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The Tribonacci and ABC Representations of Numbers are Equivalent

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Abstract

It is shown that the unique representation of positive integers in terms of tribonacci numbers and the unique representation in terms of iterated A, B and C sequences defined from the tribonacci word are equivalent. These sequences are studied in detail.

1 Introduction and Synopsis

The quintessence of many applications of the tribonacci sequence $T = \{T(n)\}_{n=0}^{\infty}$ [4] A000073 [5], [6], [2] [1] is the ternary substitution sequence $2 \to 0$, $1 \to 02$ and $0 \to 01$. Starting with 2 this generates an infinite (incomplete) binary tree with ternary node labels called $TTree$. See *Fig 1* for the first 6 levels $l = 0, 1, ..., 5$ denoted by $TTree_5$. The number of nodes on level l is the tribonacci number $T(l+2)$, for $l \geq 0$. In the limit $n \to \infty$ the last level $l = n$ of $TTree_n$ becomes the infinite self-similar tribonacci word $TWord$. The nodes on level l are numbered by $N = 0, 1, ..., T(l+2) - 1$.

The left subtree, starting with 0 at level $l = 1$ will be denoted by $TTreeL$, and the right subtree, starting with 2 at level $l = 0$ is named $TTreeR$. The number of nodes on level l of the left subtree $TTreeL$ is $T(l+1)$, for $l \in \mathbb{N}$; the number of nodes on level l of $TTreeR$ is 1 for $l = 0$ and $T(l+1)$, for $l \in \mathbb{N}$.

$TWord$ considered as ternary sequence t is given in [4] A080843 (we omit the OEIS reference henceforth if A numbers for sequences are given): $\{0, 1, 0, 2, 0, 1, 0, 0, 1, 0, 2, 0, 1, ...\}$. See also *Table 1*. This is the analogue of the binary rabbit sequence A005614 in the *Fibonacci* case. Like in the *Fibonacci* case with the complementary and disjoint *Wythoff* sequences A = A000201 and B = A001950 recording the positions of 1 and 0, respectively, in the tribonacci case the sequences A = A278040, B = A278039, and C =

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A278041 record the positions of 1, 0, and 2, respectively. These sequences start with $A = \{1, 5, 8, 12, 14, 18, 21, 25, 29, 32, ...\}$, $B = \{0, 2, 4, 6, 7, 9, 11, 13, 15, 17, ...\}$, and $C = \{3, 10, 16, 23, 27, 34, 40, 47, 54, 60, ...\}$. See also *Table 1*.

The present work is a generalization of the theorem given in the *Fibonacci* case for the equivalence of the *Zeckendorf*- and *Wythoff*- representations of numbers in [3].

Note that there are other complementary and disjoint tribonacci A, B and C sequences given in OEIS. They use the same ternary sequence t = A080843 (which has offset 0), with $0 \to a$, $1 \to b$ and $2 \to c$, however with offset 1, and record the positions of a, b and c by A = A003144, B = A003145 and C = A003146, respectively. In [2] and [1] they are called a, b, and c. This tribonacci ABC-representation is given in A317206. The relation between these sequences (we call them now a, b, c) is: $a(n) = B(n-1) + 1$, $b(n) = A(n-1) + 1$, and $c(n) = C(n-1)+1$, for $n \geq 1$. We used $B(0) = 0$ in analogy to the *Wythoff*-representation in the *Fibonacci* case.

From the uniqueness of the ternary sequence t (with offset 0) it is clear that the three sequences A, B and C cover the nonnegative integers $\mathbb{N}_0$ completely, and they are disjoint. In contrast to the *Fibonacci* case where the *Wythoff* sequences are *Beatty* sequences [7] for the irrational number φ =A001622, the golden section, and are given by $A(n) = \lfloor n\,\varphi \rfloor$ and $B(n) = \lfloor n\,\varphi^2 \rfloor$, for $n \in \mathbb{N}$ (with $A(0) = 0 = B(0)$), no such formulae for the complementary sequences A, B and C in the tribonacci case are considered. The definition given above in terms of $TWord$, or as sequence t, is not burdened by numerical precision problems.

Note that the irrational tribonacci constant $\tau = 1.83928675521416...$ =A058265, the real solution of characteristic cubic equation of the tribonacci recurrence $\lambda^3 - \lambda^2 - \lambda - 1 = 0$, defines, together with $\sigma = \dfrac{\tau}{\tau - 1} = 2.19148788395311...$ =A316711 the complementary and disjoint *Beatty* sequences $At := \lfloor n\,\tau \rfloor$ and $Bt := \lfloor n\,\sigma \rfloor$, given in A158919 and A316712, respectively.

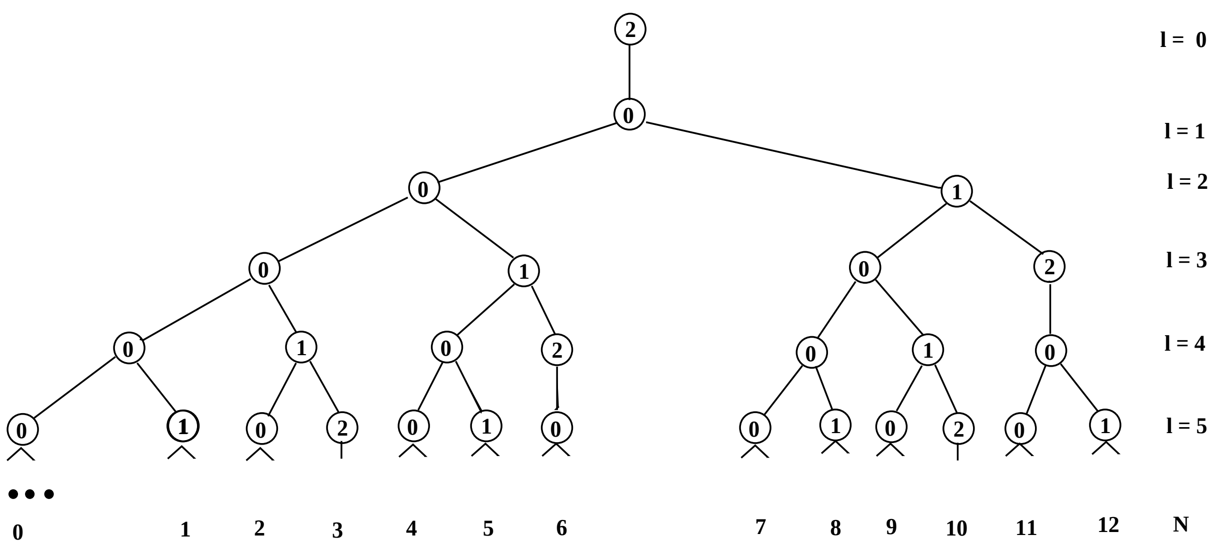

Figure 1: Tribonacci Tree TTree_5

The analogue of the unique *Zeckendorf*-representation of positive integers is the unique tribonacci-representation of these numbers.

$$(N)_T \,=\, \sum_{i=0}^{I(N)} f_i\, T(i+3), \quad f_i \,\in\, \{0,\, 1\}, \quad f_i\, f_{i+1}\, f_{i+2} \,=\, 0, \;\; f_{I(N)} \,=\, 1\,. \tag{1}$$

The sum should be ordered with falling T indices. This representation will also be denoted by (Z as a reminder of *Zeckendorf*)

$$\begin{aligned} ZT(N) &= \text{Ħ}_{i=0}^{I(N)}\, f_{I(N)-i} \\ &= f_{I(N)} f_{I(N)-1} \ldots f_0\,. \end{aligned} \tag{2}$$

The product with concatenation of symbols is here denoted by Ħ, and the concatenation symbol $\circ$ is not written. This product has to be read from the right to the left with increasing index i. This representation is given in A278038(N), for $N \,\geq\, 1$. See also *Table 3* for $ZT(N)$ for $N \,=\, 1,\, 2,\, ...,\, 100$.

E.g., $(1)_T \,=\, T(3)$, $ZT(1) \,=\, 1$; $(8)_T \,=\, T(6) + T(3)$, $ZT(8) \,=\, 1001$. The length of $ZT(N)$ is $\#ZT(N) \,=\, I(N) + 1 \,=\, \{1,\, 2,\, 2,\, 3,\, 3,\, 3,\, 4,\, 4,\, 4,\, 4,\, 4,\, 4,\, ...\} \,=\, \{\text{A278044}(N)\}_{N\,\geq\,1}$. The number of numbers n is given by $\{1,\, 2,\, 3,\, 6,\, 11,\, 20,\, 37\,...\} \,=\, \{\text{A001590}(n+2)\}_{n\,\geq\,1}$. These are the companion tribonacci numbers of $T\,=$A000073 with inputs 0, 1, 0 for $n \,=\, 0,\, 1,\, 2$, respectively.

$ZT(N)$ can be read off any finite $TTree_n$ with $T(n+2) \,\geq\, N$ after all node labels 2 have been replaced by 1. See *Figure 1* for $n = 5$ (with $2 \,\rightarrow\, 1$) and numbers $N \,=\, 0,\, 1,\, ...,\, 12$. The branch for N is read from bottom to top, recording the labels of the nodes, ending with the last 1 label. Then the obtained binary string is reversed in order to obtain the one for $ZT(N)$. *E.g.*, $N \,=\, 9$ leads to the string 0101 which after reversion becomes $ZT(9) \,=\, 1010$.

The analogue of the *Wythoff*- representation of nonnegative integers is the tribonacci ABC-representation using iterations of the sequences A, B and C.

$$(N)_{ABC} \,=\, \left(\text{Ħ}_{j=1}^{J(N)}\, X(N)_j^{k(N)_j}\right) B(0), \;\text{ with }\; N \,\in\, \mathbb{N}_0, \;\; k(N)_j \,\in\, \mathbb{N}_0\,, \tag{3}$$

again with an ordered concatenation product. Here $X(N)_j \,\in\, \{A,\, B,\, C\}$, for $j \,=\, 1,\, 2,\, ...,\, J(N) - 1$, with $X(N)_j \,\neq\, X(N)_{j+1}$, and $X(N)_{J(N)} \,\in\, \{A,\, C\}$. Powers of $X(N)_j$ are also to be read as concatenations. Concatenation means here iteration of the sequences. The exponents can be collected in $\vec{k}(N) \,:=\, (k(N)_1,\, ...,\, k(N)_{J(N)})$. For the equivalence proof only positive integers N are considered. If exponents vanish the corresponding A, B, C symbols are not present ($X(N)_j^0$ is of course not 1). If all exponents vanish, the Ħ is empty, and $N \,=\, 0$ could be represented by $(0)_{ABC} \,=\, B(0) = 0$ (but this will not be used for the equivalence proof).

E.g., $(30)_{ABC} \,=\, (BCBA)B(0) \,=\, B(C(B(A(B(0)))))$, $J(30) \,=\, 4$, $\vec{k}(30) \,:=\, (k_1,\, k_2,\, k_3,\, k_4) \,=\, (1,\, 1,\, 1,\, 1)$, $X(30)_1^{k_1} \,=\, B^1$, $X(30)_2^{k_2} = C^1$, $X(30)_3^{k_3} \,=\, B^1$, $X(30)_4^{k_4} = A^1$ (sometimes the arguments (N) are skipped).

The number of A, B and C sequences present in this representation of N is $\sum_{j=1}^{J(N)} k(N)_j + 1 =$ A316714(N), This representation is also written as

$$ABC(N) = \left(\text{Ħ}_{j=1}^{J(N)}\, x(N)_j^{k(N)_j} \right) 0, \text{with } \ k(N)_j \in \mathbb{N}_0\,, \tag{4}$$

and $x(N)_j \in \{0,\, 1,\, 2\}$, for $j = 1,\, 2,\, ...,\, J(N)-1$, with $x(N)_j \neq x(N)_{j+1}$, and $x(N)_{J(N)} \in \{1,\, 2\}$. Here $x = 0,\, 1,\, 2$ replaces $X = B,\, A,\, C$, respectively.

E.g., $ABC(0) = 0$, $J(1) = 0$ (empty product); $ABC(30) = 02010$.

For this ABC-representation see A319195. Another version is A316713 (where for a technical reason B, A, and C are represented by 1, 2 and 3 (not 0, 1 and 2), respectively). See also *Table 3* for $ABC(N)$, for $N = 1,\, 2,\, ...,\, 100$.

The number of Bs, As and Cs in the ABC-representation of N is given in sequences A316715, A316716 and A316717, respectively. The length of this representation is given in A3167174.

A) From $\mathbf{ZT(N)}$ to $\mathbf{N_{ABC}}$

For the proof of the equivalence of these two representations $(N)_T$ and $(N)_{ABC}$ for positive integers N one uses for the first part, $(N)_T \to (N)_{ABC}$, in the version $ZT(N) \to (N)_{ABC}$, the reversed word $ZT(N)$ with a concatenated 0 at the beginning and at the end. This intermediate step will be called $\widehat{ZT}(N)$. *E.g.*, $\widehat{ZT}(1) = 010$ from $ZT(1) = 1$; $\widehat{ZT}(30) = 00110010$ from $ZT(30) = 100110$. *I.e.*, $\widehat{ZT}(N) = 0\overline{ZT}(N)0$, with the reversed word $\overline{ZT}(N) := (ZT(N))_{\text{reversed}}$.

This simple definition of $\widehat{ZT}(N)$ for given N becomes somewhat complicated if a compact explicit notation is used for general $N \in \mathbb{N}$.

$$\begin{aligned}\widehat{ZT}(N) &\equiv \widehat{ZT}(N;\, P(N),\, \overrightarrow{j_A}(N,p),\, \overrightarrow{j_C}(N,p)) \\ &= 0\text{Ħ}_{p=1}^{P(N)} \left(\left(\text{Ħ}_{k=1}^{J_A(N,p)} 0^{j_{A,k}(N,p)} 1 \right) \left(\text{Ħ}_{k=1}^{J_C(N,p)} 0^{j_{C,k}(N,p)} 11 \right) \right)^p 0\,. \end{aligned} \tag{5}$$

Several explanations follow, and rules are needed to avoid the appearance of 111 in this binary word. Uniqueness requires rules for the separation between neighboring p-words.

Explanations:

1) Vanishing $p-$dependent ordered concatenation products are indicated by $J_A = 0$ or $J_C = 0$ (we omit sometimes the arguments (N,p)). In this case undefined products arise (which are here not set to 1, of course). Not both products are allowed to vanish for any p, *i.e.*, $J_A = 0 = J_C$ is forbidden.

2) Exponents of 0 indicate the multiplicity. A vanishing exponent means disappearance of the 0s. One could use another notation like $0_{j_{A,k}}$ and $0_{j_{C,k}}$.

3) The separation of consecutive p-words is done such that $P(N)$ becomes minimal. For this the following two rules apply.

i) If $J_A(N,p+1) \geq 1$ (part A present for $p+1$) then $J_C(N,p) \neq 0$.

ii) $J_C(N,p) = 0 = J_A(N,p+1)$ is forbidden.

4) To avoid the appearance of the subword 111 some rule is needed:

The exponents of 0 are collected in $\overrightarrow{j_A}$ and $\overrightarrow{j_C}$. In general they satisfy $j_{A,k} \in \mathbb{N}$ and $j_{C,k} \in \mathbb{N}$, *i.e.,* positive powers of 0 appear. But there are exceptions for which these exponents may vanish.

The first exceptions apply to the start of the $p-$product. It may start with 1 or with 11.

Exception 1

i) $j_{A,1}(N,1) \in \mathbb{N}_0$

ii) $j_{C,1}(N,1) \in \mathbb{N}_0$ if $J_A(N,1) = 0$

The remaining exception applies for $p \geq 2$. Then one has to make sure that no 111 appears in the transition from a $p-1$ to p.

Exception 2

If $J_C(N,p-1) = 0$ then $j_{A,1}(N,p) \in \mathbb{N}_0$, for $2 < p < P(N)$.

Some examples may illustrate these explanations and exceptions.

Examples 1

1) $\widehat{ZT}(N) = 0101010^2 11010$. Minimal $P(N)$ (Explanation 3) is obtained with $P(N) = 2$, $J_A(N,1) = 3$, $J_C(N,1) = 1$, $J_A(N,2) = 1$, $J_C(N,2) = 0$. Here $j_{A,1}(N,1) = 0$ (exception 1)i)). *E.g.,* the separation 01|0101|0011|010 with $P = 4$ is forbidden. For this $J_C(N,1) = 0$, $J_A(N,2) = 2$ and $J_C(N,2) = 0 = J_A(N,3)$. The last separation is the only one for the given minimal $P(N)$ solution.

2) $\widehat{ZT}(N) = 010^2 10^2 11011010$. This is an instance $P(N) = 2$, $J_A(N,1) = 2$, $J_C(N,1) = 2$, $J_C(N,2) = 0$ and

$$\overrightarrow{j_A}(N,1) = (0,2),\ \overrightarrow{j_C}(N,1) = (2,1),\ \overrightarrow{j_A}(N,2) = (1)\,.$$

Here the minimal P is 2, and exception 1)i) (start with 1) applies. Also explanation 1) is needed because for $p = 2$ part C is missing but not part A. The corresponding $ZT(N)$ is 1011011001001, with $I(N) = 12$, and $N = 1705 + 504 + 274 + 81 + 44 + 7 + 1 = 2616$.

3) $\widehat{ZT}(N) = 0110^2 11010$. Here exception 1)ii) (start with 11) occurs and $P(N) = 2$ with $J_A(N,1) = 0$ (explanation 1), $J_C(N,1) = 2$ and $J_A(N,2) = 1$, $J_C(N,2) = 0$ (explanation 1). $ZT(N)$ is 10110011, with $I(N) = 7$, and $N = 81 + 24 + 13 + 2 + 1 = 121$.

The translation from $\widehat{ZT}(N)$ to $(N)_{ABC}$ is now performed, in an intermediate step introducing two new symbols and $\bullet$ and $\times$ with the help of four substitution rules in the word $w(N)$, used here as abbreviation for $w(N) := \widehat{ZT}(N) = \text{Ħ}_{i=1}^{\#w(N)} w(N)_i$ with $\#w(N) = I(N) + 3 =$ A278044$(N) + 2$. This intermediate representation will be denoted by $(N)_{AB\bullet\times}$. The following rules depend on the neighbors of $w(N)_i$ for $i = 1, 2, ..., \#w(N) - 1$. To mark the position i, the number (letter) $w(N)_i$ to be substituted is given in the rules in boldface and underlining. $w(N)_1$ has no left neighbor denoted in the following by $\emptyset$. This $\emptyset$ is also used to signal the end of each word $w(N)$ after 10.

The four substitution rules

$$\begin{array}{llllll}
(S1) & 1\underline{\mathbf{0}}0 : \underline{\mathbf{0}} \longrightarrow \bullet & \text{and} & x\underline{\mathbf{0}}0 : \underline{\mathbf{0}} \longrightarrow B, & \text{for } x \in \{\emptyset, 0\}\,, & \\
(S2) & \underline{\mathbf{0}}11 : \underline{\mathbf{0}} \longrightarrow \times & \text{and} & \underline{\mathbf{0}}10 : \underline{\mathbf{0}} \longrightarrow A, & & \\
(S3) & \underline{\mathbf{1}}1 : \underline{\mathbf{1}} \longrightarrow \times\,, & & & & \\
(S4) & \underline{\mathbf{1}}01 : \underline{\mathbf{1}} \longrightarrow \bullet & \text{and} & \underline{\mathbf{1}}0x : \underline{\mathbf{1}} \longrightarrow B, & \text{for } x \in \{\emptyset, 0\}\,. & (6)
\end{array}$$

These rules suffice and are not in conflict which each other. Eg. $1\underline{\mathbf{1}}0$ is not needed because if the word ends in the numbers 110 then rule $(S4)$, part two, with $x = \emptyset$, applies for the substitution of the last 1 becoming a B. Otherwise it is either $1\underline{\mathbf{1}}00$ or $1\underline{\mathbf{1}}01$ in which case also $(S4)$ applies either with part two and $x = 0$ or with part one.

$E.g., w(N) = 0101010011010$ with $\#w = 13$ translates to $(N)_{AB\bullet\times} = A\bullet A\bullet AB\bullet\times\times\bullet AB$ with length $\#w - 1 = 12$.

$w(N)$ ends always in 10. This last substitution of 1 uses part two of rule $(S4)$ with $x = \emptyset$.

In the final step the translation into $(N)_{ABC}$ is obtained by omitting all $\bullet$s and substituting $\times\times \longrightarrow C$. This reduces the length to A316714(N), the one of $(N)_{ABC}$.

The preceding example thus gives $(N)_{ABC} = A^3BCAB$ which represents $N = 752$ corresponding to the given $ZT(752) = 10110010101 = 504 + 149 + 81 + 13 + 4 + 1$.

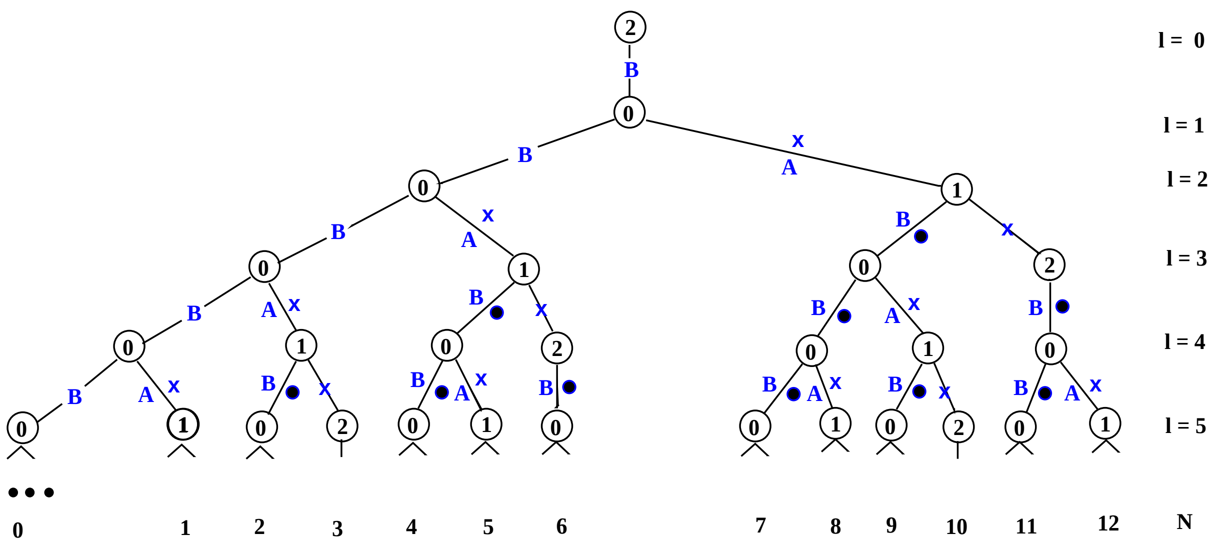

Figure 2: ABC-representation with the ABCTree$_5$

In *Figure 2* the tribonacci tree $TTree_5$ from *Figure 1* has been used with labeling the edges (branches) with symbols A, B, C, $\bullet$, $\times$ in a special way. It is called $ABCTree_5$. The new branch decorated infinite tree is denoted by $ABCTree$.

The ABC-representation of N is obtained directly by reading the branches from bottom to top. If there are two edge labels like A and $\times$, or B and $\bullet$, for an edge going out from from a node, the choice is fixed from the direction from which the previous (lower) edge reached the node. If it reached the node from the right-hand side, the label on the right-hand side of the outgoing edge has to be chosen, and similarly for the left-hand side. If one considers a finite $ABCTree_n$ with levels $l = 0, 1, ..., n$ having no incoming edges from the next level $l = n + 1$, one chooses always the left variant for the outgoing edges from nodes of the last level $l = n$ of $ABCTree_n$. The ABC-representation ends always in $AB(0)$ or $CB(0)$ which means that coming from the left subtree one stops after reaching the first node on the outermost branch with only B edges. Only in the right subtree one has to go all the way up to node 2 at level $l = 0$. The $N = 0$ case is not considered in the equivalence proof, but the tree shows that $N = 0$ would be represented by $B = B(0)$ (*Figure 2* the B emerging from the first node labeled 0 at level $l = 5$).

E.g., for $N = 8$ one has from $ABCTree_5$ the edge labels from bottom to top $A \bullet BAB$ corresponding to $ABAB = A(B(A(B(0)))) = 8$. Here one sees why the tree started with node 2 at level $l = 0$. For $N = 6$ the path is $B \times \times B \rightarrow BCB = B(C(B(0)))$ ending at node 0 at level l.

Note that if one adds a level $n + 1$ to $ABCTree_n$, thus obtaining $ABCTree_{n+1}$, the first numbers $N = 0, 1, ..., T(3+l) - 1$ related to level $l+1$ of the left subtree $TTreeL_{l+1}$ have the same ABC-representations like the those compiled starting from level n of $ABSTree_n$. This is because one stays in the left subtree $ABSTreeL_{l+1}$ and one reaches at most the 0 from level 1.

B) From $\mathbf{(N)_{ABC}}$ to $\mathbf{ZT(N)}$

The reverse part of the equivalence proof starts with the representation $(N)_{ABC}$ eq. 3, and constructs $\widehat{ZT}(N)$ eq. 5. After erasing the 0 at the beginning and end, and reversing the remaining word one obtains the binary word $ZT(N)$ eq. 2 and from this $(N)_T$ eq. 1.

The first task is to find the intermediate $(N)_{AB\bullet\times}$ version from $(N)_{ABC}$. For this one derives from the substitution rules eq. 6 how A, B and C can appear. A is reached uniquely from $\underline{\mathbf{0}}10$. For B one has to distinguish two types, called BI and BII. BI originates from a substituted $\underline{\mathbf{0}}$, either at the start from $\emptyset\underline{\mathbf{0}}0$ or from $0\underline{\mathbf{0}}0$. BII originates from a substituted $\underline{\mathbf{1}}$ either at the end from $\underline{\mathbf{1}}0\emptyset$ or from $\underline{\mathbf{1}}00$. Finally, C, represented by $\times\times$, originates from substituting 0 in $\underline{\mathbf{0}}11$ leading to $\times$, and the following substitution for 1 produces the second $\times$. Note that $\times\times$ obtained from substituting 11 would need in fact 111 which is forbidden. Therefore C can appear only from a 0110 string starting substitutions with the first 0.

Consider now $(N)_{ABC}$ from eq. 3. It turns out that the transition between the blocks of powers of A, B and C is important in order to find out the correct $(N)_{AB\bullet\times}$ representation. The final $B(0)$ in N_{ABC} will only at the end be added as a final B. There is never a final B-block for $j = J(N)$ in eq, 3 from the uniqueness requirement of the representation. The following statements then follow.

Step 1 replacements

Step1A) A block A^n, for $n \in \mathbb{N}$, (*i.e.,* $X(N)_j = A$, $k(N)_j = n$ in eq. 3), appearing alone ($J(N) = 1$) or at the end ($j = J(N)$) or if followed by a B-block is replaced by $(A\bullet)^{n-1}A$ (remember that $(A\bullet)^0$ means disappearance). The B following an A^n-block is always of type BII (in the cases $J(N) = 1$ or $j = J(N)$ this means that last omitted B is of type BII). If the A^n-block is followed by a C-block then it is replaced by $(A\bullet)^n$.

Step1B) A block B^n, for $n \in \mathbb{N}$, which can never appear alone, stays B^n if it begins with a B of type BI (especially if $X(N)_1 = B$). If the block B^n begins with a B of type BII then B^n is replaced by $B\bullet B^{n-1}$.

Step1C) A block C^n followed by an A-block is replaced by $(\bullet \times \times)^n$. If C^n is followed by a $B-$block starting with a B of type BII then it is replaced by $\times\times$. This applies also if a C-block appears alone ($J(N) = 1$). A C- block is never followed by a B-block beginning with a B of type BI.

(7)

In order to obtain the $(N)_{AB\bullet\times}$ representation one adds after these **Step1 replacements** the final B. Some examples are in order:

Examples 2

1) $N_{ABC} = B^3AB$. The starting B^3 remains B^3 because the first B is of type I (it comes from $\emptyset\underline{\mathbf{0}}0$). Because the A^1 (the last block) is followed by a B (always type II) it remains an A. After appending the omitted last B one obtains $(N)_{AB\bullet\times} = BBBAB$, *i.e.*, here no $\bullet$ appears.

2) $N_{ABC} = A^3BCAB$. The starting A-bock is replaced *via* **Step1A)** by $(A\bullet)^2A$. The following block B^1 is replaced by $B\bullet$ because the B after an A is always of type II. The next block C^1 followed by the last A-block A^1 is replaced by $\bullet\times\times$ The last A remains an A. After adding the final B one obtains $(N)_{AB\bullet\times} = A\bullet A\bullet AB\bullet\times\times\bullet AB$. This is the representation found above in part A) from $ZT(752)$

The translation from $(N)_{AB\bullet\times}$ to $\widehat{ZT}(N)$ is simply done by starting with an extra 0 and appending the $(N)_{AB\bullet\times}$ string by replacing $A \to 1$, $B \to 0$, $\bullet \to 0$ and $\times \to 1$.

In the example **1**) this produces $\widehat{ZT}(N) = 000010$. The example **2)** gives 0101010011010.

The final translation from $\widehat{ZT}(N)$ to $ZT(N)$ is then trivial: omit the two boundary 0s and reverse the remaining binary string.

The two examples give: **1**) $\overline{0001} = 1000$, which is $ZT(7)$, and **2)** $\overline{10101001101} = 10110010101$ which is $ZT(752)$. This was used above as start of the example for the proof in the other direction.

2 Equivalence of representations ZT(N) and ABC(N)

First the uniqueness of the tribonacci-representation $ZT(N)$ of eq. 2 is considered.
It is clear that every binary sequence starting with 1, without three consecutive 1s, represents some $N \in \mathbb{N}$. An algorithm for finding such a representation for every $N \in \mathbb{N}$ is given to prove the following lemma.

Lemma 1. *The tribonacci-representation $ZT(N)$ of eq.2 is unique.*

Proof:

The recurrence of the tribonacci sequence $T := \{T(l)\}_{l=3}^{\infty}$, with inputs $T(3) = 1$, $T(4) = 2$ and $T(5) = 4$, shows that this sequence is strictly increasing. Define the floor function $floor(T; n)$, for $n \in \mathbb{N}$, giving the largest member of T smaller or equal to n. The corresponding index of T will then be called $Ind(floor(T; n))$. Define the finite sequence $Nseq := \{N_j\}_{j=1}^{j_{\max}}$ recursively by

$$N_j = N_{j-1} - floor(T; N_{j-1}), \quad \text{for } j = 1, 2, ..., j_{\max}, \tag{8}$$

with $N_0 = N$ and $N_{j_{\max}} = 0$.
It is clear that this recurrence reaches always 0. Define the finite sequences $fTN := \{floor(T;\, N_j)\}_{j=0}^{j_{\max}-1}$ and $IfTN := \{Ind(fTN_j)\}_{j=0}^{j_{\max}-1}$. Then $I(N)$ in eq. 2 is given by $I(N) = IFTN_0$ and the finite sequence $fseq = \{f_{I(N)-k}\}_{k=0}^{I(N)}$ is given by

$$f_{I(N)-k} = \begin{cases} 1 & \text{if } \; I(N) - k \in IfTN\,, \\ \\ 0 & \text{otherwise}\,. \end{cases} \tag{9}$$

$\square$

Example 3 $N = 263$. $Nseq = \{263,\, 144,\, 33,\, 9\, 2,\, 0\}$, $fTN = \{149,\, 81,\, 24,\, 7,\, 2\}$, $IfTN = \{8,\, 7,\, 5,\, 3,\, 1\}$, $I(N) = 8$, $fseq = \{1,\, 1,\, 0,\, 1,\, 0,\, 1,\, 0,\, 1,\, 0\}$.
Next follows the lemma on the uniqueness of the ABC-representation given in eq. 3.

Lemma 2. *The tribonacci ABC-representation $(N)_{ABC}$ of eq.3, for $N \in \mathbb{N}_0$, is unique.*

Proof:
From the definition of the $A-$, $B-$ and $C-$sequences (each with offset 0) based on the value 1, 0 and 2, respectively, of $t(n)$, for $n \in \mathbb{N}_0$, it is clear that these sequences are disjoint and $\mathbb{N}_0$-complementary. 0 is represented by $B(0)$. Therefore the n-fold iteration $B^{[n]}(0)$ (written as $B^n(0)$) is allowed only for $n = 1$, and any representation ends in $B(0)$. Iterations acting on 0 are encoded by words over the alphabet $\{A,\, B,\, C\}$, and n-fold repetition of a letter X is written as X^n, named $X-$block, where $n = 0$ means that no such $X-$block is present. Then any word consisting of consecutive different non-vanishing $X-$blocks ending in the $B-$block B^1 represents a number $N \in \mathbb{N}_0$.
In order to prove that with such representations every $N \in \mathbb{N}_0$ is reached the following algorithm is used. Replace any number $n \in \mathbb{N}_0$, which is $n = X_n(k)$ with $X_n \in \{A,\, B,\, C\}$ and $k \in \mathbb{N}_0$, by the 2-list $L(n) = [L(n)_1,\, L(n)_2] := [X_n,\, k(n)]$. Define the recurrence

$$L(j) = [L(L(j-1)_2)_1,\, L(L(j-1)_2)_2], \quad \text{for} \;\; j = 1,\, 2,\, ...,\, j_{\max}\,, \tag{10}$$

with input $L(0) = [X_N,\, k(N)]$, and $j_{\max}$ is defined by $L(j_{\max}) = [B,\, 0]$.
Then the word is $w(N) = \prod_{j=0}^{j_{\max}} L(j)_1$ (a concatenation product), and read as iterations acting on 0 this becomes the representation $(N)_{ABC}$. The length of the word $w(N)$ is $j_{\max} + 1$.

$\square$

Example 4 $N = 38$. $L(0) = [A,\, 11]$, $L(1) = [B,\, 6]$, $L(2) = [B,\, 3]$, $L(3) = [C,\, 0]$, and $L(4) = [B, 0]$, hence $j_{\max}(38) = 4$, $w(38) = ABBCB$, and $(38)_{ABC} = ABBCB(0)$, to be read as $A(B(B(B(C(B(0))))))$.

After these preliminaries the main theorem can be stated.

Theorem. *The tribonacci-representation $ZT(N)$ of eq. 1, is equivalent to the tribonacci ABC-representation $(N)_{ABC}$ eq. 3, for $N \in \mathbb{N}$.*

Proof:
Part **A)**: The proof of the map $ZT(N) \to (N)_{ABC}$ is performed in three steps:

$$\begin{aligned}
&Step\ 1: \quad && ZT(N) && \to \widehat{ZT}(N) := 0(ZT(N)_{\text{reverse}})0\,, \\
&Step\ 2: \quad && \widehat{ZT}(N) && \to (N)_{AB\bullet\times}\,, \\
&Step\ 3: \quad && (N)_{AB\bullet\times} && \to (N)_{ABC}\,.
\end{aligned} \tag{11}$$

Step 1 is clear.
For Step 2 one uses eq. 5 and the **Explanations** 1) to 4) with **Exception** 1) and 2). See also **Example 1**. The four substitution rules $(S1)$, $(S2)$, $(S3)$ and $(S4)$ of eq. 6 are then applied to obtain $(N)_{AB\bullet\times}$. See also the example for $N = 752$ there.
In Step 3 the symbols $\bullet$ in $(N)_{AB\bullet\times}$ are omitted and the pair of symbols $\times\times$ ($\times$ always appears as a pair) is replaced by C.

Part **B)**: The proof of the map $(N)_{ABC} \to ZT(N)$ is performed also in three steps:

$$\begin{aligned}
&Step\ 1: \quad && (N)_{ABC} && \to (N)_{AB\bullet\times}\,, \\
&Step\ 2: \quad && (N)_{AB\bullet\times} && \to \widehat{ZT}(N)\,, \\
&Step\ 3: \quad && \widehat{ZT}(N) && \to ZT(N)\,.
\end{aligned} \tag{12}$$

Step 1 is a bit tricky. The representation $(N)_{ABC}$ of eq. 3 without the final $B(0)$ consists of blocks of powers of A, B or C with the restriction that a B-block never appears alone or at the end (because $B^{n+1}(0) = 0$, for $n \in \mathbb{N}$, the uniqueness of the representation would be violated). Then the **Step 1 replacements** of eq. 7 are applied to the A-, B-, and C-blocks, called there $Step1A$, $Step1B$ and $Step1C$. The omitted final B is again appended. See also **Example 2**.
In *Step* 2 the replacements $A \to 1$, $B \to 0$, $\bullet \to 0$ and $\times \to 1$ are applied and an extra 0 is added at the beginning of the thus obtained binary string. This is $\widehat{ZT}(N)$.
Step 3 is trivial: omit the two bordering 0s of $\widehat{ZT}(N)$ and reverse the binary string to obtain $ZT(N)$. □

3 Investigation of the A-, B- and C- sequences

In this section a detailed investigation of the $A-$, $B-$ and $C-$ sequences is presented.
The starting point is the infinite tribonacci word $TWord$, written as a sequence t = A080843. Its self-similarity leads to the following definitions and lemmata.

Definition 3. The tribonacci words $tw(l)$ over the alphabet $\{0,\,1,\,2\}$ of length $\#tw(l) = T(l+2)$ are defined recursively by concatenations (we omit the concatenation symbol $\circ$) as

$$tw(l) = tw(l-1)\,tw(l-2)\,tw(l-3), \quad \text{with} \quad tw(1) = 0,\, tw(2) = 01,\, tw(3) = 0102\,. \tag{13}$$

Also $tw(0) = 2$ is used.
The substitution map acting on tribonacci words and other strings with characters $\{0, 1, 2\}$ is defined as a concatenation homomorphism by $\sigma : 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$. The inverse map is $\sigma^{[-1]}$ (One replaces first each 01 and 02 then the left over 0). With σ the words $tw(l)$ are generated iteratively from $tw(0) = 2$. $\sigma(tw(l)) = tw(l+1)$, for $l \in \mathbb{N}_0$, and $\lim_{l\to\infty} \sigma^{[l]}(0) = TWord$. Self-similarity of $TWord$ means $\sigma(TWord) = TWord$.
Substrings of $TWord$ of length n, starting with the first letter (number) $t(0) = 0$, are denoted by $s_n := \text{Ħ}_{j=0}^{n-1} t(n)$. If $n = T(l+2)$, for $l \in \mathbb{N}_0$, then $s_n = tw(l)$ (the string becomes a tribonacci word), and the numbers of s_n map to the node labels of the last level of $TTree_l$ read from the left-hand side.
Also substrings of $TWord$ not starting with $t(0)$ are used, like $\hat{s}_2 = 02 = \sigma(1)$, starting with $t(2)$.

Lemma 4.

A) *With* $s_{13} = 0102010010201 = tw(5)$, $s_{11} = 01020100102$ *and* $s_7 = 0102010 = tw(4)$ *define*

$$t_1 = s_{13}s_{11}s_{13}s_7s_{13}s_{11}s_{13}s_{13}s_{11}s_{13}s_7s_{13}\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\varepsilon(t(j))}, \tag{14}$$

where $\varepsilon(0) = 13$, $\varepsilon(1) = 11$ *and* $\varepsilon(2) = 7$.
B) *With* $s_7 = 0102010 = tw(4)$, $s_6 = 010201$ *and* $s_4 = 0102 = tw(3)$ *define*

$$t_2 = s_7s_6s_7s_4s_7s_6s_7s_7s_6s_7s_4s_7\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\pi(t(j))}, \tag{15}$$

where $\pi(0) = 7$, $\pi(1) = 6$ *and* $\pi(2) = 4$.
C) *With* $s_4 = 0102 = tw(3)$, $s_3 = 010$ *and* $s_2 = 01 = tw(2) = \sigma(0)$ *define*

$$t_3 = s_4s_3s_4s_2s_4s_3s_4s_4s_3s_4s_2s_4\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\tau(t(j))}, \tag{16}$$

where $\tau(0) = 4$, $\tau(1) = 3$ *and* $\tau(2) = 2$.
D) *With* $s_2 = 01$, $\hat{s}_2 = 02$ *and* $s_1 = 0 = tw(1) = \sigma(2)$ *define*

$$t_4 = s_2\hat{s}_2s_2s_1s_2\hat{s}_2s_2s_2\hat{s}_2s_2s_2s_1\ldots \tag{17}$$

Here the string follows t *with* s_2, $\hat{s}_2$ *and* s_1 *playing the rôle of* 0, 1 *and* 2, *respectively.*
Then

$$t_1 = t_2 = t_3 = t_4 = TWord\,. \tag{18}$$

Proof:
D: The definition of $\sigma^{[-1]}$ shows that $\sigma^{[-1]}(t_4) = TWord$ Hence $t_4 = \sigma(TWord) = TWord$.
C: Because $\sigma(s_2) = s_4$, $\hat{\sigma}(s_2) = s_3$ and $\sigma(s_1) = s_2$ it follows that $t_3 = \sigma(t_4) = TWord$.
B: Because $\sigma(s_4) = s_7$, $\sigma(s_3) = s_6$ and $\sigma(s_2) = s_4$ it follows that $t_2 = \sigma(t_3) = TWord$.
A: Because $\sigma(s_7) = s_{13}$, $\sigma(s_6) = s_{11}$ and $\sigma(s_4) = s_7$ it follows that $t_1 = \sigma(t_2) = TWord$.
□

Using eq.16 a formula for sequence entry $A(n) =$ A278040(n) in terms of $z(n) := \sum_{j=0}^{n} t(j)$ is derived. This sequence $\{z(j)\}_{j=0}^{\infty}$ is given in A319198.

Proposition 5.

$$A(n) \,=\, 4\,n \,+\, 1 \,-\, z(n-1), \text{ for n} \,\in\, \mathbb{N}_0, \text{ with } \text{z}(-1) \,=\, 0\,. \tag{19}$$

Proof:
Define $\triangle A(k+1) := A(k+1) - A(k)$. Consider the word t_3 of eq. 16. The distances between the 1s in the pairs $s_4 s_3$, $s_3 s_4$, $s_4 s_2$, $s_2 s_4$ and $s_4 s_4$ are 4, 3, 4, 2, 4. Therefore, the sequence of these distances is 4, 3, 4, 2, 4, 3, 4, 4, 3, 4, 2, ... Thus, because the s-string t_2 follows the pattern of t, *i.e.*, of $TWord$,

$$\triangle A(k+1) \,=\, 4 \,-\, t(k)\,, \quad \text{for} \quad k \,=\, 0,\, 1,\, \ldots\,. \tag{20}$$

Then the telescopic sum produces the assertion, using $A(0) = 1$.

$$A(n) \,=\, A(0) \,+\, \sum_{k=0}^{n-1} \triangle A(k+1) \,=\, 1 + 4\,n \,-\, z(n-1), \quad \text{with} \quad z(-1) \,=\, 0. \tag{21}$$

□

The B-numbers A278039, giving the increasing indices k with $t(k) = 0$, come in three types: $B0$-numbers form the sequence of increasing indices k of sequence t with $t(k) = 0 = t(k+1)$. Similarly the $B1$-sequence lists the increasing indices k with $t(k) = 0$, $t(k+1) = 1$ and for the $B2$-sequence the indices k are such that $t(k) = 0$, $t(k+1) = 2$.
These numbers $B0(n)$, $B1(n)$ and $B2(n)$ are given by A319968$(n+1)$, A278040$(n) - 1$, and A278041$(n) - 1$, respectively.
Before giving proofs we define the counting sequences $z_A(n)$, $z_B(n)$ and $z_C(n)$ to be the numbers of A, B and C numbers not exceeding $n \in \mathbb{N}$, respectively. If these counting functions appear for $n = -1$ they are set to 0.
These sequences are given by A276797$(n+1)$, A276796$(n+1)$ and A276798$(n+1) - 1$ for $n \geq -1$.
Obviously,

$$z(n) \,=\, 1\,z_A(n) \,+\, 0\,z_B(n) \,+\, 2\,z_C(n) \,=\, z_A(n) \,+\, 2\,z_C(n), \quad \text{for} \quad n \,=\, -1,\, 0,\, 1,\, \ldots\,. \tag{22}$$

These counting functions are obtained by partial sums of the corresponding characteristic sequences for the $A-$, $B-$ and $C-$numbers (or $0-$, $1-$, and $2-$numbers in t), called k_A, k_B and k_C, respectively.

$$z_X(n) \,=\, \sum_{k=0}^{n} k_X(k), \quad \text{for} \quad X \,\in\, \{A,\, B,\, C\}\,. \tag{23}$$

The characteristic sequences members $k_A(n)$, $k_B(n)$ and $k_C(n)$ are given in A276794$(n+1)$, A276793$(n+1)$ and A276791$(n+1)$, for $n \in \mathbb{N}_0$, and they are, in terms of t, obviously given by

$$k_A(n) \;=\; t(n)\,(2 \,-\, t(n)), \tag{24}$$

$$k_B(n) \;=\; \frac{1}{2}\,(t(n) \,-\, 1)\,(t(n) \,-\, 2), \tag{25}$$

$$k_C(n) \;=\; \frac{1}{2}\,t(n)\,(t(n) \,-\, 1). \tag{26}$$

By definition it is trivial that (note the offset 0 of the $A,\ B,\ C$ sequences)

$$z_X(X(k)) \,=\, k\,+\,1, \quad \text{for} \quad X \,\in\, \{A,\, B,\, C\} \quad \text{and} \quad k \,\in\, \mathbb{N}\,. \tag{27}$$

Proposition 6.

For $n \,\in\, \mathbb{N}_0$:

$$\begin{aligned}
\mathbf{B0})\ B0(n) \;&=\; 13\,n\,+\,6\,-\,2\,[z_A(n-1)\,+\,3\,z_C(n-1)] \,=\, 2\,C(n)\,-\,n, && (28)\\
\mathbf{B1})\ B1(n) \;&=\; 4\,n\,-\,z(n-1) \,=\, 4\,n\,-\,[z_A(n-1)\,+\,2\,z_C(n-1)] \,=\, A(n)\,-\,1, && (29)\\
\mathbf{B2})\ B2(n) \;&=\; 7\,n\,+\,2\,-\,[z_A(n-1)\,+\,3\,z_C(n-1)] \,=\, \frac{1}{2}\,(B0(n)\,+\,n\,-\,2)\\
&=\; C(n)\,-\,1, && (30)\\
\mathbf{B})\ \ B(n) \;&=\; 2\,n\,-\,z_C(n-1)\,. && (31)
\end{aligned}$$

Proof:
B0: Part 1: Define $\triangle B0(k{+}1) \,:=\, B0(k{+}1) - B0(k)$ and consider the word t_1 of eq. 14. The distances between pairs of 00 in $s_{13}s_{11}$, $s_{11}s_{13}$, $s_{13}s_7$, s_7s_{13} and $s_{13}s_{13}$ are 13, 11, 13, 7, 13. Note that S_7 has no substring 00, however because S_7 is always followed by S_{13} the last 0 of s_7 and the first of s_{13} build the 00 pair. Similarly, in the $s_{13}s_7$ case the last 0 of s_7 is counted as a beginning of a 00 pair. Therefore, the sequence of these distances is 13, 11, 13, 7, 13, 11, 13, 13, 11, 13, 7, ... Because the s-string t_1 follows the pattern of t the defect from 13 is 0, -2, -6 if $t(k) = 0,\,1,\,2$, hence

$$\triangle B0(k+1) \,=\, 13\,-\,t(k)\,(t(k)\,+\,1)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{32}$$

The telescopic sum gives, with $B0(0) \,=\, 6$,

$$\begin{aligned}
B0(n+1) \;&=\; B0(0)\,+\,\sum_{k=0}^{n} \triangle\, B0(k+1)\\
&=\; 6\,+\,13\,(n+1)\,-\,[(1^2\,z_A(n)\,+\,2^2\,z_C(n))\,+\,z(n)]\\
&=\; 13\,n\,+\,19\,-\,2\,(z_A(n)\,+\,3\,z_C(n))\,. && (33)
\end{aligned}$$

In the last step $z(n)$ has been replaced by eq. 22. Substituting $n \,\to\, n-1$ proves the first part of **B0**. The proof of part 2 follows later from **B2**.
B1: With $\triangle B1(k+1) \,:=\, B1(k+1)\,-\,B1(k)$ and t_2 of eq. 15 one finds for the distances between consecutive 1s similar to the above argument

$$\triangle B1(k+1) \,=\, 4\,-\,t(k)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{34}$$

The telescopic sum gives, with $B1(0) = 0$,

$$B1(n+1) \,=\, 4\,(n\,+\,1)\,-\,z(n), \tag{35}$$

and with $n \,\to\, n-1$ this becomes the first part of **B1**, which shows, with eq 19, also the third one. The second part uses eq. 22.

Note that $B1(n) = A(n) - 1$ is trivial because 1 in the tribonacci word $TWord$ can only come from the substitution $\sigma(0) = 01$, and $TWord$ (and t) starts with 0. Therefore, one could directly prove **B1** from eqs. 19 and 22 without first computing $\triangle B1(k+1)$.

B2: Because 2 in $TWord$ appears only from $\sigma(1) = 02$, it is clear that $B2(n) = C(n) - 1$. Now one finds a formula for C by looking first at $\triangle\, C(k+1) := C(k+1) - C(k)$ using again t_2 of eq. 15. The distances between consecutive 2s in the five pairs $s_7 s_6$, $s_6 s_7$, $s_7 s_4$, $s_4 s_7$ and $s_7 s_7$ is 7, 6, 7, 4, 7, respectively, and

$$\triangle\, C(k+1) \,=\, 7 - \frac{1}{2}\, t(k)\,(t(k) + 1)\,, \quad \text{for} \quad k \in \mathbb{N}_0\,. \tag{36}$$

The telescopic sum leads here, using $C(0) = 3$, $z(n)$ from eq. 22 and letting $n \to n-1$, to

$$C(n) \,=\, 7\,n + 3 - [z_A(n-1) + 3\,z_C(n-1)]\,, \quad \text{for} \quad k \in \mathbb{N}_0\,. \tag{37}$$

This proves **B2**, and also the second part of **B0**.

B): Here t_4 of eq. 17 can be used. The differences of 0s in the five pairs $s_2\hat{s}_2$, $\hat{s}_2 s_2$, $s_2 s_1$, $s_1 s_2$ and $s_2 s_2$ is 2, 2, 2, 1, 2. Thus

$$\triangle\, B(k+1) := B(k+1) - B(k) \,=\, 2 - \frac{1}{2}\, t(k)\,(t(k) - 1) \,=\, 2 - k_C(n)\,, \quad \text{for} \quad k \in \mathbb{N}_0\,. \tag{38}$$

In the last step k_C from eq. 26 has been used. By telescoping, using $B(0) = 0$, eliminating $z(n-1)$ with eq. 19, and letting $n \to n-1$, proves the assertion. $\square$

Eqs. 36 and 38 show that $\triangle C(k+1) - \triangle B(k+1) = 5 - t(k)$, for $k \in \mathbb{N}_0$. Telescoping leads to the result, obtained directly from eqs. 37 and 31, with eq. 22,

$$C(n) - B(n) \,=\, 5\,n + 3 - z(n-1)\,, \quad \text{for} \quad k \in \mathbb{N}_0\,, \tag{39}$$

and with A from eq. 19 this becomes

$$C(n) - (A(n) + B(n)) \,=\, n + 2\,, \quad \text{for} \quad k \in \mathbb{N}_0\,. \tag{40}$$

This equation can be used to eliminate C from the equations.

Next the formulae for z_X for $X \in \{A, B, C\}$ are listed, valid for $n = -1, 0, 1, ...$

Proposition 7.

$$z_A(n) \;=\; 2\,B(n+1) - A(n+1) + 1\,, \tag{41}$$

$$z_B(n) \;=\; A(n+1) - B(n+1) - (n + 2), \tag{42}$$

$$z_C(n) \;=\; 2\,(n + 1) - B(n+1)\,. \tag{43}$$

Proof: Version 1. The inputs $z_X(-1) = 0$, for $X \in \{A, B, C\}$, follow from eqs. 19 and 31. The first differences $\triangle z_X(n) := z_X(n) - z_X(n-1)$ produce with the claimed formulae, and $\triangle A(n+1)$ and $\triangle B(n+1)$ from eqs. 20 and 38, the trivial results given in eqs. 24 to 26. Therefore $z_X(n)$ from eq. 23 holds.

Version 2. Besides eq. 22 the trivial formula

$$z_A(n) + z_B(n) + z_C(n) = n + 1 \tag{44}$$

can be used.
$z_A(n)$ is computed from the difference of $3\,(z_A(n-1) + 2\,z_C(n-1))$ from eq. 30, with $C(n)$ from eq. 40, and $2\,(z_A(n-1) + 3\,z_C(n-1))$ from eq. 29. This difference leads to the claim eq. 41.
$2\,z_C(n) = -A(n+1) + 4\,n + 5 - z_A(n)$ from eq. 29. Inserting the proven $z_A(n)$ formula leads to the claim eq. 43.
$z_B(n)$ can then be computed from eq. 44. □

Finally all formulae for compositions of the types $X(Y(k)+1)$ and $X(Y(k))$, for $X,\,Y \in \{A,\,B,\,C\}$ and $k \in \mathbb{N}_0$ shall be given. They are of interest in connection with the tribonacci ABC-representation given in the preceding section. For this one needs first the results for the compositions $z(X(k))$. The formulae will be given in terms of A and B (with C eliminated by eq. 40).

Proposition 8.

$$\begin{aligned} z(A(k)) &= 2\,(A(k) - B(k)) - k - 1, & (45)\\ z(B(k)) &= -A(k) + 3\,B(k) - k + 1, & (46)\\ z(C(k)) &= B(k) + 2\,k + 3. & (47) \end{aligned}$$

Proof: $z(X(k))$ will be found from the self-similarity properties given in eqs. 16, 17 and 15, for $X = A,\,B$ and C, respectively. These strings t_3, t_4 and t_2 are chosen because the relevant numbers 1, 0 and 2, respectively, appear precisely once in all $s-$substrings. For $z(X(k)) = \sum_{j=0}^{X(k)} t(j)$ one has to sum all the numbers of the first k substrings s but in the last one only the numbers up to the number standing for X are summed.
A) In the t_3 substrings $s_4 = 0102$, $s_3 = 010$ and $s_2 = 01$ the number 1 appears just once. In all three substrings the sum up to the relevant number 1 (for A) is $0 + 1 = 1$, so for the last s one has always to add 1. Because s_4, s_3 and s_2, with sums 3, 1 and 1, play the rôle of 0, 1 and 2, respectively, in t_3 one obtains $z(A(k)) = 3\,z_B(k-1) + 1\,(z_A(k-1) + z_C(k-1)) + 1$. With the identity eq. 44 this becomes $2\,z_B(k-1) + k + 1$, and with the z_B formula eq. 42 this leads to the claim eq. 45.
B) In t_4 the sums of the substrings s_2, $\hat{s}_2$, s_1 are 1, 2, 0,respectively, and because all three begin with the relevant number 0 nothing to be summed for the last s. Thus $z(B(k)) = 1\,z_B(k-1) + 2\,z_A(k-1) + 0 + 0$. Using eqs. 42 and 41 this becomes the claim.
C) In t_2 the sums are 4 for s_7, s_6 and 3 for s_4. The sums up to the relevant number 2 are 3 for each case. Therefore $z(C(k)) = 4\,(z_B(k-1) + z_A(k-1)) + 3\,z_C(k-1) + 3 = z_B(k-1) + z_A(k-1) + 3\,k + 3 = B(k) + 2\,k + 3$, with eqs. 44, 42 and 41. □

Proposition 9.

$$\begin{aligned} A(A(k)+1) &= 2\,(A(k) + B(k)) + k + 6\,, & A(A(k)) &= A(A(k)+1) - 3\,, & (48)\\ A(B(k)+1) &= A(k) + B(k) + k + 4\,, & A(B(k)) &= A(B(k)+1) - 4\,, & (49)\\ A(C(k)+1) &= 4\,A(k) + 3\,B(k) + 2\,(k + 5)\,, & A(C(k)) &= A(C(k)+1) - 2\,. & (50) \end{aligned}$$

$$
\begin{aligned}
B(A(k)+1) &= A(k) + B(k) + k + 3\,, & B(A(k)) &= B(A(k)+1) - 2\,, && (51)\\
B(B(k)+1) &= A(k) + 1\,, & B(B(k)) &= B(B(k)+1) - 2\,, && (52)\\
B(C(k)+1) &= 2\,(A(k) + B(k)) + k + 5\,, & B(C(k)) &= B(C(k)+1) - 1\,. && (53)
\end{aligned}
$$

$$
\begin{aligned}
C(A(k)+1) &= 4\,A(k) + 3\,B(k) + 2\,(k + 6)\,, & C(A(k)) &= C(A(k)+1) - 6\,, && (54)\\
C(B(k)+1) &= 2\,(A(k) + B(k)) + k + 8\,, & C(B(k)) &= C(B(k)+1) - 7\,, && (55)\\
C(C(k)+1) &= 7\,A(k) + 6\,B(k) + 4\,(k + 5)\,, & C(C(k)) &= C(C(k)+1) - 4\,. && (56)
\end{aligned}
$$

Proof:
The two versions are related by $\triangle X(n+1) = X(n+1) - X(n)$ given in eqs.20, 38, 36, for $X \in \{A, B, C\}$, respectively, and n replaced by $Y(k)$ with $Y \in \{A, B, C\}$. For $C(n)$ eq. 40 is always used.
A) This follows from $A(n+1)$ given from eq. 19 with $z(Y(k))$ from eqs. 45, 46 and 47.
B) One proves that $B(A(k)) = A(k) + B(k) + k + 1$ from which $B(A(k)+1)$ follows. With eqs. 40 and 30 this means that

$$B(A(k)) = C(k) - 1 = B2(k). \tag{57}$$

After applying z_B on both sides, using eq. 27 this is equivalent to

$$A(k) + 1 = z_B(C(k) - 1)) = z_B(C(k)). \tag{58}$$

The second equality is trivial. This is now proved. From eq. 22 $z_B(n) = n + 1 - z(n) + z_C(n)$. Hence $z_B(C(k)) = C(k) + 1 - z(C(k)) + (k + 1)$, with eq. 27. This is $C(k) - k - 1 - B(k)$ from eq. 47, and replacing $C(k)$ gives $A(k) + 1$.
One proves $B(B(k)) = A(k) + 1$ or, after application of z_B on both sides, $B(k) + 1 = z_B(A(k) - 1) = z_B(A(k))$, where the second equality is trivial. But from eqs. 44 and 27 follows $z_B(A(k)) = A(k) + 1 - (k + 1) - z_C(A(k))$. Applying eq. 43 and the just proven $B(A(k)+1)$ formula shows that

$$z_B(A(k)) = B(k) + 1. \tag{59}$$

The $B(C(k))$ claim can be written in terms of C from eqs. 40 and 28 as

$$B(C(k)) = 2\,C(k) - k = B0(k)\,. \tag{60}$$

Indeed, eqs. 31, 27 imply for $B(C(k)) = 2\,C(k) - z_C(C(k)-1) = 2\,C(k) - (z_C(C(k)) - 1) = 2\,C(k) - k$. The second equality is trivial.
C) This follows immediately from $C(n+1)$ of eq. 40 and the already proved formulae for $A(Y(k)+1)$ and $B(Y(k)+1)$.

$\square$

The collection of the results for $Z_X(Y(k))$ is, for $k \in \mathbb{N}_0$:

Proposition 10.

$$
\begin{aligned}
z_A(A(k)) &= k + 1, \\
z_A(B(k)) &= A(k) - B(k) - (k + 1) = z_C(A(k)), \\
z_A(C(k)) &= B(k) + 1.
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
z_B(A(k)) &= B(k) + 1 = z_A(C(k)) \\
z_B(B(k)) &= k + 1, \\
z_B(C(k)) &= A(k) + 1.
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
z_C(A(k)) &= A(k) - B(k) + (k + 1) = z_A(B(k)) \\
z_C(B(k) &= 2\,B(k) - A(k) + 1, \\
z_C(C(k)) &= k + 1.
\end{aligned} \tag{63}
$$

Proof:
That $z_X(X(k)) = k + 1$ has been noted already in eq. 27.
The other claims follow from the $z_X(n)$ results after replacing n by $Y(k) \neq X(k)$, and application of the formulae from *Proposition* 9. ☐

Many of the formulae from section 3 appear in [2] and [1] with the above mentioned translation between their sequences a, b, and c to our B, A, and C. For example, *Theorem 13* of [2], p. 57, for the nine twofold iterations (in our notation $X(Y(k)$ of *Proposition* 9) can be checked.

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Table 1: Sequences t, A, B, C, for n = 0, 1, ..., 79

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
t	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0	2	0	1	0
A	1	5	8	12	14	18	21	25	29	32	36	38	42	45	49	52	56	58	62	65
B	0	2	4	6	7	9	11	13	15	17	19	20	22	24	26	28	30	31	33	35
C	3	10	16	23	27	34	40	47	54	60	67	71	78	84	91	97	104	108	115	121

n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
t	0	1	0	2	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0
A	69	73	76	80	82	86	89	93	95	99	102	106	110	113	117	119	123	126	130	133
B	37	39	41	43	44	46	48	50	51	53	55	57	59	61	63	64	66	68	70	72
C	128	135	141	148	152	159	165	172	176	183	189	196	203	209	216	220	227	233	240	246

n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
t	2	0	1	0	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0
A	137	139	143	146	150	154	157	161	163	167	170	174	178	181	185	187	191	194	198	201
B	74	75	77	79	81	83	85	87	88	90	92	94	96	98	100	101	103	105	107	109
C	253	257	264	270	277	284	290	297	301	308	314	321	328	334	341	345	352	358	365	371

n	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
t	2	0	1	0	0	1	0	2	0	1	0	2	0	1	0	0	1	0	2	0
A	205	207	211	214	218	222	225	229	231	235	238	242	244	248	251	255	259	262	266	268
B	111	112	114	116	118	120	122	124	125	127	129	131	132	134	136	138	140	142	144	145
C	378	382	389	395	402	409	415	422	426	433	439	446	450	457	463	470	477	483	490	494

Table 2: ZT(N), for N = 1, 2, ..., 100

N	ZT(N)	N	ZT(N)	N	ZT(N)	N	ZT(N)	N	ZT(N)
1	1	**21**	11001	**41**	110100	**61**	1010100	**81**	10000000
2	10	**22**	11010	**42**	110101	**62**	1010101	**82**	10000001
3	11	**23**	11011	**43**	110110	**63**	1010110	**83**	10000010
4	100	**24**	100000	**44**	1000000	**64**	1011000	**84**	10000011
5	101	**25**	100001	**45**	1000001	**65**	1011001	**85**	10000100
6	110	**26**	100010	**46**	1000010	**66**	1011010	**86**	10000101
7	1000	**27**	100011	**47**	1000011	**67**	1011011	**87**	10000110
8	1001	**28**	100100	**48**	1000100	**68**	1100000	**88**	10001000
9	1010	**29**	100101	**49**	1000101	**69**	1100001	**89**	10001001
10	1011	**30**	100110	**50**	1000110	**70**	1100010	**90**	10001010
11	1100	**31**	101000	**51**	1001000	**71**	1100011	**91**	10001011
12	1101	**32**	101001	**52**	1001001	**72**	1100100	**92**	10001100
13	10000	**33**	101010	**53**	1001010	**73**	1100101	**93**	10001101
14	10001	**34**	101011	**54**	1001011	**74**	1100110	**94**	10010000
15	10010	**35**	101100	**55**	1001100	**75**	1101000	**95**	10010001
16	10011	**36**	101101	**56**	1001101	**76**	1101001	**96**	10010010
17	10100	**37**	110000	**57**	1010000	**77**	1101010	**97**	10010011
18	10101	**38**	110001	**58**	1010001	**78**	1101011	**98**	10010100
19	10110	**39**	110010	**59**	1010010	**79**	1101100	**99**	10010101
20	11000	**40**	110011	**60**	1010011	**80**	1101101	**100**	10010110

Table 3: ABC(N), for N = 1, 2, ..., 100

N	ABC(N)	N	ABC(N)	N	ABC(N)	N	ABC(N)	N	ABC(N)
1	10	**21**	1020	**41**	00120	**61**	001110	**81**	000000010
2	010	**22**	0120	**42**	1120	**62**	11110	**82**	10000010
3	20	**23**	220	**43**	0220	**63**	02110	**83**	01000010
4	0010	**24**	0000010	**44**	00000010	**64**	000210	**84**	2000010
5	110	**25**	100010	**45**	1000010	**65**	10210	**85**	00100010
6	020	**26**	010010	**46**	0100010	**66**	01210	**86**	1100010
7	00010	**27**	20010	**47**	200010	**67**	2210	**87**	0200010
8	1010	**28**	001010	**48**	0010010	**68**	0000020	**88**	00010010
9	0110	**29**	11010	**49**	110010	**69**	100020	**89**	1010010
10	210	**30**	02010	**50**	020010	**70**	010020	**90**	0110010
11	0020	**31**	000110	**51**	0001010	**71**	20020	**91**	210010
12	120	**32**	10110	**52**	101010	**72**	001020	**92**	0020010
13	000010	**33**	01110	**53**	011010	**73**	11020	**93**	120010
14	10010	**34**	2110	**54**	21010	**74**	02020	**94**	00001010
15	01010	**35**	00210	**55**	002010	**75**	000120	**95**	1001010
16	2010	**36**	1210	**56**	12010	**76**	10120	**96**	0101010
17	00110	**37**	000020	**57**	0000110	**77**	01120	**97**	201010
18	1110	**38**	10020	**58**	100110	**78**	2120	**98**	0011010
19	0210	**39**	01020	**59**	010110	**79**	00220	**99**	111010
20	00020	**40**	2020	**60**	20110	**80**	1220	**100**	021010

Here 0, 1 and 2 stand for B, A and C, respectively. *E.g.*, ABC(6) = BCB = B(C(B(0))).