Classical pattern distributions in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$

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Abstract

Classical pattern avoidance and occurrence are well studied in the symmetric group $\mathcal{S}_n$. In this paper, we provide explicit recurrence relations to the generating functions counting the number of classical pattern occurrence in the set of 132-avoiding permutations and the set of 123-avoiding permutations.

Keywords: permutation statistics, classical patterns, Catalan numbers, Dyck paths

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1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of size n. Given a sequence $w = w_1 \cdots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation that we replace the i-th smallest integer in σ with i. For example, $\mathrm{red}(4685) = 1342$. Given a permutation $\tau = \tau_1 \cdots \tau_j$ in $\mathcal{S}_j$, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$ if there exist $1 \le i_1 < \cdots < i_j \le n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. In the theory of permutation patterns, τ is called a *classical pattern*.

We let $\mathcal{S}_n(\tau)$ denote the set of permutations in $\mathcal{S}_n$ which avoid τ. If Γ is a collection of permutations, then we let $\mathcal{S}_n(\Gamma)$ denote the set of permutations in $\mathcal{S}_n$ that avoid each permutation in Γ. Let $\mathrm{occr}_\tau(\sigma)$ denote the number of pattern τ occurrences in the permutation σ. For example, the permutation $\sigma = 867943251$ avoids pattern 132, while it contains pattern 123 and $\mathrm{occr}_{123}(\sigma) = 1$ since only the subsequence $6, 7, 9$ matches pattern 123.

Classical patterns have been studied separately for a long time. It is well known that for all $n \ge 1$, $|\mathcal{S}_n(132)| = |\mathcal{S}_n(123)| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. Mansour in [9, 10] enumerated the number of permutations in $\mathcal{S}_n$ avoiding 2 classical patterns. Mansour and Vainshtein in [12, 13] enumerated the number of permutations in $\mathcal{S}_n(132)$ or $\mathcal{S}_n(123)$ that has 0 or 1 occurrence of another pattern τ. See Kitaev [7] for a comprehensive introduction to patterns in permutations. However, there is not much research about the distribution of classical patterns in $\mathcal{S}_n(\tau)$. Mansour and Vainshtein in [11] gave a continued fraction form of the generating function of the distribution of pattern $12 \cdots k$ in $\mathcal{S}_n(132)$. Very recently, Janson in [5, 6] studied patterns in random permutations avoiding the pattern 132 and 123 in a probabilistic way. Pan the authors in [15] studied consecutive pattern matches in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Given two sets of permutations $\Lambda = \{\lambda_1, \dots, \lambda_r\}$ and $\Gamma = \{\gamma_1, \dots, \gamma_s\}$, it is natural to study the distribution of classical patterns $\gamma_1, \dots, \gamma_s$ in $\mathcal{S}_n(\Lambda)$. That is, we want to study generating functions of the form

$$Q_\Lambda^\Gamma(t, x_1, \dots, x_s) := 1 + \sum_{n\geq 1} t^n Q_{n,\Lambda}^\Gamma(x_1, \dots, x_s), \tag{1}$$

where

$$Q_{n,\Lambda}^\Gamma(x_1, \dots, x_s) := \sum_{\sigma\in\mathcal{S}_n(\Lambda)} x_1^{\mathrm{occr}_{\gamma_1}(\sigma)} \cdots x_s^{\mathrm{occr}_{\gamma_s}(\sigma)}. \tag{2}$$

When $\Lambda = \{\lambda\}$ and $\Gamma = \{\gamma\}$ are singletons, we write

$$Q_\lambda^\gamma(t, x) := 1 + \sum_{n\geq 1} t^n Q_{n,\lambda}^\gamma(x) \quad \text{and} \quad Q_{n,\lambda}^\gamma(x) := \sum_{\sigma\in\mathcal{S}_n(\lambda)} x^{\mathrm{occr}_\gamma(\sigma)}. \tag{3}$$

The main goal of this paper is to study the distribution of classical patterns in 132-avoiding permutations and in 123-avoiding permutations using a recursive method.

To study the generating functions $Q_\lambda^\gamma(t, x)$ when λ is 132 or 123, we want to first study the symmetries in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$. Given a permutation σ, we denote the reverse of σ by σ^r, the complement of σ by σ^c, the reverse-complement of σ by σ^{rc}, and the inverse of σ by σ^{-1}. For example, if $\sigma = 15324$, then $\sigma^r = 42351$, $\sigma^c = 51342$, $\sigma^{rc} = 24315$, $\sigma^{-1} = 14352$.

It is clear that $\mathcal{S}_n(123)$ is closed under the operation reverse-complement, and both $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ are closed under the operation inverse. Thus we have the following lemma.
Lemma 1. *Given any permutation pattern γ,*

$$Q_{123}^\gamma(t, x) = Q_{123}^{\gamma^{rc}}(t, x) = Q_{123}^{\gamma^{-1}}(t, x), \quad Q_{132}^\gamma(t, x) = Q_{132}^{\gamma^{-1}}(t, x).$$

When γ is a pattern of length 3, we have the following corollary.
Corollary 1. *Considering the distribution of patterns of length 3, we only need to study the following 4 generating functions for $\mathcal{S}_n(132)$,*

(1) $Q_{132}^{123}(t, x)$,

(2) $Q_{132}^{213}(t, x)$,

(3) $Q_{132}^{231}(t, x) = Q_{132}^{312}(t, x)$,

(4) $Q_{132}^{321}(t, x)$,

and the following 3 generating functions for $\mathcal{S}_n(123)$,

(1) $Q_{123}^{132}(t, x) = Q_{123}^{213}(t, x)$,

(2) $Q_{123}^{231}(t, x) = Q_{123}^{312}(t, x)$,

(3) $Q_{123}^{321}(t, x)$.

It is easy to check that all the 7 generating functions are different when looking at $\mathcal{S}_8(132)$ and $\mathcal{S}_8(123)$. Our motivation of this paper is to study the 7 generating functions above, and then generalize some of the results.

The structure of this paper is as follows. In Section 2, we introduce background about permutations and two bijections between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ and Dyck paths which are useful in our computation. Then we study the length 3 pattern distributions in $\mathcal{S}_n(132)$ in Section 3 and $\mathcal{S}_n(123)$ in Section 4. In Section 5, we show two applications of our results in computing pattern popularities. In Section 6, we show the application of our results about circular permutations. Finally in Section 7, we give a summary of this paper.

2 Preliminaries

Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation written in one-line notation. The *graph* of σ, $G(\sigma)$, is obtained by placing σ_i in the i^{th} column counting from left to right and σ_i^{th} row counting from bottom to top on an $n \times n$ table for $i = 1, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1.

We define $\mathrm{inv}(\sigma) := \left|\{(i,j)|1 \leq i < j \leq n,\ \sigma_i > \sigma_j\}\right|$ to be the number of *inversions* and $\mathrm{coinv}(\sigma) := \left|\{(i,j)|1 \leq i < j \leq n,\ \sigma_i < \sigma_j\}\right|$ to be the number of *coinversions* of a permutation σ. Note that the number of inversions of a permutation is the same as the number of occurrences of pattern 21, and the number of coinversions of a permutation is the same as the number of occurrences of pattern 12. Clearly, $\mathrm{inv}(\sigma) + \mathrm{coinv}(\sigma) = \binom{n}{2}$.

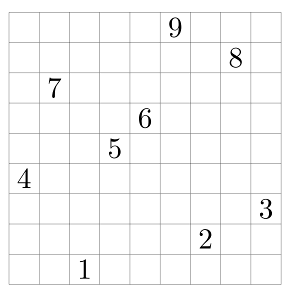


Figure 1: The graph of $\sigma = 471569283$

Given $\sigma = \sigma_1 \cdots \sigma_n$, we say that σ_j is a *left-to-right minimum* of σ if $\sigma_i > \sigma_j$ for all $i < j$. We let $\mathrm{LRmin}(\sigma)$ denote the number of left-to-right minima of σ. We shall also call each left-to-right minimum of σ a *peak*, and the remaining number *non-peaks* of σ. We can see that the permutations in Figure 3(a) and Figure 3(b) both have peaks $\{8, 6, 4, 3, 2, 1\}$.

Let $\pi = \pi_1 \cdots \pi_m \in \mathcal{S}_m$ and $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, then the *direct sum* $(\pi \oplus \sigma)$ and *skew sum* $(\pi \ominus \sigma)$ of π and σ are defined by

$$\pi \oplus \sigma := \pi_1 \cdots \pi_m (\sigma_1 + m) \cdots (\sigma_n + m), \tag{4}$$

$$\pi \ominus \sigma := (\pi_1 + n) \cdots (\pi_m + n) \sigma_1 \cdots \sigma_n. \tag{5}$$

Given an $n \times n$ square, we will label the coordinates of the columns from left to right and the coordinates of the rows from top to bottom with $0, 1, \ldots, n$ (different from the coordinates of the graph of a permutation). An (n,n)-*Dyck path* is a path made up of unit down-steps D and unit right-steps R which starts at $(0,0)$ and ends at (n,n) and stays on or below the diagonal $y = x$ (these are "down-right" Dyck paths). The set of (n,n)-Dyck paths is denoted by $\mathcal{D}_n$.

Given a Dyck path P, we let the first return of P, denoted by $\mathrm{ret}(P)$, be the smallest number $i > 0$ such that P goes through the point (i,i). For example, for $P = DDRDDRRRDDRDRDRRDR$ shown in Figure 2, $\mathrm{ret}(P) = 4$ since the leftmost point on the diagonal that P goes through is $(4,4)$.

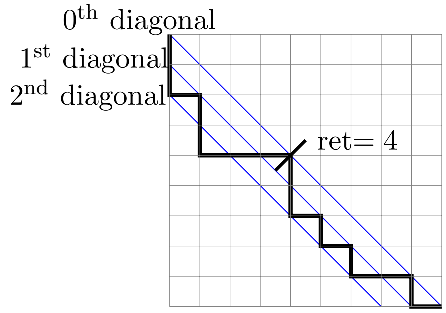


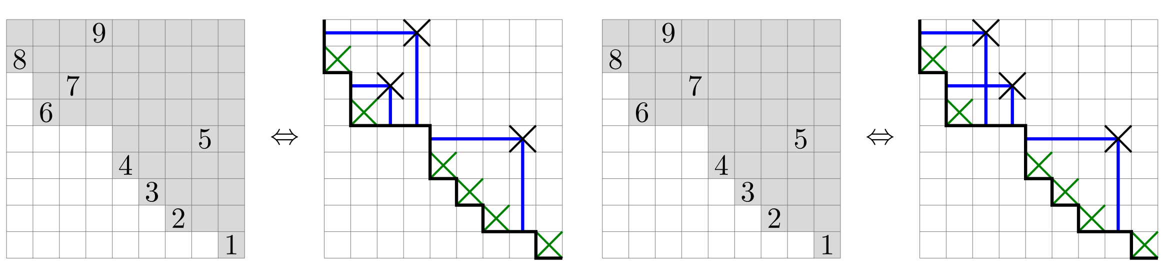
Figure 2: A $(9,9)$-Dyck path $P = DDRDDRRRDDRDRDRRDR$

We refer to positions (i,i) where P goes through as *return positions* of P. We call the full cells between P and the main diagonal *area cells*, and the cells below P *coarea cells*. Then we let $\mathrm{area}(P)$ and $\mathrm{coarea}(P)$ be the number of area cells and coarea cells of P. In the example in Figure 2, $\mathrm{area}(P) = 7$ and $\mathrm{coarea}(P) = 29$.

We shall also label the diagonals that go through corners of squares that are parallel to and below the main diagonal with $0, 1, 2, \ldots$ starting at the main diagonal, as shown in Figure 2. The *peaks* of a path P are the positions of consecutive DR steps. We can say that each peak is on a diagonal of P. In the path in Figure 2, the peaks are in the first, second, first, first, first and zeroth diagonal counting from top to bottom.

It is well known that for all $n \geq 1$, $|\mathcal{S}_n(132)| = |\mathcal{S}_n(123)| = |\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. Many bijections are known between these Catalan objects (see [18]). We use the bijection of Krattenthaler [8] between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ and the bijection of Deutsch and Elizalde [2] between $\mathcal{S}_n(123)$ and $\mathcal{D}_n$. The authors of this paper also discussed the two bijections in [16, 17] with more details.

We shall first describe the bijection Φ of Krattenthaler [8] between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$. Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$, we draw the graph $G(\sigma)$ of σ. Then, we shade the cells to the north-east of the cell that contains σ_i. $\Phi(\sigma)$ is the path that goes along the south-west boundary of the shaded cells. For example, this process is pictured in Figure 3(a) in the case where $\sigma = 867943251 \in \mathcal{S}_9(132)$. In this case, $\Phi(\sigma) = DDRDDRRRDDRDRDRRDR$.



(a) The map Φ (b) The map Ψ

Figure 3: $\mathcal{S}_n(132) \Leftrightarrow \mathcal{D}_n$, $\mathcal{S}_n(123) \Leftrightarrow \mathcal{D}_n$

The *horizontal segments* (or *segments*) of the path $\Phi(\sigma)$ are the maximal consecutive sequences of R steps in $\Phi(\sigma)$. For example, in Figure 3(a), the lengths of the horizontal segments, reading from top to bottom, are $1, 3, 1, 1, 2, 1$, and $\{6, 7, 9\}$ is the set of numbers associated with the second horizontal segment of $\Phi(\sigma)$.

The map Φ is invertible since for each Dyck path P, the peaks of P give the left-to-right minima of the 132-avoiding permutation, and the remaining numbers are uniquely determined by the left-to-right minima. More details about Φ can be found in [8]. We have the following properties for Φ from [16].

Lemma 2. *Let $P \in \mathcal{D}_n$ and $\sigma = \Phi^{-1}(P)$. Then*

(1) for each horizontal segment H of P, the set of numbers associated to H form a consecutive increasing sequence in σ and the least number of the sequence sits immediately above the first right-step of H.

(2) The number n is in the column of last right-step before the first return.

(3) Suppose that σ_i is a peak of σ and the cell containing σ_i is on the k^{th} diagonal. Then there are k elements in the graph $G(\sigma)$ in the first quadrant relative to coordinate system centered at (i, σ_i).

(4) $\mathrm{inv}(\sigma) = \mathrm{coarea}(P)$*;* $\mathrm{coinv}(\sigma) = \mathrm{area}(P)$*.*

Proof. (1), (2) and (3) are proved in Lemma 3 in [16].

For (4), it is clear that for any pair of index $i < j$, we have $\sigma_i > \sigma_j$ if and only if in path P, the i^{th} column intersects the σ_j^{th} row at a coarea cell. Thus the number of inversions of σ is equal to the coarea of P, i.e. $\mathrm{inv}(\sigma) = \mathrm{coarea}(P)$. Since $\mathrm{inv}(\sigma) + \mathrm{coinv}(\sigma) = \mathrm{area}(P) + \mathrm{coarea}(P) = \binom{n}{2}$, we have $\mathrm{coinv}(\sigma) = \mathrm{area}(P)$. $\square$

The bijection $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$ given by Deutsch and Elizalde [2] can be described in a similar way. Given any permutation $\sigma \in \mathcal{S}_n(123)$, the Dyck path $\Psi(\sigma)$ is constructed exactly as the bijection Φ. Figure 3(b) shows an example of this map, from $\sigma = 869743251 \in \mathcal{S}_9(123)$ to the Dyck path *DDRDDRRRDDRDRDRRDR*. The map Ψ is invertible because each 123-avoiding permutation has a unique left-to-right minima set. More details about Ψ can be found in [2]. We then have the following lemma from [16].

Lemma 3 ([16], Lemma 4)**.** *Let $P \in \mathcal{D}_n$ and $\sigma = \Psi^{-1}(P)$. Then*

(1) for each horizontal segment H of P, the least element of the set of numbers associated to H sits directly above the first right-step of H and the remaining numbers of the set form a consecutive decreasing sequence in σ.

(2) σ can be decomposed into two decreasing subsequences, the first decreasing subsequence corresponds to the peaks of σ and the second decreasing subsequence corresponds to the non-peaks of σ.

(3) Suppose that σ_i is a peak of σ and the cell containing σ_i is on the k^{th} diagonal. Then there are k elements in the graph $G(\sigma)$ in the first quadrant relative to coordinate system centered at (i, σ_i).

3 The functions $Q_{132}^{\gamma}(t,x)$

In this section, we give the recursions for generating functions $Q_{132}^{\gamma}(t,x)$ by examining the set $\mathcal{S}_n(132)$. We shall first look at the structure of a 132-avoiding permutation.

Given $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$, we suppose $\sigma_k = n$ is the biggest number in the permutation. The numbers $\sigma_1, \dots, \sigma_{k-1}$ must be bigger than the numbers $\sigma_{k+1}, \dots, \sigma_n$ since otherwise there will be a 132 pattern in σ. Thus we can break the permutation σ into three parts: the first $k-1$ numbers, the biggest number $\sigma_k = n$, and the last $n-k$ numbers. We let $A(\sigma) = \mathrm{red}(\sigma_1 \cdots \sigma_{k-1})$ and $B(\sigma) = \mathrm{red}(\sigma_{k+1} \cdots \sigma_n)$ be the reduction of the first $k-1$ numbers and the last $n-k$ numbers, then $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$. The left picture of Figure 3(a) is an example for $\sigma = 867943251 \in \mathcal{S}_9(132)$ with $A(\sigma) = 312$ and $B(\sigma) = 43251$. We also let $\overline{A}(\sigma) := \mathrm{red}(\sigma_1 \cdots \sigma_k)$ be the reduction of the first k numbers. The structure of σ is shown in Figure 4.

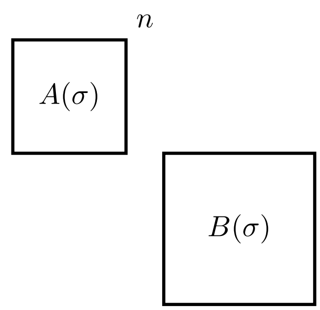


Figure 4: Structure of $\sigma \in \mathcal{S}_n(132)$

Now we count the number of occurrences of a pattern $\gamma = \gamma_1 \cdots \gamma_r \in \mathcal{S}_r(132)$ in σ. First, there are $(\mathrm{occr}_{\gamma}(A(\sigma)) + \mathrm{occr}_{\gamma}(B(\sigma)))$ occurrences of γ in parts $A(\sigma)$ and $B(\sigma)$. Then we count occurrences of γ that intersect with at least two of the three parts, $\{A(\sigma), \sigma_k, B(\sigma)\}$, of σ.

Similar to σ, we shall break γ into three parts: $A(\gamma) = \mathrm{red}(\gamma_1 \cdots \gamma_{s-1})$, $\gamma_s = r$ and $B(\gamma) = \mathrm{red}(\gamma_{s+1} \cdots \gamma_r)$. We also let $\overline{A}(\gamma) = \mathrm{red}(\gamma_1 \cdots \gamma_s)$. Let $\chi(x)$ be the function that takes value 1 when the statement x is true and 0 otherwise. Then there are

(a) $\chi(s = r) \cdot \mathrm{occr}_{A(\gamma)}(A(\sigma))$ occurrences of γ stretch over parts $A(\sigma)$ and σ_k,

(b) $\chi(s = 1) \cdot \mathrm{occr}_{B(\gamma)}(B(\sigma))$ occurrences of γ stretch over parts σ_k and $B(\sigma)$,

(c) $\chi(s < r) \cdot \mathrm{occr}_{\overline{A}(\gamma)}(A(\sigma)) \cdot \mathrm{occr}_{B(\gamma)}(B(\sigma))$ occurrences of γ stretch over parts $A(\sigma)$ and $B(\sigma)$ if $\gamma_r = r - s$,

(d) and $\chi(1 < s < r) \cdot \mathrm{occr}_{A(\gamma)}(A(\sigma)) \cdot \mathrm{occr}_{B(\gamma)}(B(\sigma))$ occurrences of γ stretch over all three parts.

Note that (c) requires $\gamma_r = r - s$, i.e. the permutation $B(\gamma)$ cannot be expressed as the skew sum of two smaller permutations. If $\gamma_r \neq r - s$, then

(c') if $\{\pi_1 \ominus \tau_1, \dots, \pi_j \ominus \tau_j\}$ is the collection of all the ways to write γ as the skew sum of two smaller permutations, then

$$\sum_{i=1}^{j} \mathrm{occr}_{\pi_i}(A(\sigma)) \cdot \mathrm{occr}_{\tau_i}(B(\sigma))$$

is the number of occurrences of γ stretch over parts $A(\sigma)$ and $B(\sigma)$.

We call the above method the *recursive counting method*, which allows us to count $\mathrm{occr}_\gamma(\sigma)$ by counting pattern occurrences from the components of σ.

3.1 The function $Q_{n,132}^{12,21}(x_1, x_2)$

As a first application of our recursive counting method, we have the following theorem first proved by Fürlinger and Hofbauer [3] in 1985 about distribution of patterns of length 2.
Theorem 1 (Fürlinger and Hofbauer). *Let* $Q_n(x_1, x_2) := Q_{n,132}^{12,21}(x_1, x_2)$ *and* $Q(t, x_1, x_2) := Q_{132}^{12,21}(t, x_1, x_2)$*, then*

$$Q_0(x_1, x_2) = 1,\ Q_n(x_1, x_2) = \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} Q_{k-1}(x_1, x_2) Q_{n-k}(x_1, x_2), \tag{6}$$

and

$$Q(t, x, 1) = 1 + tQ(t, x, 1) \cdot Q(tx, x, 1). \tag{7}$$

The theorem was initially proved using Dyck paths. Since the area and coarea of a Dyck path P correspond to the pattern 21 and 12 occurrences, we shall give a brief proof using permutations.

Proof. Equation (7) is a consequence of (6). To prove equation (6), we shall consider the distribution of pattern $\gamma = 12$ and $\tau = 21$ in $\mathcal{S}_n(132)$ using the recursive counting method.

Given $\sigma \in \mathcal{S}_n(132)$ such that $\sigma_k = n$. We have $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$. By the recursive counting method,

$$\begin{aligned} \mathrm{occr}_{12}(\sigma) &= \mathrm{occr}_{12}(A(\sigma)) + \mathrm{occr}_{12}(B(\sigma)) + \mathrm{occr}_1(A(\sigma)) \\ &= \mathrm{occr}_{12}(A(\sigma)) + \mathrm{occr}_{12}(B(\sigma)) + k - 1, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathrm{occr}_{21}(\sigma) &= \mathrm{occr}_{21}(A(\sigma)) + \mathrm{occr}_{21}(B(\sigma)) + \mathrm{occr}_1(B(\sigma)) + \mathrm{occr}_1(A(\sigma)) \cdot \mathrm{occr}_1(B(\sigma)) \\ &= \mathrm{occr}_{21}(A(\sigma)) + \mathrm{occr}_{21}(B(\sigma)) + k(n-k). \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} Q_n(x_1, x_2) &= \sum_{\sigma \in \mathcal{S}_n(132)} x_1^{\mathrm{occr}_{12}(\sigma)} x_2^{\mathrm{occr}_{21}(\sigma)} \\ &= \sum_{k=1}^{n} \sum_{\sigma \in \mathcal{S}_n(132), \sigma_k = n} x_1^{\mathrm{occr}_{12}(A(\sigma)) + \mathrm{occr}_{12}(B(\sigma)) + k - 1} x_2^{\mathrm{occr}_{21}(A(\sigma)) + \mathrm{occr}_{21}(B(\sigma)) + k(n-k)} \\ &= \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} \sum_{\sigma \in \mathcal{S}_n(132), \sigma_k = n} x_1^{\mathrm{occr}_{12}(A(\sigma))} x_2^{\mathrm{occr}_{21}(A(\sigma))} x_1^{\mathrm{occr}_{12}(B(\sigma))} x_2^{\mathrm{occr}_{21}(B(\sigma))} \\ &= \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} \sum_{\pi \in \mathcal{S}_{k-1}(132)} x_1^{\mathrm{occr}_{12}(\pi)} x_2^{\mathrm{occr}_{21}(\tau)} \sum_{\tau \in \mathcal{S}_{n-k}(132)} x_1^{\mathrm{occr}_{12}(\tau)} x_2^{\mathrm{occr}_{21}(\tau)} \\ &= \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} Q_{k-1}(x_1, x_2) Q_{n-k}(x_1, x_2). \qquad \square \end{aligned} \tag{10}$$

Using the recursive equation (6), we can use Mathematica to get

$$\begin{aligned} Q_{132}^{12}(t,x) = 1 + t + t^2(1+x) + t^3(1+2x+x^2+x^3) + t^4(1+3x+3x^2+3x^3+2x^4+x^5+x^6) \\ + t^5(1+4x+6x^2+7x^3+7x^4+5x^5+5x^6+3x^7+2x^8+x^9+x^{10}) + \cdots . \quad (11) \end{aligned}$$

3.2 The function $Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$

Let $\Gamma_2 = \{12, 21\}$, $\Gamma_3 = \{123, 213, 231, 312, 321\}$. We shall prove the following theorem of the function $Q_{n,132}^{\Gamma_2\cup\Gamma_3}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = Q_{n,132}^{12,21,123,213,231,312,321}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ which tracks all patterns of length 2 or 3 in $\mathcal{S}_n(132)$. We use the shorthand Q_n for $Q_{n,132}^{\Gamma_2\cup\Gamma_3}$.

Theorem 2. *The function $Q_{n,132}^{\Gamma_2\cup\Gamma_3}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ satisfies the recursion*

$$Q_0(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = 1, \quad (12)$$

$$\begin{aligned} Q_n(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} x_5^{(k-1)(n-k)} \\ \cdot Q_{k-1}(x_1x_3x_5^{(n-k)}, x_2x_4x_7^{(n-k)}, x_3,x_4,x_5,x_6,x_7) \cdot Q_{n-k}(x_1x_6^k, x_2x_7^k, x_3,x_4,x_5,x_6,x_7). \quad (13) \end{aligned}$$

Proof. We shall consider the distribution of the patterns $\gamma_1 = 12$, $\gamma_2 = 21$, $\gamma_3 = 123$, $\gamma_4 = 213$, $\gamma_5 = 231$, $\gamma_6 = 312$ and $\gamma_7 = 321$ in $\mathcal{S}_n(132)$.

Given $\sigma \in \mathcal{S}_n(132)$ such that $\sigma_k = n$. Like Theorem 1, we have $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$. The number of occurrences of γ_1 and γ_2 is given by equation (8) and (9). For $\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7$, we have the following from the recursive counting method.

$$\begin{aligned}
\mathrm{occr}_{123}(\sigma) &= \mathrm{occr}_{123}(A(\sigma)) + \mathrm{occr}_{123}(B(\sigma)) + \mathrm{occr}_{12}(A(\sigma)), && (14)\\
\mathrm{occr}_{213}(\sigma) &= \mathrm{occr}_{213}(A(\sigma)) + \mathrm{occr}_{213}(B(\sigma)) + \mathrm{occr}_{21}(A(\sigma)), && (15)\\
\mathrm{occr}_{231}(\sigma) &= \mathrm{occr}_{231}(A(\sigma)) + \mathrm{occr}_{231}(B(\sigma)) \\
&\quad + \mathrm{occr}_{12}(A(\sigma)) \cdot \mathrm{occr}_1(B(\sigma)) + \mathrm{occr}_1(A(\sigma)) \cdot \mathrm{occr}_1(B(\sigma)) \\
&= \mathrm{occr}_{231}(A(\sigma)) + \mathrm{occr}_{231}(B(\sigma)) + (n-k)\mathrm{occr}_{12}(A(\sigma)) + (k-1)(n-k), && (16)\\
\mathrm{occr}_{312}(\sigma) &= \mathrm{occr}_{312}(A(\sigma)) + \mathrm{occr}_{312}(B(\sigma)) + \mathrm{occr}_{12}(B(\sigma)) + \mathrm{occr}_1(A(\sigma)) \cdot \mathrm{occr}_{12}(B(\sigma)) \\
&= \mathrm{occr}_{312}(A(\sigma)) + \mathrm{occr}_{312}(B(\sigma)) + k \cdot \mathrm{occr}_{12}(B(\sigma)), \text{ and} && (17)\\
\mathrm{occr}_{321}(\sigma) &= \mathrm{occr}_{321}(A(\sigma)) + \mathrm{occr}_{321}(B(\sigma)) + \mathrm{occr}_{21}(B(\sigma)) \\
&\quad + \mathrm{occr}_1(A(\sigma)) \cdot \mathrm{occr}_{21}(B(\sigma)) + \mathrm{occr}_{21}(A(\sigma)) \cdot \mathrm{occr}_1(B(\sigma)) \\
&= \mathrm{occr}_{321}(A(\sigma)) + \mathrm{occr}_{321}(B(\sigma)) + k \cdot \mathrm{occr}_{21}(B(\sigma)) + (n-k)\mathrm{occr}_{21}(A(\sigma)). && (18)
\end{aligned}$$

Thus,

$$
\begin{aligned}
& Q_n(x_1,x_2,x_3,x_4,x_5,x_6,x_7) \\
= & \sum_{\sigma\in\mathcal{S}_n(132)} x_1^{\mathrm{occr}_{12}(\sigma)} x_2^{\mathrm{occr}_{21}(\sigma)} x_3^{\mathrm{occr}_{123}(\sigma)} x_4^{\mathrm{occr}_{213}(\sigma)} x_5^{\mathrm{occr}_{231}(\sigma)} x_6^{\mathrm{occr}_{312}(\sigma)} x_7^{\mathrm{occr}_{321}(\sigma)} \\
= & \sum_{k=1}^{n} \sum_{\pi\in\mathcal{S}_{k-1}(132)} \sum_{\tau\in\mathcal{S}_{n-k}(132)} x_1^{\mathrm{occr}_{12}(\pi)+\mathrm{occr}_{12}(\tau)+k-1} x_2^{\mathrm{occr}_{21}(\pi)+\mathrm{occr}_{21}(\tau)+k(n-k)} \\
& \cdot x_3^{\mathrm{occr}_{123}(\pi)+\mathrm{occr}_{123}(\tau)+\mathrm{occr}_{12}(\pi)} x_4^{\mathrm{occr}_{213}(\pi)+\mathrm{occr}_{213}(\tau)+\mathrm{occr}_{21}(\pi)} \\
& \cdot x_5^{\mathrm{occr}_{231}(\pi)+\mathrm{occr}_{231}(\tau)+(n-k)\mathrm{occr}_{12}(\pi)+(k-1)(n-k)} x_6^{\mathrm{occr}_{312}(\pi)+\mathrm{occr}_{312}(\tau)+k\cdot\mathrm{occr}_{12}(\tau),} \\
& \cdot x_7^{\mathrm{occr}_{321}(\pi)+\mathrm{occr}_{321}(\tau)+k\cdot\mathrm{occr}_{21}(\tau)+(n-k)\mathrm{occr}_{21}(\pi)} \\
= & \sum_{k=1}^{n} x_1^{k-1} x_2^{k(n-k)} x_5^{(k-1)(n-k)} Q_{k-1}(x_1x_3x_5^{(n-k)}, x_2x_4x_7^{(n-k)}, x_3,x_4,x_5,x_6,x_7) \\
& \cdot Q_{n-k}(x_1x_6^k, x_2x_7^k, x_3,x_4,x_5,x_6,x_7). \qquad \square
\end{aligned}
\tag{19}
$$

Using mathematical software like Mathematica, one can efficiently compute the polynomials $Q_{132}^{\Gamma_2\cup\Gamma_3}(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \sum_{n\geq 0} t^n Q_{n,132}^{\Gamma_2\cup\Gamma_3}(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ as follows.

$$
\begin{aligned}
Q_{132}^{\Gamma_2\cup\Gamma_3}&(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7) = 1+t+t^2(x_1+x_2)+t^3(x_1^3x_3+x_1^2x_2x_4+x_1x_2^2x_5+x_1x_2^2x_6+x_2^3x_7) \\
&+ t^4\big(x_1^6x_3^4 + x_1^5x_2x_3^2x_4^2 + x_1^4x_2^2x_3x_4^2x_5 + x_1^3x_2^3x_3x_5^3 + x_1^4x_2^2x_3x_4^2x_6 + x_1^2x_2^4x_5^2x_6^2 + x_1^3x_2^3x_3x_6^3 \\
&+x_1^3x_2^3x_4^3x_7 + x_1^2x_2^4x_4x_5^2x_7 + x_1^2x_2^4x_4x_6^2x_7 + x_1x_2^5x_5^2x_7^2 + x_1x_2^5x_5x_6x_7^2 + x_1x_2^5x_6^2x_7^2 + x_2^6x_7^4\big) \\
&+ t^5\big(x_1^{10}x_3^{10} + x_1^9x_2x_3^7x_4^3 + x_1^8x_2^2x_3^5x_4^4x_5 + x_1^7x_2^3x_3^4x_4^3x_5^3 + x_1^6x_2^4x_3^4x_5^6 + x_1^8x_2^2x_3^5x_4^4x_6 + x_1^6x_2^4x_3^2x_4^4x_5^2x_6^2 \\
&+ x_1^7x_2^3x_3^4x_4^3x_6^3 + x_1^4x_2^6x_3x_5^6x_6^3 + x_1^6x_2^4x_3^4x_6^6 + x_1^4x_2^6x_3x_5^3x_6^6 + x_1^7x_2^3x_3^3x_4^6x_7 + x_1^6x_2^4x_3^2x_4^5x_5^2x_7 \\
&+ x_1^5x_2^5x_3^2x_4^2x_5^5x_7 + x_1^6x_2^4x_3^2x_4^5x_6^2x_7 + x_1^5x_2^5x_3^2x_4^2x_6^5x_7 + x_1^5x_2^5x_3x_4^5x_5^2x_7^2 + x_1^4x_2^6x_3x_4^2x_5^5x_7^2 \\
&+ x_1^5x_2^5x_3x_4^5x_5x_6x_7^2 + x_1^4x_2^6x_3x_4^2x_5^4x_6x_7^2 + x_1^5x_2^5x_3x_4^5x_6^2x_7^2 + x_1^3x_2^7x_4x_5^4x_6^3x_7^2 + x_1^4x_2^6x_3x_4^2x_5x_6^4x_7^2 \\
&+ x_1^3x_2^7x_4x_5^3x_6^4x_7^2 + x_1^4x_2^6x_3x_4^2x_6^5x_7^2 + x_1^3x_2^7x_3x_5^6x_7^3 + x_1^3x_2^7x_3x_5^3x_6^3x_7^3 + x_1^3x_2^7x_3x_6^6x_7^3 + x_1^4x_2^6x_4^6x_7^4 \\
&+ x_1^3x_2^7x_4^3x_5^3x_7^4 + x_1^2x_2^8x_5^4x_6^2x_7^4 + x_1^3x_2^7x_4^3x_6^3x_7^4 + x_1^2x_2^8x_5^3x_6^3x_7^4 + x_1^2x_2^8x_5^2x_6^4x_7^4 + x_1^2x_2^8x_4x_5^4x_7^5 \\
&+x_1^2x_2^8x_4x_5^2x_6^2x_7^5 + x_1^2x_2^8x_4x_6^4x_7^5 + x_1x_2^9x_5^3x_7^7 + x_1x_2^9x_5^2x_6x_7^7 + x_1x_2^9x_5x_6^2x_7^7 + x_1x_2^9x_6^3x_7^7 + x_2^{10}x_7^{10}\big) \\
&+\cdots.
\end{aligned}
\tag{20}
$$

We can also evaluate the appropriate variables at 1 and use the relation that $\mathrm{inv}(\sigma)+\mathrm{coinv}(\sigma) = \binom{n}{2}$ to get the following corollary. We use the shorthand P_n for P_n^{γ} in the RHS of each equation.

Corollary 2. *Let* $P_n^\gamma(q,x) := \sum_{\sigma\in\mathcal{S}_n(132)} q^{\mathrm{occr}_{12}(\sigma)} x^{\mathrm{occr}_\gamma(\sigma)}$, *then*

$$\begin{aligned}
P_0^\gamma(q,x) &= 1 \quad \text{for each pattern } \gamma, && (21)\\
P_n^{123}(q,x) &= \sum_{k=1}^n q^{k-1}P_{k-1}(qx,x)P_{n-k}(q,x), && (22)\\
P_n^{213}(q,x) &= \sum_{k=1}^n q^{k-1}x^{\frac{(k-1)(k-2)}{2}}P_{k-1}(\frac{q}{x},x)P_{n-k}(q,x), && (23)\\
P_n^{231}(q,x) &= \sum_{k=1}^n q^{k-1}x^{(k-1)(n-k)}P_{k-1}(qx^{(n-k)},x)P_{n-k}(q,x), && (24)\\
P_n^{321}(q,x) &= \sum_{k=1}^n q^{k-1}x^{\frac{(n-k)(kn-4k+2)}{2}}P_{k-1}(\frac{q}{x^{n-k}},x)P_{n-k}(\frac{q}{x^k},x). && (25)
\end{aligned}$$

Then we can compute $Q_{132}^\gamma(t,x) = \sum_{n\geq 0} t^n P_n^\gamma(1,x)$ where γ has length 3 as follows.

$$Q_{132}^{123}(t,x) = 1+t+2t^2+t^3(4+x)+t^4(8+4x+x^2+x^4)+t^5(16+12x+5x^2+x^3+4x^4+2x^5+x^7+x^{10}) + t^6(32+32x+18x^2+6x^3+13x^4+10x^5+3x^6+4x^7+3x^8+5x^{10}+2x^{11}+2x^{13}+x^{16}+x^{20})+\cdots, \quad (26)$$

$$Q_{132}^{213}(t,x) = 1+t+2t^2+t^3(4+x)+t^4(8+2x+3x^2+x^3)+t^5(16+5x+6x^2+5x^3+3x^4+5x^5+2x^6) + t^6(32+12x+16x^2+11x^3+9x^4+10x^5+10x^6+5x^7+10x^8+10x^9+6x^{10}+x^{12})+\cdots, \quad (27)$$

$$\begin{aligned}
&Q_{132}^{231}(t,x) = Q_{132}^{312}(t,x)\\
&= 1+t+2t^2+t^3(4+x)+t^4(8+2x+3x^2+x^3)+t^5(16+4x+6x^2+7x^3+4x^4+2x^5+3x^6)\\
&+ t^6(32+8x+12x^2+14x^3+17x^4+7x^5+17x^6+5x^7+5x^8+8x^9+5x^10+2x^12)+\cdots, \quad (28)
\end{aligned}$$

$$Q_{132}^{321}(t,x) = 1+t+2t^2+t^3(4+x)+t^4(7+3x+3x^2+x^4)+t^5(11+5x+9x^2+3x^3+6x^4+3x^5+4x^7+x^{10}) + t^6(16+7x+15x^2+9x^3+17x^4+7x^5+10x^6+12x^7+7x^8+6x^9+7x^{10}+3x^{11}+6x^{12}+4x^{13}+5x^{16}+x^{20}) +\cdots. \quad (29)$$

3.3 Longer patterns whose distributions satisfy good recursions

We have built recursions for generating functions which give the distribution of all patterns of length 2 or 3 in $\mathcal{S}_n(132)$. This leads to a natural question – can we give recursions for the generating functions tracking any pattern in $\mathcal{S}_n(132)$ like we have done in Section 3.1 and Section 3.2. We prove that though we can always use the recursive counting method, we do not always obtain clear recursions like Theorem 1 and Theorem 2.

Let γ be a permutation pattern. We say that the distribution of the pattern γ in $\mathcal{S}_n(132)$ *satisfies a good recursion* if there exist s permutations $\gamma_1,\ldots,\gamma_s$ of length at least 2 such that the generating function $Q_n(x,x_1,\ldots,x_s) = Q_{n,132}^{\gamma;\gamma_1,\ldots,\gamma_s}(x,x_1,\ldots,x_s)$ satisfies that $Q_0(x,x_1,\ldots,x_s)=1$, and

$$Q_n(x,x_1,\ldots,x_s) = \sum_{i=1}^n q(X)Q_{i-1}(p_1(X),\ldots,p_{s+1}(X))Q_{n-i}(q_1(X),\ldots,q_{s+1}(X)) \quad (30)$$

for $n \geq 1$, where $X = \{x, x_1, \ldots, x_s\}$, and $q(X), p_1(X), \ldots, p_{s+1}(X), q_1(X), \ldots, q_{s+1}(X)$ are $2s+3$ rational functions about variables in X and the power of variables in the numerator and denominator are polynomials of n and i.

Thus, for any pattern γ whose distribution satisfies a good recursion, there is a $Q_n(x, x_1, \ldots, x_s)$ defined as above such that it can be computed directly from the functions $Q_i(x, x_1, \ldots, x_s)$ for $i = 0, \ldots, n-1$. We have the following theorem about the number of permutations in $\mathcal{S}_n(132)$ whose distribution satisfy a good recursion.

Theorem 3. *Let $\{a_n\}_{n\geq 0}$ be the integer sequence defined by*

$$a_0 = a_1 = 1, \ a_2 = 2, \ \textit{and } a_n = a_{n-1} + 2a_{n-2} + a_{n-3}. \tag{31}$$

Then the number of permutations in $\mathcal{S}_n(132)$ whose distribution satisfy a good recursion is at least a_n.

Proof. Given $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n(132)$, we define

$$\begin{aligned}
\sigma' &:= \sigma_1 \cdots \sigma_n(n+1), &(32)\\
\sigma'' &:= (n+2)\sigma_1 \cdots \sigma_n(n+1), &(33)\\
\sigma''' &:= (\sigma_1+1)\cdots(\sigma_n+1)(n+2)1 \quad \text{and} &(34)\\
\sigma'''' &:= (n+3)(\sigma_1+1)\cdots(\sigma_n+1)(n+2)1. &(35)
\end{aligned}$$

We construct a set Γ of permutation patterns as follows. We let the empty permutation $\emptyset \in \Gamma$. Next, for each permutation $\sigma \in \Gamma$, we let $\sigma', \sigma'', \sigma''', \sigma'''' \in \Gamma$. Clearly, each permutation in Γ is 132-avoiding, and the number of permutations in $\Gamma \cap \mathcal{S}_n$ is a_n based on the recursive construction of the set Γ.

From the recursive counting method, the distributions of $\sigma', \sigma'', \sigma''', \sigma''''$ satisfy a good recursion as long as the distribution of σ satisfies a good recursion. Thus the distribution of each permutation in Γ satisfies a good recursion, which proves the theorem. □

In fact, when we are counting the number of occurrences of $\gamma \in \mathcal{S}_r(132)$ in $\sigma \in \mathcal{S}_n(132)$ with recursive counting method, we shall break the pattern γ into three parts: $A(\gamma)$, r and $B(\gamma)$. γ fails to satisfy a good recursion by part (c') of the recursive counting method if $\gamma = \pi \ominus \tau$ for some permutation π, τ of length at least 2 (in this case the generating function cannot be recursively computed as equation (30) since the RHS of equation (30) never gives the product $\text{occr}_\pi(A(\sigma)) \cdot \text{occr}_\tau(B(\sigma))$). This is saying that we cannot have $|A(\gamma)| \geq 1$ and $|B(\gamma)| \geq 2$ simultaneously.

If $B(\gamma)$ is empty, then $\gamma = A(\gamma)'$. If $B(\gamma) = 1$, then $\gamma = A(\gamma)'''$. If $|B(\gamma)| \geq 2$ and $A(\gamma)$ is empty, then we shall decompose $B(\gamma)$ into three parts: $A(B(\gamma))$, $r-1$ and $B(B(\gamma))$. If $B(B(\gamma))$ is empty, then $\gamma = A(B(\gamma))''$. If $B(B(\gamma))$ is not empty, then $B(B(\gamma))$ can only be of size 1 to make γ not separable into a skew sum of two nontrivial permutations, and $\gamma = A(B(\gamma))''''$.

Thus Γ collects all the permutations satisfying a good recursion if use the recursive counting method, and there are exactly a_n permutations in $\mathcal{S}_n$ whose distributions satisfy a good recursion using the recursive counting method (there might be more permutations in $\mathcal{S}_n$ whose distributions satisfy a good recursion, but for other reasons not accessible by the recursive counting method).

The sequence $\{a_n\}_{n\geq 0} = \{1, 1, 2, 5, 10, 22, 47, 101, 217, \ldots\}$ appears in OEIS [14] as sequence A101399.

We shall give an example of a longer pattern $\gamma = 12\cdots m$ whose distribution satisfies a good recursion in the following theorem. Note that this gives a way to count the number of occurrences of $12\cdots m$ in $\mathcal{S}_n(132)$ different from Mansour and Vainshtein [11].

Theorem 4. *Given $m \geq 2$ and $n \geq 0$, let*

$$\begin{aligned} Q_{n,132}^{(m)}(x_2, x_3, \ldots, x_m) &:= \sum_{\sigma \in \mathcal{S}_n(132)} x_2^{\mathrm{occr}_{12}(\sigma)} x_3^{\mathrm{occr}_{123}(\sigma)} \cdots x_m^{\mathrm{occr}_{12\cdots m}(\sigma)} \text{ and} & (36) \\ Q_{132}^{(m)}(t, x_2, x_3, \ldots, x_m) &:= \sum_{n \geq 0} t^n Q_{n,132}^{(m)}(x_2, x_3, \ldots, x_m), & (37) \end{aligned}$$

then we have the following equations,

$$\begin{aligned} Q_{n,132}^{(m)}(x_2, \ldots, x_m) &= \sum_{k=1}^{n} x_2^{k-1} Q_{k-1,132}^{(m)}(x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,132}^{(m)}(x_2, \ldots, x_m), & (38) \\ Q_{132}^{(m)}(t, x_2, \ldots, x_m) &= 1 + t Q_{132}^{(m)}(tx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{132}^{(m)}(t, x_2, \ldots, x_m). & (39) \end{aligned}$$

Proof. We shall consider the distribution of pattern $\gamma_s = 12\cdots s$ for $s = 2, \ldots, m$ in $\mathcal{S}_n(132)$.

Given $\sigma \in \mathcal{S}_n(132)$ such that $\sigma_k = n$, we have $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$. By the recursive counting method, we have

$$\mathrm{occr}_{12\cdots s}(\sigma) = \mathrm{occr}_{12\cdots s}(A(\sigma)) + \mathrm{occr}_{12\cdots s}(B(\sigma)) + \mathrm{occr}_{12\cdots(s-1)}(A(\sigma)). \quad (40)$$

Thus,

$$\begin{aligned} Q_{n,132}^{(m)}(x_2, \ldots, x_m) &= \sum_{\sigma \in \mathcal{S}_n(132)} \prod_{i=2}^{m} x_i^{\mathrm{occr}_{12\cdots i}(\sigma)} \\ &= \sum_{k=1}^{n} \sum_{\pi \in \mathcal{S}_{k-1}(132)} \sum_{\tau \in \mathcal{S}_{n-k}(132)} \prod_{i=2}^{m} x_i^{\mathrm{occr}_{12\cdots i}(\pi)} \cdot x_i^{\mathrm{occr}_{12\cdots i}(\tau)} \cdot x_i^{\mathrm{occr}_{12\cdots(i-1)}(\pi)} \\ &= \sum_{k=1}^{n} x_2^{k-1} Q_{k-1,132}^{(m)}(x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q_{n-k,132}^{(m)}(x_2, \ldots, x_m), \quad (41) \end{aligned}$$

which proves equation (38), and (39) follows immediately. □

This theorem can be seem as a generalization of Theorem 1 of Fürlinger and Hofbauer [3].

3.4 The distribution of patterns of length 4 in $\mathcal{S}_n(132)$

Let $\Gamma_4 = \{1234, 2134, 2314, 2341, 3124, 3214, 3241, 3412, 3421, 4123, 4213, 4231, 4312, 4321\}$ be the set of patterns in $\mathcal{S}_4(132)$. By Theorem 3, there are 10 of the 14 patterns in Γ_4 satisfy good recursions. To track all the 14 patterns in $\mathcal{S}_4(132)$, we shall refine the generating function Q_n by the number of coinversions and define

$$Q_{n,i}(x_1, \ldots, x_{19}) := Q_{n,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(x, 1, x_1, \ldots, x_{19})\big|_{x^i}, \quad (42)$$

then

$$Q_{n,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(x_1, \ldots, x_{21}) = \sum_{i=0}^{\binom{n}{2}} x_1^i x_2^{\binom{n}{2}-i} Q_{n,i}(x_3, \ldots, x_{21}). \quad (43)$$

We let x_i track the occurrences of length 3 patterns and y_i to track the occurrences of length 4 patterns, then we have the following theorem which gives the recursion for the generating function $Q_{n,i}(x_1, \ldots, x_5, y_1, \ldots, y_{14})$.

Theorem 5. *The function $Q_{n,i}(x_1, \ldots, x_5, y_1, \ldots, y_{14})$ satisfies the following recursion,*

$$Q_{0,0}(x_1, \ldots, x_5, y_1, \ldots, y_{14}) = 1, \tag{44}$$

$$Q_{n,i}(x_1, \ldots, x_5, y_1, \ldots, y_{14}) = 0 \textit{ for } i < 0 \textit{ or } i > \binom{n}{2}, \tag{45}$$

$$\begin{aligned} &Q_{n,i}(x_1, \ldots, x_5, y_1, \ldots, y_{14}) = \\ &\sum_{k=1}^{n} \sum_{j=0}^{i+1-k} x_1^j x_2^{\binom{k-1}{2}-j} x_3^{(n-k)(k+j-1)} x_4^{k(i+1-k-j)} x_5^{(n-k)\left(\binom{k-1}{2}-j\right)+k\left(\binom{n-k}{2}+k+j-i-1\right)} y_4^{j(n-k)} y_7^{\left(\binom{k-1}{2}-j\right)(n-k)} \\ &\quad y_8^{(j+k-1)(i+1-k-j)} y_9^{(j+k-1)\left(\binom{n-k}{2}+k+j-i-1\right)} y_{13}^{\left(\binom{k-1}{2}-j\right)(i+1-k-j)} y_{14}^{\left(\binom{k-1}{2}-j\right)\left(\binom{n-k}{2}+k+j-i-1\right)} \\ &\qquad Q_{k-1,j}(x_1y_1y_4^{n-k}, x_2y_2y_7^{n-k}, x_3y_3y_9^{n-k}, x_4y_5y_{12}^{n-k}, x_5y_6y_{14}^{n-k}, y_1, \ldots, y_{14}) \\ &\qquad\qquad Q_{n-k,i+1-k-j}(x_1y_{10}^k, x_2y_{11}^k, x_3y_{12}^k, x_4y_{13}^k, x_5y_{14}^k, y_1, \ldots, y_{14}). \end{aligned} \tag{46}$$

Proof. We shall count the number of occurrences of 19 patterns of length 3 or 4 in $\mathcal{S}_n(132)$ using the recursive counting method. Let $\sigma \in \mathcal{S}_n(132)$ such that $\sigma_k = n$ and $\mathrm{occr}_{12}(\sigma) = i$, we have $A(\sigma) \in \mathcal{S}_{k-1}(132)$ and $B(\sigma) \in \mathcal{S}_{n-k}(132)$, and $\mathrm{occr}_{21}(\sigma) = \binom{n}{2} - i$. We shall abbreviate $\mathrm{occr}_\gamma(A(\sigma))$, $\mathrm{occr}_\gamma(B(\sigma))$ and $\mathrm{occr}_\gamma(\overline{A}(\sigma))$ to A_γ, B_γ and $\overline{A}_\gamma$ in this proof.

Similar to Theorem 2, the formulas for occurrences of patterns 123, 213, 231, 312, 321 are given by equation (14), (15), (16), (17) and (18). Then we shall look at the 14 patterns of length 4. Part (a) of the recursive counting method implies that

$$\mathrm{occr}_{\pi_1\pi_2\pi_3 4}(\sigma) = A_{\pi_1\pi_2\pi_3 4} + B_{\pi_1\pi_2\pi_3 4} + A_{\pi_1\pi_2\pi_3} \tag{47}$$

for any $\pi_1\pi_2\pi_3 \in \mathcal{S}_3(132)$. For patterns not end with 4, it follows from the recursive counting method that

$$\begin{aligned}
\mathrm{occr}_{2341}(\sigma) &= A_{2341}+B_{2341}+A_{12}\cdot B_1+A_{123}\cdot B_1 \\
&= A_{2341}+B_{2341}+j(n-k)+(n-k)A_{123}, && (48)\\
\mathrm{occr}_{3241}(\sigma) &= A_{3241}+B_{3241}+A_{21}\cdot B_1+A_{213}\cdot B_1 \\
&= A_{3241}+B_{3241}+(\binom{k-1}{2}-j)(n-k)+(n-k)A_{213}, && (49)\\
\mathrm{occr}_{3412}(\sigma) &= A_{3412}+B_{3412}+A_{12}B_{12}+A_1B_{12} \\
&= A_{3412}+B_{3412}+(j+k-1)(i+1-k-j), && (50)\\
\mathrm{occr}_{3421}(\sigma) &= A_{3421}+B_{3421}+A_{12}B_{21}+A_1AB_{21}+A_{231}B_1 \\
&= A_{3421}+B_{3421}+(n-k)A_{231}+(j+k-1)\left(\binom{n-k}{2}+k+j-i-1\right), && (51)\\
\mathrm{occr}_{4123}(\sigma) &= A_{4123}+B_{4123}B+\overline{A}_1B_{123}=A_{4123}+B_{4123}+kB_{123} && (52)\\
\mathrm{occr}_{4213}(\sigma) &= A_{4213}+B_{4213}+\overline{A}_1B_{213}=A_{4213}+B_{4213}+kB_{213} && (53)\\
\mathrm{occr}_{4231}(\sigma) &= A_{4231}+B_{4231}+\overline{A}_1B_{231}+A_{312}B_1=A_{4231}+B_{4231}+kB_{231}+(n-k)A_{312}, && (54)\\
\mathrm{occr}_{4312}(\sigma) &= A_{4312}+B_{4312}+\overline{A}_1B_{312}+A_{21}B_{12} \\
&= A_{4312}+B_{4312}+kB_{312}+\left(\binom{k-1}{2}-j\right)(i+1-k-j), && (55)\\
\mathrm{occr}_{4321}(\sigma) &= A_{4321}+B_{4321}+\overline{A}_1B_{321}\ +A_{21}B_{21}+A_{321}B_1 \\
&= A_{4321}+B_{4321}+kB_{321} \\
&\quad +(n-k)A_{321}+\left(\binom{k-1}{2}-j\right)\left(\binom{n-k}{2}+k+j-i-1\right). && (56)
\end{aligned}$$

Then, one can arrange the generating function $Q_{n,i}(x_1,\ldots,x_5,y_1,\ldots,y_{14})$ in a similar way to equation (19) to prove the recursion. □

We can still compute the polynomials $Q_{n,132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(x_1,\ldots,x_{21}) = \sum_{i=0}^{\binom{n}{2}} x_1^i x_2^{\binom{n}{2}-i} Q_{n,i}(x_3,\ldots,x_{21})$ efficiently by Mathematica as follows.

$$\begin{aligned}
Q_{132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(t,x_1,\ldots,x_{21}) &= 1+t+t^2(x_1+x_2)+t^3(x_1^3x_3+x_1^2x_2x_4+x_1x_2^2x_5+x_1x_2^2x_6+x_2^3x_7) \\
&+t^4\left(x_1^4x_{10}x_2^2x_3x_4^2x_5+x_1^3x_{11}x_2^3x_3x_5^3+x_1^4x_{12}x_2^2x_3x_4^2x_6+x_1^2x_{15}x_2^4x_5^2x_6^2+x_1^3x_{17}x_2^3x_3x_6^3\right.\\
&+x_1^3x_{13}x_2^3x_4^3x_7+x_1^2x_{14}x_2^4x_4x_5^2x_7+x_1^2x_{18}x_2^4x_4x_6^2x_7+x_1x_{16}x_2^5x_5^2x_7^2+x_1x_{19}x_2^5x_5x_6x_7^2\\
&\left.+x_1x_2^5x_{20}x_6^2x_7^2+x_2^6x_{21}x_7^4+x_1^6x_3^4x_8+x_1^5x_2x_3^2x_4^2x_9\right)+\cdots. \qquad (57)
\end{aligned}$$

4 The functions $Q_{123}^{\gamma}(t,x)$

We use the bijection $\Psi:\mathcal{S}_n(123)\to\mathcal{D}_n$ of Deutsch and Elizalde [2] to study the distribution of patterns 132 and 231 in $\mathcal{S}_n(123)$, and we generalize our result of 132 distribution to $1m\cdots 2$ in $\mathcal{S}_n(123)$. Our method does not apply for the pattern 321.

4.1 The distribution of pattern $1m\cdots 2$ in $\mathcal{S}_n(123)$

Given $\sigma \in \mathcal{S}_n(123)$. If $\sigma_{i_1}\cdots\sigma_{i_m}$ is an occurrence of pattern $1m\cdots 2$, then the number σ_{i_1} must be a left-to-right minimum of σ, otherwise there must exist a number $\sigma_a < \sigma_{i_1}$ where the index $a < i_1$, and (a, i_1, i_m) is an occurrence of pattern 123.

By the bijection $\Psi : \mathcal{S}_n(123) \to \mathcal{D}_n$, the number σ_{i_1} must be a peak of the corresponding Dyck path $\Psi(\sigma)$. Suppose that the number σ_{i_1} is on the d^{th} diagonal of $\Psi(\sigma)$, then by (3) of Lemma 3, there are d numbers to the right of σ_{i_1} that are bigger than σ_{i_1}, appear in a decreasing way. It follows immediately that there are $\binom{d}{m-1}$ occurrences of pattern $1m\cdots 2$ at the peak σ_{i_1} since any $m-1$ of the d numbers to the right of σ_{i_1} that are bigger than σ_{i_1} create a pattern $1m\cdots 2$ with the number σ_{i_1}.

Now let $c_d(\sigma)$ be the number of peaks that are on the d^{th} diagonal of $\Psi(\sigma)$, then

$$\text{occr}_{1m\cdots 2}(\sigma) = \sum_{d\geq 0} c_d(\sigma)\binom{d}{m-1}. \tag{58}$$

We also let $c_d(P)$ be the number of peaks that are on the d^{th} diagonal of a path P.

We shall define

$$Q^{(m)}_{n,123}(s, x_2, x_3, \ldots, x_m) := \sum_{\sigma\in\mathcal{S}_n(123)} s^{\text{LRmin}(\sigma)} x_2^{\text{occr}_{12}(\sigma)} x_3^{\text{occr}_{132}(\sigma)} \cdots x_m^{\text{occr}_{1m(m-1)\cdots 2}(\sigma)} \quad\text{and} \tag{59}$$

$$Q^{(m)}_{123}(t, s, x_2, x_3, \ldots, x_m) := \sum_{n\geq 0} t^n Q_{n,123}(s, x_2, x_3, \ldots, x_m), \tag{60}$$

then we have the following theorem.
Theorem 6. *Given $n \geq 0$ and $m \geq 2$,*

$$\begin{aligned} Q^{(m)}_{n,123}(s, x_2, \ldots, x_m) &= sQ^{(m)}_{n-1,123}(s, x_2, \ldots, x_m) \\ &\quad + \sum_{k=2}^{n} Q^{(m)}_{k-1,123}(sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q^{(m)}_{n-k,123}(s, x_2, \ldots, x_m) \end{aligned} \tag{61}$$

and

$$\begin{aligned} Q^{(m)}_{123}(t, s, x_2, \ldots, x_m) &= 1 + t(s-1)Q^{(m)}_{123}(t, s, x_2, \ldots, x_m) \\ &\quad + tQ^{(m)}_{123}(t, sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m)Q^{(m)}_{123}(t, s, x_2, \ldots, x_m). \end{aligned} \tag{62}$$

Proof. We enumerate the pattern occurrences using the Dyck path bijection. Given any Dyck path P, we can break the path at the first return to write P as DP_1RP_2, where P_1 is the path after the first D step before the last R step before the first return, and P_2 is the path after the first return.

Let $k = \mathrm{ret}(P)$. By equation (58), we have

$$\begin{aligned}
& Q^{(m)}_{n,123}(s, x_2, \ldots, x_m) \\
= & \sum_{\sigma\in\mathcal{S}_n(123)} s^{\sum_{d\geq 0} c_d(\sigma)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(\sigma)\binom{d}{1}} x_3^{\sum_{d\geq 0} c_d(\sigma)\binom{d}{2}} \cdots x_m^{\sum_{d\geq 0} c_d(\sigma)\binom{d}{m-1}} \\
= & \sum_{P\in\mathcal{D}_n} s^{\sum_{d\geq 0} c_d(P)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(P)\binom{d}{1}} x_3^{\sum_{d\geq 0} c_d(P)\binom{d}{2}} \cdots x_m^{\sum_{d\geq 0} c_d(P)\binom{d}{m-1}} \\
= & s \sum_{P_2\in\mathcal{D}_{n-1}} s^{\sum_{d\geq 0} c_d(P_2)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(P_2)\binom{d}{1}} \cdots x_m^{\sum_{d\geq 0} c_d(P_2)\binom{d}{m-1}} \\
& + \sum_{k=2}^{n} \sum_{P_1\in\mathcal{D}_{k-1}} s^{\sum_{d\geq 0} c_d(P_1)\binom{d+1}{0}} x_2^{\sum_{d\geq 0} c_d(P_1)\binom{d+1}{1}} \cdots x_m^{\sum_{d\geq 0} c_d(P_1)\binom{d+1}{m-1}} \\
& \cdot \sum_{P_2\in\mathcal{D}_{n-k}} s^{\sum_{d\geq 0} c_d(P_2)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(P_2)\binom{d}{1}} \cdots x_m^{\sum_{d\geq 0} c_d(P_2)\binom{d}{m-1}} \\
= & sQ^{(m)}_{n-1,123}(s, x_2, \ldots, x_m) \\
& + \sum_{k=2}^{n} \sum_{P_1\in\mathcal{D}_{k-1}} s^{\sum_{d\geq 0} c_d(P_1)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(P_1)\left(\binom{d}{0}+\binom{d}{1}\right)} \cdots x_m^{\sum_{d\geq 0} c_d(P_1)\left(\binom{d}{m-2}+\binom{d}{m-1}\right)} \\
& \cdot \sum_{P_2\in\mathcal{D}_{n-k}} s^{\sum_{d\geq 0} c_d(P_2)\binom{d}{0}} x_2^{\sum_{d\geq 0} c_d(P_2)\binom{d}{1}} \cdots x_m^{\sum_{d\geq 0} c_d(P_2)\binom{d}{m-1}} \\
= & sQ^{(m)}_{n-1,123}(s, x_2, \ldots, x_m) \\
& + \sum_{k=2}^{n} Q^{(m)}_{k-1,123}(sx_2, x_2x_3, x_3x_4, \ldots, x_{m-1}x_m, x_m) Q^{(m)}_{n-k,123}(s, x_2, \ldots, x_m). \qquad (63)
\end{aligned}$$

Equation (62) follows immediately from (61). ☐

Evaluating m at 3 gives the following corollary for the distribution of coinversion and occr_{132} statistics.

Corollary 3. $Q^{(3)}_{n,123}(s, q, x) = \sum_{\sigma\in\mathcal{S}_n(123)} s^{\mathrm{LRmin}(\sigma)} q^{\mathrm{occr}_{12}(\sigma)} x^{\mathrm{occr}_{132}(\sigma)}$ *satisfies*

$$Q^{(3)}_{0,123}(s, q, x) = 1, \quad Q^{(3)}_{n,123}(s, q, x) = sQ^{(3)}_{n-1} + \sum_{k=2}^{n} Q^{(3)}_{k-1}(sq, qx, x) Q^{(3)}_{n-k}(s, q, x). \qquad (64)$$

Further,

$$Q^{(3)}_{123}(t, s, q, x) = 1 + t(s-1)Q^{(3)}_{123}(t, s, q, x) + tQ^{(3)}_{123}(t, sq, qx, x)Q^{(3)}_{123}(t, s, q, x).$$

Then we can use the recursive formula to compute $Q^{12,132}_{123}(t, q, x) = \sum_{n\geq 0} t^n Q^{(3)}_{n,123}(1, q, x)$:

$$\begin{aligned}
Q^{12,132}_{123}(t, q, x) = {} & 1+t+(1+q)t^2+t^3(1+2q+q^2+q^2x)+t^4(1+3q+3q^2+q^3+2q^2x+2q^3x+q^4x^2+q^3x^3) \\
& +t^5(1+4q+6q^2+4q^3+q^4+3q^2x+6q^3x+3q^4x+3q^4x^2+2q^5x^2+2q^3x^3+2q^4x^3+q^6x^3+2q^5x^4+q^4x^6+q^6x^6) \\
& + \cdots . \qquad (65)
\end{aligned}$$

By looking at the coefficients of the generating functions, we find a coincidence among $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ that

$$|\{\sigma \in \mathcal{S}_n(132) : \text{occr}_{12\cdots j}(\sigma) = i\}| = |\{\sigma \in \mathcal{S}_n(123) : \text{occr}_{1j(j-1)\cdots 2}(\sigma) = i\}| \quad \text{for all } i < j. \tag{66}$$

In other words, we have the following theorem.
Theorem 7. *Let $[x^i]_Q$ denote the coefficient of x^i in function Q, then*

$$[t^n x^i]_{Q_{132}^{1\cdots j}(t,x)} = [t^n x^i]_{Q_{123}^{1j\cdots 2}(t,x)} \text{ for } i < j. \tag{67}$$

Proof. One can use the recursive equations (38) and (61) to prove the theorem using induction. Here we shall give a proof of the theorem combinatorially using the Dyck path bijections Φ and Ψ.

By equation (58), $\sigma \in \mathcal{S}_n(123)$ has i occurrences of pattern $1j(j-1)\cdots 2$ where $j > i$ if and only if the corresponding Dyck path $\Psi(\sigma)$ has i peaks on the $j-1^{\text{st}}$ diagonal and no peaks on the k^{th} diagonal for all $k \geq j$.

On the other hand, let $\pi \in \mathcal{S}_n(132)$ and $\text{occr}_{12\cdots j}(\pi) = i$. The corresponding Dyck path $\Phi(\pi)$ has no peaks on the k^{th} diagonal for all $k \geq j$. Otherwise, if π_i is on the k^{th} diagonal for some $k \geq j$, there are k numbers to the right of π_i that is greater than π_i, forming a length $k+1$ increasing subsequence with π_i. There are $\binom{k+1}{j} > i$ occurrences of pattern $12\cdots j$ in this subsequence, which lead to a contradiction.

Further, if $\pi_{\ell_1}\cdots\pi_{\ell_j}$ is an occurrence of pattern $12\cdots j$, then π_{ℓ_1} must be a peak on the $j-1^{\text{st}}$ diagonal, otherwise any peak to the right of π_{ℓ_1} that is smaller than π_{ℓ_1} is at least on the j^{th} diagonal by Lemma 2 (c), contradiction with the statement that $\Phi(\pi)$ has no peaks on the k^{th} diagonal for all $k \geq j$.

Thus, π has i occurrences of pattern $1\cdots j$ where $j > i$ if and only if the corresponding Dyck path $\Phi(\pi)$ has i peaks on the $j-1^{\text{st}}$ diagonal and no peaks on the k^{th} diagonal for all $k \geq j$, which proves the equations (66) and (67). $\square$

4.2 The distribution of pattern 231 in $\mathcal{S}_n(123)$

We give recursive formulas for generating functions of $\mathcal{S}_n(123)$ tracking the number of occurrences of pattern 231 by refining function Q_n by the number of left-to-right minima. Given $\sigma \in \mathcal{S}_n(123)$, we let $\text{linv}(\sigma)$ be the number of pairs (i,j) such that σ_i is a left-to-right minimum, σ_j is not a left-to-right minimum and $\sigma_i > \sigma_j$. For a Dyck path P, we also let $\text{linv}(P) = \text{linv}(\Psi^{-1}(\sigma))$.

Next, we define

$$D_n(s,q,x,y) := \sum_{\sigma \in \mathcal{S}_n(123)} s^{\text{LRmin}(\sigma)} q^{\text{occr}_{12}(\sigma)} x^{\text{linv}(\sigma)} y^{\text{occr}_{231}(\sigma)} \text{ and} \tag{68}$$

$$D_{n,k}(q,x,y) := \sum_{\sigma \in \mathcal{S}_n(123), \text{LRmin}(\sigma)=k} q^{\text{occr}_{12}(\sigma)} x^{\text{linv}(\sigma)} y^{\text{occr}_{231}(\sigma)}, \tag{69}$$

then $D_n(s,q,x,y) = \sum_{k=1}^{n} s^k D_{n,k}(q,x,y)$, and we have the following theorem for $D_n(s,q,x,y)$.
Theorem 8. $D_0(s,q,x,y) = D_{0,0}(q,x,y) = 1$. *For any $n, k \geq 1$,*

$$D_{n,1}(q,x,y) = q^{n-1}, \; D_{n,n}(q,x,y) = 1, \; D_{n,k}(q,x,y) = 0 \textit{ for } k > n, \textit{ and} \tag{70}$$

$$
\begin{aligned}
D_{n,k}(q,x,y) &= x^{n-k}D_{n-1,k-1}(q,x,y) + q^k D_{n-1,k}(q,xy,y)\\
&\quad + \sum_{i=2}^{n-1} \sum_{j=\max(1,k+i-n)}^{\min(i-1,k-1)} q^j x^{j(n-i-k+j)} y^{j(n-i)} D_{i-1,j}(qy^{n-i},xy,y) D_{n-i,k-j}(q,x,y). \quad (71)
\end{aligned}
$$

Proof. Given $\sigma \in \mathcal{S}_n(123)$ such that $\mathrm{LRmin}(\sigma) = k$, we let $P = \Psi(\sigma)$ be the corresponding Dyck path which has k peaks by Lemma 3. Suppose that $\mathrm{ret}(\sigma) = i$, then like Theorem 6, we can write $P = DP_1RP_2$, where P_1 is a Dyck path of size $i-1$ and P_2 is a Dyck path of size $n-i$.

If $i = 1$, then P_1 is empty, and DP_1R is a peak on the main diagonal, thus P_2 should be a Dyck path of size $n-1$ with $k-1$ peaks. There are $n-k$ extra linvs between the first peak and the $n-k$ non-peaks in P_2, thus the contribution of this case is $x^{n-k}D_{n-1,k-1}(q,x,y)$.

If $i = k$, then P_2 is empty, and $P = DP_1R$. P_1 should be a Dyck path of size $n-1$ with k peaks. There are k more inversions of $\Psi^{-1}(DP_1R)$ than $\Psi^{-1}(P_1)$, and $\mathrm{linv}(P_1)$ more occurrences of pattern 231 in $\Psi^{-1}(DP_1R)$ than $\Psi^{-1}(P_1)$, thus the contribution of this case is $q^k D_{n-1,k}(q,xy,y)$.

If $1 < i < n$, then both P_1 and P_2 are not empty. Suppose that there are j peaks in P_1, then there are $k-j$ peaks in P_2. Other than the statistics counted inside P_1 and P_2, there are j more inversions of $\Psi^{-1}(DP_1R)$ than $\Psi^{-1}(P_1)$, $j(n-i-k+j)$ extra linvs between P_1 and P_2, and $j(n-i) + (n-i)\mathrm{occr}_{12}(\Psi^{-1}(P_1)) + \mathrm{linv}(P_1)$ extra occurrences of pattern 231, thus the contribution of this case is $q^j x^{j(n-i-k+j)} y^{j(n-i)} D_{i-1,j}(qy^{n-i},xy,y) D_{n-i,k-j}(q,x,y)$.

Summing over all the cases gives (71). ☐

Then we can compute $Q_{123}^{12,231}(t,q,x) = \sum_{n\geq 0} t^n D_n(1,q,1,x)$ using the recursive formula:

$$
\begin{aligned}
Q_{123}^{12,231}(t,q,x) &= 1+t+(1+q)t^2+t^3(1+q+2q^2+qx)+t^4(1+q+2q^2+2q^3+q^4+qx+2q^3x+qx^2+3q^2x^2)\\
&\quad + t^5(1+q+2q^2+2q^3+3q^4+2q^6+qx+2q^3x+2q^5x+qx^2+3q^2x^2+5q^4x^2+2q^5x^2\\
&\quad + qx^3+q^2x^3+4q^3x^3+q^4x^3+3q^2x^4+4q^3x^4+q^4x^4)+\cdots. \quad (72)
\end{aligned}
$$

5 Applications in pattern popularity

Let S be a set of permutations and γ be a permutation pattern. The *popularity* of γ in S, $f_S(\gamma)$, is defined by

$$f_S(\gamma) = \sum_{\sigma\in S} \mathrm{occr}(\gamma). \quad (73)$$

Let

$$
\begin{aligned}
F_\gamma(t) &= \sum_{n\geq 0} f_{\mathcal{S}_n(132)}(\gamma)t^n \quad \text{and} \quad (74)\\
G_\gamma(t) &= \sum_{n\geq 0} f_{\mathcal{S}_n(123)}(\gamma)t^n, \quad (75)
\end{aligned}
$$

Bóna [1] and Homberger [4] studied the popularity of length 2 or 3 patterns in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$.

Theorem 9 (Bóna and Homberger)**.** *Let* $C(t)=\sum_{n\geq 0} C_n t^n$ *be the generating function of Catalan numbers. Then*

$$\begin{aligned} F_{12}(t) &= \frac{t^2C^3(t)}{(1-2tC(t))^2}, && (76)\\ G_{12}(t) &= \frac{tC^2(t)}{1-2tC(t)}. && (77) \end{aligned}$$

In this section, we shall give two applications of the results in Section 3 and Section 4 about pattern popularity in the following theorem.

Theorem 10. *Let* $m>2$ *be an integer. Then*

$$\begin{aligned} F_{12\cdots m}(t) &= \frac{tC(t)F_{12\cdots(m-1)}(t)}{1-2tC(t)}, && (78)\\ G_{1m\cdots 2}(t) &= \frac{tC(t)G_{1(m-1)\cdots 2}(t)}{1-2tC(t)}. && (79) \end{aligned}$$

Proof. Equation (78) is a consequence of equation (39). It follows from the definition of $Q_{132}^{(m)}(t,x_2,\ldots,x_m)$ that

$$F_{12\cdots m}(t)=\left.\frac{\partial Q_{132}^{(m)}(t,x_2,\ldots,x_m)}{\partial x_m}\right|_{x_2=\cdots=x_m=1}. \tag{80}$$

Taking partial derivative of equation (39) over x_m and evaluating $x_2,\ldots,x_m$ at 1 give

$$\begin{aligned} F_{12\cdots m}(t) &= \left.\frac{\partial Q_{132}^{(m)}(t,x_2,\ldots,x_m)}{\partial x_m}\right|_{x_2=\cdots=x_m=1}\\ &= t\left(\frac{\partial Q_{132}^{(m)}(t,x_2,\ldots,x_m)}{\partial x_m}Q_{132}^{(m)}(t,x_2,\ldots,x_m)\right.\\ &\quad +\frac{\partial Q_{132}^{(m)}(t,x_2,\ldots,x_m)}{\partial x_{m-1}}Q_{132}^{(m)}(t,x_2,\ldots,x_m)\\ &\quad \left.\left.+\frac{\partial Q_{132}^{(m)}(t,x_2,\ldots,x_m)}{\partial x_m}Q_{132}^{(m)}(t,x_2,\ldots,x_m)\right)\right|_{x_2=\cdots=x_m=1}\\ &= t\left(C(t)F_{12\cdots(m-1)}(t)+2C(t)F_{12\cdots m}(t)\right), \end{aligned} \tag{81}$$

which implies equation (78).

Equation (79) is a consequence of equation (62) and can be proved in a similar way. We shall omit the proof of equation (79). ☐

6 Circular pattern distribution in $\mathcal{CS}_n(1243)$ and $\mathcal{CS}_n(1324)$

We first define circular permutations. A *circular permutation* is a permutation with only one cycle. We use the cycle notation for circular permutations in this section. Let $\mathcal{CS}_n$ denote the set of circular permutations of size n, then for any $\sigma = (\sigma_1 \cdots \sigma_n) \in \mathcal{CS}_n$, σ can also be expressed as $(\sigma_i \cdots \sigma_n \sigma_1 \cdots \sigma_{i-1})$ for any $i = 1, \ldots, n$.

Next, we define pattern avoidance of circular permutation. For any permutation $\tau \in \mathcal{S}_j$, we say that a circular permutation $\sigma = (\sigma_1 \cdots \sigma_n) \in \mathcal{CS}_n$ *circularly avoids* the pattern τ if each expression $\sigma_i \cdots \sigma_n \sigma_1 \cdots \sigma_{i-1}$ (for $i = 1, \ldots, n$) avoids the pattern τ. We let $\mathcal{CS}_n(\tau)$ denote the set of circular permutations that circularly avoid the pattern τ. Let $\mathrm{coccr}_\tau(\sigma)$ denote the number of circular occurrences of pattern τ in circular permutation σ.

$\mathcal{CS}_n(\tau)$ is the empty set when τ has length ≤ 2 and $n \geq 2$. $\mathcal{CS}_n(\tau)$ is also trivial when τ is of length 3. We shall study circular pattern distribution in $\mathcal{CS}_n(\tau)$ where τ is of length 4.

There are 6 circular permutations of length 4. By the reverse (or complement) action, we have $|\mathcal{CS}_n(1234)| = |\mathcal{CS}_n(1432)|$, $|\mathcal{CS}_n(1243)| = |\mathcal{CS}_n(1342)|$ and $|\mathcal{CS}_n(1324)| = |\mathcal{CS}_n(1423)|$. Thus by symmetry, we only need to study circular pattern distribution in $\mathcal{CS}_n(1234)$, $\mathcal{CS}_n(1243)$ and $\mathcal{CS}_n(1324)$. We will extend our result on classical permutation pattern distribution in $\mathcal{S}_n(132)$ to circular pattern distribution in $\mathcal{CS}_n(1243)$ and $\mathcal{CS}_n(1324)$.

6.1 Circular pattern distribution in $\mathcal{CS}_n(1243)$

Given $\sigma = (\sigma_1 \cdots \sigma_n) \in \mathcal{CS}_n$, without loss of generality, we let $\sigma_1 = 1$. Let $\widetilde{\sigma} = \mathrm{red}(\sigma_2 \cdots \sigma_n)$, then $\sigma \in \mathcal{CS}_n(1243)$ if and only if $\widetilde{\sigma} \in \mathcal{S}_{n-1}(132, 3124, 4312)$, and we can count the number of circular pattern occurrences of σ from the permutation $\widetilde{\sigma}$.

Next, we consider circular pattern distribution in circular permutations. When the circular pattern is of length 2, we have $\mathrm{coccr}_{12}(\sigma) = \mathrm{coccr}_{21}(\sigma) = \binom{n}{2}$ for all $\sigma \in \mathcal{CS}_n$, which is trivial. We study all the 2 circular patterns of length 3 and all the 5 nontrivial circular patterns of length 4 in $\mathcal{CS}_n(1243)$. For $\sigma \in \mathcal{CS}_n(1243)$ and $\widetilde{\sigma}$ defined as before, we have

$$\begin{aligned}
\mathrm{coccr}_{123}(\sigma) &= \mathrm{occr}_{12}(\widetilde{\sigma}) + \mathrm{occr}_{123}(\widetilde{\sigma}) + \mathrm{occr}_{312}(\widetilde{\sigma}) + \mathrm{occr}_{231}(\widetilde{\sigma}), && (82)\\
\mathrm{coccr}_{132}(\sigma) &= \mathrm{occr}_{21}(\widetilde{\sigma}) + \mathrm{occr}_{213}(\widetilde{\sigma}) + \mathrm{occr}_{321}(\widetilde{\sigma}), && (83)\\
\mathrm{coccr}_{1234}(\sigma) &= \mathrm{occr}_{123}(\widetilde{\sigma}) + \mathrm{occr}_{1234}(\widetilde{\sigma}) + \mathrm{occr}_{4123}(\widetilde{\sigma}) + \mathrm{occr}_{3412}(\widetilde{\sigma}) + \mathrm{occr}_{2341}(\widetilde{\sigma}), && (84)\\
\mathrm{coccr}_{1324}(\sigma) &= \mathrm{occr}_{213}(\widetilde{\sigma}) + \mathrm{occr}_{3241}(\widetilde{\sigma}), && (85)\\
\mathrm{coccr}_{1342}(\sigma) &= \mathrm{occr}_{231}(\widetilde{\sigma}) + \mathrm{occr}_{2134}(\widetilde{\sigma}) + \mathrm{occr}_{4213}(\widetilde{\sigma}) + \mathrm{occr}_{3421}(\widetilde{\sigma}), && (86)\\
\mathrm{coccr}_{1423}(\sigma) &= \mathrm{occr}_{312}(\widetilde{\sigma}) + \mathrm{occr}_{2314}(\widetilde{\sigma}) + \mathrm{occr}_{4231}(\widetilde{\sigma}), && (87)\\
\mathrm{coccr}_{1432}(\sigma) &= \mathrm{occr}_{321}(\widetilde{\sigma}) + \mathrm{occr}_{3214}(\widetilde{\sigma}) + \mathrm{occr}_{4321}(\widetilde{\sigma}). && (88)
\end{aligned}$$

Let

$$\begin{aligned}
&P_{n,1243}(y_{123}, y_{132}, y_{1234}, y_{1324}, y_{1342}, y_{1423}, y_{1432}) := \\
&\quad : \sum_{\sigma \in \mathcal{CS}_n(1243)} y_{123}^{\mathrm{coccr}_{123}(\sigma)} y_{132}^{\mathrm{coccr}_{132}(\sigma)} y_{1234}^{\mathrm{coccr}_{1234}(\sigma)} y_{1324}^{\mathrm{coccr}_{1324}(\sigma)} y_{1342}^{\mathrm{coccr}_{1342}(\sigma)} y_{1423}^{\mathrm{coccr}_{1423}(\sigma)} y_{1432}^{\mathrm{coccr}_{1432}(\sigma)} \qquad (89)
\end{aligned}$$

and recall the function $Q_{n,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(x_1, \ldots, x_{21})$ defined in (43), then Theorem 11 follows.

Theorem 11. *Given any $n \geq 1$,*

$$\begin{aligned}
&P_{n,1243}(y_{132}, y_{1234}, y_{1324}, y_{1342}, y_{1423}, y_{1432}) \\
&\quad = Q_{n-1,132}^{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4}(y_{123}, y_{132}, y_{123}y_{1234}, y_{132}y_{1324}, y_{123}y_{1342}, y_{123}y_{1423}, y_{132}y_{1432}, \\
&\qquad y_{1234}, y_{1342}, y_{1423}, y_{1234}, 0, y_{1432}, y_{1324}, y_{1234}, y_{1342}, y_{1234}, y_{1342}, y_{1423}, 0, y_{1432}). \qquad (90)
\end{aligned}$$

Thus, one can compute $P_{n,1243}(y_{132}, y_{1234}, y_{1324}, y_{1342}, y_{1423}, y_{1432})$ directly from $Q_{n,132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(x_1, \ldots, x_{21})$.

6.2 Circular pattern distribution in $\mathcal{CS}_n(1324)$

Given $\sigma = (\sigma_1 \cdots \sigma_n) \in \mathcal{CS}_n$, without loss of generality, we let $\sigma_n = n$. Let $\overline{\sigma} = \sigma_1 \cdots \sigma_{n-1}$, then $\sigma \in \mathcal{CS}_n(1324)$ if and only if $\overline{\sigma} \in \mathcal{S}_{n-1}(132, 3241)$, and we can count the number of circular pattern occurrences of σ from the permutation $\overline{\sigma}$.

Similar to $\mathcal{CS}_n(1243)$, we study all the 2 circular patterns of length 3 and all the 5 nontrivial circular patterns of length 4 in $\mathcal{CS}_n(1324)$. For $\sigma \in \mathcal{CS}_n(1324)$ and $\overline{\sigma}$ defined as before, we have

$$\begin{aligned}
\mathrm{coccr}_{123}(\sigma) &= \mathrm{occr}_{12}(\widetilde{\sigma}) + \mathrm{occr}_{123}(\widetilde{\sigma}) + \mathrm{occr}_{312}(\widetilde{\sigma}) + \mathrm{occr}_{231}(\widetilde{\sigma}), && (91)\\
\mathrm{coccr}_{132}(\sigma) &= \mathrm{occr}_{21}(\widetilde{\sigma}) + \mathrm{occr}_{213}(\widetilde{\sigma}) + \mathrm{occr}_{321}(\widetilde{\sigma}), && (92)\\
\mathrm{coccr}_{1234}(\sigma) &= \mathrm{occr}_{123}(\widetilde{\sigma}) + \mathrm{occr}_{1234}(\widetilde{\sigma}) + \mathrm{occr}_{4123}(\widetilde{\sigma}) + \mathrm{occr}_{3412}(\widetilde{\sigma}) + \mathrm{occr}_{2341}(\widetilde{\sigma}), && (93)\\
\mathrm{coccr}_{1243}(\sigma) &= \mathrm{occr}_{312}(\widetilde{\sigma}) + \mathrm{occr}_{1243}(\widetilde{\sigma}) + \mathrm{occr}_{4312}(\widetilde{\sigma}) + \mathrm{occr}_{2431}(\widetilde{\sigma}), && (94)\\
\mathrm{coccr}_{1342}(\sigma) &= \mathrm{occr}_{213}(\widetilde{\sigma}) + \mathrm{occr}_{1342}(\widetilde{\sigma}) + \mathrm{occr}_{3421}(\widetilde{\sigma}), && (95)\\
\mathrm{coccr}_{1423}(\sigma) &= \mathrm{occr}_{231}(\widetilde{\sigma}) + \mathrm{occr}_{1423}(\widetilde{\sigma}) + \mathrm{occr}_{4231}(\widetilde{\sigma}), && (96)\\
\mathrm{coccr}_{1432}(\sigma) &= \mathrm{occr}_{321}(\widetilde{\sigma}) + \mathrm{occr}_{1432}(\widetilde{\sigma}) + \mathrm{occr}_{4321}(\widetilde{\sigma}). && (97)
\end{aligned}$$

Let

$$\begin{aligned}
&P_{n,1324}(y_{123}, y_{132}, y_{1234}, y_{1243}, y_{1342}, y_{1423}, y_{1432}) := \\
&\qquad \sum_{\sigma \in \mathcal{CS}_n(1243)} y_{123}^{\mathrm{coccr}_{123}(\sigma)} y_{132}^{\mathrm{coccr}_{132}(\sigma)} y_{1234}^{\mathrm{coccr}_{1234}(\sigma)} y_{1243}^{\mathrm{coccr}_{1243}(\sigma)} y_{1342}^{\mathrm{coccr}_{1342}(\sigma)} y_{1423}^{\mathrm{coccr}_{1423}(\sigma)} y_{1432}^{\mathrm{coccr}_{1432}(\sigma)}, \qquad (98)
\end{aligned}$$

then Theorem 12 follows.
Theorem 12. *Given any $n \geq 1$,*

$$\begin{aligned}
&P_{n,1324}(y_{132}, y_{1234}, y_{1243}, y_{1342}, y_{1423}, y_{1432}) \\
&\quad = Q_{n-1,132}^{\Gamma_2\cup\Gamma_3\cup\Gamma_4}(y_{123}, y_{132}, y_{123}y_{1234}, y_{132}y_{1342}, y_{123}y_{1423}, y_{123}y_{1243}, y_{132}y_{1432}, y_{1234}, \\
&\qquad y_{1342}, y_{1423}, y_{1234}, y_{1243}, y_{1432}, 0, y_{1234}, y_{1342}, y_{1234}, y_{1342}, y_{1423}, y_{1243}, y_{1432}). \qquad (99)
\end{aligned}$$

7 Summary and future work

We obtained the recursion tracking all patterns of length 2, 3 or 4 in $\mathcal{S}_n(132)$. In fact, it is possible to give a recursion for the generating function tracking patterns of any length in $\mathcal{S}_n(132)$ if we do enough refinement of the functions Q_n.

On $\mathcal{S}_n(123)$, we only track 2 patterns of length 2, 2 of 3 patterns of length 3 and the special pattern $1m(m-1)\cdots 2$. The recursions in $\mathcal{S}_n(123)$ tend to be more complicated than those in $\mathcal{S}_n(132)$, and we are not able to get a recursion for the pattern 321.

We also adapt our method to circular permutations. We define patter avoidance in circular permutations, and we are able to track all circular patterns of length 3 or 4 in $\mathcal{CS}_{1243}$ and $\mathcal{CS}_{1324}$. The pattern distribution problems in the circular permutation class $\mathcal{CS}_{1234}$ remain to be solved.

We notice other equalities of coefficients of generating functions Q^{γ}_{132} and Q^{γ}_{123} except equation (67). For example, the number of permutations in $\mathcal{S}_n(123)$ with one occurrence of pattern 231 is equal to the number of permutations in $\mathcal{S}_n(231)$ with one occurrence of pattern 123, which is equal to $2n-5$; the number of permutations in $\mathcal{S}_n(132)$ with one occurrence of pattern 3412 is equal to the number of permutations in $\mathcal{S}_n(132)$ with one occurrence of pattern 2341, which is one less than the $2n-5^{\text{th}}$ Fibonacci number.

We have not studied sets of permutations avoiding patterns of length bigger than 3, and circular patterns of length bigger than 4. We shall study the problems in the future.

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References

[1] M. Bóna, Surprising symmetries in objects counted by Catalan numbers, *The Electronic Journal of Combinatorics*, **19, no.1** (2012), 62.

[2] E. Deutsch and S. Elizalde, A simple and unusual bijection for Dyck paths and its consequences, *Annals of Combinatorics*, **7, no.3** (2003), 281–297.

[3] J. Fürlinger and J. Hofbauer, q-Catalan numbers, *Journal of Combinatorial Theory, Series A*, **40, no.2** (1985), 248–264.

[4] C. Homberger, Expected patterns in permutation classes, arXiv preprint, arXiv:1206.0320 (2012).

[5] S. Janson, Patterns in random permutations avoiding the pattern 132, *Combinatorics, Probability and Computing*, **26, no.1** (2017), 24–51.

[6] S. Janson, Patterns in random permutations avoiding the pattern 321, arXiv:1709.08427 (2017).

[7] S. Kitaev, Patterns in permutations and words, *Springer Science & Business Media* (2011).

[8] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Advances in Applied Mathematics*, **27, no.2–3** (2001), 510–530.

[9] T. Mansour, Permutations avoiding a pattern from S_k and at least two patterns from S_3, *Ars Combinatorica*, **62** (2001), 227–239.

[10] T. Mansour, 321-avoiding permutations and Chebyshev polynomials, *Third Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities*, Vienna, Austria, September 13–17, 2004.

[11] T. Mansour and A. Vainshtein, Restricted permutations, continued fractions, and Chebyshev polynomials, *The Electronic Journal of Combinatorics*, **7** (2000), #R17.

[12] T. Mansour and A. Vainshtein, Restricted 132-avoiding permutations, *Advances in Applied Mathematics*, **26** (2001), 258–269.

[13] T. Mansour and A. Vainshtein, Restricted permutations and Chebyshev polynomials, *Seminaire Lotharingien de Combinatoire*, **47** (2002), #B47c.

[14] The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org (2005).

[15] R. Pan, D. Qiu and J. Remmel, Counting consecutive pattern matches in $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$, arXiv preprint, arXiv:1809.01384 (2018).

[16] D. Qiu and J. Remmel, Quadrant marked mesh patterns in 123-avoiding permutations, *Discrete Mathematics & Theoretical Computer Science*, **19, no.2** (2018), #12.

[17] D. Qiu and J. Remmel, Patterns in words of ordered set partitions, arXiv preprint, arXiv:1804.07087 (2018).

[18] R. P. Stanley, Catalan Addendum, http://www-math.mit.edu/r̃stan/ec/catadd.pdf (2013).