q^n} \right)^3 = \frac{1}{q} + 744 + 196884q + O(q^2) \quad (55)$$

which is invariant under $z \mapsto (az + b)/(cz + d)$ with integers satisfying $ad - bc = 1$.

We remark that analytic continuation of (53) to $n = 0$ gives $R_{1,1}(0) = 24$, which differs from the constant term 744 in (55). Our work concerns only the values of $R_{N,M}(n)$ for integers $n > 0$.

The congruence subgroup $\Gamma_0(N)$ is the group of Möbius transformations with $N|c$. The Hauptmodul

$$\frac{\eta_2\eta_6^5}{\eta_1^5\eta_3} = q + 5q^2 + 19q^3 + 61q^4 + 174q^5 + 455q^6 + 1112q^7 + \dots \quad (56)$$

of $\Gamma_0(6)$ has a Fourier coefficient $R_{6,1}(n)/72 = A128638(n)$, which we are now able to evaluate at large n . We found that this Fourier coefficient is odd if and only if the core (i.e. the square-free part) of n is a divisor of 6. We determined the probably prime values of $A128638(n)$ for $n \in [1, 900000000]$ and found these occur at surprisingly few values of n , namely these: 2, 3, 4, 9, 32, 48, 324, 578, 864, 121032, 940896, 11723776, 88360000, 180848704, 198443569.

We remark that $A128638(900000000)$, with 66832 decimal digits, would be rather hard to compute in the absence of a Rademacher-sum formula.

3.2 Further examples of integer sequences

We found several integer sequences of the form $R_{N,M}(n)/D$, with $\gcd(N, M) = 1$, $N > M > 1$ and integer D , as for example in Table 2.

N	M	D	sequence
3	2	3^7	8, 339, 6552, 82796, 790896, 6171606, 41232064, 243306300,...
4	3	2^{10}	33, 1800, 42412, 633024, 7003278, 62405984, 471069624, 3114275328,...
5	2	5^3	12, 197, 1824, 12426, 68780, 327819, 1391472, 5383270, 19289244,...
6	5	432	145, 10085, 286435, 5004925, 63619086, 642751655, 5445694040,...
7	2	7^2	8, 81, 504, 2476, 10248, 37590, 125328, 387384, 1123992, 3092369,...
8	3	2^7	9, 132, 1132, 7200, 37566, 169648, 685368, 2532096, 8688909,...
9	2	3^4	2, 15, 72, 287, 984, 3051, 8704, 23286, 58968, 142677, 331728,...
10	3	80	6, 63, 418, 2139, 9216, 35004, 120594, 384147, 1146842, 3241083,...
11	8	11^2	234, 11950, 266994, 3812019, 40551362, 348772038, 2548265460,...
12	5	72	25, 435, 4255, 30255, 174126, 859305, 3766760, 15014775, 55334545,...
13	2	13	4, 21, 72, 222, 600, 1509, 3536, 7902, 16860, 34740, 69264, 134412,...
14	5	56	17, 229, 1852, 11213, 55998, 243084, 946991, 3382221, 11242933,...
15	7	45	67, 1398, 15919, 128386, 826187, 4509396, 21688133, 94244610,...
16	3	2^5	3, 18, 76, 264, 810, 2264, 5880, 14400, 33583, 75132, 162180, 339296,...
17	3	17	5, 26, 107, 352, 1045, 2814, 7091, 16842, 38225, 83260, 175329,...
18	5	36	10, 95, 580, 2770, 11226, 40340, 132080, 401255, 1145740, 3104412,...
19	2	19	1, 4, 10, 25, 55, 116, 229, 440, 809, 1455, 2541, 4354, 7300, 12050,...

Table 2 Examples of integer sequences $R_{N,M}(n)/D$

We identified some of the generating functions, as follows

$$\sum_{n>0} R_{3,2}(n)q^n = \frac{3^7 \eta_3^{12}}{\eta_1^{12}} \left(8 + \frac{3^5 \eta_3^{12}}{\eta_1^{12}} \right) \tag{57}$$

$$\sum_{n>0} R_{5,2}(n)q^n = \frac{5^3 \eta_5^6}{\eta_1^6} \left(12 + \frac{5^3 \eta_5^6}{\eta_1^6} \right) \tag{58}$$

$$\sum_{n>0} R_{7,2}(n)q^n = \frac{7^2 \eta_7^4}{\eta_1^4} \left(8 + \frac{7^2 \eta_7^4}{\eta_1^4} \right) \tag{59}$$

$$\sum_{n>0} R_{9,2}(n)q^n = \frac{3^4 \eta_9^3}{\eta_1^3} \left(2 + \frac{3^2 \eta_9^3}{\eta_1^3} \right) \tag{60}$$

$$\sum_{n>0} R_{13,2}(n)q^n = \frac{13 \eta_{13}^2}{\eta_1^2} \left(4 + \frac{13 \eta_{13}^2}{\eta_1^2} \right) \tag{61}$$

$$\sum_{n>0} R_{16,3}(n)q^n = 8 \left(\frac{\eta_2^{18}}{\eta_1^{12} \eta_4^6} - 1 \right). \tag{62}$$

Moreover,

$$\sum_{n>0} R_{18,2}(n)q^n = 4 \left(\frac{\eta_2^6 \eta_3^2}{\eta_1^6 \eta_6^2} - 1 \right) \tag{63}$$

$$\sum_{n>0} R_{27,2}(n)q^n = 3 \left(\frac{\eta_3^4}{\eta_1^3 \eta_9} - 1 \right) \tag{64}$$

$$\sum_{n>0} R_{32,3}(n)q^n = 4 \left(\frac{\eta_2^2 \eta_4^4}{\eta_1^4 \eta_8^2} - 1 \right) \quad (65)$$

$$\sum_{n>0} R_{36,5}(n)q^n = 6 \left(\frac{\eta_2^5 \eta_3^5}{\eta_1^7 \eta_6^3} - 1 \right) \quad (66)$$

$$\sum_{n>0} R_{48,7}(n)q^n = 6 \left(\frac{\eta_2^8 \eta_3^4}{\eta_1^8 \eta_4^2 \eta_6^2} - 1 \right) \quad (67)$$

$$\sum_{n>0} R_{64,3}(n)q^n = 2 \left(\frac{\eta_2 \eta_4^2}{\eta_1^2 \eta_8} - 1 \right) \quad (68)$$

$$\sum_{n>0} R_{64,7}(n)q^n = 4 \left(\frac{\eta_2^7}{\eta_1^6 \eta_8} - 1 \right). \quad (69)$$

When $N > 1$ has genus 0, $R_{N,M}(n)$ is an integer sequence generated by a polynomial of degree M in the eta quotient that generates $R_{N,1}(n)$. Thus, for example, we may compute the coefficient of q^n in $(\eta_{25}/\eta_1)^P$ from a linear combination of Rademacher-type formulas for $R_{25,M}(n)$ with $M \in [1, P]$, using polynomials in $g = \sum_{n>0} R_{25,1}(n)q^n = 5\eta_{25}/\eta_1$ as follows:

$$\sum_{n>0} R_{25,2}(n)q^n = 2g + g^2 \quad (70)$$

$$\sum_{n>0} R_{25,3}(n)q^n = 6g + 3g^2 + g^3 \quad (71)$$

$$\sum_{n>0} R_{25,4}(n)q^n = 12g + 10g^2 + 4g^3 + g^4 \quad (72)$$

$$\sum_{n>0} R_{25,5}(n)q^n = 25g + 25g^2 + 15g^3 + 5g^4 + g^5 = \frac{5^3 \eta_5^6}{\eta_1^6} \quad (73)$$

$$\sum_{n>0} R_{25,6}(n)q^n = 42g + 60g^2 + 44g^3 + 21g^4 + 6g^5 + g^6 \quad (74)$$

$$\sum_{n>0} R_{25,7}(n)q^n = 77g + 126g^2 + 119g^3 + 70g^4 + 28g^5 + 7g^6 + g^7 \quad (75)$$

$$\sum_{n>0} R_{25,8}(n)q^n = 120g + 260g^2 + 288g^3 + 210g^4 + 104g^5 + 36g^6 + 8g^7 + g^8. \quad (76)$$

3.3 Genus 1

The genus $g_0(N)$ of $\Gamma_0(N)$ is computed in Pari-GP by a procedure

```
g0(N)={local(f=factor(N),t=vector(4,k,1),p,r,n);
for(k=1,matsize(f)[1],p=f[k,1];r=f[k,2];n=p^r;
t[1]*=n*(1+1/p);t[2]*=if(n==2,1,if(p%4==1,2));
t[3]*=if(n==3,1,if(p%3==1,2));
t[4]*=if(r%2,2*p^((r-1)/2),(p+1)*p^(r/2-1));
```

$$1+t [1] / 12-t [2] / 4-t [3] / 3-t [4] / 2 ; \}$$

that combines 4 multiplicative functions [18, 25, 31].

N	primes $M < 1000$
11	19, 29, 199, 569, 809
14	5, 11, 23, 71, 101, 263, 503
15	7, 23, 31, 79, 167, 431, 479, 983
17	3, 11, 47, 359, 967
19	2, 23, 257, 449, 509, 521, 641
20	11, 131, 251, 491, 599
21	23, 31, 47, 71, 127, 367, 383, 743
24	7, 47, 191, 383, 439
27	$M = 2 \pmod 3$
32	$M = 3 \pmod 4$
36	$M = 5 \pmod 6$
49	$M = 3, 5, 6 \pmod 7$

Table 3 Primes M such that $R_{N,M}(n)$ is an integer sequence for N with genus 1

We conjecture that only when N has genus 0 is $R_{N,1}(n)$ an integer sequence. To deal with genus 1, we introduced the additional parameter M into $R_{N,M}(n)$. For each level N with genus 1, we specify in Table 3 the prime values of $M < 1000$, coprime to N , for which $R_{N,M}(n)$ is an integer sequence.

At genus 1, the criterion for whether $R_{N,M}(n)$ forms an integer sequence is provided by the Fourier expansion of the unique weight 2 cusp form of level N , which we denote by $f_N = \sum_{M>0} C_{N,M}q^M$. Specifically,

$$f_{11} = (\eta_1 \eta_{11})^2 \tag{77}$$

$$f_{14} = \eta_1 \eta_2 \eta_7 \eta_{14} \tag{78}$$

$$f_{15} = \eta_1 \eta_3 \eta_5 \eta_{15} \tag{79}$$

$$f_{17} = q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + 2q^{10} - 2q^{13} + \dots \tag{80}$$

$$f_{19} = q - 2q^3 - 2q^4 + 3q^5 - q^7 + q^9 + 3q^{11} + 4q^{12} - 4q^{13} \dots \tag{81}$$

$$f_{20} = (\eta_2 \eta_{10})^2 \tag{82}$$

$$f_{21} = q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + 3q^8 + q^9 + 2q^{10} + 4q^{11} - q^{12} + \dots \tag{83}$$

$$f_{24} = \eta_2 \eta_4 \eta_6 \eta_{12} \tag{84}$$

$$f_{27} = (\eta_3 \eta_9)^2 \tag{85}$$

$$f_{32} = (\eta_4 \eta_8)^2 \tag{86}$$

$$f_{36} = \eta_6^4 \tag{87}$$

$$f_{49} = q + q^2 - q^4 - 3q^8 - 3q^9 + 4q^{11} - q^{16} - 3q^{18} + 4q^{22} \dots \tag{88}$$

with explicit formulas for $N = 17, 19, 21, 49$ given below in (97,98,99).

For N with genus 1, we found that $R_{N,M}(n)$ is an integer sequence if and only if $C_{N,M} = 0$. Moreover

$$\bar{R}_{N,M}(n) = R_{N,M}(n) - C_{N,M}R_{N,1}(n) \quad (89)$$

is always an integer sequence, with $\bar{R}_{N,1}(n) = 0$, by construction.

With $G_{N,M} = \sum_{n>0} \bar{R}_{N,M}(n)q^n$, we found at $N = 21$ that

$$7 \left(\frac{\eta_3 \eta_7^3}{\eta_1^3 \eta_{21}} - 1 \right) = G_{21,2}, \quad (90)$$

$$3^3 \left(\frac{\eta_3^7 \eta_7}{\eta_1^7 \eta_{21}} - 1 \right) = G_{21,4} + G_{21,3} + 2G_{21,2}, \quad (91)$$

$$7^2 \left(\frac{\eta_3^2 \eta_7^6}{\eta_1^6 \eta_{21}^2} - 1 \right) = G_{21,4} + 2G_{21,3} + 5G_{21,2}, \quad (92)$$

$$\frac{3^3 7 \eta_3^4 \eta_{21}^2}{\eta_1^6} = G_{21,4} - 2G_{21,3} - G_{21,2}, \quad (93)$$

$$\frac{3^3 7^2 \eta_3 \eta_{21}^5}{\eta_1^5 \eta_7} = G_{21,4} - 5G_{21,3} + 5G_{21,2}, \quad (94)$$

$$\frac{3^3 7 \eta_3^6 \eta_7^2}{\eta_1^8} = G_{21,5} - 2G_{21,2}, \quad (95)$$

$$\frac{3^3 7^2 \eta_3^3 \eta_7 \eta_{21}^3}{\eta_1^7} = G_{21,5} - 3G_{21,4} + 4G_{21,2}. \quad (96)$$

N	$C_{N,2}$	$C_{N,3}$	E_N
11	-2	-1	$Y^2 + 6XY + 121Y = X^3 + 38X^2 + 363X$
14	-1	-2	$Y^2 + 3XY + 56Y = X^3 + 25X^2 + 168X$
15	-1	-1	$Y^2 + 3XY + 45Y = X^3 + 23X^2 + 135X$
17	-1	0	$Y^2 + 3XY + 34Y = X^3 + 18X^2 + 85X$
19	0	-2	$Y^2 + 19Y = X^3 + 16X^2 + 76X$
20	0	-2	$Y^2 + 20Y = X^3 + 13X^2 + 60X$
21	-1	1	$Y^2 + 3XY + 21Y = X^3 + 13X^2 + 42X$
24	0	-1	$Y^2 + 12Y = X^3 + 11X^2 + 36X$
27	0	0	$Y^2 + 9Y = X^3 + 9X^2 + 27X$
32	0	0	$Y^2 + 8Y = X^3 + 6X^2 + 16X$
36	0	0	$Y^2 + 6Y = X^3 + 6X^2 + 12X$
49	1	0	$Y^2 - 3XY = X^3 + 3X^2 + 7X$

Table 4 Elliptic curves E_N for N with genus 1

For each level N with genus 1, we found that $(X, Y) = (G_{N,2}, G_{N,3})$ is a point on an elliptic curve E_N specified in Table 4 and verified up to $O(q^{20000})$. Moreover $G_{N,M} = P_0(X) + P_1(X)Y$ where P_0 and P_1 are polynomials with degrees not exceeding $M/2$ and $(M-3)/2$, respectively.

With $N = 21$, relations (90,91,92) show that $X = G_{21,2}$ is determined by an eta quotient and that $Y = G_{21,3}$ is determined by 3 eta quotients. The transformation

$(X, Y) = (x - 5, y - x - 3)$ yields a minimal model $y^2 + xy = x^3 - 4x - 1$, whose small coefficients were noted in [19].

Applying the `ellak` procedure of `Pari-GP` to E_N , we reproduce the Fourier coefficients of $f_N = \sum_{M>0} C_{N,M} q^M$. Thanks to work recorded at OEIS, we are able to provide formulas for f_N in the 4 cases where a single eta quotient does not suffice, namely for $N = 17, 19, 21, 49$:

$$f_{17} = \eta_1 \eta_{17} (\psi_2 \phi_{17} - \psi_{34} \phi_1), \quad f_{19} = (\psi_4 \phi_{38} - \psi_1 \psi_{19} + \psi_{76} \phi_2)^2, \quad (97)$$

$$f_{21} = \frac{3(\eta_7^3 \eta_9^3 + \eta_1 \eta_7^2 \eta_9^2 \eta_{63} + \eta_1^2 \eta_7 \eta_9 \eta_{63}^2)}{2\eta_3 \eta_{21}} - \frac{\eta_3^4 \eta_7^2}{2\eta_1^2} + \frac{7\eta_3 \eta_7 \eta_{21}^3}{2\eta_1} - \frac{3\eta_3^4 \eta_7 \eta_{63}}{2\eta_1 \eta_9}, \quad (98)$$

$$f_{49} = \theta_{7,14}^3 (q\theta_{21,28} + q^2\theta_{14,35} - q^4\theta_{7,42}), \quad (99)$$

$$\psi_n = \frac{\eta_{2n}^2}{\eta_n}, \quad \phi_n = \frac{\eta_{2n}^5}{\eta_n^2 \eta_{4n}^2}, \quad \theta_{a,b} = \sum_{n=-\infty}^{\infty} (-q^a)^{(n^2+n)/2} (-q^b)^{(n^2-n)/2}, \quad (100)$$

with (99) recorded in [12]. At $N = 49$, we have complex multiplication, with $C_{49,p} = 0$ for prime $p = 3, 5, 6 \pmod{7}$. Moreover we have a pair of eta quotients,

$$x = \frac{\eta_{49}}{\eta_1} = \frac{G_{49,2}}{7}, \quad y = \frac{\eta_7^4}{\eta_1^4} = \frac{G_{49,7}}{7^2}, \quad (101)$$

with Fourier coefficients given by Rademacher sums. The latter is determined by $G_{49,2}$ and $G_{49,3}$. Hence the elliptic curve E_{49} provides an algebraic relation between these eta quotients, namely

$$(2y - 7x - 35x^2 - 49x^3)^2 = (4x + 21x^2 + 28x^3)(1 + 7x + 7x^2)^2. \quad (102)$$

At $N = 21$, we found 2937 eta quotients whose Fourier coefficients are linear combinations of $R_{21,M}(n)$ with $M \leq 50$. Including the unit quotient, the tally of 2938 is the coefficient of x^{50} in the generating function

$$T_{21}(x) = \frac{1 - x + x^2 - x^3 + 2x^4}{(1-x)^2(1-x^4)^2} \quad (103)$$

which predicts a total of 22126 eta quotients with Fourier coefficients determined by $R_{21,M}(n)$ for $M \leq 100$. We have identified all of these.

At $N = 36$, with 9 divisors, the corresponding tallies of eta quotients are spectacularly large. Using Padé approximants, the generating function was found to be

$$T_{36}(x) = \frac{H(x) + x^{18}H(1/x)}{(1-x)^4(1-x^3)^2(1-x^4)^2(1-x^{12})}, \quad (104)$$

$$H(x) = 1 - 3x + 6x^2 - 3x^3 + 6x^4 + x^5 + x^6 + 4x^7 + 4x^8 + x^9, \quad (105)$$

giving 49307076 eta quotients with Fourier coefficients determined by $R_{36,M}(n)$ for $M \leq 50$ and 8204657877 for $M \leq 100$. We were able to identify all of these, by tak-

ing products of 78 eta quotients found at $M \leq 12$ and eliminating redundancies. Using more refined methods, we also validated the monstrous tally of 180919436828 for $M \leq 150$.

From the denominator of $T_{36}(x) = \sum_{n \geq 0} (c(M) + 1)x^M$ it is clear that the number $c(M)$ of non-trivial eta quotients determined by our procedures may be found by polynomial interpolation at integers with the same residue modulo 12. We denote the result by $c(M) = p(M) + q_r(M)/12$ for $M = r \pmod{12}$, with a leading polynomial

$$p(n) = \frac{(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)((n+6)^2-7)}{1935360} - 1 \quad (106)$$

of degree 8 and sub-dominant terms that are at most quadratic:

$$q_0(n) = q_3(n) + q_4(n) + 5, \quad (107)$$

$$q_1(n) = q_5(n) = q_7(n) = q_{11}(n) = 0, \quad (108)$$

$$q_2(n) = q_{10}(n) = \frac{2(n+6)^2-5}{512}, \quad (109)$$

$$q_3(n) = q_9(n) = \frac{(n+3)(n+9)}{9}, \quad (110)$$

$$q_4(n) = q_8(n) = q_2(n) + \frac{(n+4)(n+8)}{16}, \quad (111)$$

$$q_6(n) = q_2(n) + q_3(n) + 1. \quad (112)$$

The situation at the prime levels $N = 11, 17, 19$ is rather different. Here we have a wealth of Rademacher sums but only one eta quotient. Consider the case $N = 19$. Since $f_{19} = q + O(q^3)$, we have $C_{19,2} = 0$ and hence $R_{19,2}(n)$ yields integers. As noted, the sequence $R_{19,2}(n)/19$ begins with

$$1, 4, 10, 25, 55, 116, 229, 440, 809, 1455, 2541, 4354, 7300, 12050, \dots$$

This sequence may be developed using η_{19}^4/η_1^4 , which is determined by $G_{19,2}$ and $G_{19,3}$. The elliptic curve relating the latter pair gives an algebraic relation between $G_{19,2}$ and the eta quotient, namely

$$s^3/e_{19} = 1 + 8s + 19e_{19}, \quad s = G_{19,2}/19, \quad e_{19} = \eta_{19}^4/\eta_1^4, \quad (113)$$

from which the expansion of $s = q + O(q^2)$ is easily developed, iteratively.

Similarly, at $N = 11$ and $N = 17$ we obtain the algebraic relations

$$t^5/e_{11} = 1 + 13t + 34t^2 + 11^2e_{11}, \quad t = G_{11,2}/11^2, \quad e_{11} = \eta_{11}^{12}/\eta_1^{12}, \quad (114)$$

$$u^4/e_{17} = 16 + 64u + 34u^2 - 17^2e_{17}, \quad u = G_{17,2}/17, \quad e_{17} = \eta_{17}^6/\eta_1^6, \quad (115)$$

and hence develop the expansions of $t = q + O(q^2)$ and $u = 2q + O(q^2)$.

Intermediate between the plethora of eta quotients at $N = 36$, with 9 divisors, and their relative scarcity at $N = 11, 17, 19, 49$, with less than 4 divisors, sit the remaining genus 1 levels, $N = 14, 15, 20, 21, 24, 27, 32$. Proceeding as for $N = 21$,

we found the generating functions

$$T_{14}(x) = \frac{1 + x^3 + 2x^4 + x^6 + x^7}{(1-x)(1-x^2)(1-x^3)(1-x^6)}, \quad (116)$$

$$T_{15}(x) = T_{21}(x) = \frac{1 - x + x^2 - x^3 + 2x^4}{(1-x)^2(1-x^4)^2}, \quad (117)$$

$$T_{20}(x) = \frac{1 - x + x^2 + 4x^3 + 2x^4 + 3x^6 + x^7 + x^8}{(1-x)^2(1-x^2)^2(1-x^3)(1-x^6)}, \quad (118)$$

$$T_{24}(x) = \frac{(1+x^3)(h(x) + x^6h(1/x))}{(1-x)^3(1-x^2)^3(1-x^4)^2}, \quad h(x) = 1 - 2x + 3x^2 + 2x^3, \quad (119)$$

$$T_{27}(x) = \frac{1 - x^{11}}{(1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^6)}, \quad (120)$$

$$T_{32}(x) = \frac{1 - x + x^2 + 2x^3 + x^4 - x^5 + x^6}{(1-x)^2(1-x^2)^2(1-x^4)^2}, \quad (121)$$

with the coefficient of x^m in $T_N(x)$ giving the number of eta quotients whose Fourier coefficients are determined by linear combinations of the Rademacher sums $R_{N,M}(n)$ with $M \leq m$.

3.4 Rational Rademacher sums

There are 5 levels with genus greater than 0 for which it appears that the Rademacher sums $R_{N,M}(n)$ are rational for all positive integers M and n , namely $N = 27, 32, 36, 49$ with genus 1 and $N = 64$ with genus 3. At genus 1, we convert these rationals to the integers

$$\bar{R}_{N,M}(n) = R_{N,M}(n) - C_{N,M}R_{N,1}(n), \quad (122)$$

which vanish at $M = 1$. The rationals $R_{64,M}(n)$ do not form integer sequences for $M = 1, 2, 5 \pmod{8}$. To remedy this, we define

$$g_{8k+r} = \sum_{n>0} (R_{64,8k+r}(n) - c_{k,r}R_{64,r}(n)) q^n, \quad (123)$$

$$c_{k,1} = C_{32,8k+1}, \quad c_{k,2} = C_{32,4k+1}, \quad c_{k,5} = -C_{32,8k+5}/2, \quad (124)$$

with $k \geq 0$, $r \in [1, 8]$ and $c_{k,r} = 0$ for $r = 3, 4, 6, 7, 8$. Then g_M has integer Fourier coefficients, which vanish for $M = 1, 2, 5$. Eta quotients appear in

$$\frac{g_3}{2} = \frac{\eta_2 \eta_4^2}{\eta_1^2 \eta_8} - 1, \quad \frac{g_4}{8} = \frac{\eta_2 \eta_{16}^2}{\eta_1^2 \eta_8}, \quad \frac{g_6}{4} = \frac{\eta_2^2 \eta_4^4}{\eta_1^4 \eta_8^2} - 1, \quad \frac{g_7}{4} = \frac{\eta_2^7}{\eta_1^6 \eta_8} - 1, \quad \frac{g_8}{32} = \frac{\eta_2^2 \eta_8^4}{\eta_1^4 \eta_4^2}. \quad (125)$$

Moreover, $g_6 = (4 + g_3)g_3$, $g_7 = 2g_3 + (2 + g_3)g_4$, $g_8 = (4 + g_4)g_4$.

We found that $(X, Y) = (g_3 + 2, g_4 + 2)$ is a point on the genus 3 curve

$$Y(Y^2 + 4) = X^4 \quad (126)$$

and that $g_M = P_0(X) + P_1(X)Y + P_2(X)Y^2$, with P_n a polynomial of degree at most $\lfloor (M - 4n)/3 \rfloor$. Every eta quotient of that form has Fourier coefficients that are linear combinations of Rademacher sums at $N = 64$.

3.5 Genus 2

It is instructive to compare the genus 3 case $N = 64$ with the genus 2 case $N = 50$. For the latter we construct integer sequences as follows:

$$\bar{R}_{50,M}(n) = \begin{cases} R_{50,M}(n) - d(M)R_{50,1}(n), & \text{for } M = 1, 4 \pmod{5}, \\ R_{50,M}(n) + d(M)R_{50,2}(n), & \text{for } M = 2, 3 \pmod{5}, \\ R_{50,M}(n), & \text{for } M = 0 \pmod{5}, \end{cases} \quad (127)$$

$$\sum_{M>0} d(M)q^M = f_{50} = q - q^2 + q^3 + q^4 - q^6 + 2q^7 - q^8 - 2q^9 + O(q^{11}), \quad (128)$$

where f_{50} is the weight 2 level 50 Hecke eigenform whose Fourier coefficients $d(M)$ are obtained from the L-series of the elliptic curve $y(y+x+1) = x^3 - x - 2$. Then $G_{50,M} = \sum_{n>0} \bar{R}_{50,M}(n)q^n$ vanishes by construction at $M = 1, 2$ and yields eta quotients at $M = 3, 5$, with

$$\frac{G_{50,3}}{5} = \frac{\eta_2 \eta_{25}^2}{\eta_1^2 \eta_{50}} - 1, \quad \frac{G_{50,5}}{20} = \frac{\eta_2 \eta_{10}^3}{\eta_1^3 \eta_5}. \quad (129)$$

The Fourier coefficients of $G_{50,4}$ are also identified by an eta quotient: $\bar{R}_{50,4}(n)/10$ is the coefficient of q^{2n} in the Fourier expansion of η_{25}/η_1 .

We found that $(X, Y) = (G_{50,3}, G_{50,4})$ is a point on the curve

$$Y^3 + 4(X+5)Y^2 + 2(X+5)(X+10)Y = X(X+5)(X^2 + 8X + 20) \quad (130)$$

which Sage confirmed as having genus 2.

Proceeding similarly for $(X, Y) = (G_{N,3}, G_{N,4})$ we found the curves

$$Y^3 + 55Y^2 - 2(X^2 + 11X - 484)Y = X(X^3 + 34X^2 + 473X + 2904) \quad (131)$$

$$Y^3 + 2(4X + 69)Y^2 + (9X^2 + 460X + 4761)Y = X(X^3 + 55X^2 + 1035X + 6348) \quad (132)$$

$$Y^3 + 8(X + 13)Y^2 + 4(X + 13)(3X + 52)Y = X(X + 13)(X^2 + 28X + 208) \quad (133)$$

$$Y^3 + 21Y^2 + (5X^2 + 70X + 392)Y = X(X + 14)(X^2 + 14X + 56) \quad (134)$$

$$Y^3 + 2(2X + 29)Y^2 - (4X^2 - 58X - 841)Y = X(X^3 + 31X^2 + 406X + 1682) \quad (135)$$

$$Y^3 + 2(4X + 31)Y^2 + (11X^2 + 217X + 961)Y = X(X^3 + 31X^2 + 310X + 961) \quad (136)$$

$$Y^3 - 8XY^2 - X(2X + 259)Y = X(X^3 - 10X^2 + 148X + 1369) \quad (137)$$

at $N = 22, 23, 26, 28, 29, 31, 37$, respectively. All have genus 2.

3.6 Genus 3

We found these genus 3 curves for $N = 30, 33, 34, 35, 39, 40, 41, 43, 45, 48, 64$:

$$\begin{aligned} & Y^2(Y - 2X)(Y - 3X) + X(7X^2 - 30X + 75)Y \\ &= X^5 + 25X(4X - 5), \text{ with } (X, Y) = (G_{30,4} + 5, G_{30,5} + 10), \end{aligned} \quad (138)$$

$$\begin{aligned} & Y^4 + X(5X - 11)Y^2 - X^2(4X - 11)Y \\ &= X^3(X^2 - 11X + 22), \text{ with } (X, Y) = (G_{33,4} + 11, G_{33,5} + 11), \end{aligned} \quad (139)$$

$$\begin{aligned} & Y^4 + 10XY^3 + X(21X - 221)Y^2 + 2X(3X^2 - 119X + 867)Y \\ &= X(X^4 - 2X^3 + 51X^2 - 578X + 4913), \text{ with } (X, Y) = (G_{34,4} + 17, G_{34,5} + 17), \end{aligned} \quad (140)$$

$$\begin{aligned} & Y^4 + 10(X - 3)Y^3 + (31X^2 - 210X + 800)Y^2 + (12X^3 + 25X^2 + 200X - 2000)Y \\ &= X(X^4 - 5X^3 - 15X^2 - 200X - 2000), \text{ with } (X, Y) = (G_{35,4} + 20, G_{35,5} + 10), \end{aligned} \quad (141)$$

$$\begin{aligned} & Y^4 + 5XY^3 + 3X(X + 13)Y^2 - X(19X^2 - 234X + 507)Y \\ &= X(X^4 - 14X^3 + 234X^2 - 1690X + 2197), \text{ with } (X, Y) = (G_{39,4} + 13, G_{39,5} + 13), \end{aligned} \quad (142)$$

$$Y^4 = X(X + 5)(X^2(X + 4) - 4Y^2), \text{ with } (X, Y) = (G_{40,4}, G_{40,5}), \quad (143)$$

$$\begin{aligned} & Y^4 + (10X - 41)Y^3 + X(30X - 451)Y^2 + X^2(11X - 1681)Y \\ &= X^3(X^2 + 70X + 2214), \text{ with } (X, Y) = (G_{41,4}, G_{41,5} + 41), \end{aligned} \quad (144)$$

$$\begin{aligned} & 32Y^4 + (40X + 43)Y^3 + (94X^2 + 1591X + 9245)Y^2 + X(49X^2 + 946X + 5547)Y \\ &= X^3(X^2 + 21X + 129), \text{ with } (X, Y) = (G_{43,3}, G_{43,5} - 2G_{43,3}), \end{aligned} \quad (145)$$

$$(Y^2 + 5X)^2 = X^3(X^2 - Y), \text{ with } (X, Y) = (G_{45,4} + 5, G_{45,5} + 5), \quad (146)$$

$$Y^4 = X^3(X - 3)(X - 4), \text{ with } (X, Y) = (G_{48,4} + 6, G_{48,5} + 6), \quad (147)$$

$$Y(Y^2 + 4) = X^4, \text{ with } (X, Y) = (G_{64,3} + 2, G_{64,4} + 2), \quad (148)$$

where the final curve at $N = 64$ was already given in (126) and was obtained by subtractions that make $G_{64,M}$ vanish at $M = 1, 2, 5$. At $N = 43$, the subtractions make $G_{43,M}$ vanish at $M = 1, 2, 4$. In all other cases with genus 3, $G_{N,M}$ vanishes for $M = 1, 2, 3$.

3.7 Genus 4

At $N = 81$, we found that $(X, Y) = (G_{81,5}, G_{81,6})$ lies on the genus 4 curve

$$Y^3(Y+3)^2 + 3(X+3)^3(Y+3)(2Y+3) = (X+3)^6, \quad \frac{Y}{3} = \frac{\eta_3^4}{\eta_1^3\eta_9} - 1. \quad (149)$$

The Fourier coefficients of $X/9$ form an integer sequence beginning with

$$1, 2, 4, 7, 13, 21, 35, 55, 87, 132, 200, 295, 434, 625, 897, 1267, 1782, 2475, \dots$$

for $n = 1$ to 18. The general term is given by Rademacher sums as

$$\frac{R_{81,5}(n) + R_{81,2}(n)}{9} = \frac{\exp(4\pi\sqrt{5n}/9)}{27(4n^3/5)^{1/4}} \left(1 - \frac{27}{32\pi\sqrt{5n}} + O(1/n) \right). \quad (150)$$

The Fourier coefficients of $Y/9$ form an integer sequence beginning with

$$1, 3, 6, 13, 24, 45, 77, 132, 216, 351, 552, 861, 1313, 1986, 2952, 4354, 6336, \dots$$

for $n = 1$ to 17. The general term is given by a Rademacher sum

$$\frac{R_{81,6}(n)}{9} = \frac{\exp(4\pi\sqrt{6n}/9)}{27(2n^3/3)^{1/4}} \left(1 - \frac{27}{32\pi\sqrt{6n}} + O(1/n) \right). \quad (151)$$

3.8 Genus 5

At $N = 72$, with genus 5, we obtain integer Fourier coefficients in

$$G_{72,M} = \sum_{n>0} \bar{R}_{72,M}(n)q^n, \quad \bar{R}_{72,M}(n) = R_{72,M}(n) - \sum_{r=1,2,3,5,7} p_r(M)R_{72,r}(n), \quad (152)$$

$$\eta_{12}^2\eta_{18}^4/\eta_{36}^2 = \sum_{n>0} p_1(n)q^n, \quad \eta_{12}^4 = \sum_{n>0} p_2(n)q^n, \quad \eta_6\eta_{12}\eta_{18}\eta_{36} = \sum_{n>0} p_3(n)q^n, \quad (153)$$

$$\eta_6^2\eta_{36}^4/\eta_{18}^2 = \sum_{n>0} p_5(n)q^n, \quad \eta_6^4 = \sum_{n>0} (p_1(n) - 4p_7(n))q^n. \quad (154)$$

Then $G_{72,M}$ vanishes for $M = 1, 2, 3, 5, 7$. Moreover $(G_{72,4}, G_{72,6})$ is a point on the elliptic curve E_{36} , while $(G_{72,6}, G_{72,9})$ lies on E_{24} . Eliminating $G_{72,6}$, we obtain a genus 5 curve from the resultant:

$$(Y^2 + 12Y + 34 - 2(X+2)^3)^2 = ((X+2)^3 + 1)((X+2)^3 - 2)^2, \quad (155)$$

$$X = G_{72,4} = \frac{6\eta_2\eta_3\eta_{18}^2}{\eta_1^2\eta_6\eta_9}, \quad Y = G_{72,9} = \frac{6\eta_2^5\eta_3^2\eta_6}{\eta_1^6\eta_4\eta_{12}} - 6. \quad (156)$$

The Fourier coefficients of $X/6$ are the integers $R_{18,1}(n)/6 = A128129(n)$. The Fourier coefficients of $Y/12$ form an integer sequence beginning with

$$3, 11, 33, 87, 210, 473, 1008, 2055, 4035, 7674, 14196, 25629, 45282, 78472, \dots$$

for $n = 1$ to 14. The general term is given by Rademacher sums as

$$\frac{R_{24,3}(n) + R_{24,1}(n)}{12} = \frac{\exp(\pi\sqrt{2n})}{24(2n^3)^{1/4}} \left(1 - \frac{3}{8\pi\sqrt{2n}} + O(1/n) \right). \quad (157)$$

3.9 Genus 6

Moving on to the genus 6 case $N = 121$, we determined that subtractions of $R_{121,r}(n)$ are needed for the 6 values $r = 1, 2, 3, 4, 6, 11$. The coefficients of these subtractions are determined by four new forms and two old forms of weight 2 and level 121. The new forms are the L-series of the elliptic curves $y^2 + xy + y = x^3 + x^2 - 30x - 76$, $y^2 + y = x^3 - x^2 - 7x + 10$, $y^2 + xy = x^3 + x^2 - 2x - 7$ and $y^2 + y = x^3 - x^2 - 40x - 221$. The old forms are $(\eta_1 \eta_{11})^2$ and $(\eta_{11} \eta_{121})^2$. The first two non-zero integer series are

$$\bar{R}_{121,5}(n) = R_{121,5}(n) - R_{121,4}(n) - R_{121,3}(n), \quad \bar{R}_{121,7}(n) = R_{121,7}(n) + R_{121,6}(n). \quad (158)$$

We expect their generators, $G_{121,5}$ and $G_{121,7}$, to define a curve of genus 6 with degree 7 in $G_{121,5}$ and degree 5 in $G_{121,7}$. This is indeed the case. We found that $(X, Y) = (G_{121,5}/11, G_{121,7}/11)$ is a point on the curve

$$\begin{aligned} Y^5 - 20XY^4 + 5X(10X - 9)Y^3 &= X(132X^3 - 64X^2 - 33X + 31)Y^2 \\ &\quad + X(33X^4 + 95X^3 - 48X^2 - 5X + 9)Y \\ &\quad + X(121X^6 - 66X^5 + 23X^4 + 18X^3 - 9X^2 + 1) \end{aligned} \quad (159)$$

which Sage confirmed as having genus 6. Since integer combinations of Rademacher sums are computable with ease, we are able to validate this curve up to $O(q^{1000})$ in a matter of seconds.

3.10 Genus 7

In the genus 7 case $N = 100$, we determined that subtractions of $R_{100,r}(n)$ are required for the 7 values $r = 1, 2, 3, 4, 5, 7, 9$, with coefficients determined by 6 old forms of weight 2 and level 100 and a new form

$$f_{100} = q + 2q^3 - 2q^7 + q^9 - 2q^{13} + 6q^{17} - 4q^{19} - 4q^{21} - 6q^{23} - 4q^{27} + 6q^{29} + O(q^{31}) \quad (160)$$

which is the L-series of the elliptic curve $y^2 = x^3 - x^2 - 33x + 62$.

We found that $(X, Y) = (G_{100,6} + 5, G_{100,15} + 10)$ lies on the genus 7 curve

$$\begin{aligned} Y^6 = & X(3X^4 - 90X^3 + 415X^2 - 560X + 200)Y^4 \\ & - X(3X^9 + 20X^8 - 350X^7 + 1795X^6 - 4790X^5 \\ & + 7805X^4 - 8350X^3 + 7325X^2 - 4625X + 1375)Y^2 \\ & + X(X^2 + 2X + 5)(X^4 - 6X^3 + 14X^2 - 10X + 5)^2(X^4 - 5X^3 + 15X^2 - 25X + 25), \end{aligned} \quad (161)$$

with $X/5 = \eta_2 \eta_{25}^2 / (\eta_1^2 \eta_{50})$ and the Fourier coefficients of $Y/10$ given by

$$1, 6, 26, 88, 258, 686, 1688, 3904, 8594, 18142, 36946, 72952, 140184, 262948, \dots$$

for $n = 0$ to 13, and in general by $(R_{100,15}(n) + 2R_{100,5}(n))/10$ for $n > 0$.

3.11 Genus 8

When $N = p^2$ with prime $p = 12k + 1$, the genus of $I_0(N)$ is given by $g_0(N) = 3k(4k - 1) - 1$. Setting $k = 1$, we obtain $g_0(169) = 8$. Moreover $N = 169$ is the largest level with genus 8. We devised a procedure of 8 subtractions that gives integer sequences $\bar{R}_{169,M}(n)$ by subtracting multiples of $R_{169,r}(n)$, with $r = 1, 2, 3, 4, 5, 6, 8, 9$. The subtraction coefficients are determined by 8 modular forms of level 169 and weight 2, of which two have Fourier coefficients in $\mathbf{Q}(\sqrt{3})$. The rest have coefficients in the cubic number fields $x(x^2 - 1) = \pm(1 - 2x^2)$.

Our first non-zero integer sequence occurs at $M = 7$, where

$$\bar{R}_{169,7}(n) = R_{169,7}(n) - R_{169,6}(n) - R_{169,5}(n) + R_{169,2}(n) \quad (162)$$

is the coefficient of q^n in $G_{169,7} = 13\eta_{169}/\eta_1$. There is no subtraction at $M = 13$, where $G_{169,13} = G_{13,1} = 13\eta_{13}^2/\eta_1^2$.

We found that $(X, Y) = (G_{169,7}/13, G_{169,13}/13)$ is point on a genus 8 curve with degree 13 in X and degree 7 in Y , namely

$$Y^7 = 143XY^6 + 156X(39X^2 - 17X + 3)Y^5 + X \sum_{k=2}^6 P_k(X)Y^{6-k}, \quad (163)$$

$$P_2(X) = 26(3211X^4 - 2249X^3 + 819X^2 - 173X + 19), \quad (164)$$

$$P_3(X) = 26(21970X^6 - 18759X^5 + 8619X^4 - 2743X^3 + 663X^2 - 111X + 10), \quad (165)$$

$$P_4(X) = 26(169X^4 - 104X^3 + 39X^2 - 8X + 1)(507X^4 - 169X^3 + 26X^2 - 13X + 3), \quad (166)$$

$$P_5(X) = 13(371293X^{10} - 371293X^9 + 199927X^8 - 81289X^7)$$

$$\begin{aligned}
& +28561X^6 - 8619X^5 + 2197X^4 - 481X^3 + 91X^2 - 13X + 1), \quad (167) \\
P_6(X) = & 4826809X^{12} - 4826809X^{11} + 2599051X^{10} - 1113879X^9 + 428415X^8 \\
& - 142805X^7 + 41743X^6 - 10985X^5 + 2535X^4 - 507X^3 + 91X^2 - 13X + 1. \quad (168)
\end{aligned}$$

3.12 Genus 13

Finally, we studied $N = 144$, with genus 13. Integer sequences are obtained after subtractions at $r = 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 19$, with coefficients determined by 11 old forms and two new forms. The old forms were already encountered as eta quotients at $N = 48$ and $N = 72$. They determine the subtractions for $R_{144,M}(n)$ when M is divisible by 3 or 2. The new forms relate to the subtractions that are needed when M is coprime to 6. One of these new forms is the eta quotient

$$f_{144a} = \frac{\eta_{12}^{12}}{\eta_4^6 \eta_{24}^4} = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} + \dots \quad (169)$$

The other has Fourier coefficients determined by the L-series of the elliptic curve $y^2 = x^3 + 6x + 7$, which gives

$$f_{144b} = q + 2q^5 + 4q^{11} - 2q^{13} - 2q^{17} + 4q^{19} - 8q^{23} - q^{25} - 6q^{29} - 8q^{31} + \dots \quad (170)$$

We reduced this to a telling combination of 5 eta quotients with weight 2:

$$f_{144b} = \frac{\eta_{24}^3 \eta_{36}^2 \eta_{72}}{\eta_{48} \eta_{144}} + \frac{2\eta_{12} \eta_{24}^2 \eta_{72}^4}{\eta_{36} \eta_{48} \eta_{144}} + \frac{4\eta_{12}^2 \eta_{48} \eta_{72} \eta_{144}}{\eta_{24}} - \frac{2\eta_{12} \eta_{24}^3 \eta_{72} \eta_{144}^2}{\eta_{36} \eta_{48}^2} + \frac{4\eta_{12}^2 \eta_{144}^4}{\eta_{72}^2} \quad (171)$$

with $C_1(M) + 2C_5(M) + 4C_{11}(M) - 2C_{13}(M) + 4C_{19}(M)$ giving the coefficient of q^M in (171), where $C_r(M)R_{144,r}(n)$ is subtracted from $R_{144,M}(n)$ to make $\bar{R}_{144,M}(n)$ an integer sequence. The subtraction at $r = 7$ is determined by the eta quotient in (169) where the coefficient of q^M is $C_1(M) + 4C_7(M) + 2C_{13}(M) - 8C_{19}(M)$. Hence we determine all the subtractions by eta quotients.

We are left with 6 values of $M < 20$ for which $G_{144,M} = \sum_{n>0} \bar{R}_{144,M}(n)q^n$ is non-zero, namely $M = 8, 12, 15, 16, 17, 18$. To produce a genus 13 curve we should choose a coprime pair of M values. The simplest choice is the pair $(8, 15)$. We know that $(G_{144,8}, G_{144,18}) = (G_{72,4}, G_{72,9})$ gives a point on the genus 5 curve (155) found at $N = 72$. Moreover $(G_{144,15}, G_{144,18}) = (G_{48,5}, G_{48,6})$ gives a point on a genus 3 curve that is not hard to determine. Then, by taking a resultant to eliminate $G_{144,18}$, we determined that $(X, Y) = (G_{144,8} + 2, G_{144,15} + 6)$ lies on the genus 13 curve

$$(Y^4 - 8(X^3 + 1)^2)^2 = (X^3 + 1)(X^6 + 20X^3 - 8)^2 \quad (172)$$

neatly parametrized by eta quotients as follows

$$\frac{X}{2} = \frac{\eta_2^3 \eta_3}{\eta_1^3 \eta_6}, \quad \frac{Y}{6} = \frac{\eta_2^2 \eta_6^4}{\eta_1^4 \eta_{12}^2}. \quad (173)$$

3.13 Remarks

Remark 1. After we completed this work, Yajun Zhou kindly called our attention to [17, Theorem 8.12]. The methods used there may be capable of furnishing proofs of some of our empirical findings in Section 3, following the approach that Knopp [22] attributes to Rademacher [28] as an “entirely fresh viewpoint”, namely by adopting formulas (53,54) as definitions of Fourier coefficients of objects $G_{N,M}$ and demonstrating that the latter have the required modular properties. At genus 0, with a unique Hauptmodul, that could furnish a proof of Table 1. At higher genera, more work might be needed.

Remark 2. We conclude this section with a note on the approach in [32,33] to modular curves. In [33, Section 4.1], Yifan Yang gives modular curves, up to level $N = 50$, that are parametrized by quotients of “generalized” Dedekind eta functions [32], in the many cases where the eta function itself is insufficient to solve the problem. Moreover his q -expansions are highly singular as $q \rightarrow 0$. Our approach was quite different. We began with an explicit formula (53) that reproduces, at $M = 1$, the Fourier coefficients of the genus 0 eta quotients taken as “canonical” Hauptmoduln in [24, Table 8], which vanish as $q \rightarrow 0$. At genus 1, after subtraction of the non-integer sequence $R_{N,1}(n)$, we obtained $G_{N,M} = \sum_{n>0} \bar{R}_{N,M}(n)q^n$ as Fourier series with integer coefficients, vanishing at $q = 0$. Then $G_{N,2}$ and $G_{N,3}$ parametrize our modular curve. We were able to extend this to higher genera. It may be that our explicit Fourier coefficients are capable of reproducing those of Yang’s “generalized” Dedekind eta quotients, after performing a Fricke involution $z \mapsto -1/(Nz)$ on his Ansätze. We have not investigated this, since it lay outside the remit of our title.

4 Conclusions

1. Eta quotients on $\Gamma_0(6)$, with 4 cusps, neatly solve the equal-mass two and three loop sunrise problems, whose differential equations with respect to the external energy have 4 singular points. This cannot continue, since at higher loops there is more than one pseudo-threshold.
2. Atkin-Lehner transformations on $\Gamma_0(6)$ yield optimal nomes.
3. For the on-shell problem, Eichler integrals of eta quotients on $\Gamma_0(6)$ yield Bessel moments at 4 and 6 loops that are periods or quasi-periods.
4. Rademacher sums yield the Fourier coefficients of a Hauptmodul for $\Gamma_0(6)$ and for all other levels of genus 0.
5. After subtractions determined by weight 2 cusp forms, they yield the Fourier coefficients of vast numbers of eta quotients.

6. They yield the Fourier coefficients of parametrizations of modular curves, irrespective of whether the Fourier series are eta quotients.

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