# The Hessenberg matrices and Catalan and its generalized numbers 

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#### Abstract

We present determinantal representations of the Catalan numbers, $k$-Fuss-Catalan numbers, and its generalized number. The entries of the normalized Hessenberg matrices are the binomial coefficients that related with the enumeration of lattice paths.


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## 1 Introduction

An upper Hessenberg matrix $H_{n}=\left(h_{i j}\right), i, j=1,2, \cdots, n$, is a special kind of square matrix, such that $h_{i, j}=0$ for $i>j+1$. Ulrich Tamm [10] give the concept of the Hessenberg matrix in a normalized form, i.e. $h_{i+1, i}=1$ for $i=1, \cdots, n$.

$$
H_{n}=\left(\begin{array}{cccccc}
h_{1,1} & h_{1,2} & h_{1,3} & \cdots & \cdots & h_{1, n} \\
1 & h_{2,2} & h_{2,3} & \cdots & \cdots & h_{2, n} \\
& 1 & h_{3,3} & \cdots & \cdots & h_{3, n} \\
& & \ddots & \ddots & & \\
& & & \ddots & \ddots & h_{n-1, n} \\
& & & & 1 & h_{n, n}
\end{array}\right) \text {, }
$$

The so-called Pascal matrix $P=\left(\binom{i}{j}\right)_{i, j \geq 0}$ is a triangular array of the binomial coefficients. We can generalize Pascal matrix to Hessenberg matrix which
elements are binomial coefficients. It is known that there are a lot of relations between determinants of matrices and well-known number sequences (see 6] [12] and references therein). In this paper, we give a normalized Hessenberg matrices representation of the famous Catalan and its generalized number.

Lattice paths are omnipresent in enumerative combinatorics, since they can represent a plethora of different objects. Especially, lattice paths from $(0,0)$ to $(x, y)$ with $E=(1,0)$ step and $N=(0,1)$ step that never go above the line $L: y=k x$, have been models in many combinatorial problems. Let $n$ be a positive integer. It is well-known that when $k$ is a positive integer, the number of lattice paths from $(0,0)$ to $(n, k n)$ which may touch but never rise above $L$ is $\frac{1}{k n+1}\binom{(k+1) n}{n}$. In particular, when $k=1$, the number of lattice paths of length $n$ is the $n$th Catalan number

$$
C_{n}=\frac{1}{n+1}\binom{2 n}{n}, \text { for } \mathrm{n} \geq 0
$$

The first Catalan numbers for $n=0,1,2,3, \ldots$ are
$1,1,2,5,14,42,132,429,1430,4862,16796, \ldots$ (sequence A000108 in the OEIS).

In general for positive integer $k$, in paper [1], it is called as $k$-Fuss-Catalan numbers $F_{n}$. The purpose of this paper is that we will give a determinant of normalized Hessenberg matrix in Section 2 for positive integer, and in Section 3 for rational $k$. In Section 4, we give the iterative method to evaluate the determinant. Subsequent reduction of Hessenberg matrix to a triangular matrix can be achieved through iterative procedures, this is a fast method to evaluate the determinant of upper normalized Hessenberg matrix.

In [8], it is to count paths in a region that is delimited by nonlinear upper and lower boundaries. Let $a_{1} \leq a_{2} \leq \cdots \leq a_{n}$, and $b_{1} \leq b_{2} \leq \cdots \leq b_{n}$ be integers with $a_{i} \geq b_{i}$. We abbreviate $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{b}=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$. Let $L\left(0, b_{1}\right) \rightarrow\left(n, a_{n}\right)$ denote the set of all lattice paths from $\left(0, b_{1}\right)$ to $\left(n, a_{n}\right)$ satisfying the property that for all $i=1,2, \ldots, n$ the height of the $i$-th horizontal step is in the interval $\left[b_{i}, a_{i}\right]$. Theorem 10.7.1 in [8] gives a formula for counting these paths, we restate it as follows.

Theorem 1 [8] Let $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{b}=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be integer sequences with $a_{1} \leq a_{2} \leq \cdots \leq a_{n}, b_{1} \leq b_{2} \leq \cdots \leq b_{n}$, and $a_{i} \geq b_{i}$, $i=1,2, \ldots, n$. The number of all paths from $\left(0, b_{1}\right)$ to $\left(n, a_{n}\right)$ satisfying the property that for all $i=1,2, \ldots, n$ the height of the $i$-th horizontal step is between $b_{i}$ and $a_{i}$ is given by

$$
\left|L\left(\left(0, b_{1}\right) \rightarrow\left(n, a_{n}\right): \mathbf{b} \leq y \leq \mathbf{a}\right)\right|=\operatorname{det}_{1 \leq i, j \leq n}\left(\binom{a_{i}-b_{j}+1}{j-i+1}\right)
$$

## 2 The $k$-Fuss-Catalan numbers

Given a positive number $k$, a $k$-Fuss-Catalan path of length $n$ is a path from $(0,0)$ to $(n, k n)$ using east steps $(1,0)$ and north steps $(0,1)$ such that it stays weakly below the line $y=k x$. The number of all $k$-Fuss-Catalan paths of length $n$ is given by the $k$-Fuss-Catalan numbers [1],

$$
F_{n}=\frac{1}{k n+1}\binom{(k+1) n}{n},
$$

and Armstrong [5] enumerates the number of $k$-Fuss-Catalan paths of given type. We can give another formula to $F_{n}$ by the following theorem.

Theorem 2 Suppose $k$, $n$ are positive integers. Let

$$
a_{i}=k(i-1), \text { for } i=1, \cdots, n
$$

Then the $k$-Fuss-Catalan numbers $F_{n}$ is equal to

$$
F_{n}=\frac{1}{k n+1}\binom{(k+1) n}{n}=\operatorname{det}_{1 \leq i, j \leq n}\left(\binom{a_{i}+1}{j-i+1}\right) .
$$

PROOF. Let $a_{i}=k(i-1)$, for $i=1, \cdots, n$, and let

$$
\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)=(k, 2 k, \cdots, k(n-1))
$$

and $\mathbf{b}=(0,0, \cdots 0)$. Applying Theorem 1, we find the number of lattice paths is $\underset{1 \leq i, j \leq n}{\operatorname{det}}\left(\binom{a_{i}+1}{j-i+1}\right)$.

It might be interesting to have a formula for enumerating Fuss-Catalan paths using Theorem 5. We implement an evaluation by a procedure of Maple as follows.
$>$ with(combinat);
with(LinearAlgebra);
ffT $:=\operatorname{proc}(n, r)$
local $a, k, j, i, A$;
$k:=r^{*} n$;
for $j$ to $k$ do $a[j]:=r^{*}(j-1)$ end do;
$A:=\operatorname{Matrix}(n, n,(i, j)->\operatorname{binomial}(a[i]+1, j-i+1))$;
[A, Determinant(A)];
end proc;
For $k=3$, we can illustrate the identity by a Maple command:
$>$ for $i$ from 1 to 18 do ffT(i, 3)[2];binomial((3+1)*i, i)/(3*i+1); end do;
$1,4,22,140,969,7084,53820,420732,3362260,27343888, \cdots$

For $k=2$, one can get Ternary number [9,

$$
T_{n}=\frac{1}{2 n+1}\binom{3 n}{n}=\left|\begin{array}{ccccc}
\binom{1}{1} & & & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \\
\\
& \binom{5}{0}\binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \ldots \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots \\
& & & \binom{2 n-1}{0} & \binom{2 n-1}{1}
\end{array}\right| .
$$

To illustrate the identity, we use the following Maple command:
$>$ for $i$ from 1 to 18 do ffT(i, 2)[2];binomial $((2+1) * i, i) /\left(2{ }^{*} i+1\right)$; end do;
$1,3,12,55,273,1428,7752,43263,246675,1430715, \cdots$

In particular, when $k=1$, we get an identity about Catalan number,

$$
C_{n}=\frac{1}{n+1}\binom{2 n}{n}=\left|\begin{array}{llllll}
\binom{1}{1} & & & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & \\
& \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{2} & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & \ddots \\
& & & \binom{n}{0} & \binom{n}{1}
\end{array}\right| .
$$

$>$ for $i$ from 1 to 18 do ffT(i, 1)[2];binomial $\left((2+1){ }^{*} i, i\right) /\left(2{ }^{*} i+1\right)$; end do;

Namely, there are

$$
\begin{aligned}
& C_{1}=|1|=1 ; C_{2}=\left|\begin{array}{l}
\binom{1}{1} \\
\binom{2}{0}
\end{array}\right|=2 \text { ( } 12 \text {; }
\end{aligned}
$$

## 3 The generalized Fuss-Catalan numbers

When $k=\frac{r}{m}$ is rational, here $r$ and $m$ are coprime positive integers, it is shown in [4] that the number of lattice paths from $(0,0)$ to $(m n, r n)$ that may touch but never rise above the line $y=\frac{r}{m} x$, is $\sum_{a} \prod_{a_{i}} \frac{F_{i}^{a_{i}}}{\alpha_{i}!}$, where $F_{i}=\frac{1}{i(m+r)}\binom{i(m+r)}{i m}$ and the sum $\sum_{a}$ is taken over all sequences of non-negative integers $a=\left(a_{1}, a_{2}, \cdots\right)$ such that $\sum_{i=1}^{\infty} i a_{i}=n$. We give another representation of determinant of normalized upper Hessenberg matrix as the follows.

Theorem 3 Suppose $m$, $r$ are coprime positive integers. Let

$$
a_{i}=r\left\lfloor\frac{i-1}{m}\right\rfloor+\left\lfloor\frac{r}{m}\left(i-m\left\lfloor\frac{i-1}{m}\right\rfloor-1\right)\right\rfloor, \text { for } i=1, \cdots, m n .
$$

Then the generalized Fuss-Catalan numbers $W_{n}$, which is the number of lattice paths from $(0,0)$ to ( $m n, r n$ ) that may touch but never rise above the line $y=\frac{r}{m} x$, is equal to

$$
W_{n}=\operatorname{det}_{1 \leq i, j \leq m n}\left(\binom{a_{i}+1}{j-i+1}\right) .
$$

PROOF. Let $a_{i}=r\left\lfloor\frac{i-1}{m}\right\rfloor+\left\lfloor\frac{r}{m}\left(i-m\left\lfloor\frac{i-1}{m}\right\rfloor-1\right)\right\rfloor$, for $i=1, \cdots, n m$, and let $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{b}=(0,0, \cdots 0)$. Applying Theorem 1, we find the number of lattice paths is $\underset{1 \leq i, j \leq n}{\operatorname{det}}\left(\binom{a_{i}+1}{j-i+1}\right)$.

For example, the generalized Fuss-Catalan numbers $W_{1}$, which is the number of lattice paths from $(0,0)$ to $(7,16)$ that may touch but never rise above the
line $y=\frac{16}{7} x$, is equal to

For the general, we implement the evaluation of the determinant of Hessenberg matrices by a procedure of Maple as follows.

```
\(>\) with(combinat);
with(LinearAlgebra);
FjsDyckpath \(:=\operatorname{proc}(m, r, n)\)
local \(c, k, j, i, C\);
\(k:=m^{*} n\);
for \(j\) to \(k\) do
if \(j<=m\) then
\(c[j]:=\operatorname{floor}\left(r^{*}(j-1) / m\right)\)
else
\(c[j]:=\operatorname{floor}\left(r^{*}\left(j-m^{*}\right.\right.\) floor \(\left.\left.((j-1) / m)-1\right) / m\right)+r^{*}\) floor \(((j-1) / m)\);
end if
    end do;
\(C:=\operatorname{Matrix}\left(m^{*} n, m^{*} n,(i, j)->\operatorname{binomial}(c[i]+1, j-i+1)\right.\);
[C, Determinant (C)];
end proc
```

$>$ for $i$ to 16 do FjsDyckpath(7, 16, i) end do

## 4 Evaluate the determinants

We can find that above determinants are of upper normalized Hessenberg matrices, which entries are binomial coefficients. The row operations may be applied to the matrix from the first row to the last row: adding the negative reciprocal multiple of the entry $a_{i i}$ of the $i$ th row to the succeeding row. This can reduce the matrix to strictly upper triangular, so we evaluate the deter-
minant of the corresponding upper normalized Hessenberg matrices. This can reduce the matrix to a strictly upper triangular, so the determinant of upper normalized Hessenberg matrices is equal to the product of main diagonal elements.

For example,


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