FACTORIZATION THEOREMS FOR RELATIVELY PRIME DIVISOR SUMS, GCD SUMS AND GENERALIZED RAMANUJAN SUMS

HAMED MOUSAVI
MAXIE D. SCHMIDT

Abstract. We generalize recent matrix-based factorization theorems for Lambert series generating functions generating the coefficients $(f * 1)(n)$ for some arithmetic function f. Our new factorization theorems provide analogs to these established expansions generating sums of the form $\sum_{d:(d,n)=1} f(d)$ (type I) and the Anderson-Apostol sums $\sum_{d|(m,n)} f(d)g(n/d)$ (type II) for any arithmetic functions f and g. Our treatment of the type II sums includes a matrix-based factorization method relating the partition function $p(n)$ to arbitrary arithmetic functions f. We also conclude the last section of the article by directly expanding new formulas for an arithmetic function g by the type II sums using discrete Fourier transforms for functions over inputs of greatest common divisors and by suitably defined orthogonal polynomial sequences whose weight function we can define by an inverse Laplace (Mellin) transform involving the partition function $p(n)$. There are numerous applications and special cases of our new results which we are able to cite as examples in the article. Particular cases of the applications we give in the article include new identities for Euler's totient function, the Ramanujan sums $c_q(n)$, the generalized sum-of-divisors functions, the Mertens function which is the summatory function of the Möbius function, and the cyclotomic polynomials.

1. Introduction

1.1. **Motivation.** We are motivated by considering the breakdown of the partial sums of an arithmetic function $f(d)$ whose average order we would like to estimate into sums over the pairwise disjoint sets of component indices $d \leq x$:

$$\sum_{d\leq x} f(d) = \sum_{\substack{d=1\\(d,x)=1}}^{x} f(d) + \sum_{d|x} f(d) + \sum_{\substack{d=1\\1<(d,x)<x}}^{x} f(d). \tag{1}$$

In particular, in evaluating the partial sums of an arithmetic function $f(d)$ over all $d \leq x$, we wish to break the terms in these partial sums into three sets: those d relatively prime to x, the d dividing x, and the somewhat less "round" set of indices d which are neither relatively prime to x nor proper divisors of x. In particular, if we let f denote an arithmetic function, we define the remainder terms in our average order expansions as follows:

$$\widetilde{S}_f(x) = \sum_{d\leq x} f(d) - \sum_{\substack{d=1\\(d,x)=1}}^{x} f(d) - \sum_{d|x} f(d). \tag{2}$$

For instance, when $x = 24$ we have that

$$\widetilde{S}_f(24) = f(9) + f(10) + f(14) + f(16) + f(18) + f(20) + f(21) + f(22).$$

Date: Monday 22^{nd} October, 2018.
2010 *Mathematics Subject Classification.* 11N64; 11A25; 05A17.
Key words and phrases. divisor sum; totient function; matrix factorization; Möbius inversion; partition function.

We observe that the last divisor sum terms in (2) correspond to the coefficients of powers of q in the Lambert series generating function over f in the following form considered in the next subsection:

$$\sum_{d|x} f(d) = [q^x] \left(\sum_{n \geq 1} \frac{f(n)q^n}{1-q^n} \right), \ |q| < 1.$$

We can see that the average order sums on the left-hand-side of (1) correspond to the hybrid of divisor and relatively prime divisor sums of the form

$$\sum_{n \leq x} f(d) = \sum_{m|x} \sum_{\substack{k=1 \\ \left(k, \frac{x}{m}\right)=1}}^{\frac{x}{m}} f(km) = \sum_{m|x} \sum_{\substack{k=1 \\ (k,m)=1}}^{m} f\left(\frac{kx}{m}\right).$$

We study and prove new results relating both variants of the sums expanding the right-hand-side of the previous equation to restricted partitions and special partition functions. Namely, a combination of the results we prove in Section 2 and the Lambert series factorization theorem results summarized in the next subsection allow us to write[1]

$$\sum_{n \leq x} f(d) = \sum_{d|x+1} s_{x,n} \left[\sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} (-1)^{\left\lceil \frac{i}{2} \right\rceil} p(n-j) \chi_{1,k}(j-k-G_i) \left[j-k-G_i \geq 1 \right]_\delta \cdot f\left(\frac{(x+1)k}{n} \right) \right],$$

where $[n=k]_\delta = \delta_{n,k}$ denotes Iverson's convention, $G_j := \frac{1}{2} \lceil j/2 \rceil \lceil (3j+1)/2 \rceil$ denotes the sequence of interleaved, or generalized pentagonal numbers, the triangular sequence $s_{n,k} := [q^n](q;q)_\infty \frac{q^k}{1-q^k}$ corresponds to the difference of restricted partition functions discussed in the next subsection, and $p(n)$ is the classical partition function. The analysis of the asymptotic properties of these sums is a central topic in the study of the behavior of arithmetic functions, analytic number theory, and in applications such as algorithmic analysis. Our new results connect variants of such sums over multiplicative functions with the distinctly additive flavor of the theory of partitions.

1.2. **Variantions on recent work.** There is a fairly complete and extensive set of expansions providing identities related to these Lambert series generating functions and their matrix factorizations in the form of so-termed "Lambert series factorization theorems" studied by Merca and Schmidt in 2017–2018 [15, 9, 11]. These results provide factorizations for a Lambert series generating function over the arbitrary arithmetic function f expanded in the form of

$$L_f(q) := \sum_{n \geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k) \right) q^n, \tag{3}$$

where $s_{n,k} = s_o(n,k) - s_e(n,k)$ is independent of f and is defined as the difference of the functions $s_{o/e}(n,k)$ which respectively denote the number of k's in all partitions of n into an odd (even) number of distinct parts. These so-termed factorization theorems, which effectively provide a matrix-based expansion of an

[1] For natural numbers $k \geq 1$, we use the notation $\chi_{1,k}(n)$ to denote the *principal Dirichlet character* (modulo k) which is defined explicitly for all $n \geq 1$ by [12, §27.8]

$$\chi_{1,k}(n) = \begin{cases} 1, & \text{if } (n,k) = 1; \\ 0, & \text{if } (n,k) > 1. \end{cases}$$

ordinary generating function for the divisor sums of the type enumerated by Lambert series expansions, connect the additive theory of partitions to the more multiplicative constructions of power series generating functions found in other branches on number theory. In these cases, it appears that it is most natural, in some sense, to expand these sums via the factorizations defined in (3) since the matrix entries (and their inverses) are also partition-related. It then leads us to the question of what other natural, or cannonical analogous expansions can be formed for other more general variants of the above divisor sums.

More generally, we can form analogous matrix-based factorizations of the generating functions of the next summation sequences provided that these transformations are invertible:

$$\sum_{n\geq 1}\left(\sum_{\substack{k=1\\k\in\mathcal{A}_n}}^{n}\sum_{\mathcal{A}_n\subseteq[1,n]} f(k)\right)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n} v_{n,k}(\mathcal{A})f(k)\right)q^n,$$

The sums in (4) below are refered to as *type I* and *type II sums* in the next subsections as the special cases of the generalized factorizations defined by the previous equation when $\mathcal{A}_n := \{d : 1 \leq d \leq n, (d,n)=1\}$ and $\mathcal{A}_n := \{d : 1 \leq d \leq n, d|(k,n)=1\}$ for some $1\leq k\leq n$, respectively.

$$\begin{aligned} T_f(x) &= \sum_{\substack{d=1\\(d,x)=1}}^{x} f(d) \\ L_{f,g,k}(x) &= \sum_{d|(k,x)} f(d)g\left(\frac{x}{d}\right) \end{aligned} \tag{4}$$

In particular, we define the following preliminary constructions for the factorizations of the Lambert-like series whose coefficients are respectively given by the sums in the previous two definitions:

$$T_f(x) = [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k}f(k)\cdot q^n + f(1)\cdot q\right) \tag{5a}$$

$$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left[\sum_{m=1}^{k} L_{f,g,m}(k)w^m\right]\cdot q^n\right),\ w\in\mathbb{C}. \tag{5b}$$

We focus on the special expansions of each factorization type in Section 2 and Section 3, respectively, though we note that other related variants of these expansions are possible.

1.3. **Applications of our new results.**

1.3.1. *Forms of the type I sums.* The identities and theorems we prove for the general sum case defined by (4) in Section 2 can be useful in constructing new identities for well-known functions which have not yet been discovered, and hence, are not well explored yet. We give a few notable examples of summation identities which express classical functions and combinatorial objects in new ways below to illustrate the stylistic components to our new methods before we set out to prove them for the general case later in this article.

We obtain the following identities for Euler's totient function where and where $[n=k]_\delta \equiv \delta_{n,k}$ denotes *Iverson's convention* based on our new constructions:

$$\phi(n) = \sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i \geq 1\right]_\delta + [n=1]_\delta$$

$$\phi(n) = \sum_{\substack{d=1\\(d,n)=1}}^{n}\left(\sum_{k=1}^{d+1}\sum_{i=1}^{d}\sum_{j=0}^{k} p(i+1-k)(-1)^{\lceil j/2\rceil}\phi(k-G_j)\mu_{d,i}\left[k-G_j \geq 1\right]_\delta\right).$$

To give another related example that applies to classical multiplicative functions, recall that we have a known representation for the Möbius function given as an exponential sum in terms of powers of the n^{th} primitive roots of unity given by [3, §16.6]

$$\mu(n) = \sum_{\substack{d=1\\(d,n)=1}}^{n} \exp\left(2\pi\imath \frac{d}{n}\right).$$

The *Mertens function*, $M(x)$, is defined as the summatory function over the Möbius function $\mu(n)$ for all $n \leq x$. Using the definition of the Möbius function as one of our type I sums defined above, we have new expansions for the Mertens function given by (*cf.* Corollary 3.15)

$$M(x) = \sum_{1\leq k<j\leq n\leq x}\left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i \geq 1\right]_\delta e^{2\pi\imath k/n}\right).$$

Finally, we can form another related *polynomial* sum of the type indicated above when we consider the logarithm of the *cyclotomic polynomials* leads to the sums

$$\begin{aligned}\log \Phi_n(z) &= \sum_{\substack{1\leq k\leq n\\(k,n)=1}} \log\left(z - e^{2\pi\imath k/n}\right)\\ &= \sum_{1\leq k<j\leq n}\left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i \geq 1\right]_\delta \log\left(z-e^{2\pi\imath k/n}\right)\right).\end{aligned}$$

1.3.2. *Forms of the type II sums.* The sums $L_{f,g,k}(n)$ are sometimes refered to as *Anderson-Apostol sums* named after the authors who first defined them (*cf.* [2, §8.3] [1]). Other variants and generalizations of these sums are studied in the references [5, 7]. There are many number theoretic applications of the periodic sums factorized in this form. For example, the famous form of Ramanujan's sum $c_q(n)$ is expressed as the following right-hand-side divisor sum [4, §IX]:

$$c_q(n) = \sum_{\substack{d=1\\(d,n)=1}}^{n} e^{2\pi\imath dn/q} = \sum_{d|(q,n)} d\cdot\mu(q/d).$$

The applications of our new results to Ramanujan's sum include the expansions

$$c_n(x) = [w^x]\left(\sum_{k=1}^{n} u_{n,k}^{(-1)}(\mu,w)\sum_{j\geq 0}(-1)^{\lceil j/2\rceil}\mu(k-G_j)\right)$$

$$= \sum_{k=1}^{n} \left(\sum_{d|(n,x)} d \cdot p(n/d-k) \right) \sum_{j \geq 0} (-1)^{\lceil j/2 \rceil} \mu(k - G_j),$$

where the inverse matrices $u_{n,k}^{(-1)}(\mu, w)$ are expanded according to Proposition 3.1. We then immediately have the following new results for the next special expansions of the generalized sum-of-divisors functions for $\Re(s) > 0$:

$$\begin{aligned} \sigma_s(n) &= n^s \zeta(s+1) \times \sum_{i=1}^{\infty} \sum_{k=1}^{i} \left(\sum_{d|(n,i)} d \cdot p(i/d-k) \right) \sum_{j \geq 0} \frac{(-1)^{\lceil j/2 \rceil} \mu(k-G_j)}{i^{s+1}} \\ &= n^s \zeta(s+1) \times \sum_{i=1}^{n} \sum_{k \geq 0} \frac{\mu\left(\frac{nk+i}{(n,i)}\right)}{(nk+i)^{s+1}}. \end{aligned}$$

Section 3.3 expands the left-hand-side function $g(x)$ in (5b) by considering a new indirect method involving the sums $L_{f,g,k}(n)$ using discrete Fourier transforms of functions of the greatest common divisor studied in [6, 16]. This method allows us to study the factorization forms in (5b) by bypassing the complicated forms of the ordinary matrix coefficients $u_{n,k}(f, w)$ which we expand in Corollary 3.4 of Section 3.1. These discrete Fourier series methods lead to the next key result proved in Theorem 3.13 that

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} \varphi(d) f(d) (k/d)^2 g(k/d),$$

where $e(x) = \exp(2\pi\imath \cdot x)$ is standard notation for the exponential function.

1.4. **Significance of our new results.** Our new results provide generating function expansions for the type I and type II sums in the form of matrix-based factorization theorems. The matrix products involved in expressing the coefficients of these generating functions for arbitrary arithmetic functions f and g are closely related to the partition function $p(n)$. The known Lambert series factorization theorems proved in the references and which are summarized in the subsections on variants above demonstrate the flavor of the matrix-based expansions of these forms for ordinary divisor sums of the form $(f * 1)(n) = \sum_{d|n} f(d)$. Our extensions of these factorization theorem approaches in the context of the new forms of the type I and type II sums similarly relate special arithmetic functions in number theory to partition functions and more additive branches of number theory. The last results proved in Section 3.3 are expanded in the spirit of these matrix factorization constructions using discrete Fourier transforms of functions (and sums of functions) evaluated at greatest common divisors. We pay special attention to illustrating our new results with many relevant examples and new identities expanding famous special number theoretic functions throughout the article.

2. Factorization theorems for sums of the first type

2.1. **Inversion relations.** We begin our exploration here by expanding an inversion formula which is analogous to Möbius inversion for ordinary divisor sums. We prove the following result which is the analog to the sequence inversion relation provided by the Möbius transform in the context of our sums over the integers relatively prime to n [13, *cf.* §2, §3].

Proposition 2.1 (Inversion Formula)**.** *For all $n \geq 2$, there is a unique lower triangular sequence, denoted by $\mu_{n,k}$, which satisfies the inversion relation*

$$g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) \qquad \Longleftrightarrow \qquad f(n) = \sum_{d=1}^{n} g(d+1)\mu_{n,d}.$$

Moreover, if we form the matrix $(\mu_{i,j})_{1 \leq i,j \leq n}$ for any $n \geq 2$, we have that the inverse sequence satisfies

$$\mu_{n,k}^{(-1)} = [(n+1,k)=1]_\delta \, [k \leq n]_\delta \, .$$

Proof. Consider the $(n-1) \times (n-1)$ matrix

$$([(i,j)=1]_\delta)_{1 \leq i,j < n} \, ,$$

which effectively corresponds to the formula on the left-hand-side of the first equation by applying the matrix to the vector of $[f(1) \; f(2) \; \cdots f(n)]^T$ and extracting the n^{th} column as our stated formula. Since $\gcd(n, n-1) = 1$ for all $n > 1$, we see that this matrix is lower triangular with ones on its diagonal. Thus the matrix is non-singular and its unique inverse, which we denote by $(\mu_{i,j})_{1 \leq i,j < n}$, leads to the sum on the right-hand-side of the first equation when we shift $n \mapsto n+1$. The second equation restates the form of the first matrix when we perform the shift of $n \mapsto n+1$ as on the right-hand-side of the first equation. $\square$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0
−1	0	2	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	−1	0	1	0	0	0	0	0	0	0	0
1	0	−1	1	0	−1	1	−1	−1	1	0	0	0	0	0	0	0
−1	0	1	0	0	0	−1	0	0	0	1	0	0	0	0	0	0
1	0	−1	0	0	0	1	0	0	−1	−1	1	0	0	0	0	0
3	0	−2	0	−2	0	2	0	−1	0	−1	0	1	0	0	0	0
−3	0	1	0	3	0	−1	−1	1	0	0	0	−1	1	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	−1	0	1	0	0
1	0	0	0	−2	0	0	1	0	0	1	−1	1	−1	−1	1	0
−3	0	2	0	2	0	−2	0	1	0	0	0	−1	0	0	0	1

$\mu_{n,k}$ *for* $1 \leq n,k < 18$

Figure 2.1. *Inversion formula coefficient sequences*

Remark 2.2. Figure 2.1 provides a listing of the relevant analogs to the Möbius function in the context of the Möbius transform of the ordinary divisor sum over an arithmetic function from the proposition. We not know of a comparatively simple closed-form function for the sequence of $\mu_{n,k}$ [17, *cf.* A096433]. However, we readily see by construction that the sequence and its inverse satisfy

$$\sum_{\substack{d=1\\(d,n)=1}}^{n} \mu_{d,k} = 0$$

$$\sum_{\substack{d=1\\(d,n)=1}}^{n} \mu_{d,k}^{(-1)} = \phi(n),$$

where $\phi(n)$ is Euler's totient function. The first columns of the corresponding sums in the previous equation performed over the columns index k for fixed n appear in the integer sequences database as the entry [17, A096433].

2.2. **Exact formulas for the factorization matrices.** The next result is key to proving the exact formulas for the matrix sequences, $t_{n,k}$ and $t_{n,k}^{(-1)}$, and their expansions by the partition functions defined in the introduction. We prove the following result first as a lemma which we will use in the proof of Theorem 2.4 given below. The first several rows of the matrix sequence $t_{n,k}$ and its inverse implicit to the factorization theorem in (5) are tabulated in Figure 2.2 for intuition on the formulas we prove in the next proposition and following theorem.

Lemma 2.3 (A Convolution Identity for Relatively Prime Integers)**.** *For all natural numbers $n \geq 2$ and $k \geq 1$ with $k \leq n$, we have the following expression for the indicator function of whether (n,k) forms a pair of relatively prime integers:*

$$\sum_{j=1}^{n} t_{j,k} p(n-j) = \chi_{1,k}(n).$$

Equivalently, we have that

$$t_{n,k} = \sum_{i=0}^{n} (-1)^{\lceil i/2 \rceil} \chi_{1,k}(n-G_i) \left[n - G_i \geq k+1\right]_{\delta}.$$

Proof. We begin by noticing that the right-hand-side expression in the statement of the lemma is equal to $\mu_{n,k}^{(-1)}$ by the construction of the sequence in Proposition 2.1. Next, we see that the factorization in (5a) is equivalent to the expansion

$$\sum_{d=1}^{n-1} f(d) \mu_{n,d}^{(-1)} = \sum_{j=1}^{n} \sum_{k=1}^{j} p(n-j) t_{j,k} \cdot f(k).$$

Since $\mu_{n,k}^{(-1)} = [(n+1,k)=1]_{\delta}$, we may take the coefficients of $f(k)$ on each side of the previous equation for each $1 \leq k < n$ to establish the claimed result. The equivalent statement of the first result follows by a generating function argument applied to the product that generates the left-hand-side Cauchy product in the first equation. $\square$

1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0
−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	−1	−2	−1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	−1	−1	−1	1	0	0	0	0	0	0	0
1	0	−1	0	−1	−1	0	1	0	0	0	0	0	0
1	1	1	0	−2	0	−1	−1	1	0	0	0	0	0
1	0	1	0	1	1	−1	0	0	1	0	0	0	0
1	1	0	1	1	0	−1	−1	−2	−1	1	0	0	0
0	0	1	0	1	0	0	0	0	0	0	1	0	0
0	1	1	1	1	0	−1	0	0	−1	−1	−1	1	0

(i) $t_{n,k}$

1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0	0
4	3	2	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
5	3	2	2	1	1	1	0	0	0	0	0	0
4	4	3	1	1	1	0	1	0	0	0	0	0
15	11	8	5	4	2	1	1	1	0	0	0	0
−1	−1	−1	1	0	0	1	0	0	1	0	0	0
32	24	18	12	9	6	4	3	2	1	1	0	0
−6	−4	−3	−1	−1	0	0	0	0	0	0	1	0
24	17	13	12	8	7	6	3	2	2	1	1	1

(ii) $t_{n,k}^{(-1)}$

Figure 2.2. *The factorization matrices,* $t_{n,k}$ *and* $t_{n,k}^{(-1)}$*, for* $1 \leq n, k < 14$

Theorem 2.4 (Exact Formulas for the Factorization Matrix Sequences)**.** *For integers* $n, k \geq 1$*, the two invertible lower triangular factorization sequences defining the expansion of* (5a) *satisfy exact formulas* *given by*

$$t_{n,k} = \sum_{j=0}^{n} (-1)^{\lceil j/2 \rceil} \chi_{1,k}(n-k-G_j) \left[n-k-G_j \geq 1\right]_{\delta} \tag{i}$$

$$t_{n,k}^{(-1)} = \sum_{d=1}^{n} p(d-k) \mu_{n,d}, \tag{ii}$$

where we define the sequence of interleaved pentagonal numbers G_j as in the introduction.

Proof of (i). It is plain to see by the considerations in our construction of the factorization theorem that both matrix sequences are lower triangular. Thus, we need only consider the cases where $n \leq k$. By a convolution of generating functions, the identity in Lemma 2.3 shows that

$$t_{n,k} = \sum_{j=k}^{n} [q^{n-j}](q;q)_\infty \cdot \left[(j+1,k)=1\right]_\delta .$$

Then shifting the index of summation in the previous equation implies (i). □

Proof of (ii). To prove (ii), we consider the factorization theorem when $f(n) := t_{n,r}^{(-1)}$ for some fixed $r \geq 1$. We then expand (5a) as

$$\begin{aligned}
\sum_{\substack{d=1 \\ (d,n)=1}}^{n} t_{d,r}^{(-1)} &= [q^n] \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n-1} t_{n,k} \cdot t_{k,r}^{(-1)} \cdot q^n \\
&= \sum_{j=1}^{n} p(n-j) \times \sum_{k=1}^{j-1} t_{j,k} t_{k,r}^{(-1)} \\
&= \sum_{j=1}^{n} p(n-j) \left[r = j-1\right]_\delta \\
&= p(n-1-r).
\end{aligned}$$

Hence we may perform the inversion by Proposition 2.1 to the left-hand-side sum in the previous equations to obtain our stated result. □

Remark 2.5 (Relations to the Lambert Series Factorization Theorems)**.** We notice that by inclusion-exclusion applied to the right-hand-side of (5a), we may write our matrices $t_{n,k}$ in terms of the triangular sequence expanded as differences of restricted partitions in (3). For example, when $k := 12$ we see that

$$\sum_{n \geq 12} \left[(n,12)=1\right]_\delta q^n = \frac{q^{12}}{1-q} - \frac{q^{12}}{1-q^2} - \frac{q^{12}}{1-q^3} + \frac{q^{12}}{1-q^6}.$$

In general, when $k > 1$ we can expand

$$\sum_{n \geq k} \left[(n,k)=1\right]_\delta q^n = \sum_{d|k} \frac{q^k \mu(d)}{1-q^d}.$$

Thus we can relate the triangles $t_{n,k}$ in this article to the $s_{n,k}$ employed in the expansions from the references as follows:

$$t_{n,k} = \begin{cases} s_{n,k}, & k = 1; \\ \sum_{d|k} s_{n+1-k+d,d} \cdot \mu(d), & k > 1. \end{cases}$$

2.3. **Completing the proofs of the main applications.** We remark that as in the Lambert series factorization results from the references [9], we have three primary expansion types of identities that we primarily consider for any fixed choice of the arithmetic function f in the forms of

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} p(n-j) t_{j-1,k} f(k) + f(1) \left[n=1\right]_{\delta} \tag{6a}$$

$$\sum_{k=1}^{n-1} t_{n-1,k} f(k) = \sum_{j=1}^{n} \sum_{\substack{d=1 \\ (d,j)=1}}^{j} [q^{n-j}](q;q)_{\infty} \cdot f(d) - [q^{n-1}](q;q)_{\infty} \cdot f(1), \tag{6b}$$

and the corresponding inverted formula providing that

$$f(n) = \sum_{k=1}^{n} t_{n,k}^{(-1)} \left(\sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\lceil \frac{j}{2} \rceil} T_f(k+1-G_j) - [q^k](q;q)_{\infty} \cdot f(1) \right). \tag{6c}$$

Now the applications cited in the introduction follow immediately and require no further proof than to cite these results for the respective special cases of f. We provide other similar corollaries and examples of these factorization theorem results below.

Example 2.6 (Sum-Of-Divisors Functions)**.** For any $\alpha \in \mathbb{C}$, the last expansion identity in (6) also implies the following new formula for the generalized sum-of-divisors functions, $\sigma_\alpha = \sum_{d|n} d^\alpha$:

$$\sigma_\alpha(n) = \sum_{d|n} \sum_{k=1}^{d} t_{d,k}^{(-1)} \left(\sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\lceil \frac{j}{2} \rceil} \phi_\alpha(k+1-G_j) - [q^k](q;q)_{\infty} \right).$$

In particular, when $\alpha := 0$ we obtain the next identity for the divisor function $d(n)$ expanded in terms of Euler's totient function.

$$d(n) = \sum_{d|n} \sum_{k=1}^{d} t_{d,k}^{(-1)} \left(\sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\lceil \frac{j}{2} \rceil} \varphi(k+1-G_j) - [q^k](q;q)_{\infty} \right).$$

Remark 2.7. There are also numerous noteworthy applications of the expansions of the type-I sums we have proved in this section in the context of exact (and asymptotic) expansions of named partition functions. For instance, Rademacher's exact series formula for the partition function $p(n)$ involves Dedekind sums implicitly expanded through sums of this type. Similarly, an asymptotic approximation for the named special function $q(n) = [q^n](-q;q)_{\infty}$ [17, A000009] which counts the number of partitions of n into distinct parts involves an infinite series over modified Bessel functions and nested Kloosterman sums [12, §26.10(vi)].

We have not attempted to study the usefulness of our new finite sums in these contexts in deconstructing asymptotic properties in these more famous examples of partition formulas. A detailed treatment is nonetheless suggested as an exercise to readers which may unravel some undiscovered combinatorial twists to the expansions of such sums.

Example 2.8 (Menon's Identity and Related Arithmetical Sums)**.** We can use our new results proved in this section to expand new identities for known closed-forms of special arithmetic sums. For example, *Menon's identity* [18] states that

$$\varphi(n)d(n) = \sum_{\substack{1\leq k\leq n\\(k,n)=1}} \gcd(k-1,n),$$

where $\varphi(n)$ is Euler's totient function and $d(n)=\sigma_0(n)$ is the divisor function. We can then expand the right-hand-side of Menon's identity as follows:

$$\varphi(n)d(n) = \sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i\geq 1\right]_{\delta}\gcd(k-1,n).$$

Another closely related identity considered by Tóth in [18] is that for any arithmetic function f we have the identity (*cf.* [8])

$$\sum_{\substack{1\leq k\leq n\\(k,n)=1}} f\left(\gcd(k-1,n)\right) = \varphi(n)\cdot\sum_{d|n}\frac{(\mu*f)(d)}{\varphi(d)}.$$

We can use our new formulas to write a gcd-related recurrence relation for f in two steps. First, we observe that the right-hand-side divisor sum in the previous equation is expanded by

$$\begin{aligned}\sum_{d|n}\frac{(\mu*f)(d)}{\varphi(d)} &= \frac{1}{\varphi(n)}\cdot\sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i\geq 1\right]_{\delta} f(\gcd(k-1,n))\\ &\quad + f(1)\left[n=1\right]_{\delta}.\end{aligned}$$

Next, by Möbius inversion and noting that the Dirichlet inverse of $\mu(n)$ is $\mu*1=\varepsilon$, we can express $f(n)$ as follows:

$$\begin{aligned}f(n) = \sum_{d|n}\sum_{r|d}\sum_{j=0}^{r}\sum_{k=1}^{j-1}\sum_{i=0}^{j}\Bigg[& p(r-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i)\left[j-k-G_i\geq 1\right]_{\delta}\times\\ &\times f(\gcd(k-1,r))\frac{\varphi(d)}{\varphi(r)}\mu\left(\frac{d}{r}\right)\Bigg] + f(1)\cdot\sum_{d|n}\varphi(d)\mu(d).\end{aligned}$$

3. Factorization theorems for sums of the second type

3.1. **Formulas for the inverse matrices.** It happens that in the case of the series expansions we defined in (5b) of the introduction, the corresponding terms of the inverse matrices $u_{n,k}^{-1}(f,w)$ satisfy considerably simpler formulas that the ordinary matrix entries themselves. We first prove a partition-related explicit formula for these inverse matrices in Proposition 3.1 and then discuss several applications of this result.

Proposition 3.1 (Formulas for the Inverse Matrix Sequence $u_{n,k}^{-1}(f,w)$)**.** *For all $n \geq 1$ and $1 \leq k \leq n$, any fixed arithmetic function f, and $w \in \mathbb{C}$, we have that*

$$u_{n,k}^{(-1)}(f,w) = \sum_{m=1}^{n} \left(\sum_{d|(m,n)} f(d) p(n/d-k) \right) w^m.$$

Proof. Let $1 \leq r \leq n$ and for some suitably chosen arithmetic function g define

$$u_{n,r}^{(-1)}(f,w) := \sum_{m=1}^{n} L_{f,g,m}(n) w^m. \tag{i}$$

By directly expanding the series on the right-hand-side of (5b), we obtain that

$$\begin{aligned} g(n) &= \sum_{j=0}^{n} \left(\sum_{k=1}^{j} u_{j,k}(f,w) \cdot u_{k,r}^{(-1)}(f,w) \right) p(n-j) \\ &= \sum_{j=0}^{n} p(n-j) \, [j=r]_{\delta} = p(n-r). \end{aligned}$$

Hence the choice of the function g which satisfies (i) above is given by $g(n) := p(n-r)$. The claimed expansion of the inverse matrices then follows. $\square$

Corollary 3.2 (A New Formula for Ramanujan Sums)**.** *For any natural numbers $x, m \geq 1$, we have that*

$$c_x(m) = \sum_{k=1}^{x} \sum_{d|(m,x)} dp\left(\frac{x}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\lceil \frac{j}{2} \rceil} \mu\left(k - G_j\right).$$

Proof. We in fact prove the following more general identity:

$$L_{f,g,m}(n) = \sum_{k=1}^{n} \sum_{d|(m,n)} f(d) p\left(\frac{n}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\lceil \frac{j}{2} \rceil} g\left(k - G_j\right). \tag{7}$$

Since the coefficients on the left-hand-side of the next equation correspond to a right-hand-side matrix product as

$$[q^n](q;q)_\infty \sum_{m \geq 1} g(m) q^m = \sum_{k=1}^{n} u_{n,k}(f,w) \sum_{m=1}^{k} L_{f,g,m}(k) w^m,$$

we can invert the matrix product on the right to obtain that

$$\sum_{m=1}^{k} L_{f,g,m}(k) w^m = \sum_{k=1}^{n} \left(\sum_{m=1}^{n} \sum_{d|(n,m)} f(d) p\left(\frac{n}{d} - k\right) \cdot w^m \right) \times [q^k](q;q)_\infty \sum_{m \geq 1} g(m),$$

so that by comparing coefficients of w^m for $1 \leq m \leq n$, we obtain (7). The Ramanujan sums are the special case of this identity where $f(n) := n$ is the identity function and $g(n) := \mu(n)$ is the Möbius function. $\square$

Remark 3.3. We define the following shorthand notation:

$$\widehat{L}_{f,g}(n;w) := \sum_{m=1}^{n} L_{f,g,m}(n) w^m.$$

In this notation we have that $u_{n,k}^{(-1)}(f,w) = \widehat{L}_{f,p(\cdot-k)}(n;w)$. Moreover, if we let the polynomials $T_n(x) := 1+x+x^2+\cdots+x^{n-1} = \frac{x^n-1}{x-1}$, then we have expansions of these sums as convolved ordinary divisor sums "twisted" by a semi-inseparable polynomial term of the form

$$\begin{aligned} \widehat{L}_{f,g}(n;w) &= \sum_{d|n} w^d f(d) T_{n/d}(w^d) g\left(\frac{n}{d}\right) \\ &= (w^n-1) \times \sum_{d|n} \frac{w^d}{w^d-1} f(d) g\left(\frac{n}{d}\right). \end{aligned} \tag{8}$$

The Dirichlet inverse of these divisor sums is also not difficult to express, though we will not give its formula here. These sums lead to a first formula for the more challenging expressions for the ordinary matrix entries $u_{n,k}(f,w)$ given by the next corollary.

Corollary 3.4 (A Formula for the Ordinary Matrix Entries)**.** *To distinguish notation, let $\widehat{P}_{f,k}(n;w) := \widehat{L}_{f(n),p(n-k)}(n;w)$. For $n \geq 1$ and $1 \leq k < n$, we have the following formula:*

$$\begin{aligned} u_{n,k}(f,w) = &-\frac{(1-w)^2}{w^2 \cdot (1-w^n)(1-w^k) \cdot f(1)^2} \Bigg(\widehat{P}_{f,k}(n;w) \\ &+ \sum_{m=1}^{n-k-1} \left(\frac{w-1}{wf(1)}\right)^m \left[\sum_{k \leq i_1 < \cdots < i_m < n} \frac{\widehat{P}_{f,k}(i_1;w) \widehat{P}_{f,i_1}(i_2;w) \widehat{P}_{f,i_2}(i_3;w) \cdots \widehat{P}_{f,i_{m-1}}(i_m;w) \widehat{P}_{f,i_m}(n;w)}{(1-w^{i_1})(1-w^{i_2}) \cdots (1-w^{i_m})} \right] \Bigg) \end{aligned}$$

When $k = n$, we have that

$$u_{n,n}(f,w) = \frac{1-w}{w(1-w^n) \cdot f(1)}.$$

Proof. This follows inductively from the inversion relation between the coefficients of a matrix and its inverse. For any invertible lower triangular $n \times n$ matrix $(a_{i,j})_{1 \leq i,j \leq n}$, we can express a non-recursive formula for the inverse matrix entries as follows:

$$a_{n,k}^{(-1)} = \frac{1}{a_{n,n}} \left(-\frac{a_{n,k}}{a_{k,k}} + \sum_{m=1}^{n-k-1} (-1)^{m+1} \left[\sum_{k \leq i_1 < \cdots < i_m < n} \frac{a_{i_1,k} a_{i_2,i_1} a_{i_3,i_2} \cdots a_{i_m,i_{m-1}} a_{n,i_m}}{a_{k,k} a_{i_1,i_1} a_{i_2,i_2} \cdots a_{i_m,i_m}} \right] \right) [k < n]_\delta + \frac{[k=n]_\delta}{a_{n,n}}. \tag{9}$$

The proof of our result is then just an application of the formula in (9) when $a_{n,k} := u_{n,k}^{-1}(f,w)$. While the identity in (9) is not immediately obvious from the known inversion formulas between inverse matrices in the form of

$$a_{n,k}^{(-1)} = \frac{[n=k]_\delta}{a_{n,n}} - \frac{1}{a_{n,n}} \sum_{j=1}^{n-k-1} a_{n,j} a_{j,k}^{(-1)},$$

the result is easily obtained by induction on n so we do not prove it here. ∎

3.2. **Formulas for simplified variants of the ordinary matrices.** In Corollary 3.4 we proved an exact, however somewhat implicit and unsatisfying, expansion of the ordinary matrix entries $u_{n,k}(f,w)$ by sums of weighted products of the inverse matrices $u_{n,k}^{(-1)}(f,w)$ expressed in closed form through Proposition 3.1. We will now develop the machinery needed to more precisely express the ordinary forms of these matrices first for general cases of the indeterminate indexing parameter $w \in \mathbb{C}$.

$\frac{1}{\widehat{f}(1)}$	0	0	0	0	0
$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0	0
$\frac{\widehat{f}(2)}{\widehat{f}(1)^2}-\frac{\widehat{f}(3)}{\widehat{f}(1)^2}-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0
$\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3}+\frac{\widehat{f}(2)}{\widehat{f}(1)^2}+\frac{\widehat{f}(3)}{\widehat{f}(1)^2}-\frac{\widehat{f}(4)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3}+\frac{\widehat{f}(3)}{\widehat{f}(1)^2}+\frac{\widehat{f}(4)}{\widehat{f}(1)^2}-\frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3}+\frac{2\widehat{f}(3)\widehat{f}(2)}{\widehat{f}(1)^3}+\frac{\widehat{f}(4)}{\widehat{f}(1)^2}+\frac{\widehat{f}(5)}{\widehat{f}(1)^2}-\frac{\widehat{f}(6)}{\widehat{f}(1)^2}+\frac{1}{\widehat{f}(1)}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2}-\frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$

Table 3.1. *The simplified matrix entries $\widehat{u}_{n,k}(f,w)$ for $1 \leq n,k \leq 6$.*

Remark 3.5 (Simplifications of the Matrix Terms)**.** Using the formula for the coefficients of $u_{n,k}(f,w)$ in (5b) expanded by (8), we can simplify the form of the matrix entries we seek closed-form expressions for in the next calculations. In particular, we make the following definitions for $1 \leq k \leq n$:

$$\begin{aligned} \widehat{f}(n) &:= \frac{w^n}{w^n-1} f(n) \\ \widehat{u}_{n,k}(f,w) &:= (w^k-1)u_{n,k}(f,w). \end{aligned}$$

Then an equivalent formulation of finding the exact formulas for $u_{n,k}(f,w)$ is to find exact expressions expanding the triangular sequence of $\widehat{u}_{n,k}(f,w)$ satsifying

$$\sum_{\substack{j\geq 0 \\ n-G_j>0}} (-1)^{\lceil \frac{j}{2} \rceil} g(n-G_j) = \sum_{k=1}^{n} \widehat{u}_{n,k}(f,w) \sum_{d|k} \widehat{f}(d) g\left(\frac{n}{d}\right).$$

We will obtain precisely such formulas in the next few results. Table 3.1 provides the first few rows of our simplified matrix entries.

Definition 3.6 (Special Multiple Convolutions)**.** For $n, j \geq 1$, we define the following nested j-convolutions of the function $\widehat{f}(n)$ [10]:

$$\operatorname{ds}_j(n) = \begin{cases} (-1)^{\delta_{n,1}} \widehat{f}(n), & \text{if } j=1; \\ \sum\limits_{\substack{d|n \\ d>1}} \widehat{f}(d) \operatorname{ds}_{j-1}\left(\frac{n}{d}\right), & \text{if } j \geq 2. \end{cases}$$

n	$D(n)$	n	$D(n)$	n	$D(n)$
2	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	7	$-\frac{\widehat{f}(7)}{\widehat{f}(1)^2}$	12	$\frac{2\hat{f}(3)\hat{f}(4)+2\hat{f}(2)\hat{f}(6)-\hat{f}(1)\hat{f}(12)}{\hat{f}(1)^3}-\frac{3\hat{f}(2)^2\hat{f}(3)}{\hat{f}(1)^4}$
3	$-\frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	8	$\frac{2\widehat{f}(2)\widehat{f}(4)-\widehat{f}(1)\widehat{f}(8)}{\widehat{f}(1)^3}-\frac{\widehat{f}(2)^3}{\widehat{f}(1)^4}$	13	$-\frac{\hat{f}(13)}{\hat{f}(1)^2}$
4	$\frac{\widehat{f}(2)^2-\widehat{f}(1)\widehat{f}(4)}{\widehat{f}(1)^3}$	9	$\frac{\widehat{f}(3)^2-\widehat{f}(1)\widehat{f}(9)}{\widehat{f}(1)^3}$	14	$\frac{2\hat{f}(2)\hat{f}(7)-\hat{f}(1)\hat{f}(14)}{\hat{f}(1)^3}$
5	$-\frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	10	$\frac{2\widehat{f}(2)\widehat{f}(5)-\widehat{f}(1)\widehat{f}(10)}{\widehat{f}(1)^3}$	15	$\frac{2\hat{f}(3)\hat{f}(5)-\hat{f}(1)\hat{f}(15)}{\hat{f}(1)^3}$
6	$\frac{2\widehat{f}(2)\widehat{f}(3)-\widehat{f}(1)\widehat{f}(6)}{\widehat{f}(1)^3}$	11	$-\frac{\hat{f}(11)}{\hat{f}(1)^2}$	16	$\frac{\hat{f}(2)^4}{\hat{f}(1)^5}-\frac{3\hat{f}(4)\hat{f}(2)^2}{\hat{f}(1)^4}+\frac{\hat{f}(4)^2+2\hat{f}(2)\hat{f}(8)}{\hat{f}(1)^3}-\frac{\hat{f}(16)}{\hat{f}(1)^2}$

Table 3.2. *The multiple convolution function $D(n)$ for $2 \leq n \leq 16$.*

Then we define our primary multiple convolution function of interest as

$$D(n) := \sum_{j=1}^{n} \frac{\mathrm{ds}_{2j}(n)}{\widehat{f}(1)^{2j+1}}.$$

For example, the first few cases of $D(n)$ for $2 \leq n \leq 16$ are computed in Table 3.2. The examples in the table should clarify precisely what multiple convolutions we are definining by the function $D(n)$. Namely, a signed sum of all possible ordinary k Dirichlet convolutions of $\widehat{f}$ with itself evaluated at n.

Lemma 3.7. *We claim that for all $n \geq 1$*

$$(D * \widehat{f})(n) = -\frac{\widehat{f}(n)}{\widehat{f}(1)} + \varepsilon(n).$$

Proof. We note that the statement of the lemma is equivalent to showing that

$$\left(D + \frac{\varepsilon}{\widehat{f}(1)}\right)(n) = \widehat{f}^{-1}(n). \tag{10}$$

A general recursive formula for the inverse of $\widehat{f}(n)$ is given by [2]

$$\widehat{f}^{-1}(n) = \left(-\frac{1}{\widehat{f}(1)} \sum_{\substack{d \mid n \\ d>1}} \widehat{f}(d) \widehat{f}^{-1}(n/d)\right) [n>1]_{\delta} + \frac{1}{\widehat{f}(1)} [n=1]_{\delta}.$$

This definition is almost how we defined $\mathrm{ds}_j(n)$ above. Let's see how to modify this recurrence relation to obtain the formula for $D(n)$. We can recursively substitute in the formula for $\widehat{f}^{-1}(n)$ until we hit the point where successive substitutions only leave the base case of $\widehat{f}^{-1}(1) = 1/\widehat{f}(1)$. This occurs after $\Omega(n)$ substitutions where $\Omega(n)$ denotes the number of prime factors of n counting multiplicity. We can write the nested formula for $\mathrm{ds}_j(n)$ as

$$\mathrm{ds}_j(n) = \widehat{f}_{\pm} * \underbrace{\left(\widehat{f} - \widehat{f}(1)\varepsilon\right) * \cdots * \left(\widehat{f} - \widehat{f}(1)\varepsilon\right)}_{j-1 \text{ factors}}(n),$$

where we define $\widehat{f}_{\pm}(n) := \widehat{f}(n)\,[n>1]_\delta - \widehat{f}(1)\,[n=1]_\delta$. Next, define the nested k-convolutions $C_k(n)$ recursively by

$$C_k(n) = \begin{cases} \widehat{f}(n) - \widehat{f}(1)\varepsilon(n), & \text{if } k=1; \\ \sum\limits_{d|n} \left(\widehat{f}(d) - \widehat{f}(1)\varepsilon(d)\right) C_{k-1}(n/d), & \text{if } k \geq 2. \end{cases}$$

Then we can express the inverse of $\widehat{f}(n)$ using this definitition as follows:

$$\widehat{f}^{-1}(n) = \sum_{d|n} \widehat{f}(d) \left[\sum_{j=1}^{\Omega(n)} \frac{C_{2k}(n/d)}{\widehat{f}(1)^{\Omega(n)+1}} - \frac{\varepsilon(n/d)}{\widehat{f}(1)^2}\right].$$

Then based on the initial conditions for $k=1$ ($j=1$) in the definitions of $C_k(n)$ and $\mathrm{ds}_j(n)$, we see that the function in (10) is in fact the inverse of $\widehat{f}(n)$. □

Proposition 3.8. *For all $n \geq 1$ and $1 \leq k \leq n$, we have that*

$$\sum_{i=0}^{n-1} p(i)\widehat{u}_{n-i,k}(f,w) = D\left(\frac{n}{k}\right)[n \equiv 0 \bmod k]_\delta + \frac{1}{\widehat{f}(1)}[n=k]_\delta.$$

Proof. We notice that Lemma 3.7 implies that

$$\varepsilon(n) = \left(\left(D + \frac{\varepsilon}{\widehat{f}(1)}\right) * \widehat{f}\right)(n),$$

where $\varepsilon(n)$ is the multiplicative identity for Dirichlet convolutions. The last equation implies that

$$g(n) = \left(\left(D + \frac{\varepsilon}{\widehat{f}(1)}\right) * \widehat{f} * g\right)(n). \tag{i}$$

Additionally, we know by the expansion of (5b) and that $\widehat{u}_{n,n}(f,w) = 1/\widehat{f}(1)$ that we also have the expansion

$$g(n) = \sum_{k\geq 1}\left[\sum_{j=0}^{n-1} p(j)\widehat{u}_{n-j,k}\right]\sum_{d|k}\widehat{f}(d)g(k/d). \tag{ii}$$

So we can equate (i) and (ii) to see that

$$\sum_{j=0}^{n-1} p(j)\widehat{u}_{n-j,k} = D\left(\frac{n}{k}\right)[k|n]_\delta + \frac{[n=k]_\delta}{\widehat{f}(1)}.$$

This establishes our claim. □

Corollary 3.9 (An Exact Formula for the Ordinary Matrix Entries)**.** *For all $n \geq 1$ and $1 \leq k \leq n$, we have that*

$$\widehat{u}_{n,k}(f,w) = \sum_{\substack{j\geq 0 \\ n-G_j>0}} (-1)^{\lceil \frac{j}{2} \rceil}\left(D\left(\frac{n-G_j}{k}\right)[n-G_j \equiv 0 \bmod k]_\delta + \frac{1}{\widehat{f}(1)}[n-G_j=k]_\delta\right).$$

Proof. This is an immediate consequence of Proposition 3.8 by noting that the generating function for $p(n)$ is $(q;q)_\infty^{-1}$ and that

$$(q;q)_\infty = \sum_{j\geq 0} (-1)^{\lceil \frac{j}{2} \rceil} q^{G_j}. \qquad \square$$

3.3. **The general matrices expressed through discrete Fourier transforms.** The proof of the key formula result given in Theorem 3.13 builds on several key ideas for discrete Fourier transforms of the greatest common divisor function $(k,n) \equiv \gcd(k,n)$ developed in [6]. We adopt the common convention that the function $e(x)$ denotes the exponential function $e(x) := e^{2\pi\imath x}$. Throughout the remainder of this section we take $k \geq 1$ to be fixed and consider divisor sums of the following form which are periodic with respect to k:

$$L_{f,g,k}(n) := \sum_{d|(n,k)} f(d) g\left(\frac{n}{d}\right).$$

In [6] these sums are called k-convolutions of f and g. We will first need to discuss some terminology related to discrete Fourier transforms.

A *discrete Fourier transform* (DFT) maps a finite sequence of complex numbers $\{f[n]\}_{n=0}^{N-1}$ onto their associated Fourier coefficients $\{F[n]\}_{n=0}^{N-1}$ defined according to the following reversion formulas relating these sequences:

$$\begin{aligned} F[k] &= \sum_{n=0}^{N-1} f[n] e\left(-\frac{kn}{N}\right) \\ f[k] &= \frac{1}{N} \sum_{n=0}^{N-1} F[k] e\left(\frac{kn}{N}\right). \end{aligned}$$

The discrete Fourier transform of functions of the greatest common divisor, which we will employ repeatedly to prove Theorem 3.13 below, is summarized by the formula in the next lemma [6, 16].

Lemma 3.10 (Typical relations between periodic divisor sums and Fourier series)**.** *if we take the any two arithmetic functions f and g, we can express the periodic divisor sums of the forms*

$$s_k(f,g;n) = \sum_{d|(n,k)} f(d) g(k/d) = \sum_{m=1}^{k} a_k(f,g;m) \cdot e^{2\pi\imath \cdot mn/k}, \tag{11a}$$

where the discrete Fourier coefficients in the second equation are given by

$$a_k(f,g;m) = \sum_{d|(m,k)} g(d) f(k/d) \cdot \frac{d}{k}. \tag{11b}$$

Proof. For a proof of these relations consult the references [2, §8.3] [12, *cf.* §27.10]. These relations are also related to the gcd-transformations proved in [6, 16]. $\square$

Lemma 3.11 (DFT of Functions of the Greatest Common Divisor)**.** *Let h be any arithmetic function. For natural numbers $m \geq 1$, the discrete fourier transform of h defined by*

$$\widehat{h}[a](m) := \sum_{k=1}^{m} h(\gcd(k,m))\, e\left(\frac{ka}{m}\right),$$

is given by $\widehat{h}[a] = h * c_{-}(a)$ *where*

$$c_m(a) := \sum_{\substack{k=1 \\ \gcd(k,m)=1}}^{m} e\left(\frac{ka}{m}\right).$$

The function $c_m(a)$ *defined by the previous equation is Ramanujan's sum expanded by the divisor sums in Corollary 3.2 of the last subsection.*

Definition 3.12 (Notation and Special Exponential Sums)**.** In what follows, we denote the ℓ^{th} Fourier coefficient with respect to k of the function $L_{f,g,k}(n)$ by $a_{k,\ell}$ which is well defined since $L_{f,g,k}(n) = L_{f,g,k}(n+k)$ is periodic in k. We then have an expansion of this function in the form of

$$L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell} e\left(\frac{\ell n}{k}\right),$$

where we can compute these coefficients directly from $L_{f,g,k}(n)$ according to the formula

$$a_{k,\ell} = \sum_{n=0}^{k-1} L_{f,g,k}(n) e\left(-\frac{\ell n}{k}\right).$$

We also notice that these Fourier coefficients are given explicitly in terms of the f and g by the formulas cited in (11) of the introduction.

Theorem 3.13. *For all arithmetic functions* f, g *and natural numbers* $k \geq 1$*, we have that*

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} \varphi(d) f(d) (k/d)^2 g(k/d),$$

where $\varphi(n)$ *is Euler's totient function.*

Proof. We notice that the left-hand-side of the claim is equivalent to the divisor sum over the d^{th} Fourier coefficients with respect to k in the form of

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} d \cdot a_{k,d} \mu(k/d),$$

where the Fourier coefficients in this expansion are given by (11) [2, §8.3] [12, §27.10]. In particular, we have that

$$a_{k,d} = k \cdot \sum_{r|(k,d)} g(r) f(k/r) \frac{r}{k}.$$

The left-hand-side of our expansion then becomes (*cf.* (12) below)

$$\begin{aligned}
\sum_{d|k} d \cdot a_{k,d} \mu(k/d) &= \sum_{d|k} \sum_{r|d} r g(r) f\left(\frac{k}{r}\right) \cdot d\mu\left(\frac{k}{d}\right) \\
&= \sum_{d=1}^{k} [d|k]_{\delta}\, d\mu\left(\frac{k}{d}\right) \times \sum_{r|d} r g(r) f\left(\frac{k}{r}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r|k} rg(r) f\left(\frac{k}{r}\right) \times \sum_{d=1}^{k/r} dr \cdot \mu\left(\frac{k}{dr}\right) [d|k]_\delta \\
&= \sum_{r|k} r^2 g(r) \cdot f\left(\frac{k}{r}\right) \varphi\left(\frac{k}{r}\right).
\end{aligned}$$

We notice that while the exponential sums in the original statement of the claim are desirable in expanding applications, this direct expansion is difficult to manipulate algebraically. Therefore, we have swapped out the exponential sum for the known divisor sum formula for the Fourier coefficients implicit in the theorem statement in order to prove our key result. □

Corollary 3.14 (An Exact Formula for $g(n)$)**.** *For any $n \geq 1$ and arithmetic functions f, g we have the formula*

$$g(n) = \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j \cdot L_{f,g,r}(d)}{d^2} e\left(-\frac{rj}{d}\right) \mu(d/j) y(n/d),$$

where $y(n) = (\varphi f \operatorname{Id}_{-2})^{-1}(n)$ is the Dirichlet inverse of $\varphi(n)/n^2$.

Proof. We first divide both sides of the result in the theorem by k^2. Then we apply a convolution with the left-hand-side of the formula in Theorem 3.13 with $y(n)$ defined as above to obtain the exact expansion for $g(n)$. □

Corollary 3.15 (The Mertens Function)**.** *For all $x \geq 1$, the Mertens function defined in the introduction is expanded by Ramanujan's sum as*

$$M(x) = \sum_{d=1}^{x} \sum_{n=1}^{\left\lfloor \frac{x}{d} \right\rfloor} \sum_{r=0}^{d-1} \left(\sum_{j|d} \frac{j}{d} \cdot e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) \right) \frac{c_d(r)}{d} y(n),$$

where $y(n) = (\varphi \operatorname{Id}_{-1})^{-1}(n)$ is the Dirichlet inverse of $\varphi(n)/n$.

Proof. We begin by citing Theorem 3.13 in the special case corresponding to $L_{f,g,k}(n)$ a Ramanujan sum for $f(n) = n$ and $g(n) = \mu(n)$. Then we sum over the left-hand-side $g(n)$ in the theorem result to obtain the initial summation identity for $M(x)$ given by

$$M(x) = \sum_{n \leq x} \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) y\left(\frac{n}{d}\right). \tag{i}$$

We can then apply the identity that for any arithmetic functions h, u, v we can interchange nested divisor sums as[2]

$$\sum_{k=1}^{n} \sum_{d|k} h(d) u(k/d) v(k) = \sum_{d=1}^{n} h(d) \left(\sum_{k=1}^{\lfloor \frac{n}{d} \rfloor} u(k) v(dk) \right). \tag{12}$$

Application of this identity in (i) leads to the first form for $M(x)$ stated above. ☐

Corollary 3.16 (Euler's Totient Function). *For any $n \geq 1$ we have*

$$\varphi(n) = n \cdot \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right).$$

Additionally, we have the following expansion of the average order sums for $\varphi(n)$ given by

$$\sum_{2 \leq n \leq x} \varphi(n) = \sum_{d=1}^{x} \sum_{r=0}^{d-1} \frac{c_d(r)}{2d} \left\lfloor \frac{x}{d} \right\rfloor \left(\left\lfloor \frac{x}{d} \right\rfloor - 1 \right) \times \sum_{j|d} j e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right).$$

Proof. We consider the formula in Theorem 3.13 with $f(n) := n$ and $g(n) := \mu(n)$. Since the Dirichlet inverse of the Möbius function is $\mu * 1 = \varepsilon$, we obtain our result by convolution and multiplication by the factor of n. The average order identity follows from the first expansion by applying (12). ☐

3.4. **An approach via polynomials and orthogonality relations.** In Corollary 3.9 of Section 3.2 we proved an exact formula for the modified ordinary matrix entries $\widehat{u}_{n,k}(f, w)$ defined by the simplifications of the original $u_{n,k}(f, w)$ from (5b) in Remark 3.5 (*cf.* Table 3.1). We proved the exact formula for $\widehat{u}_{n,k}(f, w)$ in the previous subsection using a more combinatorial argument involving the multiple convolutions of the function $D(n)$ constructed recursively in Definition 3.6 (see Table 3.2). In this section we define a sequence of related polynomials $P_j(w; t)$ whose coefficients are the corresponding simplified forms of the inverse matrices which provide us with the identity (*cf.* Proposition 3.17)

$$\sum_{k=1}^{n} \widehat{u}_{n,k}(f, w) \cdot P_k(w; t) = t^n.$$

which we may integrate against in our constructions below. We then find and prove the form of a weight function $\omega(t)$ which provides us with the orthogonality condition

$$\int_{|t|=1, t \in \mathbb{C}} \omega(t) P_i(w; t) P_j(w; t) dt = c_i(w) \, [i = j]_\delta \,, \tag{13a}$$

where we define the right-hand-side coefficients by

$$c_n(w) := \int_{|t|=1, t \in \mathbb{C}} \omega(t) \left(P_n(w; t) \right)^2 dt. \tag{13b}$$

[2] We also have a related identity which allows us to interchange the order of summation in the Anderson-Apostol sums of the following form for any natural numbers $x \geq 1$ and arithmetic functions $f, g, h : \mathbb{N} \to \mathbb{C}$:

$$\sum_{d=1}^{x} f(d) \sum_{r|(d,x)} g(r) h\left(\frac{d}{r}\right) = \sum_{r|x} g(r) \sum_{d=1}^{x/r} h(d) f\left(\gcd(x, r) d\right).$$

This consitruction, which we develop and make rigorous below, provides us with another method by which we may exactly extract the form of the simplified matrices $\widehat{u}_{n,k}(f,w)$. Namely, we have that for all $n \geq 1$ and $1 \leq k \leq n$ the operation

$$\widehat{u}_{n,k}(f,w) = \frac{1}{c_k(w)} \int_{|t|=1, t\in\mathbb{C}} \omega(t) t^n P_k(w;t) dt, \tag{13c}$$

yields an exact formula for our matrix entries of interest here. We now develop the requisite machinery to prove that this construction holds.

Proposition 3.17 (A Partition-Related Polynomial Sum)**.** *Let an arithmetic function f be fixed and for an indeterminate $w \in \mathbb{C}$ let $\widehat{f}(n)$ denote*

$$\widehat{f}(n) := \frac{w^n}{w^n-1} f(n).$$

For natural numbers $j \geq 1$ and any indeterminate w, let the polynomials

$$P_j(w;t) := \sum_{i=1}^{j} \left(\sum_{d|j} \widehat{f}(d) p\left(\frac{j}{d} - i\right)\right) t^i. \tag{14}$$

Then for all $n \geq 1$ we have that

$$\sum_{k=1}^{n} \widehat{u}_{n,k}(f,w) \cdot P_k(w;t) = t^n.$$

Proof. The claim is equivalent to proving that for each $n \geq 1$, we have that

$$(w^n - 1) \cdot P_n(w;t) = \sum_{k=1}^{n} u_{n,k}^{(-1)}(f,w) \cdot t^k. \tag{15}$$

Notice that the previous equation also implies that

$$P_n(w;t) = \sum_{k=1}^{n} \widehat{u}_{n,k}^{(-1)}(f,w) \cdot t^k = \sum_{k=1}^{n} \left(\sum_{d|n} \widehat{f}(d) p\left(\frac{n}{d} - k\right)\right) t^k = \sum_{d|n} \widehat{f}(d) \times \sum_{i=0}^{\frac{n}{d}-1} p(i) \cdot t^{\frac{n}{d}-i}. \tag{16}$$

Now finally, for each $1 \leq k \leq n$, we can expand the coefficients of the left-hand-side as

$$\begin{aligned}
[t^k]P_n(w;t) &= \sum_{d|n} \frac{(w^n-1)w^d}{w^d-1} f(d) p(n/d-k) \\
&= \sum_{d|n} f(d) p(n/d-k) \left(\sum_{i=1}^{n/d} w^{id}\right) \\
&= \sum_{m=1}^{n} \left(\sum_{d|m} f(d) p(n/d-k) \left[d|n\right]_{\delta}\right) w^m && (\mathrm{m} = \mathrm{id}) \\
&= \sum_{m=1}^{n} \left(\sum_{d|(m,n)} f(d) p(n/d-k)\right) w^m.
\end{aligned}$$

Hence by the formula for the inverse matrices given in Proposition 3.1, we have proved our claim. □

Proposition 3.18 (Another Matrix Formula)**.** *For $n \geq 1$ and $1 \leq k \leq n$, we have the following formula for the simplified matrix entries where the coefficients c_k are given by* (13b)*:*

$$\widehat{u}_{n,k}(f, w) = \sum_{\substack{j \geq 0 \\ n-G_j>0 \\ k \mid n-G_j}} (-1)^{\lceil \frac{j}{2} \rceil} \cdot \widehat{f}^{-1}\left(\frac{n-G_j}{k}\right).$$

Proof. According to the last expansion in (16), we have that

$$\sum_{i=0}^{n-1} p(i) t^{n-i} = \left(\widehat{f}^{-1} * P\right)(n),$$

or equivalently that

$$t^n = \sum_{\substack{j \geq 0 \\ n-G_j>0}} (-1)^{\lceil \frac{j}{2} \rceil} \cdot \left(\widehat{f}^{-1} * P\right)(n-G_j).$$

Then by substituting the previous equation into (13c) we have our result. □

Theorem 3.19. *Suppose that the form of the sequence of $\{c_k(\omega)\}_{k\geq 1}$ is given. Let $D_{f(n)}(n) := \mathrm{DTFT}(f)|_n$ denote the discrete-time Fourier transform of f. Then writing $t := e^{\imath u}$ for $0 \leq u \leq 2\pi$, we have the following exact expression for the weight function $\omega(t)$ which depends only on the prescribed sequence of $c_k(\omega)$:*

$$\omega\left(e^{\frac{\imath u}{2}}\right) = 2 \cdot D\left(\sum_{G_r<i} (-1)^{\lceil \frac{r}{2} \rceil} \sum_{G_l<i-G_r} (-1)^{\lceil \frac{l}{2} \rceil} \left((c_- * \widehat{f}^{-1}) * \widehat{f}^{-1}\right)(i-G_l-G_r)\right)(u).$$

Proof. We have that

$$c_i(\omega) = \sum_{d|i} \widehat{f}(d) \times \sum_{r=0}^{\frac{i}{d}-1} p(r) \times \sum_{c|i} \widehat{f}(c) \times \sum_{l=0}^{\frac{i}{c}-1} p(l) \times \int_{|t|=1} \omega(t) t^{\frac{i}{d}+\frac{i}{c}-r-l} dt.$$

For the $t := e^{\imath u}$ and $0 \leq u \leq 2\pi$ defined above, let $h(u) = \omega(e^{\imath u})$. By a direct appeal to Möebius inversion we see that

$$((c_- * \widehat{f}^{-1}) * \widehat{f}^{-1})(i) = \sum_{r=0}^{i-1} \sum_{l=0}^{i-1} p(r) p(l) D^{-1}_{h(u)}(2i-l-r).$$

Then we can obtain that

$$\sum_{r=0}^{i-1} p(r) D^{-1}_{h(u)}(2i-r) = \sum_{G_l<i} (-1)^{\lceil \frac{l}{2} \rceil} \left((c_- * \widehat{f}^{-1}) * \widehat{f}^{-1}\right)(i-G_l),$$

and

$$D^{-1}_{h(u)}(2i) = \sum_{G_r<i} (-1)^{\lceil \frac{r}{2} \rceil} \sum_{G_l<i-G_r} (-1)^{\lceil \frac{l}{2} \rceil} \left((c_- * \widehat{f}^{-1}) * \widehat{f}^{-1}\right)(i-G_l-G_r).$$

Thus by taking DTFT of both sides we arrive at the formula

$$\frac{1}{2} h\left(\frac{u}{2}\right) = D\left(\sum_{G_r<i} (-1)^{\lceil \frac{r}{2} \rceil} \sum_{G_l<i-G_r} (-1)^{\lceil \frac{l}{2} \rceil} \left((c_- * \widehat{f}^{-1}) * \widehat{f}^{-1}\right)(i-G_l-G_r)\right)(u)$$

Since $h(u) = \omega(e^{\imath u})$ this proves our key formula. $\square$

4. Conclusions

We have proved several new expansions of the type I and type II sums defined by (4) for any prescribed arithmetic functions f and g. Our new results proved in the article include treatments of the expansions of these two sum types by both matrix-based factorization theorems and analogous identities formulated through discrete Fourier transforms of special function sums. The type I sums implicitly define many special number theoretic functions and sequences by exponential sum variants of this type. Perhaps the most notable canonical example of this sum type is given by Euler's totient function which counts the number of integers relatively prime to a natural number n. The Möbius function also has a representation in the form of a type I sum.

The type II sums form an alternate flavor of the ordinary divisor sums enumerated by Lambert series generating functions and the Dirichlet convolutions of two arithmetic functions f and g. These sums are sometimes refered to as Anderson-Apostol sums, or k-convolutions in the references. The prototypical example of sums of this type are given by the Ramanujan sums $c_q(n)$ which form expansions of many other special number theortic functions by composition and infinite series. Our results provide new and useful expansions that characterize common and important classes of sums that arise in applications. Our results are unique in that we are able to relate partition functions to the expansions of these general classes of sums in both cases.

References

[1] T. M. Apostol, Arithmetical properties of generalized Ramanujan sums, *Pacific J. Math.* **41** (1972).
[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.
[3] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 2008.
[4] G. H. Hardy, *Ramanujan: Twelve lectures on subjects suggested by his life and work*, AMS Chelsea Publishing, 1999.
[5] S. Ikeda, et. al. Sums of products of generalized Ramanujan sums, *J. Integer Seq.* **19** (2016).
[6] van der Kamp, Peter H. On the Fourier transform of the greatest common divisor. No. arXiv: 1201.3139. 2012.
[7] I. Kiuchi, On sums of averages of generalized Ramanujan sums, *Tokyo J, Math.* **40** (2017).
[8] I. Kiuchoi and S. S. Eddin, On sums of weighted averages of gcd-sum functions, https://arxiv.org/abs/1801.03647 (2018).
[9] M. Merca, The Lambert series factorization theorem, *The Ramanujan Journal*, pp. 1–19 (2017).
[10] M. Merca and M. D. Schmidt, Factorization Theorems for Generalized Lambert Series and Applications, *The Ramanujan Journal*, to appear (2018).
[11] M. Merca and M. D. Schmidt, Generating Special Arithmetic Functions by Lambert Series Factorizations, *Contributions to Discrete Mathematics*, to appear (2018).
[12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.
[13] J. Riordan, *Combinatorial Identities*, Wiley, 1968.
[14] M. D. Schmidt, Exact Formulas for the Generalized Sum-of-Divisors Functions, https://arxiv.org/abs/1705.03488, submitted (2018).
[15] M. D. Schmidt, New recurrence relations and matrix equations for arithmetic functions generated by Lambert series, *Acta Arithmetica* **181** (2018).
[16] W. Schramm, The Fourier transform of functions of the greatest common divisors, *INTEGERS* **8** (2008).
[17] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, 2017, https://oeis.org/.
[18] L. Tóth, Menons identity and arithmetical sums representing functions of several variables, https://arxiv.org/abs/1103.5861 (2011).

E-mail address: hmousavi6@gatech.edu, maxieds@gmail.com, mschmidt34@gatech.edu

Georgia Institute of Technology, School of Mathematics, 117 Skiles Building, 686 Cherry Street NW, Atlanta, GA 30332, USA