

# On the modular covariance properties of composite fermions on the torus

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In this work we show that the composite fermion construction for the torus geometry is modular covariant. We show that this is the case both before and after projection, and that modular covariance properties are preserved under both exact projection and under JK projection which was recently introduced by Pu, Wu, and Jain (PRB 96, 195302 (2017)). It is crucial for the modular properties to hold that the CF state is a proper state, *i.e.* that there are no holes in the occupied  $\Lambda$ -levels.

## I. INTRODUCTION

In the study of the fractional quantum hall effect, a prominent role has been played by the construction of trial wave functions, dating back to Laughlin's wave function<sup>Lau83</sup> more than three decades ago. The first formulation, which was for a finite quantum liquid on an infinite plane, was subsequently generalized to both a sphere<sup>Hal83</sup> and torus<sup>HR85</sup>. In the following years other trial wave functions were also constructed on the three geometries, such as the Pfaffian<sup>MR91,Mor98,RH00</sup>. These three geometries: plane, sphere and torus, have since then been the canonical playground for fractional quantum hall trial wave functions.

In this work, we build upon and extend recent developments<sup>PWJ17</sup> in constructing Jain-Kamilla projected wave functions for composite fermions on the torus geometry. Composite fermions (CF), are straight forward to write down in unprojected form<sup>Jai89,DJ92,Her13</sup> on all the three above mentioned geometries. To obtain physical wave functions, *i.e.* that reside in the lowest Landau level (LLL), the CF wave functions however need to be projected onto that LLL. This can be achieved either analytically<sup>GJ84</sup> or via the Jain-Kamilla (JK) projection<sup>JK97</sup>. The former is exact, but numerically inefficient, and the latter is an uncontrolled approximation, but numerically fast. It was early understood how to perform both of these projections on the plane and sphere, but the torus geometry proved more difficult, mainly due to technical difficulties with the non-trivial interplay of boundary conditions and the action of derivatives on quasi-periodic wave functions. The first successful attempts in this direction was taken by Ref. Her13 for the analytical projection.

In a parallel development, trial wave functions on the torus was also developed for the Jain series with the help of CFT<sup>HSB<sup>+</sup>08,FHS14,HHSV17</sup> techniques. Recently DMRG methods have also been extended to the cylinder geometry for the Laughlin state<sup>ZM12</sup>, its quasi-particles<sup>KAD<sup>+</sup>18</sup>, and states higher up in the Hierarchy<sup>ZMPR15</sup>. See also the construction of quasi-particles for the Laughlin state on the Torus<sup>GST16</sup>.

Recently however, Pu, Wu and Jain<sup>PWJ17</sup> (PWJ) managed to extend the JK projection-scheme to also encom-

pass the torus. The same techniques were later used in Ref. PFJ18 to study the composite fermion Fermi liquid, which had previously been examined by other numerical techniques<sup>RH00,WGRH17,FMSS18,GWRH18</sup>.

In comparison to the other two geometries, the torus comes with an extra parameter  $\tau$ , which controls its geometry. The parameter  $\tau$  is important since multiple values of  $\tau$  may correspond to the same physical geometry. This redundancy poses additional physical constraints on the trial wave functions that are not present on the plane and sphere, where it's sufficient to respect the boundary conditions. It is therefore of great importance that wave functions defined on the torus, not only have correct boundary conditions, but also that wave functions at different (but physically equivalent)  $\tau$  span the same space of wave functions. The mapping from one value of  $\tau$  to a physically equivalent value is a modular transformation and comes in two flavors; the  $T$ -transform which sends  $\tau \rightarrow \tau+1$  and the  $S$ -transform which sends  $\tau \rightarrow -\frac{1}{\tau}$ . The former is a remapping of the torus lattice vectors, and the latter is a rotation that interchanges the order of the vectors. Sets of wave functions that span the same physical space before and after a the above mentioned modular transformations have the property of modular covariance. The modular covariance property was of great importance to compute e.g Hall viscosity<sup>Rea09,Rea08,RR11,FHS14</sup>.

The property of modular covariance is not guaranteed simply because appropriate boundary conditions are imposed. This was made clear in Ref. FHS14, where it was shown that the primary CFT correlation functions used to construct hierarchy wave functions have the correct modular properties, but that the naive introduction of a regularized derivative (as was previously done in Ref. HSB<sup>+</sup>08) broke modular covariance. The authors could find another regularization which restored the modular covariance and as a positive side effect also significantly improved the overlap with the coulomb ground state.

The property of modular covariance has never been proven for the composite fermion states, neither before nor after projection, and that is what we will do in this paper. On the route there we will also present some (hopefully) useful reformulations and results of the PWJ approach. The paper is organized as follows: In Section (II) we introduce the torus geometry and the single parti-

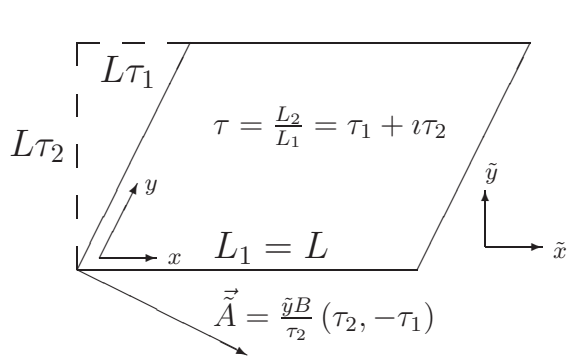


Figure 1. The relationship between the Cartesian coordinates  $(\tilde{x}, \tilde{y})$  and the dimensionless coordinates  $(x, y)$ . In the figure one can also see that  $\tau_1$  is interpreted as the skewness, and  $\tau_2$  as the aspect ratio, of the torus. Note how the  $\tau$ -gauge vector potential  $\vec{A}$  is perpendicular to the vector  $\vec{\tau} = (\tau_1, \tau_2)$ . The area of the torus is fixed to be  $L^2\tau_2 = 2\pi N_\phi \ell_B^2$ .

cle wave functions. In Section III we discuss the Girvin-Jach rule in  $\tau$ -gauge. In Section IV we briefly discuss the CF construction on the torus and in Section (V) we discuss the modification of the Girvin-Jach rule that is necessary to obtain periodic boundary conditions for the Jain-Kamilla projection. In section (VI) we derive the covariance properties for unprotected as well as exactly projected and PWJ projected CF:s and show that they all satisfy modular covariance. We end with a discussion and outlook in Section (VII). Detailed derivations are deferred to the Appendices.

## II. THE TORUS AND ITS WAVE FUNCTIONS

In this section we give a short recapitulation regarding the torus. This also serves to define the notation that is used later in the paper. The torus is defined by two axes  $\mathbf{L}_1, \mathbf{L}_2$  on the plane and we will adopt the conventions that in complex coordinates the axes are  $L_1 = L$  and  $L_2 = \tau L$ , where  $\tau = \tau_1 + i\tau_2$ . For coordinates we use the (unusual) convention that  $z = \tilde{x} + i\tilde{y} = L(x + \tau y)$  where  $\tilde{x}, \tilde{y}$  are the physical euclidean (dimensional) coordinates and  $x, y$  are the reduced (dimensionless) coordinates. The reduced coordinates  $x, y \in [0, 1]$ , defined on the unit square are convenient since  $x = 1$  ( $y = 1$ ) corresponds to  $z = L_1$  ( $z = L_2$ ). The two torus axes span an area  $|\mathbf{L}_1 \times \mathbf{L}_2| = \tau_2 L^2 = 2\pi N_\phi \ell^2$  where  $\ell = \sqrt{\frac{\hbar c}{B}}$  is the magnetic length and  $N_\phi$  is the number of magnetic fluxes that penetrate it's surface. See Fig. 1 for an illustration of the coordinates and gauge choice.

The single particle Hamiltonian is

$$H = \sum_{i=\tilde{x}, \tilde{y}} \frac{1}{2m} (p_j - e\tilde{A}_j)^2, \quad (1)$$

where  $p_j = i\hbar\partial_j$  and  $\vec{A} = \sum_{i=\tilde{x}, \tilde{y}} \tilde{A}_i \hat{i}$  is a vector potential

satisfying  $\nabla \times \vec{A} = B\hat{z}$ . We will choose to work in the  $\tau$ -gauge, where the vector potential is  $\vec{A} = \frac{\tilde{y}B}{\tau_2}(\tau_2, -\tau_1)$ , which is perpendicular to the vector  $\vec{\tau}$ . In reduced coordinates the vector potential simplifies to  $\vec{A} = (2\pi N_\phi y B, 0)$  which is explicitly  $\tau$ -independent. In this work we will work exclusively in  $\tau$ -gauge, as it is especially convenient to handle modular transformations and boundary conditions at arbitrary  $\tau$ . The Hamiltonian in (1) can be diagonalized by introducing ladder operators, yielding the form  $H = \hbar\omega_B (a^\dagger a + \frac{1}{2})$ , where  $\omega_B = \frac{eB}{m}$ . The ladder operators in  $\tau$ -gauge are

$$a_\tau = \sqrt{2} \left( \frac{\tau L}{2\ell} y + \ell\partial_z \right) \\ a_\tau^\dagger = \sqrt{2} \left( \frac{\bar{\tau} L}{2\ell} y - \ell\partial_z \right), \quad (2)$$

and satisfy  $[a_\tau^\dagger, a_\tau] = 1$ . Physical wave functions  $\psi(z)$ , are quasi-periodic and obey the boundary conditions

$$\psi(z + L_j) = e^{\lambda_j(z, \bar{z})} \psi(z) \quad (3)$$

where  $\lambda_j(z, \bar{z})$  depends on the gauge choice  $\vec{A}$ . For  $\tau$ -gauge this is  $\lambda_j = \delta_{j,2} 2\pi N_\phi x$ .

In  $\tau$ -gauge, the shift operator and magnetic translation operators are

$$\tilde{t}(\alpha L + \beta\tau L) = e^{\alpha\partial_x + \beta\partial_y} \quad (4) \\ t(\alpha L + \beta\tau L) = e^{\alpha\partial_x + \beta\partial_y + i2\pi\beta N_\phi x}$$

where it is the latter that defines periodic boundary conditions  $t(L_j)\psi = e^{i\phi_j}\psi$ . Note how we differentiate between the shift operator  $\tilde{t}(\tau L)$  and the full magnetic translation operator  $t(\tau L)$  such that  $t(\tau L) = e^{i2\pi N_\phi x} \tilde{t}(\tau L)$ . General LLL wave functions in  $\tau$ -gauge take the form  $\psi^{(A)}(z) = e^{i2\pi A y^2} f(z)$  where  $f(z)$  is a holomorphic function and  $A$  counts the number of magnetic fluxes through the torus. The above formula is particularly useful since if  $\psi^{(A)}$  and  $\psi^{(B)}$  are wave functions with boundary conditions  $\phi_A$  and  $\phi_B$  then  $\psi^{(A+B)} = \psi^{(A)} \cdot \psi^{(B)}$  is automatically a wave function with boundary conditions  $\phi_A + \phi_B$ .

The operator  $t\left(\frac{L}{N_\phi}\right)$  commutes with the operator  $t(\tau L)$ , and consequently can be used to defined a basis of  $N_\phi$  linearly independent states. The single particle orbitals in the lowest Landau level – in a basis that diagonalizes  $t\left(\frac{L}{N_\phi}\right)$  – can be written as

$$\phi_i^{(N_\phi)} = \frac{1}{\sqrt{\ell L \sqrt{\pi}}} e^{i\pi\tau N_\phi y^2} \vartheta_{\frac{i}{N_\phi}, 0} \left( \frac{N_\phi z}{L} \middle| N_\phi \tau \right) \\ = \frac{1}{\sqrt{\ell L \sqrt{\pi}}} \sum_{k \in \mathbb{Z} + \frac{i}{N_\phi}} e^{i\pi N_\phi \tau (y+k)^2} e^{i2\pi N_\phi k x}. \quad (5)$$

The function  $\vartheta_{a,b}(z|\tau) = \sum_{k \in \mathbb{Z} + a} e^{i\pi\tau k^2} e^{i2\pi k(z+b)}$  is a generalized Jacobi theta function. The orbitals for the

higher Landau levels are obtained by application of the raising operators as  $\phi_{j,n} = \frac{(a^\dagger)^n}{\sqrt{n!}} \phi_j$ . The explicit expression for the  $n$ :th Landau level orbitals, also as eigenstates of  $t \left( \frac{L}{N_\phi} \right)$ , are

$$\begin{aligned} \phi_{i,n}^{(N_\phi)} &= \mathcal{N}_n \sum_{k \in \mathbb{Z} + \frac{i}{N_\phi}} e^{i\pi N_\phi \tau (y+k)^2} e^{i2\pi N_\phi k x} H_n \left( \frac{\tau_2 L}{\ell^2} (y+k) \right) \\ &= \mathcal{N}_n e^{i\pi N_\phi \tau y^2} \sum_{k \in \mathbb{Z} + \frac{i}{N_\phi}} e^{i\pi N_\phi \tau k^2} e^{i2\pi N_\phi k \frac{\tilde{y}}{L}} H_n \left( \tilde{y} + \tau_2 L k \right), \end{aligned} \quad (6)$$

where  $H_n$  is a Hermite polynomial and  $\mathcal{N}_n = \frac{1}{\sqrt{2^n n! L \sqrt{\pi}}}$ . Note the appearance of the physical  $\tilde{y} = L\tau y$  in the argument of the Hermite polynomial. We will refer to  $f_{i,n}^{(N_\phi)}$  as the holomorphic polynomial in  $\phi_{i,n}^{(N_\phi)} = e^{i\pi N_\phi \tau y^2} f_{i,n}^{(N_\phi)}$ , and we will often drop the momentum index  $i$  for brevity.

In recent papers<sup>WGRH17,GWRH18,Hal18a,Hal18b</sup>, Haldane has been advocating the use of Weierstrass  $\sigma$ -functions over the traditionally used  $\vartheta$ -functions. In this paper we follow in that tradition and define a generalized  $\sigma$ -function in  $\tau$ -gauge as

$$\sigma_{a,b}^{(n)}(z) = e^{i\pi \tau n y^2} \vartheta_{a,b} \left( \frac{nz}{L} \middle| n\tau \right). \quad (7)$$

Comparing with (5) we have *e.g.* that  $\sigma_{\frac{i}{N_\phi},0}^{(N_\phi)} = \sqrt{\ell L \sqrt{\pi}} \phi_i^{(N_\phi)}$ . The Weierstrass functions build in the quasi-period boundary conditions and thus transform under coordinate changes as

$$\begin{aligned} \sigma_{a,b}^{(n)}(z+L) &= e^{i2\pi a n} \sigma_{a,b}^{(n)}(z) \\ \sigma_{a,b}^{(n)}(z+L\tau) &= e^{-i2\pi(na+b)} \sigma_{a,b}^{(n)}(z), \end{aligned}$$

in accordance with (3). With this definition, one may also rewrite the  $q$ -fold degenerate Laughlin's state on the torus, which is  $\vartheta$ -form is

$$\begin{aligned} \psi_{\frac{1}{q}} &= \mathcal{N}(\tau) e^{i\pi \tau N_\phi \sum_i y_i^2} \vartheta_{0,0} \left( \frac{q \sum z_i}{L} \middle| q\tau \right) \\ &\quad \times \prod_{i < j} \vartheta_{\frac{1}{2}, \frac{1}{2}} \left( \frac{z_i - z_j}{L} \middle| \tau \right)^q. \end{aligned}$$

In Weierstrass form this is the more compact

$$\psi_{\frac{1}{q}} = \mathcal{N}(\tau) \sigma_{0,0}^{(q)} \left( \sum_i z_i \right) \prod_{i < j} \sigma_{\frac{1}{2}, \frac{1}{2}}^{(1)}(z_i - z_j)^q. \quad (8)$$

The normalization factor is here chosen to be  $\mathcal{N}(\tau) = \frac{[\sqrt{\tau_2} \eta^2(\tau)]^{\frac{qN_\phi}{2}}}{\eta(\tau)^{\frac{qN_\phi(N_\phi-1)}{2} + 1}}$  as suggested in Ref. Rea09. This normalization ensures that the Laughlin state transforms under  $S$ -transformations as  $\psi_{\frac{1}{q}} \rightarrow \left( \frac{\tau}{|\tau|} \right)^{\frac{N_\phi q}{2}} \psi_{\frac{1}{q}}$ . This normalization is the correct one (up to  $\tau$ -independent scale factors) as along as the torus is large enough<sup>Fre16</sup>.

### III. GIRVIN-JACH PROJECTION IN $\tau$ -GAUGE

In their work in Ref. GJ84, Girvin and Jach introduced the classic rule for LLL projection, namely that  $\bar{z} \rightarrow 2\partial_z$ . What might not be obvious is that this is a gauge dependent rule, and is only guaranteed to hold in symmetric gauge. In this section we review the Girvin-Jach projection trick<sup>GJ84</sup> and then extend it to  $\tau$ -gauge. We begin by reminding ourselves of the argument goes in symmetric gauge, before we turn to the  $\tau$ -gauge. The ladder operators in symmetric gauge are

$$\begin{aligned} a_s &= \sqrt{2} \left( \frac{z}{4} + \partial_{\bar{z}} \right) \\ a_s^\dagger &= \sqrt{2} \left( \frac{\bar{z}}{4} - \partial_z \right). \end{aligned} \quad (9)$$

where the  $s$  denotes the symmetric gauge choice. The equation for  $a^\dagger$  can be rewritten as  $\bar{z} = \frac{4a_s^\dagger}{\sqrt{2}} + 4\partial_z$ , which allows us to write

$$\begin{aligned} \bar{z} e^{-\frac{\bar{z}\bar{z}}{4}} f(z) &= \left( \frac{4a_s^\dagger}{\sqrt{2}} + 4\partial_z \right) e^{-\frac{\bar{z}\bar{z}}{4}} f(z) \\ &= \frac{4a_s^\dagger}{\sqrt{2}} e^{-\frac{\bar{z}\bar{z}}{4}} f(z) + e^{-\frac{\bar{z}\bar{z}}{4}} (4\partial_z - \bar{z}) f(z), \end{aligned}$$

or equivalently

$$\bar{z} e^{-\frac{\bar{z}\bar{z}}{4}} f(z) = \sqrt{2} a_s^\dagger e^{-\frac{\bar{z}\bar{z}}{4}} f(z) + 2e^{-\frac{\bar{z}\bar{z}}{4}} \partial_z f(z).$$

Applying the LLL projection kills the  $a^\dagger$  term and we have  $P_{LLL} \bar{z} e^{-\frac{\bar{z}\bar{z}}{4}} f(z) = 2e^{-\frac{\bar{z}\bar{z}}{4}} \partial_z f(z)$  which amounts to the famous rule<sup>GJ84</sup>  $\bar{z} \rightarrow 2\partial_z$ , where it is understood that the derivative does not act on the exponential  $e^{-\frac{\bar{z}\bar{z}}{4}}$ . Here, since  $[a_s^\dagger, \bar{z}] = 0$  the argument can also be iterated to higher powers of  $\bar{z}$  as  $\bar{z}^n \rightarrow (2\partial_z)^n$ . In  $\tau$ -gauge, due to (2), the same equation reads  $\frac{\bar{\tau}}{2} Ly = \frac{a_s^\dagger}{\sqrt{2}} + \partial_z$ , which becomes the equation

$$\frac{\bar{\tau}}{2} Ly G_\tau f(z) = \frac{a_s^\dagger}{\sqrt{2}} G_\tau f(z) + G_\tau \left( \partial_z + \frac{L\tau y}{2} \right) f(z).$$

after acting on  $G_\tau = e^{i\pi \tau N_\phi y^2}$ . The above equation may be rewritten as

$$\tau_2 Ly G_\tau f(z) = \frac{a_s^\dagger}{\sqrt{2}} G_\tau f(z) + G_\tau (i\partial_z) f(z).$$

After projection (and the  $a_s^\dagger$  term is killed) this becomes  $\tilde{y} = \tau_2 Ly \rightarrow i\partial_z$ , with the understanding that  $\partial_z$  does not act on the Gaussian factor  $e^{i\pi \tau N_\phi y^2}$ . The rule for  $\tilde{y}$  can however not be extended directly to higher powers of  $\tilde{y}$  since  $[\tilde{y}, a_s^\dagger] \neq 0$ . Instead due to this noncommutativity the projection rule reads

$$\tilde{y}^n \rightarrow \frac{1}{(-2i)^n} H_n(\partial_z) \quad (10)$$

where  $H_n$  is a Hermite polynomial. A proof and an extended discussion can be found in Appendix A. We wish to stress that since the  $P_{\text{LLL}}$  operator only involves  $a$  and  $a^\dagger$  operators, that act between LL:s, it trivially commutes with the operators within any LLL. This has the important consequence that if a wave function satisfies the boundary conditions before projection, it will automatically do so also after projection.

### A. LLL projection as an operator

Here we develop a formalism where we view the LLL projection as an operator action on holomorphic LLL wave functions. To be concrete, we consider a general state (*e.g.* basis state)  $\phi_n^{(M)}$  in the  $n$ :th LL defined for  $M$  fluxes, that is multiplied with an arbitrary LLL wave function  $\psi^{(N_\phi - M)}$  defined for  $N_\phi - M$  fluxes. The power of the  $\tau$ -gauge formalism and reduced coordinates is that the product of  $\phi_n^{(M)}\psi^{(N_\phi - M)}$  (when expressed in reduced coordinates) is automatically a proper wave function at  $N_\phi$  fluxes, since the different magnetic lengths  $\ell$  of the two wave functions are automatically renormalized.

The product  $\phi_n^{(M)}\psi^{(N_\phi - M)}$  can now be written as

$$\phi_n^{(M)}\psi^{(N_\phi - M)} = e^{i\pi\tau N_\phi y^2} f_n^{(M)} f^{(N_\phi - M)}$$

where  $\psi^{(N_\phi - M)} = e^{i\pi\tau(N_\phi - M)y^2} f^{(N_\phi - M)}$  is separated into its Gaussian and holomorphic factor, and the same for  $\phi_n^{(M)} = e^{i\pi\tau M y^2} f_n^{(M)}$ . When applying  $P_{\text{LLL}}$  on this combined wave function we can use the fact that only  $\phi_n^{(M)}$  is non-holomorphic and promote  $f_n^{(M)}$  to a differential operator acting on  $f^{(N_\phi - M)}$  as

$$P_{\text{LLL}}\phi_n^{(M)}\psi^{(N_\phi - M)} = e^{i\pi\tau N_\phi y^2} \hat{f}_n^{(M)} f^{(N_\phi - M)}.$$

The operator  $\hat{f}_n^{(M)}$  can after some transformations (see Appendix B) be rewritten as

$$\hat{f}_n^{(M)} = \sum_{k=0}^n \binom{n}{k} (M\partial_z)^{n-k} \left[ (-N_\phi\partial_z)^k f_0 \right], \quad (11)$$

where  $f_0 = e^{-i\pi M y^2} \frac{a^n}{\sqrt{n!}} \phi_n$  is the LLL version of  $f_n$ , and where an scale factor of  $(2i)^n \frac{N_n}{N_0 M^n}$  has been suppressed<sup>Not</sup>. The derivative within square brackets acts only on  $f_0$ .

We may symbolically write the operator in (11) as

$$\hat{f}_n^{(M)} = \left( M\partial_z - N_\phi \tilde{\partial}_z \right)^n f_0, \quad (12)$$

where the operator  $\tilde{\partial}_z$  is understood to act only on  $f_0$  and thus has the property  $\tilde{\partial}_z f_0 f = f \tilde{\partial}_z f_0$ . We may also introduce the derivative operator  $\hat{\partial}_z$  which does not act on  $f_0$  at all and can be defined as  $\hat{\partial}_z f_0 f = f_0 \hat{\partial}_z f$ . Using

that these two operators have the identity  $\partial_z = \tilde{\partial}_z + \hat{\partial}_z$  and that the three operators  $\partial_z, \tilde{\partial}_z, \hat{\partial}_z$  all commute, we may rewrite (12) as

$$\hat{f}_n = \left( N_\phi \hat{\partial}_z - (N_\phi - M) \partial_z \right)^n f_0, \quad (13)$$

and

$$\hat{f}_n = \left( M \hat{\partial}_z - (N_\phi - M) \tilde{\partial}_z \right)^n f_0, \quad (14)$$

where especially (14) will be useful later. This is also the form that was found by PWJ. For brevity we will also introduce the operator  $\hat{D} = M \hat{\partial}_z - (N_\phi - M) \tilde{\partial}_z$  such that (14) can be written in shorthand as  $\hat{f}_n = \hat{D}^n f_0$ .

### B. Periodic boundary conditions of $\hat{f}_n$

To set the stage for the discussion of the PWJ projection in the later sections we now prove that  $\hat{f}_n$  indeed provides for periodic boundary conditions. We know that an  $A$  flux wave function  $\psi^{(A)} = e^{i\pi\tau A y^2} f^{(A)}$  should obey the relation  $t(\tau L) \psi^{(A)} = \psi^{(A)}$  (assuming p.b.c). Removing the factor  $e^{i\pi\tau A y^2}$  and the gauge factor  $e^{i2\pi A x}$  we see that this implies that  $\tilde{t}(\tau L) f^{(A)} = e^{-i2\pi A(z + \frac{x}{2})} f^{(A)} \tilde{t}(\tau L)$ . This means that the relation

$$\left( \tilde{t}(\tau L) \hat{f}_n \right) e^{-i2\pi(N_\phi - M)(z + \frac{x}{2})} = e^{-i2\pi N_\phi(z + \frac{x}{2})} \hat{f}_n \quad (15)$$

should hold for  $\hat{f}_n$ . Note here that  $\left[ \tilde{t}(\tau L), \hat{D} \right] = 0$ , but that  $\tilde{t}(\tau L) f_0 = e^{-i2\pi M(z + \frac{x}{2})} f_0 \tilde{t}(\tau L)$ . This means that when  $\tilde{t}(\tau L)$  acts on  $f_0$  it will produce the factor  $e^{-i2\pi M(z + \frac{x}{2})}$ . This factor will be acted upon by  $\tilde{\partial}_z$  in eqn. (14), effectively causing the shift  $\tilde{\partial}_z \rightarrow \tilde{\partial}_z - i2\pi M$ . Likewise when  $\tilde{t}(\tau L)$  acts on  $f^{(N_\phi - M)}$  it produces the factor  $e^{-i2\pi(N_\phi - M)(z + \frac{x}{2})}$ , which when pulled through  $\hat{\partial}_z$  causes the shift  $\hat{\partial}_z \rightarrow \hat{\partial}_z - i2\pi(N_\phi - M)$ . Since the two shifts are simple constant they commute and we have

$$\begin{aligned} \hat{D} &= M \hat{\partial}_z - (N_\phi - M) \tilde{\partial}_z \\ &\rightarrow M \left( \hat{\partial}_z - i2\pi(N_\phi - M) \right) \\ &\quad - (N_\phi - M) \left( \tilde{\partial}_z - i2\pi M \right) = \hat{D}, \end{aligned} \quad (16)$$

when the exponentials are pulled through  $\hat{D}$ . This shows that  $\hat{D}$  is invariant and proves (15).

We mention in passing that we may define  $\hat{\phi}_n^{(M)} = e^{i\pi\tau N_\phi y^2} \hat{f}_n^{(M)} e^{-i\pi\tau(N_\phi - M)y^2}$  which is an operator that has proper operator boundary conditions. This operator may be expressed as

$$\hat{\phi}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} T(n, k) \chi^k D^{n-2k} \phi_0 \quad (17)$$



where  $T(k, n)$  is the triangle of Bessel numbers<sup>Inc18</sup> and  $\chi = \frac{M(N-M)\tau}{4i\tau_2}$ . This has been confirmed by Mathematica up to  $n = 8$ , and we assume it holds for general  $n$ . See Appendix (C) for details.

#### IV. COMPOSITE FERMIONS ON THE TORUS

In this section we briefly introduce the CF construction on the torus at filling fraction  $\nu = \frac{n}{2pn+1}$  and discuss how the expected degeneracy of  $q = 2pn + 1$  comes about. A generic CF wave functions may be written on the form

$$\psi_{CF} = P_{LLL}\chi_n\psi_{\nu=\frac{1}{2p}} \quad (18)$$

where  $\chi_n$  is a Slater-determinant of occupied CF-orbitals given by (6), where the CF-flux is  $M$ . If  $\psi_{CF}$  represents a ground state at filling fraction  $\nu = \frac{n}{2pn+1}$ , then  $nM = N_e$  and  $N_\phi = M + 2pN_e$ , meaning that the  $n$  lowest CF  $\Lambda$ -levels are filled. As  $\psi_{\nu=\frac{1}{2p}}$  contains a center of mass piece and a Jastrow factor (see eqn. (8)) we may pull the Jastrow factor into the determinant and write

$$\psi_{CF} = P_{LLL}\sigma_a^{(2p)}(Z) \cdot \mathcal{A} \left\{ \prod_j \phi_j(z_j) \cdot J_j^p(z) \right\}.$$

Here,  $\mathcal{A}$  is an antisymmetrizer of the coordinates that plays the same role as the determinant, and  $J_j(z) = \prod_{k \neq j} \sigma_{\frac{1}{2}, \frac{1}{2}}^{(1)}(z_{jk})$ . The subscript  $a$  on  $\sigma_a^{(2p)}(Z)$  is labeling one of the  $2p$  states of  $\psi_{\nu=\frac{1}{2p}}$  and the subscript  $j$  on  $\phi_j$  contains for brevity both the LL-index and the orbital index. We will later see that it is crucial for the PWJ projection recipe that the CF state is a proper state. A proper CF state has the property that there are no holes in the filling of the  $\Lambda$ -levels, in the sense that if the orbital  $\phi_{j,n}$  is occupied (with  $n > 0$ ), then also the orbital  $\phi_{j,n-1}$  is occupied.

##### A. Notes on the multiplicity of the wave functions

Here we mention for completeness how the correct degeneracy of the CF states is counted. It is well known that for a LL with partial filling  $\nu = \frac{p}{q}$  ( $p, q$  being relatively prime) every state is at least  $q$ -fold degenerate on the torus<sup>Hal85</sup> (with higher degeneracy for non-abelian states). To show this degeneracy explicitly for the CF states, we make use of the many-body translation operator commutations relations  $T^{(A)}(\alpha\tau L)t^{(A)}(L) = e^{i2\pi A\alpha}$ , where the  $(A)$  denotes that the wave functions act on  $A$ -flux wave functions. The many body operators are

$$T^{(A)}(\gamma) = \prod_{j=1}^{N_e} t_j^{(A)}(\gamma),$$

where  $t_j^{(A)}(\gamma)$  is the magnetic translation operator in (4) acting on coordinate  $j$ . We next define the translated state  $\psi_{CF}^{(\alpha)} = T^{(N_\phi)}(\alpha\tau L)\psi_{CF}$ . If we assume that  $\psi_{CF}$  has periodic boundary conditions then  $\psi_{CF}^{(\alpha)}$  will also have periodic boundary conditions when  $e^{i2\pi N_\phi \alpha} = 1$ , which happens first when  $\alpha = \frac{1}{N_\phi} = \frac{n}{N_e(2pn+1)}$ . Naively one might expect that there should be  $\frac{1}{\alpha} = N_\phi$  degenerate states from this argument, which is clearly wrong. To get the correct counting, one has to also take into account that the trivial cycle (*i.e.* the cycle that sends  $\psi_{CF}^{(\alpha)} = \psi_{CF}^{(0)}$ ) is not  $\alpha = 1$ , but is determined by the trivial cycles of  $\psi_{\nu=1}^{2p}$  and  $\chi_n$ . The trivial cycle for  $\psi_{\nu=1}$  is  $\alpha' = \frac{1}{N_e}$  since that cycles the states  $\phi_{j,0}^{(N_e)} \rightarrow \phi_{j+1,0}^{(N_e)}$  leaving the  $\psi_{\nu=1}$  invariant. In a similar manner, the trivial cycle for  $\chi_n$  is  $\alpha'' = \frac{1}{M} = \frac{n}{N_e}$  since it sends  $\phi_{j,k}^{(M)} \rightarrow \phi_{j+1,k}^{(M)}$  in the determinant. We thus see that  $(2pn+1)\alpha = n\alpha' = \alpha''$  which shows that  $2pn+1 = q$  applications of  $\alpha$  are needed to obtain trivial cycles for the two sub-factors. This shows that the degeneracy of  $\psi_{CF}$  is  $q = 2pn+1$  as expected.

#### V. MODIFIED JK PROJECTION

We now discuss the modification to (14) that is necessary to obtain JK projected wave functions that respect the periodic boundary conditions. In a naive implementation of the JK projection we would move the projector into the determinant and perform the LLL projection on each term of the determinant. On the plane and sphere this is an uncomplicated procedure, but of the torus this is highly nontrivial since the boundary conditions of the factor  $J_j(z)$  depends on the other  $k \neq j$  coordinates. Nevertheless we may be bold and stipulate that we can still use (13), and then hope for the best. In that case we first extract the Gaussian factors and write

$$\psi_{JK} = \sigma_a^{(2p)}(Z) e^{i\pi\tau(N_\phi \sum_j y_j^2 - 2pY^2)} \times \mathcal{A} \left\{ \prod_j \hat{f}_j \cdot F_j^p(z) \right\}, \quad (19)$$

where now  $\hat{f}_j$  only acts on the function

$$F_j^p(z) = J_j^p e^{-i\pi\tau p \sum_{k \neq j} (y_j - y_k)^2}. \quad (20)$$

Here, and below, we use the abbreviations  $Y = \sum_j y_j$ ,  $X = \sum_j x_j$  and  $Y_j = Y - y_j = \sum_{k \neq j} y_k$ ,  $X_j = X - x_j = \sum_{k \neq j} x_k$ . Here the number of fluxes in  $\hat{f}_j \cdot F_j^p(z)$  is  $N_\phi = M + p(N_e - 1)$  instead of  $N_\phi = M + 2pN_e$ . To see why this does not work, and also determine what does, we follow the reasoning of PWJ and introduce a modification of  $\hat{f}_j$  that is  $\tilde{f}_n = \tilde{D}^n f_0$ , where

$$\tilde{D} = \alpha M \hat{\partial}_z - (N_\phi - M) \tilde{\partial}_z = \alpha M \hat{\partial}_z - 2pN_e \tilde{\partial}_z. \quad (21)$$

For  $\alpha = 1$  then  $\tilde{D} = \hat{D}$  and  $\hat{f}_j = \tilde{f}_j$ , but we will soon see that the choice  $\alpha = 2$  will be necessary. We begin with reviewing the relevant transformations. Acting with  $\tilde{t}_j(\tau L)$  on  $F_l$  produces (after we have dropped some constant factors)

$$\begin{aligned} \tilde{t}_j(\tau L) F_j(z) &\propto e^{-i2\pi \prod_{k \neq j} (z_j - z_k)} \prod_{k \neq j} \vartheta_1(z_j - z_k | \tau) \tilde{t}_j(\tau L) \\ &= e^{-i2\pi[(N_e-1)z_j - Z_j]} F_j \tilde{t}_j(\tau L) \end{aligned} \quad (22)$$

and

$$\begin{aligned} \tilde{t}_j(\tau L) F_{l \neq j}(z) &\propto e^{-i2\pi(z_l - z_j)} \prod_{k \neq l} \vartheta_1(z_l - z_k | \tau) \tilde{t}_j(\tau L) \\ &= e^{-i2\pi(z_l - z_j)} F_{l \neq j} \tilde{t}_j(\tau L). \end{aligned} \quad (23)$$

depending on if  $j = l$  or not. We now apply the translation operator  $\tilde{t}_j(\tau L)$  on  $\hat{f}_j F_j$ . For brevity we suppress the factors of  $e^{-i2\pi[(N_e-1)z_j - Z_j]}$  and  $e^{-i2\pi(z_l - z_j)}$  coming from (22) and (23) as well as the phase  $e^{i2\pi z_j M}$  coming from  $f_0^{(M)}$ . We obtain, by an analogs calculation to the one in (16) that

$$\begin{aligned} \hat{f}_n^{(M)} F_j^p &\rightarrow \left( -2N_e p \left( \tilde{\partial}_{z_j} - i2\pi M \right) + \alpha M \left( \hat{\partial}_{z_j} - i2\pi p (N_e - 1) \right) \right)^n f_0 F_j^p \\ &= \left( \tilde{D} + i2\pi p M [2N_e - \alpha (N_e - 1)] \right)^n f_0 F_j^p. \end{aligned}$$

for  $j = l$ . For  $j \neq l$  we instead have

$$\begin{aligned} \hat{f}_n^{(M)} F_{l \neq j} &\rightarrow \left( -2N_e p \tilde{\partial}_{z_j} + \alpha M \left( \hat{\partial}_{z_j} + i2\pi p \right) \right)^n f_0 F_l^p \\ &= \left( \tilde{D} + \alpha p M i2\pi \right)^n f_0 F_l^p. \end{aligned}$$

The important observation here is that both  $\alpha p M i2\pi$  and  $i2\pi p M [2N_e - \alpha (N_e - 1)]$  are constants, but they are only equal when  $\alpha = 2$ . It is crucial that the transformation  $\tilde{D}^n \rightarrow (\tilde{D} + \text{const})^n$  is the same for all coordinates, since the shift  $i\pi 4pM$  can then be removed by row addition if the CF state is a proper state. Otherwise the cancellation will not work.

As a minimal example lets consider the simple case of a determinant consisting of only  $N_e = 2$  particles; one in the  $n = 0$  LL and one in the  $n = 1$  LL. The entries in (19) are then  $DfF$  and  $fF$ , which gives determinant

$$\begin{vmatrix} D_1 f_1 F_1 & D_2 f_2 F_2 \\ f_1 F_1 & f_2 F_2 \end{vmatrix},$$

where the subscripts labels the coordinates of the two

particles. If we assume that  $D_1 \rightarrow D_1 + \alpha$  and  $D_2 \rightarrow D_2 + \beta$  under the action of  $t_1(\tau L)$ , we then have

$$\begin{aligned} \begin{vmatrix} D_1 f_1 F_1 & D_2 f_2 F_2 \\ f_1 F_1 & f_2 F_2 \end{vmatrix} &\rightarrow \begin{vmatrix} (D_1 + \alpha) f_1 F_1 & (D_2 + \beta) f_2 F_2 \\ f_1 F_1 & f_2 F_2 \end{vmatrix} \\ &= \begin{vmatrix} D_1 f_1 F_1 & (D_2 + (\beta - \alpha)) f_2 F_2 \\ f_1 F_1 & f_2 F_2 \end{vmatrix}. \end{aligned}$$

It is evident that the determinant is only invariant under the transformation if  $\alpha = \beta$ .

## VI. MODULAR COVARIANCE

We are now in a positions to study the modular covariance properties of the PWJ wave functions. For this purpose (and to simplify the discussion somewhat) we assume that we are considering one of the CF ground states at filling fraction  $\nu = \frac{n}{2pn+1}$ . That is, we assume that we have a state with  $n$  filled  $\Lambda$ -levels, and everything above unoccupied. In this work we will focus on the  $S$ -transform,  $\tau \rightarrow -\frac{1}{\tau}$  since that is the more complicated of the two.

Before we deal with the many-body state, let us review how single particle orbitals transform under the  $S$ -transform. An  $S$ -transform sends  $\tau \rightarrow -\frac{1}{\tau}$  and affects the LLL single particle orbitals from (5) as

$$\phi_{i,0}^{(M)} \left( x, y, -\frac{1}{\tau} \right) = e^{-i2\pi M y x} \sqrt{\frac{\tau}{|\tau|}} \frac{1}{\sqrt{M}} \sum_k e^{i2\pi \frac{kj}{M}} \phi_{k,0}^{(M)} (-y, x, \tau). \quad (24)$$

The higher order higher LL orbitals in (6) similarly transform as

$$\phi_{i,n}^{(M)} \left( x, y, -\frac{1}{\tau} \right) = e^{-i2\pi M y x} \left( \frac{\tau}{|\tau|} \right)^{n+\frac{1}{2}} \frac{1}{\sqrt{M}} \sum_k e^{i2\pi \frac{kj}{M}} \phi_{k,n}^{(M)} (-y, x, \tau). \quad (25)$$

In the above equations we note that  $\tau \rightarrow -\frac{1}{\tau}$  effectively sends  $y \rightarrow x \rightarrow -y$  and maps  $\phi_{i,n}^{(M)}$  into a Fourier sum  $\sum_k e^{i2\pi \frac{ki}{M}} \phi_{k,n}^{(M)}$ . We may also identify the factor  $e^{-i2\pi Myx}$  as the gauge transformation related with the coordinate change and  $\left(\frac{\tau}{|\tau|}\right)^{n+\frac{1}{2}}$  can be interpreted as the conformal weight of the orbital. The extra factors of  $\frac{\tau}{|\tau|}$  for  $n > 0$  can be understood by noting that the derivative operator  $\partial_z^{(x,y,\tau)}$  transforms as

$$\begin{aligned} \partial_z^{(x,y,-\frac{1}{\tau})} &= \frac{|\tau|}{\bar{\tau}L2i\tau_2} (\bar{\tau}\partial_y + \partial_x) \\ &= \left(\frac{\tau}{|\tau|}\right) \partial_z^{(-y,x,\tau)}. \end{aligned}$$

In this calculation we used that  $\tau_2 \rightarrow \frac{\tau_2}{|\tau|^2}$ ,  $L = \sqrt{\frac{2\pi N_\phi}{\tau_2}} \rightarrow |\tau| \sqrt{\frac{2\pi N_\phi}{\tau_2}} = |\tau|L$  and that  $\bar{\tau} \rightarrow -\frac{1}{\bar{\tau}} = \frac{-\tau}{|\tau|^2}$ . Thus, the ladder operator  $a_\tau^\dagger(x,y,\tau) = -\sqrt{2} \left(\partial_z - \frac{\bar{\tau}}{2}Ly\right)$  transforms as

$$\begin{aligned} a_\tau^\dagger(x,y,-\tau) &= -\sqrt{2} \left(\frac{\tau}{|\tau|}\right) \left(\partial_z^{(-y,x,\tau)} + \frac{1}{2}Ly\right) \\ &= e^{-i2\pi Myx} \left(\frac{\tau}{|\tau|}\right) a_\tau^\dagger(-y,x,\tau) e^{i2\pi Myx}. \end{aligned} \quad (26)$$

By applying (26) and (24) to  $\phi_{j,n} = a_\tau^{\dagger n} \phi_{j,0}$  then (25) is directly obtained. From this also follows that  $J_j^p$ , which is used in the CF construction will transform trivially under  $\tau \rightarrow -\frac{1}{\tau}$  since they contain a product of  $\sigma_{\frac{1}{2},\frac{1}{2}}^{(1)}(z_{ij})$  which are an  $M = 1$  representation.

### A. A modular invariant CF wave function

To make the discussion in the following subsection a little bit cleaner, we spend some time in this section defining a CF wave function that transforms trivially under  $\mathcal{S}$ -transforms in its unprojected form. We do this, since if we can find one wave function  $\psi$  that transforms trivially under  $\mathcal{S}$  we can then build the family of  $q$ -fold degenerate wave functions from this template, as eigenstates of either  $T_1 = T\left(\frac{L}{N_\phi}\right)$  or  $T_2 = T\left(\frac{\tau L}{N_\phi}\right)$ .

In practice, if we assume that  $\psi$  is a wave function that is modular invariant, we may define the states  $\psi_j = T_2^j P_{1,0}\psi$  and  $\varphi_l = T_1^{-l} P_{2,0}\psi$ . Here

$$P_{m,l} = \frac{1}{q} \sum_j e^{i2\pi \frac{-lj}{q}} T_m^j,$$

is a projector onto the basis defined by  $T_m$  and it satisfies  $\sum_{l=1}^q P_{m,l} = 1$  and  $P_{m,l} P_{m,k} = P_{m,l} \delta_{l,k}$ . Since  $T_1 T_2 =$

$T_2 T_1 e^{i2\pi \frac{L}{q}}$  we have that

$$\begin{aligned} \psi_l &\propto \frac{1}{\sqrt{q}} \sum_j e^{i2\pi \frac{-plj}{q}} \varphi_j \\ \varphi_j &\propto \frac{1}{\sqrt{q}} \sum_j e^{i2\pi \frac{plj}{q}} \psi_j \end{aligned}$$

where the  $\propto$  is inserted since  $\psi_l$  and  $\varphi_j$  might not be properly normalized with respect to each other. Now, by applying the  $\mathcal{S}$ -transform, which transforms  $T_1 \rightarrow T_2 \rightarrow T_2^{-1}$ , we find that

$$\begin{aligned} \psi_l &= T_2^j P_{1,0}\psi \\ &\rightarrow T_1^{-j} P_{2,0}\psi \\ &= \varphi_l \propto \frac{1}{\sqrt{q}} \sum_j e^{i2\pi \frac{plj}{q}} \psi_j \end{aligned}$$

which shows that the set of  $q$  wave functions  $\psi_l$  is closed under  $\mathcal{S}$ . It thus remains to be seen that  $\psi$  transforms trivially under  $\mathcal{S}$ .

### B. Unprojected CF

According to the argument of the previous section, it is sufficient to show modular covariance if we can find one CF-wave function that is invariant under the  $\mathcal{S}$ -transform. For this purpose we note that if we choose  $\psi_{\nu=1}^{2p}$  instead of  $\psi_{\nu=\frac{1}{2p}}$  in (18) then the center of mass part and the Jastrow factors  $\sigma_{\frac{1}{2},\frac{1}{2}}(z_{ij})$  are all manifestly invariant under these transformations (up to constant factors and phases). The determinant  $\chi_n$  can be made invariant in two different but equivalent ways. The first is to argue that if one fills a  $\Lambda$ -levels completely, it will also be filled after the  $\mathcal{S}$ -transform, thus ensuring the invariance. The second, which will make the later discussion of the PWJ projection much cleaner, is to build  $\chi_n$  from orbitals that themselves are invariant under the  $\mathcal{S}$ -transform.

By choosing the  $\chi_n$  orbitals from the Lattice coherent states

$$\rho_{n,m}(z) = \sigma_{\frac{1}{2},\frac{1}{2}} \left( z - \frac{1}{M} (n + \tau m) \right)^M, \quad (27)$$

one can ensure that each orbital is invariant under  $\mathcal{S}$ . These states were introduced by Haldane in Ref. HR85 as a possible way to construct maximally localized wave functions and were later studied in detail in Ref. Fre13. They have the property that they have all their zeroes at the same position  $z = L(n + \tau m)$ , and transform as  $\rho_{n,m} \rightarrow \rho_{m,-n}$  under modular transformations. By constructing the states  $\tilde{\rho}_{n,m} = \rho_{n,m} + \rho_{m,-n} + \rho_{-n,-m} + \rho_{-m,n}$  we ensure that all the orbitals transform trivially under the  $\mathcal{S}$ -transform. These are examples of eigenstates for certain finite rotations. The LCS forms an over complete  $M \times M$  lattice of states and there are thus roughly

$M^2/4$  acceptable choices for  $\tilde{\rho}_{n,m}$ . Since  $M^2/4 > M$  these states are enough to fill the lowest of the  $\Lambda$ -levels and thus all also of the higher  $\Lambda$ -levels by the action of raising operators.

### C. Exactly projected CF

To prove that the exactly projected states have good modular properties, it is sufficient to show that the modular transformation commutes with the projector  $P_{\text{LLL}}$ . This is straight forward due to equations (24) and (25). These equations namely show that the modular transformation never mixes states between landau levels, and thus trivially commutes with the Landau level projection.

A more formal proof of the same is to note that  $P_{\text{LLL}}$  can formally be written as  $P_{\text{LLL}} = \prod_{n=1}^{\infty} \left(1 - \frac{a^\dagger a}{n}\right)$  where  $a^\dagger, a$  are the operators in (2). Using the result from (26) we see that  $a^\dagger a_\tau \rightarrow e^{-i2\pi Myx} a^\dagger a_\tau e^{i2\pi Myx}$  only contains the overall gauge transformation  $e^{i2\pi Myx}$ , and so  $P_{\text{LLL}} \rightarrow e^{-i2\pi Myx} P_{\text{LLL}} e^{i2\pi Myx}$  under the  $\mathcal{S}$  transform. This shows that  $P_{\text{LLL}}$  commutes with  $\mathcal{S}$  up to the ever present gauge transformation.

### D. PWJ projected CF

We now turn our attention to the PWJ projected CF state, where we are especially interested in the transformation properties of  $f_n$  and  $F_j^p$  as defined in (20) and (21). We here assume, following the discussion in (VIA) and (VIB) that  $f_0$  is chosen from the set of Lattice coherent states (27). For  $f_j \equiv f_0(z_j)$  and  $F_j$  we have the respective transformations (again with constant faces removed)

$$f_j^{(M)} \rightarrow e^{i\pi\tau M z_j^2} f_j^{(M)}$$

and

$$\begin{aligned} F_j &\rightarrow e^{i\tau\pi \sum_l (z_j - z_l)^2} F_j = e^{i\tau\pi (N_e z_j^2 - 2z_j Z + \sum_l z_l^2)} F_j \\ &= e^{i\tau\pi ((N_e - 1)z_j^2 - 2z_j Z_j + \sum_{l \neq j} z_l^2)} F_j. \end{aligned}$$

The combined transformation is thus

$$\begin{aligned} \tilde{D}_j^n f_j F_j^p &\rightarrow \tilde{D}_j^n \left( e^{i\pi\tau M z_j^2} f_j \right) \\ &\times \left( e^{i\tau\pi p ((N_e - 1)z_j^2 - 2z_j Z_j + \sum_{l \neq j} z_l^2)} F_j^p \right). \end{aligned}$$

Let us first consider the simplest case of  $n = 1$  where we define  $\gamma = e^{i\pi\tau M z_j^2} e^{i\tau\pi p ((N_e - 1)z_j^2 - 2z_j Z_j + \sum_{l \neq j} z_l^2)}$ . This yields

$$\begin{aligned} D_j f_j F_j^p &\rightarrow \gamma^{-1} \left( -2p N_e \tilde{\partial}_{z_j} + 2M \hat{\partial}_{z_j} \right) \left( e^{i\pi\tau M z_j^2} f_j \right) \\ &\times \left( e^{i\tau\pi p ((N_e - 1)z_j^2 - 2z_j Z_j + \sum_{l \neq j} z_l^2)} F_j^p \right) \\ &= -2p N_e \left( \tilde{\partial}_{z_j} + i2\pi\tau M z_j \right) f_j F_j^p \\ &\quad + 2M \left( \hat{\partial}_{z_j} + i\tau\pi 2p ((N_e - 1)z_j - Z_j) \right) f_j F_j^p \\ &= D_j f_j F_j^p - i4\pi\tau M p Z f_j F_j^p, \end{aligned}$$

where we see an extra term  $-i\tau\pi 4M p Z f_j F_j^p$  appearing. This term can then be removed under row addition of the determinant. This is since it is proportional to  $Z f_j F_j^p$  and  $Z$  is independent of the  $j$  index.

For general  $n$  we cannot use the trick employed above since  $[\partial_z, [\partial_z, z^2]] = 2 \neq 0$ , which means that the factors  $\tilde{\partial}_{z_j} \rightarrow \tilde{\partial}_{z_j} + i2\pi\tau M z_j$  and  $\hat{\partial}_{z_j} \rightarrow \hat{\partial}_{z_j} + i\tau\pi 2p ((N_e - 1)z_j - Z_j)$  can only in the  $n = 1$  case be directly combined to  $\tilde{D} \rightarrow \tilde{D} - i\tau\pi 4M p Z$ . For the  $n = 2$  case, one may after some algebra conclude that

$$\begin{aligned} \tilde{D}^2 &\rightarrow \tilde{D}^2 \\ &\quad + i\pi\tau 8M p Z \tilde{D} \\ &\quad - 16 (\pi\tau M p)^2 Z^2 \\ &\quad - 8i\pi\tau M p (M (N_e - 1) + N_e^2 p). \end{aligned}$$

Here we see that we still only get terms that depend on  $Z$  and  $\tilde{D}$ , and they can all be removed by row addition. By Mathematica calculations we have confirmed up to  $n = 10$ , and we believe it holds in general, that the general transformation that takes place is

$$D^n \rightarrow \sum_{k=0}^n \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} A_{k,l} Z^{k-2l} D^{n-k} \alpha^k \beta^l.$$

The constants in the expression are  $\alpha = i\pi\tau 4M p$ ,  $\beta = \frac{M(N_e - 1) + N_e^2 p}{i\pi\tau 2M p}$  and  $A_{k,l}$  is defined as

$$\begin{aligned} A_{k,0} &= \binom{n}{k} \\ A_{k,l} &= -\frac{A_{2l,l-1}}{l} \binom{n-2l}{k-2l}. \end{aligned}$$

Again, since the extra terms that are generated are all proportional to powers of  $Z$  and  $\tilde{D}$ , they can all be removed by row-addition in the determinant is the CF state is proper. This proves that the PWJ wave functions transform nicely under modular transformations.

## VII. DISCUSSION

In this paper we have shown explicitly that the CF wave functions have proper modular properties on the



torus. As part of this work we have also reformulated the PWJ method in  $\tau$ -tau gauge, which is the natural gauge choice for the torus. We have along the way exposed a series of analytical expressions for the projected states that we hope will be useful for future studies of composite fermions on the torus. One limitation of the original PWJ formulation is that it is not applicable for reverse flux states, and we especially hope that this is a step in extending the PWJ method to this class of CF wave functions.

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$U_{s \rightarrow \tau} = e^{-i\pi N_{\phi} xy}$  in such a way that

$$U_{s \rightarrow \tau} G_s f_s(z) = G_{\tau} f_{\tau}(z),$$

where  $G_{\tau} = e^{i\pi\tau N_{\phi} y^2}$  and  $f_{\tau}(z) = e^{-\frac{z^2}{4}} f_s(z)$  is also a holomorphic polynomial. Technically, also  $P_{\text{LLL}}$  is gauge dependent but we suppress that in the analysis below. Applying  $U_{s \rightarrow \tau}$  to the left and right hand sides of eqn. (A1) now gives

### Appendix A: The LLL projection of $\tilde{y}$

We know from the work of Grivin and Jach<sup>GJ84</sup> that we may write the LLL projection as

$$P_{\text{LLL}} \bar{z} G_s f_s(z) = G_s (2\partial_z) f_s(z), \quad (\text{A1})$$

where  $G_s = e^{-\frac{z\bar{z}}{4}}$  and  $f_s(z)$  is a polynomial in  $z$ . Any wave function in symmetric gauge can be transformed into  $\tau$ -gauge with the action of the unitary operator

$$P_{\text{LLL}} \bar{z} G_{\tau} f_{\tau}(z) = G_{\tau} e^{-\frac{z^2}{4}} (2\partial_z) e^{\frac{z^2}{4}} f_{\tau}(z), \quad (\text{A2})$$

which after pulling the  $\partial_z$  through the  $e^{\frac{z^2}{4}}$  gives  $P_{\text{LLL}} \bar{z} G_{\tau} f_{\tau}(z) = G_{\tau} (2\partial_z + z) f_{\tau}(z)$ . Finally after moving  $G_{\tau} z f_{\tau}(z)$  to the left hand side and using that  $P_{\text{LLL}} G_{\tau} z f_{\tau}(z) = G_{\tau} z f_{\tau}(z)$  while identifying  $\bar{z} - z = -2i\tilde{y}$ , we have

$$P_{\text{LLL}} \tilde{y} G_{\tau} f_{\tau}(z) = G_{\tau} (i\partial_z) f_{\tau}(z), \quad (\text{A3})$$

just as in the main text. The generalization to higher powers of  $\tilde{y}$  is straight forward since we can write

$$\begin{aligned} P_{\text{LLL}} \tilde{y}^n G_{\tau} f_{\tau}(z) &= \frac{1}{(-2i)^n} P_{\text{LLL}} (\bar{z} - z)^n G_{\tau} f_{\tau}(z) \\ &= \frac{1}{(-2i)^n} P_{\text{LLL}} \sum_{k=0}^n \binom{n}{k} \bar{z}^k (-z)^{n-k} G_{\tau} f_{\tau}(z) \\ &= \frac{1}{(-2i)^n} G_{\tau} \sum_{k=0}^n \binom{n}{k} e^{-\frac{z^2}{4}} (2\partial_z)^k e^{\frac{z^2}{4}} (-z)^{n-k} f_{\tau}(z) \\ &= \frac{1}{(-2i)^n} G_{\tau} \sum_{k=0}^n \binom{n}{k} (2\partial_z + z)^k (-z)^{n-k} f_{\tau}(z). \end{aligned} \quad (\text{A4})$$

For the second row we expanded  $(\bar{z} - z)^n$ , for the third we used the rule  $\bar{z}^k \rightarrow (2\partial_z)^k$  and for the last row that  $e^{-\frac{z^2}{4}} (2\partial_z)^k e^{\frac{z^2}{4}} = (2\partial_z + z)^k$ . We will now prove that eqn. A4 can be rewritten as the more elegant

$$P_{\text{LLL}} \tilde{y}^n G_{\tau} f_{\tau}(z) = \frac{1}{(-2i)^n} G_{\tau} H_n(\partial_z) f_{\tau}(z)$$

where  $H_n(x)$  is a Hermite polynomial. The proof uses that the Hermite polynomial satisfies the relation  $H_{n+1}(x) = 2xH_n(x) - H'_n(x)$ . Since  $H_n$  has an operator  $\partial_z$  as argument, we can implement the derivative with respect to  $\partial_z$  as  $\frac{\partial}{\partial \partial_z} H_n(\partial_z) = [H_n(\partial_z), z]$ . We then get

the equation

$$H_{n+1}(\partial_z) = 2\partial_z H_n(\partial_z) - [H_n(\partial_z), z] \quad (\text{A5})$$

where we propose that

$$H_n(\partial_z) \stackrel{?}{=} \sum_{k=0}^n \binom{n}{k} (2\partial_z + z)^k (-z)^{n-k} \quad (\text{A6})$$

is a solution. We construct a proof by induction. First we show that  $H_1(\partial_z) = (-z) + (2\partial_z + z) = 2\partial_z$  is trivially true. After some algebra we can show that (A4) satisfies the recursion relation (A5). This is since

$$\begin{aligned}
& 2\partial_z H_n(\partial_z) - [H_n(\partial_z), z] \\
&= \sum_{k=0}^n \binom{n}{k} \left[ 2\partial_z (2\partial_z + z)^k (-z)^{n-k} + (2\partial_z + z)^k (-z)^{n+1-k} + z (2\partial_z + z)^k (-z)^{n-k} \right] \\
&= \sum_{k=0}^n \binom{n}{k} (2\partial_z + z)^{k+1} (-z)^{n-k} + \sum_{k=0}^n \binom{n}{k} (2\partial_z + z)^k (-z)^{n+1-k} \\
&= \left\{ \sum_{k=1}^{n+1} \binom{n}{k-1} + \sum_{k=0}^n \binom{n}{k} \right\} (2\partial_z + z)^k (-z)^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} (2\partial_z + z)^k (-z)^{n+1-k} = H_{n+1}(\partial_z),
\end{aligned}$$

where on line four we used that  $\binom{n}{n+1} = \binom{n}{-1} = 0$ . This concludes the proof.

### Appendix B: The projection operator

In this section we investigate the effect of the GJ trick on the  $n$ :th LL wave function  $\phi_{j,n}^{(M)}$  as defined in (6), where it is also understood that this is always multiplied with a  $N_\phi - M$  flux wave function. If we strip of the leading Gaussian we have the wave function

$$f_{j,n}^{(M)} = \mathcal{N}_n \sum_{k \in \mathbb{Z} + \frac{i}{N_\phi}} e^{i\pi M \tau k^2} H_n(\tilde{y} + \tau_2 Lk) e^{i2\pi M k \frac{\tilde{z}}{L}} \quad (\text{B1})$$

As mentioned in the previous section, we cannot simply replace  $\tilde{y} \rightarrow i\partial_z$ , but the rule is rather that  $\tilde{y}^n \rightarrow \frac{1}{(-2i)^n} H_n(\partial_z)$ . By expanding the Hermite polynomial in powers of  $\tilde{y} + \tau_2 Lk$  we have

$$\begin{aligned}
H_n(\tilde{y} + \tau_2 Lk) &= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} g_{n,m} (\tilde{y} + \tau_2 Lk)^{n-2m} \\
&= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} g_{n,m} \sum_{r=0}^{n-2m} \binom{n-2m}{r} \tilde{y}^{n-2m-r} (\tau_2 Lk)^r,
\end{aligned}$$

where we used the expansion  $H_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} g_{n,m} x^{n-2m}$  and  $g_{n,m} = \frac{n!(-1)^m 2^{n-2m}}{m!(n-2m)!}$ . We note that we can write  $(\tau_2 Lk)^r e^{i2\pi M k \frac{\tilde{z}}{L}} \cdot 1 = \left(-i\partial_z \frac{N_\phi}{M}\right)^r e^{i2\pi M k \frac{\tilde{z}}{L}} \cdot 1$ . This allows us to write (B1) as

$$\begin{aligned}
\hat{f}_{j,n}^{(M)} &= \frac{\mathcal{N}_n}{\mathcal{N}_0} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} g_{n,m} \sum_{r=0}^{n-2m} \binom{n-2m}{r} \times \\
&\quad \times \tilde{y}^{n-2m-r} \left[ \left(-i\partial_z \frac{N_\phi}{M}\right)^r f_{j,0} \right],
\end{aligned}$$

where the  $[\dots]$  signifies that the derivative does not act outside of the square bracket box. After the projection step this becomes

$$\begin{aligned}
\hat{f}_{j,n}^{(M)} &= \frac{\mathcal{N}_n}{\mathcal{N}_0} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} g_{n,m} \sum_{r=0}^{n-2m} \binom{n-2m}{r} \times \\
&\quad \times \frac{H_{n-2m-r}(\partial_z)}{(-2i)^{n-2m-r}} \left[ \left(-\partial_z \frac{N_\phi}{M}\right)^r f_{j,0} \right].
\end{aligned}$$

However, due to a clever re-summation of Hermite polynomials (which we will not demonstrate) we have the much cleaner result

$$\hat{f}_{j,n}^{(M)} = \sum_{k=0}^n \binom{n}{k} M^{n-k} \partial_z^{n-k} \left[ (-N_\phi)^k \partial_z^k f_{j,0} \right], \quad (\text{B2})$$

where we have once again dropped  $\frac{(2i)^n \mathcal{N}_n}{M^n \mathcal{N}_0}$  just as in the main text.

### Appendix C: Operators with periodic boundary conditions

Similarly to the relation  $e^{i\pi\tau M y^2} f_{j,n}^{(M)} = \phi_{j,n}^{(M)}$  in the main text, we may now define an operator equivalent of the LLL projector  $e^{i\pi\tau N_\phi y^2} \hat{f}_{j,n}^{(M)} = \hat{\phi}_{j,n}^{(M)} e^{i\pi\tau(N_\phi - M)y^2}$  for a general  $n$ :th Landau level. We may express  $\hat{\phi}_{j,n}^{(M)}$  as a series expansion in

$$\hat{g}_n = \hat{f}_n^{(M)} \Big|_{f_0 \rightarrow \phi_0} = \left( M\partial_z - \tilde{\partial}_z N_\phi \right)^n \phi_0, \quad (\text{C1})$$

where we simply replace the  $f_0$  in  $\hat{f}_{j,n}^{(M)}$  by  $\phi_0$ . It is straight forward to show that the operator  $\hat{g}_n$  satisfies the desired periodicity boundary conditions

$$e^{i2\pi N_\phi x \tilde{t}(\tau L)} \hat{g}_n = \hat{g}_n \tilde{t}(\tau L) e^{i2\pi(N_\phi - M)x}$$

by repeating the arguments that were used in conjunction with eqn. (16). The only difference is that now its an exponential of  $x$  and not  $z$  that is considered. However, since  $[\partial_z, [\partial_z, x]] = 0 = [\partial_z, [\partial_z, z]]$  the calculation is identical.

Considering now the function

$$\hat{\phi}_n^{(M)} = e^{i\pi\tau N_\phi y^2} \hat{f}_n^{(M)} e^{-i\pi\tau(N_\phi - M)y^2},$$

we can use (C1) to argue that  $\hat{\phi}_n^{(M)} \neq \hat{g}_n$  but that there will also be sub leading terms proportional to  $\hat{g}_{n-2}, \hat{g}_{n-4}, \dots, \hat{g}_0$ . Unlike the arguments that were used in conjunction with eqn. (16) we are now pulling exponentials of  $y^2$  through  $\hat{D}$ , and since  $[\partial_z, [\partial_z, y^2]] \neq 0$  the shifts of  $\partial_z$  and  $\tilde{\partial}_z$  cannot be applied independently. This is what leads to the sub leading terms. If we define  $\chi = M(N - M)N_\phi \frac{\pi\tau}{2i\tau_2^2 L^2} = \frac{M(N-M)\tau}{4i\tau_2}$  then we may explicitly show that

$$\begin{aligned} \hat{\phi}_1 &= \hat{g}_1 \\ \hat{\phi}_2 &= \hat{g}_2 + 1\chi\hat{g}_0 \\ \hat{\phi}_3 &= \hat{g}_3 + 3\chi\hat{g}_1 \\ \hat{\phi}_4 &= \hat{g}_4 + 6\chi\hat{g}_2 + 3\chi^2\hat{g}_0 \\ \hat{\phi}_5 &= \hat{g}_5 + 10\chi\hat{g}_3 + 15\chi^2\hat{g}_1 \\ \hat{\phi}_6 &= \hat{g}_6 + 15\chi\hat{g}_4 + 45\chi^2\hat{g}_2 + 15\chi^3\hat{g}_0 \\ \hat{\phi}_7 &= \hat{g}_7 + 21\chi\hat{g}_5 + 105\chi^2\hat{g}_3 + 105\chi^3\hat{g}_1 \\ \hat{\phi}_8 &= \hat{g}_8 + 28\chi\hat{g}_6 + 210\chi^2\hat{g}_4 + 420\chi^3\hat{g}_2 + 105\chi^4\hat{g}_0. \end{aligned}$$

This may be summarized as

$$\hat{\phi}_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} T(n, k) \chi^k \hat{g}_{n-2k}, \quad (\text{C2})$$

where  $T(n, k)$  is the triangle of Bessel numbers (OEIS series A100861)<sup>Inc18</sup>.