# Surreal Birthdays and Their Arithmetic 

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## 1. INTRODUCTION

I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games IS math.

John Horton Conway
Surreal numbers, invented by John Horton Conway [1], and named by Knuth in "Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness" [2], are appealing because they provide a single construction for all of the numbers we are familiar with and many others - the reals, rationals, hyperreals and ordinals - with only the use of set theory. Thus they provide an underpinning idea of what a number really is. And they have an elegant and complex structure that is worthy of study in its own right.

Surreal numbers aren't numbers as we are taught in grade school, but they have many of the same properties. The tricky thing is that they are defined recursively from the very start.

Recursion is like the joke: "an American, an Englishman and an Australian walk into a bar, and one of them says
"an American, an Englishman and an Australian walk into a bar, and one of them says,
"an American, an Englishman and an Australian walk into a bar, and one of them says, ..."
A recursive definition means that a surreal number is defined in terms of other surreals and so on. The break in this circularity that makes it possible to get a foothold is that each turn of the circle progresses inexorably towards 0 , which can be defined ab initio. To continue the Latin the surreal numbers violate the dictum ex nihlo nihil fit, or "from nothing comes nothing." From 0 springs forth all other numbers.

The definition that performs this miracle is as follows: a surreal number $x$ consists of an ordered pair of two sets of surreal numbers (call them the left and right sets, $X_{L}$ and $X_{R}$, respectively) such that no member of the left set is $\geq$ any of the members of the right set. We write $x=\left\{X_{L} \mid X_{R}\right\}$ for such a surreal.

This seems a difficult definition. We haven't even defined $\geq$, and yet are using it in the definition. Everything resolves because we can always work with empty sets. The starting point - the first surreal number - is $\{\emptyset \mid \emptyset\}$ (where $\emptyset$ is the empty set). A careful reading of the definition says that no elements of one can be $\geq$ the other, but as there are no elements, the comparison is automatically true. We call this number $\overline{0}$.

Then on the "first day" a new generation of surreals can be created, building on $\overline{0}$. On the second day we create a second generation and so on. Each has a meaning corresponding to traditional numbers in order to have a consistent interpretation with

[^0]respect to standard mathematical operators such as addition. The construction is elegant, and surprisingly general, and leads naturally to the idea of the "birthday" of a surreal numbers being literally the day on which it is born.

A natural question then is, when we perform arithmetic on surreal numbers, what is the birthday of the result? This paper answers that question.
2. THE SURREAL NUMBERS AND THEIR BIRTHDAYS We won't describe the surreals in detail here; there are several good tutorials or books, e.g., [1-6]. In particular, Tøndering [3] and Simons [6] provide excellent introductions. We need provide a little background though. For instance, notation varies a little: here we denote numbers in lower case, and sets in upper case, with the convention that $X_{L}$ and $X_{R}$ are the left and right sets of $x$, and we write a form as $x=\left\{X_{L} \mid X_{R}\right\}$. I'd like to make a surrealist/computer-science joke here, namely ' $\mid$ ' is not a pipe, but a conjunction between Magritte and Unix might be considered too obscure even for a paper on surreal numbers. More prosaically, it is common to omit empty sets, but I prefer writing $\emptyset$ explicitly because it is a little clearer when writing complicated sequences.

In much of the literature the idea of a surreal number is interwoven with its form. This is best explained by an analogy to rational numbers. We can write a rational number in many ways, e.g., $1 / 2=2 / 4$. That is, we have many forms of the same number. Likewise, a surreal number can have many forms. Here, we work with the form because it is in terms of these that Conway's surreal arithmetic operators are defined. The distinction requires a clarification of the notion of equality. Following Keddie [7] we call two forms identical if they are the same form (i.e., have identical left and right sets), and equal if they have the same value (i.e., they denote the same number). We shall distinguish these two cases by writing equality of value as equivalence, $\equiv$, and identity by $==$. A single equal sign will be reserved for conventional numbers.

The first surreal number form to be defined is $\overline{0} \stackrel{\text { def }}{=}\{\emptyset \mid \emptyset\}$. We call this number $\overline{0}$, because it will turn out to be the additive identity (the 0 of conventional arithmetic). All other numbers are defined from this point, following a construction to be laid out below. The line over the 0 denotes that this is a special, canonical form of zero.

The second two surreal number forms, the numbers we can define on Day 1 , immediately after creating $\overline{0}$, are

$$
\overline{1} \stackrel{\text { def }}{=}\{\overline{0} \mid \emptyset\} \text { and } \overline{-1} \stackrel{\text { def }}{=}\{\emptyset \mid \overline{0}\} \text {. }
$$

This notation, however, hides some of the structure of the surreals. To see them in all their glory we should write

$$
\overline{1} \stackrel{\text { def }}{=}\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \text { and } \overline{-1} \stackrel{\text { def }}{=}\{\emptyset \mid\{\emptyset \mid \emptyset\}\},
$$

but no doubt you can see that this will quickly result in a very complicated expressions. We will resolve this by drawing pictures such as in Figure 1(a). The figure shows each surreal as a node in a graph. It is (almost) a connected Directed Acyclic Graph (DAG) with links showing how each surreal is constructed from its parents. But a DAG, by itself, would loose information. The graph would only specify parents, not left and right parents. So in displaying the DAG, we show a box for each surreal number, with the value given in the top section, and the left and right sets shown in the bottom left and right sections, respectively. From each member of each set we show a link to its box, and its parents in turn: a red link indicates a left parent, and blue right. The advantage of the DAG is that it shows the whole recursive structure of a surreal.


Figure 1. DAGs depicting the forms of two surreal numbers. Each box represents a surreal number with the value given in the top section, and the left and right sets shown in the bottom left and right sections, respectively. The arrows show the form of each parent surreal, with its own recursive structure. Note that there are equivalent forms, e.g., $\{\emptyset \mid \emptyset\} \equiv\{\overline{-1} \mid 1\}$, even within a single DAG.

Most aspects of surreals are defined recursively. For instance $x \geq y$ (which we need even in the definition) means that no member of $X_{L}$ is greater than or equal to $y$, and no member of $Y_{R}$ is less than or equal to $x$. It might be hard to see how to use this in the definition when it is also defined in terms of surreals (which in turn use the definition), but this is the nature of surreal operations: they are recursive, not just in terms of themselves, but each definition in turn uses others at lower levels. In any case, it is now relatively easy to check that $\overline{-1} \leq \overline{0} \leq \overline{1}$, and we can define further comparisons, for instance $x \equiv y$ means $x \geq y$ and $y \geq x$.

Once we have defined $\overline{ \pm 1}$, we can proceed to define yet more surreals. Figure 1(b) shows another form equivalent to zero, i.e., $\{\overline{-1} \mid \overline{1}\} \equiv\{\emptyset \mid \emptyset\}==\overline{0}$. The graph shows that a value can reappear at multiple places in the structure of the form: in this case 0 appears both at the top and the bottom of the DAG. The information characterising the surreal form is not its value, or even the values of its subsets, but the structure of the whole DAG that describes it. So the two ' 0 ' nodes in the graph are different (non-identical) surreal forms that just happen to have the same value.

Each number is actually an infinite equivalence class of forms, so we need to have standard, canonical forms, at least to bootstrap later work. The standard construction (called the Dali function by Tondering [3]) maps dyadic numbers $\mathbb{D}:=\left\{n / 2^{k} \mid\right.$ $n, k$ integers $\}$ to (finite) surreals, and is defined recursively by $d: \mathbb{D} \rightarrow \mathbb{S}$ where

$$
d(x)= \begin{cases}\{\emptyset \mid \emptyset\}, & \text { if } x=0,  \tag{1}\\ \{d(n-1) \mid \emptyset\}, & \text { if } x=n, \text { a positive integer, } \\ \{\emptyset \mid d(n+1)\}, & \text { if } x=n, \text { a negative integer, } \\ \left\{\left.d\left(\frac{n-1}{2^{k}}\right) \right\rvert\, d\left(\frac{n+1}{2^{k}}\right)\right\}, & \text { if } x=n / 2^{k} \text { for } k>0 \text { and } n \text { odd. }\end{cases}
$$

For convenience, we denote canonical forms through the shorthand of placing a line above the number. All other forms are defined by their DAG, or in terms of canonicals.

This recursive construction is often illustrated as a tree $[4,8-10]$ showing the numbers that are created in each generation and their position on the real number line. However, that is misleading. The (non-integer) Dali surreals have two parents, and the resultant structure of dependency in the recursion is the DAG shown in Figure 2. We can see from this, for instance, that each canonical form has either one or two parents.

The construction of surreals leads to the notion of birthdays: take $\overline{0}$ to be born on


Figure 2: The dyadic DAG, i.e., the recursive structure of the dyadic surreal numbers, up to birthday/generation 3.

Day 0 , and $\overline{ \pm 1}$ to be born on Day 1 , and so on, then we can assign a birthday to all surreals. I prefer the term generation over birthday if only because it links up to the notion of parents and children more cleanly. The birthday can also be seen as how deeply you must recurse through the DAG structure to get to $\overline{0}$. We say a surreal is older if it comes from an earlier generation, i.e., it has a smaller (earlier) birthday. We formalise notions these below.

Definition 1. We refer to the elements of the left and right sets of a surreal form $x$ as its parents (note there may be other than two parents), and $x$ as their child. The birthday or generation of $\{\emptyset \mid \emptyset\}$ is 0 , and the generation of all other finite surreal forms is 1 greater than that of their youngest parent.

If we denote the parents of $x$ by $X_{P}=X_{L} \cup X_{R}$, then the generation/birthday function of a surreal number form $x$ is given by

$$
\begin{equation*}
g(x)=\sup _{x_{p} \in X_{P}} g\left(x_{p}\right)+1 \tag{2}
\end{equation*}
$$

where $g(\{\emptyset \mid \emptyset\})=0$. For example, in Figure $1(\mathrm{~b}) g(\{\overline{-1} \mid \overline{1}\})=g(\overline{1})+1=2$.
Surreal forms with equivalent values can come from different generations, so knowing the value of a surreal tells us only a lower bound on its generation (namely the generation of the canonical form of that surreal). Thus, some questions arise in regard to birthdays: e.g., can we derive birthdays for standard surreal constructs?

We call these problems in birthday arithmetic.
We will start by deriving the birthday of the canonical forms as it provides an example of the standard proof structure for many surreal arguments. Simons states the result [6, p.27] but only as a minor note within a larger result. As in many proofs in this domain, it is inductive. Throughout this we use $g(x)$ as shorthand for $g(d(x))$.
Lemma 1. The generation/birthday of the canonical form of dyadic $x=n / 2^{k}$, which is in irreducible form (or lowest terms) is

$$
g(x)=\lceil|x|\rceil+k
$$

where $\lceil x\rceil$ denotes the ceiling function of $x$ (the smallest integer larger than $x$ ).
Proof. The statement is true for $x=\overline{0}$ because $g(\overline{0})=0$ by definition. The negative case can be treated by considering $g(-x)$ (see $\S 3$ ), so we only consider $x>0$ here. The integer case is trivial (see Figure 2) so we focus on the case $n$ odd and $k>0$.

Assume for the purpose of induction that the lemma is true for all parents of $x$.
From Definition (1) such a dyadic has exactly two parents, and hence (2) reduces to

$$
g\left(\frac{n}{2^{k}}\right)=\max \left\{g\left(\frac{n-1}{2^{k}}\right), g\left(\frac{n+1}{2^{k}}\right)\right\}+1
$$

For $n$ odd and $k>0$ both $x=n / 2^{k}$ and $(n+1) / 2^{k}$ have the same ceiling (call it $m$ ), and $\left\lceil(n-1) / 2^{k}\right\rceil \leq m$, so by the inductive hypothesis

$$
g\left(\frac{n}{2^{k}}\right)=g\left(\frac{n+1}{2^{k}}\right)+1
$$

Now $n+1$ is even so we can simplify $(n+1) / 2^{k}=\ell / 2^{k-1}$, and by the inductive hypothesis the theorem is true for the parents of $x$, so we must have

$$
g\left(\frac{n+1}{2^{k}}\right)=m+k-1 .
$$

Hence $g\left(n / 2^{k}\right)=m+k$.
Intrinsic to this proof (and others) is the fact that $\overline{0}$ is the starting point for the construction of the surreals, and therefore is the ultimate ancestor of all surreals.
3. ADDITION AND SUBTRACTION In order for the surreals to fulfil their role as "numbers" they must be able to play all the tricks of numbers, for instance, we must be able to do arithmetic. Conway defined addition and subtraction for surreal numbers, and showed these satisfy the conditions required, but they are actually operations on the forms. Let us examine them in detail below.

Addition The standard definition of addition on surreal forms [1,3] is

$$
x+y \stackrel{\text { def }}{=}\left\{X_{L}+y \cup x+Y_{L} \mid X_{R}+y \cup x+Y_{R}\right\}
$$

Notation is often abbreviated, and so you will sometimes set operations simplified, $e . g .,\{x, y\} \cup A$ is written $\{x, y, A\}$. Also, in this and other definitions we implicitly extend the operators to sets, or combinations of sets and set with surreals, e.g., $x+y$ is comprised of terms like $X_{L}+y$. Operations on sets are applied to each member [3]:

$$
\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}+y \stackrel{\text { def }}{=}\left\{x_{1}+y, x_{2}+y, \ldots, x_{n}+y\right\}
$$

and operations on empty sets result in empty sets, i.e., $x+\emptyset=\emptyset$.
Many of the texts on surreals provide proofs that addition satisfies all of the usual requirements, e.g., associativity, commutativity and so on e.g., $[1,3]$. For instance, 0 is the additive identity [1,3]

$$
x+0 \equiv 0+x \equiv x
$$

but this is a statement about values not forms: $x+0$ is not (in general) identical to x , except for the canonical zero, i.e., $x+\overline{0}==x$. For instance, consider the following
addition of $\overline{1 / 2}+0$ noting carefully the bars indicating which terms are canonical.

$$
\begin{aligned}
\{\overline{0} \mid \overline{1}\}+\{\overline{-1} \mid \overline{1}\} & ==\{\overline{0}+0, \overline{1 / 2}+\overline{(-1)} \mid \overline{1}+\overline{0}, \overline{1 / 2}+\overline{1}\} \\
& =\{0,-1 / 2 \mid \overline{1}, 3 / 2\},
\end{aligned}
$$

which is not identical to $\{\overline{0} \mid \overline{1}\}$.
Another instructive example is $\overline{2}+\overline{2}$. In this case $X_{R}=Y_{R}=\emptyset$ and so the rightset of $\overline{2}+\overline{2}$ will also be $\emptyset$, thus

$$
\begin{aligned}
\overline{2}+\overline{2} & ==\{\overline{1}+\overline{2}, \overline{2}+\overline{1} \mid \emptyset\} \\
& ==\{\overline{3} \mid \emptyset\} \\
& ==\overline{4} .
\end{aligned}
$$

The result is the canonical form of 4 . Naively, we might expect that addition of canonical forms would always lead to the same. However this is not true. We can play with such hypotheses using the SurrealNumbers package ${ }^{1}$ in the programming language Julia. For instance, the above calculation can be performed using the commands

```
julia> using SurrealNumbers
julia> x = dali(2) // set x to the canonical form of 2
julia> y = x + x // calculate }\overline{2}+\overline{2
```

A more complicated example with a non-canonical result is

$$
\overline{1}+\overline{1 / 2}==\{\{\{\emptyset \mid \emptyset\} \mid\{\{\emptyset \mid \emptyset\} \mid \emptyset\}\},\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \mid\{\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \mid \emptyset\}\} .
$$

Figure 3 shows this, and two other forms with value $3 / 2$. The figure makes it easy to see that the latter two forms are non-canonical, and we see in (c) a different generation as well.

That brings us to the nub of the problem - can we calculate the generation/birthday of the sum purely from the generation of the inputs? Simons proves [6, p.25] that $g(x+y) \leq g(x)+g(y)$, but we present a small set of examples in Table 1, and in all of these (and every other case tested) we find $g(x+y)=g(x)+g(y)$. The question then is, can we prove that equality is always the case? The answer follows.

|  |  | $\overline{0}$ | $\overline{1 / 2}$ | $\overline{3 / 4}$ | $\overline{1}$ | $\overline{2}$ | $x$ |
| :---: | :---: | ---: | ---: | ---: | ---: | ---: | :--- |
| $y$ | $g(y)$ | 0 | 2 | 3 | 1 | 2 | $g(x)$ |
| $\overline{0}$ | 0 | 0 | 2 | 3 | 1 | 2 |  |
| $\overline{1 / 2}$ | 2 | 2 | 4 | 5 | 3 | 4 |  |
| $\overline{3 / 4}$ | 3 | 3 | 5 | 6 | 4 | 5 |  |
| $\overline{1}$ | 1 | 1 | 3 | 4 | 2 | 3 |  |
| $\overline{2}$ | 2 | 2 | 4 | 5 | 3 | 4 |  |

Table 1. The birthday table $g(x+y)$, i.e., the birthdays of the sums of the $x$ and $y$ values in the columns and rows. Note that they are additive, i.e., $g(x+y)=g(x)+g(y)$.

[^1]

Figure 3. Graphs depicting three forms for the surreal number 3/2. It is interesting that the form in (b) appears as a subgraph of the form in (c). An open question concerns whether this is true in general: is the canonical form always a subgraph of any alternative form of the same number?

Theorem 1 (Birthday addition theorem). For two surreal numbers $x$ and $y$

$$
g(x+y)=g(x)+g(y)
$$

where $g(\cdot)$ is the birthday/generation function.
Proof. Birthday addition is trivially true if $x$ or $y=\overline{0}$. Presume that the theorem is true for all combinations of the parents of summands $x$ and $y$ and the summands themselves (excepting $x+y$ itself). Apply Definition (2) to addition and we get

$$
g(x+y)=\sup _{x_{p}, y_{p}}\left\{g\left(x+y_{p}\right)+1, g\left(y+x_{p}\right)+1\right\}
$$

Now, by the inductive hypothesis, birthday addition is true for all parents and their combinations so the above reduces to

$$
\begin{aligned}
g(x+y) & =\sup _{x_{p}, y_{p}}\left\{g(x)+g\left(y_{p}\right)+1, g(y)+g\left(x_{p}\right)+1\right\} \\
& =\max \left\{g(x)+\sup _{y_{p}}\left\{g\left(y_{p}\right)\right\}+1, g(y)+\sup _{x_{p}}\left\{g\left(x_{p}\right)\right\}+1\right\} \\
& =\max \{g(x)+g(y), g(x)+g(y)\} \\
& =g(x)+g(y)
\end{aligned}
$$

Thus by induction the result.

Negation and subtraction Negation and subtraction are defined together as

$$
-x \stackrel{\text { def }}{=}\left\{-X_{R} \mid-X_{L}\right\} \text { and } x-y \stackrel{\text { def }}{=} x+(-y)
$$

Subtraction looks as simple as addition, but can complicate matters more than one might think with respect to surreal forms because the number of empty sets in the results change, and we end up with more complicated expressions. For instance $\overline{1}-\overline{1}$ is equivalent to 0 (this is the form that is illustrated in Figure 1(b)), but is not identical to $\overline{0}$. Thus $-x$ is the additive inverse of $x$ in terms of equivalence, but not identity.

However, here we are interested in the birthday/generation of the output. We can construct a very simple inductive proof that $g(-x)=g(x)$, using the definition above. Incidentally, this concludes the proof of Lemma 1.

From this, the definition of subtraction and Theorem 1 we immediately get the following corollary.

Corollary 1 (Birthday subtraction corollary). For two surreal numbers $x$ and $y$

$$
g(x-y)=g(x)+g(y)
$$

where $g(\cdot)$ is the birthday/generation function.
4. MULTIPLICATION If you thought addition and subtraction were complicated then fasten your seat belts. Multiplication is defined by

$$
\begin{align*}
x y \stackrel{\text { def }}{=}\left\{\left\{X_{L} y+x Y_{L}-X_{L} Y_{L}\right\} \cup\left\{X_{R} y+x Y_{R}-X_{R} Y_{R}\right\}\right. \\
\left.\left\{X_{L} y+x Y_{R}-X_{L} Y_{R}\right\} \cup\left\{X_{R} y+x Y_{L}-X_{R} Y_{L}\right\}\right\} \tag{3}
\end{align*}
$$

We need to be clear about exactly what each term in this definition means because it does not follow the typical convention for expressions of this form. Consider the term $\left\{X_{L} y+x Y_{L}-X_{L} Y_{L}\right\}$; this means create a set by taking all pairs $x_{i} \in X_{L}$ and $y_{j} \in Y_{L}$ and combining as follows:

$$
\left\{X_{L} y+x Y_{L}-X_{L} Y_{L}\right\} \stackrel{\text { def }}{=}\left\{x_{i} y+x y_{j}-x_{i} y_{j} \mid \forall x_{i} \in X_{L} \text { and } y_{j} \in Y_{L}\right\}
$$

where each of the products in the set above is another surreal multiplication and the additions are surreal additions.

Another quick example is informative: let's work quickly through $\overline{2} \times \overline{2}$. Once again $X_{R}=Y_{R}=\emptyset$, and so only one term of the four in the multiplication is non-empty:

$$
\begin{aligned}
\overline{2} \times \overline{2} & ==\{\overline{1} \times \overline{2}+\overline{2} \times \overline{1}-\overline{1} \times \overline{1} \mid \emptyset\} \\
& ==\{\overline{2}+\overline{2}-\overline{1} \mid \emptyset\} \\
& ==\{\overline{4}-\overline{1} \mid \emptyset\} \\
& \equiv\{3 \mid \emptyset\},
\end{aligned}
$$

where we exploit the fact that $\overline{1}$ is the multiplicative identity for surreal forms, and $\overline{0}$ is the multiplicative annihilator, i.e., $\overline{0} x=\overline{0}$ for all $x$, a fact seen by considering that

$$
\overline{0} x==\left\{\emptyset y+\overline{0} Y_{L}-\emptyset Y_{L}, \emptyset y+\overline{0} Y_{R}-\emptyset Y_{R} \mid \emptyset y+\overline{0} Y_{L}-\emptyset Y_{L}, \emptyset y+\overline{0} Y_{R}-\emptyset Y_{R}\right\},
$$



Figure 4. Graph depicting the form of $\overline{2} \times \overline{3}$. Inside this we frequently see subforms that might superficially appear identical, e.g., $4=\{3 \mid 5\}$ appears three times, but these are not identical forms, which is implicit in their dependent DAG. Interestingly, the DAG also includes even some negative values, which might be unexpected in a product of positive integers.
and noting that operations with $\emptyset$ result in $\emptyset$. The proof of the identity of $\overline{1}$ is very similar though it requires an inductive step.

We also use a minor result we have not bothered to prove that addition of canonical forms of non-negative integers results in canonical forms, however, subtraction does not. The result of $\overline{2} \times \overline{2}$ looks like the canonical form for 4 , but we are taking liberties by reducing ' 3 ' to its short hand. This isn't the canonical 3 , and so the shorthand is here misleading. In fact if we write this out in full we get (as in [11])

$$
\begin{aligned}
\overline{2} \times \overline{2}== & \{\{\{\{\{\{\emptyset \mid\{\emptyset \mid \emptyset\}\} \mid\{\{\emptyset \mid \emptyset\} \mid \emptyset\}\} \mid\{\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \mid \emptyset\}\} \mid \\
& \{\{\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \mid \emptyset\} \mid \emptyset\}\} \mid\{\{\{\{\{\emptyset \mid \emptyset\} \mid \emptyset\} \mid \emptyset\} \mid \emptyset\} \mid \emptyset\}\} \mid \emptyset\} .
\end{aligned}
$$

Gonshor proves a Weak Birthday Multiplication Theorem [12, Theorem 6.2], namely that $g(x y) \leq 3^{g(x)+g(y)}$. Simons [6] conjectures a much tighter bound that $g(x y) \leq$ $g(x) g(y)$, and states that there are no obvious counter-examples. It was this hypothesis that largely motivated this investigation.

The problem with Simons' statement is that there are very few examples at all. Multiplication is very complex to do in practice. As far as I am aware only a few multiplications are explicitly tabulated! Several places show that simple multiplicative identities hold, but few go any further. With the help of the SurrealNumbers package we can calculate other products (they quickly become too complicated to do with pen and paper). The result of $\overline{2} \times \overline{3}$ is shown in DAG form in Figure 4. From this we can calculate $g(\overline{2} \times \overline{3})=12$. Unfortunately this breaks Simons' conjecture, which suggests the bound should be 6. It also suggests that Gonshor's bound is very weak.

Table 2 shows a table for the birthday/generation of products. The black numbers are those that we have calculated (in multiple ways) using the Julia package. Note that the only surreal forms with birthdays 0 and 1 are $\overline{0}$ and $\pm 1$ so the first two rows and columns of this table are trivially true.

Patterns appear in the results, e.g., $g(\overline{2} \times \bar{m})=m(m+1)$, but the general pattern is not so trivial. For instance, the product $g(\overline{3} \times \overline{3})=31$ is prime, and so the resulting birthdays are not even the product of a function the underlying birthdays!

The results below explain the pattern observed in the black values of the table, and can be extrapolated to provide the gray values.

|  |  | $m=g(y)$ |  |  |  |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | :--- |
|  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |  |
|  | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $=m$ |
| $n=g(x)$ | 2 | 0 | 2 | 6 | 12 | 20 | 30 | 42 | $=m(m+1)$ |
|  | 3 | 0 | 3 | 12 | 31 | 64 | 115 | 188 |  |
|  | 4 | 0 | 4 | 20 | 64 | 160 | 340 | 644 |  |
|  | 5 | 0 | 5 | 30 | 115 | 340 | 841 | 1826 |  |
|  | 6 | 0 | 6 | 42 | 188 | 644 | 1826 | 4494 | $f(n, m)$ |

Table 2. The values of $g(x y)$ in relation to $n=g(x)$ and $m=g(y)$. Roman values have been verified through multiple instances of the products of (not just canonical) forms from the same generation. Italic are extrapolated using Theorem 2. The right-hand column expresses the pattern, where known.

Lemma 2. If the generation of $x y$ is given by a function $f$ of the generations of $x$ and $y$, i.e., $g(x y)=f(g(x), g(y))$, then the function will be symmetric, i.e., $f(n, m)=$ $f(m, n)$, and strictly increasing.

Proof. The function must be symmetric by the commutativity of multiplication. Now find the "maximal" parent of $x$, i.e., that is the parent $x_{p}^{(\max )}$ that has the maximum generation so that $g(x)=g\left(x_{p}^{(\max )}\right)+1$. If the generation of $x y$ is given by a function $f(g(x), g(y))$ then

$$
g(x y)=f\left(g\left(x_{p}^{(\max )}\right)+1, g(y)\right) .
$$

Also $x y$ contains parents containing products of all pairs of parents of $x$ and $y$, and so $g(x y)>g\left(x_{p} y\right)$ for all $x_{p} \in X_{P}$. Choosing the maximal parent we get

$$
f\left(g\left(x_{p}^{(\max )}\right)+1, g(y)\right)>f\left(g\left(x_{p}^{(\max )}\right), g(y)\right),
$$

and as we can find examples $x$ for any $n=g(x)$, this must be true for all values of $n$, and likewise $m$, and hence the function is increasing.

Theorem 2 (Birthday multiplication theorem). The generation of $x y$ is given by a function $f(\cdot, \cdot)$ of the generations of $x$ and $y$, i.e., $g(x y)=f(g(x), g(y))$, where the function $f(\cdot, \cdot)$ satisfies the recurrence relation
$f(n, m)= \begin{cases}0, & \text { if } m=0, \\ 0, & \text { if } n=0, \\ f(n, m-1)+f(n-1, m)+f(n-1, m-1)+1, & \text { otherwise },\end{cases}$
for $n, m \in \mathbb{Z}^{+}$.
Proof. The only number $x$ with $g(x)=0$ is $\overline{0}$ which is the multiplicative annihilator, i.e., $\overline{0} \times x=\overline{0}$ for all $x$, and hence the bounding case $n=0$ and by symmetry $m=0$.

For $n, m>0$ we start from the definition of multiplication (3), which contains terms $x_{p} y+x y_{p}-x_{p} y_{p}$ for all pairs of parents ( $x_{p}, y_{p}$ ), and hence

$$
g(x y)=\sup _{\left(x_{p}, y_{p}\right)} g\left(x_{p} y+x y_{p}-x_{p} y_{p}\right)+1
$$

$$
=\sup _{\left(x_{p}, y_{p}\right)}\left[g\left(x_{p} y\right)+g\left(x y_{p}\right)+g\left(x_{p} y_{p}\right)\right]+1
$$

by the Birthday Addition Theorem, and its Subtraction corollary.
Assume (for inductive purposes) that $g\left(x_{p} y\right)=f\left(g\left(x_{p}\right), g(y)\right)$, and similarly for the other such terms, and so
$f(g(x), g(y))=\sup _{\left(x_{p}, y_{p}\right)}\left[f\left(g\left(x_{p}\right), g(y)\right)+f\left(g(x), g\left(y_{p}\right)\right)+f\left(g\left(x_{p}\right), g\left(y_{p}\right)\right)\right]+1$,
We can choose the respective parents $x_{p}$ and $y_{p}$ independently. By the previous theorem the function $f(n, m)$ must be increasing. Hence the above sum will be maximised when we choose $\left(x_{p}, y_{p}\right)$ such that $g\left(x_{p}\right)$ and $g\left(y_{p}\right)$ are both individually maximised. In this case note the definition of $g(x)$ in (2), and hence

$$
\begin{aligned}
& f(g(x), g(y)) \\
& \quad=\sup _{x_{p}} f\left(g\left(x_{p}\right), g(y)\right)+\sup _{y_{p}} f\left(g(x), g\left(y_{p}\right)\right)+\sup _{\left(x_{p}, y_{p}\right)} f\left(g\left(x_{p}\right), g\left(y_{p}\right)\right)+1 \\
& \quad=f(g(x)-1, g(y))+f(g(x), g(y)-1)+f(g(x)-1, g(y)-1)+1
\end{aligned}
$$

where existence of the suprema is required by the construction of the surreals.
Functions defined by the recurrence relationship of Theorem 2 have been studied by Fredman [13], and are summarised in [14]. Table 2 shows values that have been derived empirically via multiplication of surreal forms, and (in grey) values that have been derived from the recurrence. Unfortunately, as the depth of recursion is given by the birthday, and multiplication is built from recursive application of multiple recursive operations, combined across the two surreal forms, we have not been able to pursue complicate surreal multiplications with resulting generations beyond around 100, though it is noteworthy that many of the empirical values in the table were first extrapolated using the recursion, before being verified computationally.

Theorem 2 leads to a number of immediate corollaries regarding asymptotic growth of surreals birthdays in particular cases.

Corollary 2. The birthday/generation of $\bar{n}^{2}$ takes values $0,1,6,31,160,841,4494, \ldots$, which grow as

$$
g\left(\bar{n}^{2}\right) \sim a \lambda^{n} / \sqrt{n}
$$

where $\lambda=3+2 \sqrt{2} \approx 5.83$ and $a=2^{-9 / 4} \sqrt{\lambda / \pi} \approx 0.29$.
Proof. The values $g\left(\bar{n}^{2}\right)$ are the main diagonal of $f(n, m)$, which are given in $[13,15]$.

Corollary 3. The birthday/generation of powers of $\overline{2}$ takes values $2,6,42,18006, \ldots$, which grow as $\left\lfloor c^{\left(2^{n}\right)}\right\rfloor$ for $c=1.597910218031873178338070118157 \ldots$

Proof. The generation of $\overline{2}^{n}$ is

$$
g\left(\overline{2} \times \overline{2}^{n-1}\right)=f\left(2, g\left(\overline{2}^{n-1}\right)\right)
$$

where $f(2, n)=n(n+1)$ from Theorem 2, and hence

$$
g\left(\overline{2}^{n}\right)=g\left(\overline{2}^{n-1}\right)\left(g\left(\overline{2}^{n-1}\right)+1\right)
$$

Now the sequence $g_{n}=g_{n-1}\left(g_{n-1}+1\right)$ is known [16], and follows $g_{n}=\left\lfloor c^{\left(2^{n}\right)}\right\rfloor$.

The outstanding feature of both of these corollaries is the very high rate of growth. The size and complexity of calculations involving these forms grows even faster than the generation of the output due to the complicated set of recursive operations built on top of each other, and hence the complexity of larger computations.
5. CONCLUSION This paper derived rules for birthday arithmetic (in particular, addition, subtraction and multiplication) for surreal number forms.

The notable absentee from this list is division. Naively, division is the reciprocal of multiplication, and so should be no harder, i.e., $x / y=x \times(1 / y)$. But division is in fact quite different. Only dyadic surreals have finite representations. Thus we can represent $1 / 2,15 / 16$ and so on exactly with a finite form. However, non-dyadic real and even simple rationals are not finite. Thus numbers even as simple as $1 / 3$ do not have a finite representation. So to apply the multiplication theorem to division, we need a formula to calculate the birthday of a reciprocal. This remains to be found.

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[^1]:    ${ }^{1}$ The SurrealNumbers toolkit (v0.1.1) is released under the MIT license, and available at https:// github.com/mroughan/SurrealNumbers.jl

