TILING-BASED MODELS OF PERIMETER AND AREA

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Abstract. We consider polygonal tilings of certain regions and use these to give intuitive definitions of tiling-based perimeter and area. We apply these definitions to rhombic tilings of Elnitsky polygons, computing sharp bounds and average values for perimeter tiles in convex centrally symmetric $2n$-gons. These bounds and values have Coxeter-theoretic implications for the commutation classes of the longest element in the symmetric group. We also classify the permutations whose polygons have minimal perimeter, defined in two different ways, and we conclude by looking at some of these questions in the context of domino tilings of rectangles, giving a recursive formula and generating function for one family. Throughout the work, we contrast the tiling-based results that we obtain with classical contour-based isoperimetric results.

1. Introduction

Isoperimetric problems involve relationships between area and perimeter, and their study has a long history in geometry and analysis (see, for example, [16, 23, 24]). One can phrase this type of problem from two vantagepoints: fix the area and try to minimize perimeter, or fix the perimeter and try to maximize area. The "answer" to the classical *contour-based* version of this problem, which was long suspected and has been known for over a century, is a circle.

Theorem 1.1 (Classical contour-based isoperimetric theorem). Let Ω be a region bounded by a simple closed curve. The Lebesgue measures for the length $P(\Omega)$ of that closed curve and the area $A(\Omega)$ of its enclosure Ω, are related by

$$4\pi A(\Omega) \leq P(\Omega)^2,$$

with equality if and only if the curve is a circle and Ω is a disk.

Put another way, in the contour-based setting, among regions with fixed area, circles have the minimum perimeter.

Discrete versions of isoperimetric problems exist in the literature, particularly within the context of graph theory (see, for example, [5, 13, 30]). Another discrete-flavored field of isoperimetric research concerns least-perimeter (under the Lebesgue measure) tilings of the plane, subject to possible tile restrictions, as discussed in [6, 12].

In recent work with Duchin, we studied a frequent application of the isoperimetric problem to political science, and analyzed important flaws in that usage [8]. More precisely, in legal and political science literature, isoperimetric quotients (see Definition 6.1) of geographic regions are computed using contour-based measures of area and perimeter. In that literature, these quotients are known as *Polsby-Popper scores*, and they serve as a metric of *compactness*,

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or “shape quality,” of electoral districts and districting plans. These scores, and their relative rankings, have legal ramifications and electoral impact. However, contour-based calculations rely inherently on the way regions and curves are drawn on a surface, whereas, as discussed in [8], the political redistricting problem begins, implicitly, with a graph of census units and their geographical/graphical adjacency. Any notion of “area” or “perimeter” for a collection of those units (such as those forming a district) should acknowledge this underlying structure. In particular, it should recognize that any contour-based interpretation has actually been imposed upon the underlying discrete data. As such, a shape analysis of districts warrants a correspondingly discrete set of tools, rather than the ones that have heretofore been applied. As discussed in [8], forcing contour-based measures of area and perimeter in a discrete setting — as Polsby-Popper scoring of political districts has historically done — raises important concerns. In addition to the issues discussed in [8], we show here that there are natural definitions of area and perimeter in discrete tiling-based frameworks that do not necessarily exhibit the same behavior as those in the contour-based setting.

This paper takes the discretization of isoperimetric questions in a tiling-based direction. We turn our attention to polygonal tilings of bounded regions, and study the relationship between natural notions of “area” and “perimeter” in this context. We will use rhombic tilings of so-called “Elnitsky polygons” as our primary mechanism for this analysis. Such tilings of this family of polygons have algebraic significance, and the area and perimeter metrics that we develop for them will have significance as well. Another major area of tiling research concerns domino tilings, and we conclude this paper by turning our isoperimetric attention in that direction. As we will show, perimeter in tiling-based settings behaves in notably different ways than it does in contour-based ones.

In our work here, we will consider *bounded*, *nonempty*, and *contiguous* regions. This makes sense for several reasons. First, there is nothing to say about an empty region. Second, the area (respectively, perimeter) of an arbitrary region can be computed by summing the areas (respectively, perimeters) of its connected components — in both a contour-based setting and in the discrete setting that we study here. Third, our contiguity requirement echoes the contiguity requirement of many political districting guidelines, which, as studied in [8], is one of the main motivations of this work. Indeed, when we fix area and look at perimeter behavior in these tiling-based models, that behavior is interesting and not always what one might have assumed from the contour-based results.

In Section 2 of this work, we present motivating material from the theory of Coxeter groups. This provides algebraic context to this work, as discussed in Section 3, namely in Corollary 3.5. In Sections 4 and 5, we look at tling-based perimeter questions for regular $2n$-gons, both in the extreme and average cases. We give sharp upper and lower bounds for the number of perimeter tiles in rhombic tilings of these regions in Theorems 4.5 and 4.6, and present the Coxeter-theoretic significance of these results in Corollary 4.7. The average cases is discussed in Theorem 5.9 via a recursive formula, with data given in Table 2. Section 6 turns these isoperimetric questions to arbitrary permutations, and the main results of that section are Theorems 6.8 and 6.9. Following this work with rhombic tilings, we turn briefly to domino tilings of rectangles in Section 7, computing extreme and average values (Propositions 7.4 and 7.6, and Corollary 7.7), and demonstrating that the result of the contour-based isoperimetric theorem does not carry over directly to this tiling-based setting. We conclude the paper with a discussion of possible topics for future research.

2. Background: reduced words, commutation classes, Elnitsky's bijection

In this section, we introduce the Coxeter-theoretic objects and terminology that will be required for our work. The reader is referred to [4] for more details.

Coxeter groups are generated by simple reflections. Minimally long expressions of a Coxeter group element in terms of these simple reflections are *reduced decompositions* of that element. For a group element w, we write $R(w)$ for the collection of these reduced decompositions. The set $R(w)$ can be partitioned by the *commutation* relation $\sim$, identifying reduced decompositions when they differ only by a sequence of commutation moves. The result is the collection $C(w) = R(w)/\sim$ of *commutation classes* of w.

For the purposes of this paper, we are concerned with the finite Coxeter group of type A: the *symmetric group*.

Definition 2.1. Let $\mathfrak{S}_n$ be the symmetric group on $[n] := \{1, \ldots, n\}$. For $i \in [n-1]$, let s_i be the simple reflection transposing i and $i+1$. The elements of $\mathfrak{S}_n$ are *permutations*, and we can write a permutation w as a product of these generators or in *one-line notation* as $w(1)w(2)\cdots w(n)$. The minimal length of a product of simple reflections needed to represent w is the *length* of w, denoted $\ell(w)$, and this is equal to the number of *inversions* in w: $\ell(w) = \#\{i < j : w(i) > w(j)\}$.

Permutations are maps, and so we interpret their compositions as follows: ws_i transposes the values in positions i and $i+1$ in w, while s_iw transposes the positions of the values i and $i+1$ in w.

Example 2.2. Consider the permutation $42153 \in \mathfrak{S}_5$. It has eleven reduced decompositions

$$\begin{aligned}R(42153) = \{&s_1s_3s_2s_1s_4, s_1s_3s_2s_4s_1, s_1s_3s_4s_2s_1, s_3s_1s_2s_1s_4, s_3s_1s_2s_4s_1, s_3s_1s_4s_2s_1,\\ &s_3s_2s_1s_2s_4, s_3s_2s_1s_4s_2, s_3s_2s_4s_1s_2, s_3s_4s_1s_2s_1, s_3s_4s_2s_1s_2\},\end{aligned}$$

and two commutation classes

$$\begin{aligned}C(42153) = \Big\{&\{s_3s_2s_1s_2s_4, s_3s_2s_1s_4s_2, s_3s_2s_4s_1s_2, s_3s_4s_2s_1s_2\},\\ &\{s_1s_3s_2s_1s_4, s_1s_3s_2s_4s_1, s_1s_3s_4s_2s_1, s_3s_1s_2s_1s_4, s_3s_1s_2s_4s_1, s_3s_1s_4s_2s_1, s_3s_4s_1s_2s_1\}\Big\}.\end{aligned}$$

Reduced decompositions, commutation classes, and related objects, are of interest from both algebraic and enumerative perspectives. Examples of work on these topics include [1, 2, 3, 10, 11, 14, 17, 22, 26, 27, 28, 31]. There is a wealth of interesting mathematics in these objects, and many open questions. In [10], Elnitsky gave a bijection between the commutation classes $C(w)$ and the rhombic tilings of a polygon $X(w)$. We explored that relationship further in [27, 28]. Elnitsky's bijection provides important context for the work of this paper. We take time now to describe it and to give examples of its utility.

Definition 2.3. For a permutation $w \in \mathfrak{S}_n$, the polygon $X(w)$ is an equilateral $2n$-gon defined as follows: starting at the topmost vertex, label the sides $1, \ldots, n, w(n), \ldots, w(1)$ in counterclockwise order; the first (leftmost) n of these sides (labeled $1, \ldots, n$) form half of a convex $2n$-gon; the remaining n sides are drawn so that sides are parallel if and only if they have the same label. This $X(w)$ is *Elnitsky's polygon* for w. A polygon obtained in this way for some permutation is an *Elnitsky polygon*.

The keen reader might make several observations about $X(w)$. First, if $\{w(1), \ldots, w(r)\} = \{1, \ldots, r\}$ for some $r < n$, then the left and right edges of the "polygon" $X(w)$ would share (at least) a vertex. This would mean that the interior of $X(w)$ is not contiguous. Hence, throughout this paper, we make the following requirements to ensure nonempty and contiguous polygonal area.

Requirement 2.4. Throughout this paper, if $w \in \mathfrak{S}_n$ is a permutation, then

(a) $n > 1$, and
(b) $\{w(1), \ldots, w(r)\} \neq \{1, \ldots, r\}$ for all $r < n$.

Second, no angles have been specified in Definition 2.3, besides requiring that the left half of the region be convex. In fact, the requirement that $X(w)$ be equilateral is also unnecessary, and could be replaced by requiring that same-labeled sides be congruent (as well as parallel). The only impact of this is that we will tile Elnitsky's polygon by equilateral tiles (rhombi) instead of by more general ones (parallelograms). This is a superficial distinction and has no impact on the mathematics. Indeed, the angular freedom means that for a given w, there are infinitely many ways to draw $X(w)$. However, because they are all combinatorially equivalent, we will refer to all such possibilities as "$X(w)$" without distinction.

Example 2.5. Continuing Example 2.2, Elnitsky's polygon $X(42153)$ appears in Figure 1.

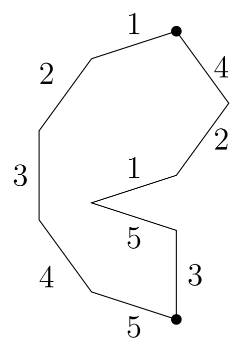


FIGURE 1. Elnitsky's polygon $X(42153)$.

To prevent drawings from becoming cluttered, we will omit the edge labels in $X(w)$. However, we will compensate by marking the top and bottom vertices of the polygon throughout this work, and the side labels can be recovered using Definition 2.3. It will be helpful for us to refer to the left and right halves of $X(w)$ by name, and we will use the marked top and bottom vertices to make this convention.

Definition 2.6. The counterclockwise path from the top vertex to the bottom vertex in an Elnitsky polygon will be called the *leftside* boundary of the polygon, and the clockwise path from the top vertex to the bottom vertex will be called the *rightside* boundary. Note that the top and bottom vertices are in both the leftside and rightside boundaries of $X(w)$.

We endow a tiling and its tiles with top/bottom/left/right directionality that is consistent with that of $X(w)$.

Elnitsky's work gives a bijection between commutation classes of a permutation and rhombic tilings of these polygons.

Definition 2.7. For a permutation w, the set $T(w)$ consists of the rhombic tilings of $X(w)$ in which all tile edges are congruent and parallel to edges of $X(w)$. Throughout this work, the phrase *rhombic tiling*, in reference to an Elnitsky polygon $X(w)$, will be understood to refer to an element of $T(w)$.

For $w \in \mathfrak{S}_n$, the allowable tiles in elements of $T(w)$ mean that any shortest path from the top vertex of $X(w)$ to the bottom vertex has exactly n edges. We use this to describe the vertical position of tiles appearing in these tilings.

Definition 2.8. Fix a permutation w and a tiling $T \in T(w)$. Let t be a tile in T. If the topmost vertex of t is d edges from the top vertex of $X(w)$ in any shortest path, then we will say that the tile t has *depth* $d + 1$.

Thus, for $w \in \mathfrak{S}_n$, the depth fiunction may take values between 1 and $n - 1$.

We are now ready to state Elnitsky's bijection.

Theorem 2.9 ([10, Theorem 2.2])**.** Fix a permutation $w \in \mathfrak{S}_n$ and a tiling $T \in T(w)$. There are $\ell(w)$ tiles in T, where $\ell(w)$ is the length of w. Label the tiles $1, 2, \ldots, \ell(w)$ so that the rightmost edges of a tile t are shared with the rightside boundary of $X(w)$ and/or with tiles whose labels are less than the label of t. (Equivalently, the rightmost edges of a tile t are not shared with any tiles whose labels are greater than the label of t.) This labeling corresponds to a reduced decomposition

$$s_{i_{\ell(w)}} \cdots s_{i_2} s_{i_1} \in R(w),$$

where $i_a = d$ if the tile labeled a has depth d. Although a given tiling may admit multiple labelings, all reduced decompositions obtained from them will be in the same commutation class. Moreover, the correspondence

$$T \mapsto s_{i_{\ell(w)}} \cdots s_{i_2} s_{i_1} / \sim$$

is a bijection between $T(w)$ and $C(w)$.

Giving more detail to the conclusion of Theorem 2.9, any differences between the reduced decompositions produced by label variation can be entirely addressed via commutations of the group generators, and the converse implication holds as well. In this way, a rhombic tiling of $X(w)$ produces an entire (and unique) commutation class of $R(w)$. We note that Elnitsky's bijection goes beyond the statement given here, addressing graphs produced by $T(w)$ and $C(w)$, but the details of that extension are not relevant to this work.

Example 2.10. Continuing Example 2.2, there are two rhombic tilings of $X(42153)$. The tiling depicted in Figure 2(a) corresponds to the first commutation class in Example 2.2 (which consisted of four reduced decompositions), and the tiling depicted in Figure 2(b) corresponds to the second (which had seven reduced decompositions). More precisely, the four reduced decompositions in the commutation class

$$\{s_3 s_2 s_1 s_2 s_4, s_3 s_2 s_1 s_4 s_2, s_3 s_2 s_4 s_1 s_2, s_3 s_4 s_2 s_1 s_2\}$$

correspond to the four labelings of the tiling in Figure 2(a). These four labelings are depicted, respectively, in Figure 3.

The definition of Elnitsky's polygon and its rhombic tilings has additional implications for what sorts of tiles may appear in elements of $T(w)$.

Definition 2.11. If the edges of a tile t in $T \in T(w)$ are parallel to the sides labeled a and b in $X(w)$, then we will say that the *edge labels* of t are $\{a, b\}$.

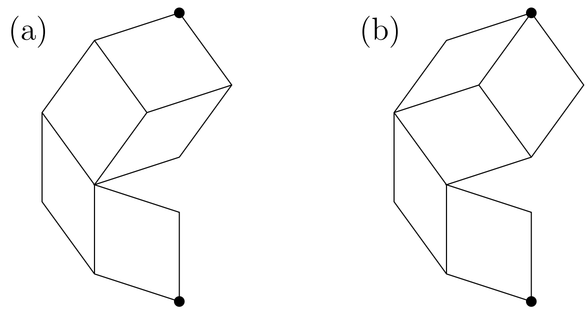


FIGURE 2. The rhombic tilings $T(42153)$, discussed in Example 2.10.

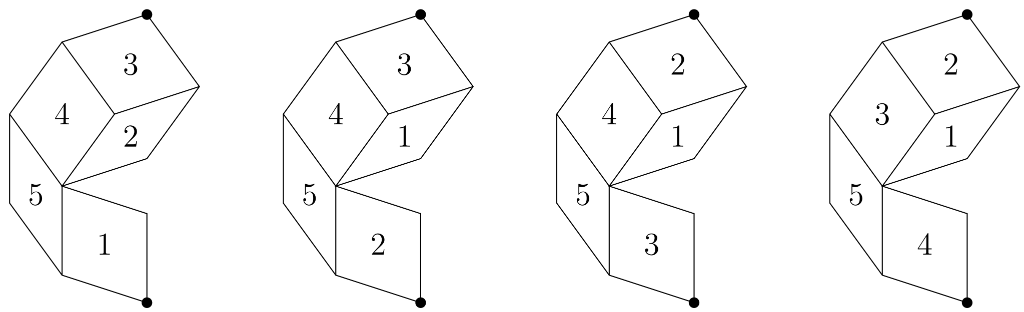


FIGURE 3. The four permitted labelings of the tiling in Figure 2(a). These labelings correspond to the four elements in one of the commutation classes of $R(42153)$.

In Elnitsky's bijection, each tile corresponds to an inversion in the permutation w. Moreover, if $i < j$ and $w(i) > w(j)$, then each $T \in T(w)$ contains a tile whose edge labels are $\{w(i), w(j)\}$.

Corollary 2.12. For any permutation w and any $T \in T(w)$, no two tiles in T have the same edge labels.

Proof. For each $\{w(i) > w(j)\}$ such that $i < j$, there is a tile in T with edge labels $\{w(i), w(j)\}$. Since the number of tiles in T is $\ell(w)$, which is equal to the number of such $\{w(i) > w(j)\}$, each pair of such edge labels appears exactly once and no other pairs of edge labels appear at all. $\square$

3. Perimeter tiles and their Coxeter-theoretic significance

Consider a simple closed curve and the region that it encloses. If that region is tiled (partitioned) by some collection of shapes, then we can identify the tiles that intersect the boundary nontrivially. Suppose that the curve, region, and tiling are *discrete* in the sense that all shapes (region and tiles) are polygons, and tile edges only ever meet other edges in their entirety. Then there is an intuitive tiling-based distance metric in this setting: each tile edge has "discrete length" one, and the "discrete length" of a path is the number of tile edges that the path contains. Similarly, there is an intuitive notion of "discrete area," as defined below.

Definition 3.1. Let R be a region with polygonal boundary, and T a tiling of R in which all tiles are polygons and tile edges only ever meet other edges in their entirety. The *area* of this tiling of R is the number of tiles in T.

Note that if the number of tiles in a tiling of R is independent of the tiling itself, as will be the case with all tilings that we study in this paper, then the notion of area in Definition 3.1 will not depend on the tiling T.

Because the regions and tiles that we study are all polygons, the interesting question about tiling behavior along the boundary of R is: how many tiles share multiple edges with the boundary of R? In other words, conspicuous tile/boundary intersection occurs when there is prolonged (multi-edge) overlap.

Definition 3.2. Let R be a region with polygonal boundary, and T a tiling of R in which all tiles are polygons and tile edges only ever meet other edges in their entirety. A tile in T that shares a path of discrete length one with the boundary of R, and shares no longer path with that boundary, is a *weak perimeter tile*. A tile in T that shares a path of discrete length at least two with the boundary of R is a *strong perimeter tile*. The focus of this paper is on strong perimeter tiles, and we will henceforth refer to strong perimeter tiles as *perimeter* tiles, distinguishing weak perimeter tiles by use of the modifier.

Our interest in (strong) perimeter tiles is due to the fact, mentioned above, that a tile has exceptional boundary interaction when that tile is positioned so that it shares more than one consecutive edge with the boundary of R.

We now specialize to the tilings $T(w)$ of Elnitsky's polygon $X(w)$ for permutations w. In a moment, we will discuss the algebraic significance of perimeter tiles in this setting, which will reinforce the mathematical significance of (strong) perimeter tiles. First, however, we will classify those perimeter tiles in four, possibly overlapping, ways.

Definition 3.3. Fix a permutation w and consider a rhombic tiling of Elnitsky's polygon $X(w)$. If, in such a tiling, a perimeter tile t includes ...

- two edges from the leftside boundary of $X(w)$, then t is a *left-perimeter tile*;
- two edges from the rightside boundary of $X(w)$, then t is a *right-perimeter tile*;
- the two boundary edges to the left and right of the top vertex of $X(w)$, then t is a *top-perimeter tile*;
- the two boundary edges to the left and right of the bottom vertex of $X(w)$, then t is a *bottom-perimeter tile*.

The *type* of a perimeter tile is its classification as left-, right-, top-, and/or bottom-. We may use the term *side-perimeter* to refer to tiles whose types are left- or right-.

We demonstrate Definition 3.3 with the tiling depicted previously in Figure 2(a).

Example 3.4. The tiling of $X(42153)$ shown in Figure 2(a) has one left-perimeter tile, two right-perimeter tiles, one top-perimeter tile, and one bottom-perimeter tile. The bottom-perimeter tile is also a right-perimeter tile. These type classifications are indicated in Figure 4.

The Coxeter-theoretic significance of perimeter tiles follows directly from Elnitsky's bijection (Theorem 2.9).

Corollary 3.5. Fix a permutation $w \in \mathfrak{S}_n$ and consider the tilings $T(w)$. There exists a tiling containing a perimeter tile t of depth d such that t is a ...

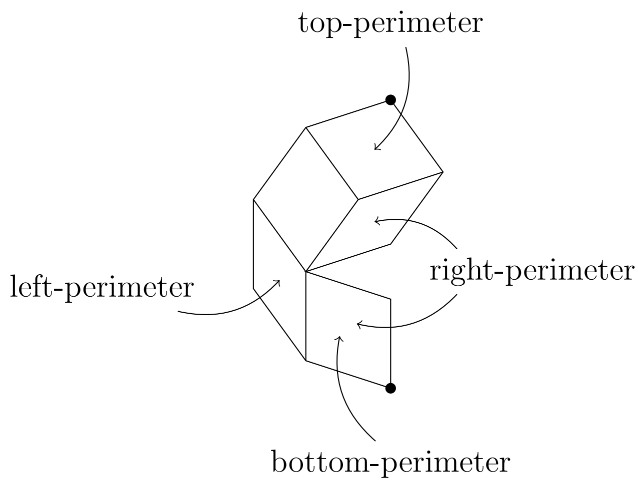


FIGURE 4. An element of $T(42153)$ with its four perimeter tiles labeled by type.

- left-perimeter tile if and only if $\ell(s_d w) < \ell(w)$; equivalently, $w^{-1}(d) > w^{-1}(d+1)$; equivalently, there exists a reduced decomposition of w in which s_d is the leftmost letter in the product.
- right-perimeter tile if and only if $\ell(ws_d) < \ell(w)$; equivalently, $w(d) > w(d+1)$; equivalently, there exists a reduced decompositions of w in which s_d is the rightmost letter in the product.

Moreover, there exists a tiling containing a ...

- top-perimeter tile if and only if there exists a commutation class $C \in C(w)$ in which there is exactly one s_1 in each of the reduced decompositions in C.
- bottom-perimeter tile if and only if there exists a commutation class $C \in C(w)$ in which there is exactly one s_{n-1} in each of the reduced decompositions in C.

The implications of Corollary 3.5 can be demonstrated by the ongoing example.

Example 3.6. Let $w = 42153$. In the tilings of $X(w)$ (see Figure 2), there exist left-perimeter tiles of depths 1 and 3, right-perimeter tiles of depths 1, 2, and 4, top-perimeter tiles, and bottom-perimeter tiles. The left-perimeter tiles indicate that the only leftmost letters that appear in reduced decompositions of w are s_1 and s_3. The right-perimeter tiles indicate that the only rightmost letters that appear in reduced decompositions of w are s_1, s_2, and s_4. The top-perimeter tile indicates that there is a commutation class in which all reduced decompositions contain exactly one s_1, and the bottom-perimeter tiles indicate that there are two commutation classes in which all reduced decompositions contain exactly one s_4. Moreover, because one tiling of $X(w)$ has both a top- and a bottom-perimeter tile, there is a commutation class in which each element contains exactly one s_1 and exactly one s_4. These conclusions confirm the details of Example 2.2.

It is clear that the perimeter tiles in elements of $T(w)$ have Coxeter-theoretic significance. The purpose of this paper is to understand when, how, and how often these perimeter tiles occur.

We start with an easy observation about an upper bound for the number of perimeter tiles that can occur in any tiling. This observation will be tightened later in the paper.

Proposition 3.7. For any permutation $w \in \mathfrak{S}_n$ and any $T \in T(w)$, there are at most n perimeter tiles in T.

Proof. The polygon $X(w)$ has $2n$ edges. Each perimeter tile accounts for at least two of these edges, and no edge of $X(w)$ can belong to more than one tile. Therefore there can be at most n perimeter tiles in T. □

For an initial lower bound on the number of perimeter tiles that can appear in a tiling, we draw an immediate consequence from Elnitsky's theorem and Corollary 3.5.

Corollary 3.8. For any permutation w and any $T \in T(w)$, there is at least one left-perimeter tile and at least one right-perimeter tile in T.

Note that Corollary 3.8 does not require that those left- and right-perimeter tiles be distinct. However, it is clear from Theorem 2.9 and Requirement 2.4(b) that the only situation in which distinct left- and right-perimeter tiles cannot be found is when $w = 21 \in \mathfrak{S}_2$.

4. Perimeter tiles for the longest element: extremal cases

We devote this section and the next to understanding perimeter tiles that appear in $X(w_0^{(n)})$, where

$$w_0^{(n)} = n(n-1)\cdots 321$$

is the longest element in $\mathfrak{S}_n$. Recall Requirement 2.4(a), that $n > 1$ throughout this discussion. The element $w_0^{(n)}$ and its properties are of particular Coxeter-theoretic interest, and have been studied often (see, for example, [19, 25] as well as the works cited previously).

Definition 4.1. For a permutation w and a rhombic tiling $T \in T(w)$, let

$$\mathsf{perim}(T)$$

be the number of (strong) perimeter tiles in T. When tile type is relevant, the functions $\mathsf{L\text{-}perim}$, $\mathsf{R\text{-}perim}$, $\mathsf{T\text{-}perim}$, and $\mathsf{B\text{-}perim}$ will be used in the obvious way. We say that the value of $\mathsf{perim}(T)$ measures the *strong perimeter* of the tiling T. If all $T \in T(w)$ have the same strong perimeter, then this value is the *strong perimeter* of w.

Notice that for any $w \in \mathfrak{S}_n$ and $T \in T(w)$, $\mathsf{L\text{-}perim}(T)$ and $\mathsf{R\text{-}perim}(T)$ are integers between 0 and $n-1$, while $\mathsf{T\text{-}perim}(T)$ and $\mathsf{B\text{-}perim}(T)$ are indicator variables taking values 0 or 1.

The total number of weak and strong perimeter tiles in a tiling $T \in T(w)$ (that is, the number of tiles sharing paths of positive length with the boundary of $X(w)$) is less than or equal to

$$2n - \mathsf{perim}(T),$$

depending on whether any weak or strong perimeter tiles in T share more than the minimally required number of edges with the boundary of $X(w)$. In the case of weak perimeter tiles, this refers to tiles that intersect the boundary in two nonconsecutive edges, necessarily on the leftside and rightside boundaries of $X(w)$. There is certainly a case to be made for studying this quantity. However, as it depends so closely on the statistic perim, and because of the Coxeter-theoretic relevance of (strong) perimeter tiles discussed in Section 3, we will focus our attention on the statistic perim in this paper.

Our intention is to understand the statistic $\mathsf{perim}(T)$, particularly when $w = w_0^{(n)}$. When n is clear from context, we may write $w_0 := w_0^{(n)}$. Because $X(w_0^{(n)})$ is a convex centrally symmetric $2n$-gon, we can make use of its symmetries in our arguments, and we start this section with an elementary, but important, observation about regular polygons. It simplifies the argument, and does not change the result, to assume, for the moment, that $X(w_0)$ is equiangular.

Lemma 4.2. Consider a regular $2n$-gon X and the set $T(X)$ of rhombic tilings of X, in which the sides of each rhombus are congruent and parallel to sides of X. For any $T \in T(X)$, let T' be the result of rotating T about the center of X by an integer multiple of $(2\pi)/(2n)$ radians. Then $T' \in T(X)$.

Proof. This follows from the rotational symmetry of X. □

Although Elnitsky's polygon $X(w_0)$ need not be equiangular, the result of Lemma 4.2 applies: rhombic tilings of $X(w_0)$ can be rotated to produce other (not necessarily distinct) rhombic tilings of $X(w_0)$. An example of this is depicted in Figure 5.

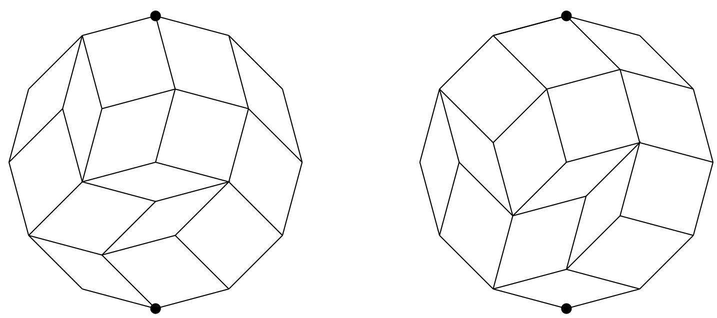


FIGURE 5. Two rhombic tilings of $X(w_0^{(6)})$, which differ by a rotation.

Proposition 4.3. Over all rhombic tilings of $X(w_0^{(n)})$, the following quantities are equal:

- the total number of top-perimeter tiles that appear,
- the total number of bottom-perimeter tiles that appear,
- the total number of left-perimeter tiles of depth d that appear, for $d \in [n-1]$, and
- the total number of right-perimeter tiles of depth d that appear, for $d \in [n-1]$.

Proof. Suppose, without loss of generality, that $X(w_0)$ is equiangular. Then, by Lemma 4.2, there is nothing special about a perimeter tile appearing at, say, depth 1 along the rightside boundary of $X(w_0)$, and so no perimeter location is preferred to any other amongst all rhombic tilings of $X(w_0)$. □

Put another way, Proposition 4.3 says that if L_d-perim and R_d-perim are indicator variables that detect depth d left-, and depth d right-perimeter tiles, respectively, then

$$\sum_{T \in T(w_0)} \mathsf{T}\text{-}\mathsf{perim}(T) = \sum_{T \in T(w_0)} \mathsf{B}\text{-}\mathsf{perim}(T) = \sum_{T \in T(w_0)} \mathsf{L}_d\text{-}\mathsf{perim}(T) = \sum_{T \in T(w_0)} \mathsf{R}_d\text{-}\mathsf{perim}(T),$$

for any $d \in [n-1]$. In other words, the total number of each type of perimeter tile (at fixed depth, if the type is left- or right-) is dependent only on n, and not on the type itself. We demonstrate Proposition 4.3 using the eight rhombic tilings of $X(w_0^{(4)})$.

Example 4.4. Among the eight rhombic tilings of $X(w_0^{(4)})$ (depicted in Figure 6), there are three appearances each of top-perimeter tiles, bottom-perimeter tiles, left-perimeter tiles of any fixed depth, and right-perimeter tiles of any fixed depth.

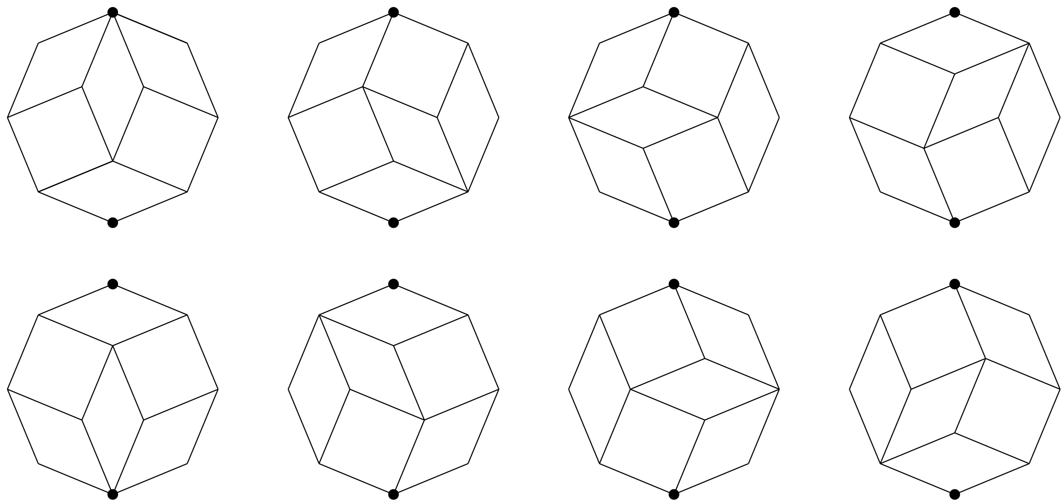


FIGURE 6. The eight rhombic tilings of $X(w_0^{(4)})$.

Our first step in the study of perimeter tiles for w_0 will be to bound (sharply) the number of perimeter tiles that may appear in elements of $T(w_0)$. By Corollary 3.5, this will have implications for commutation classes of $R(w_0)$.

One perspective on Corollary 3.8, in light of Lemma 4.2, is that at least one perimeter tile appears in its entirety among any n consecutive boundary edges of $X(w_0^{(n)})$. In other words, there cannot be $n-1$ consecutive boundary edges that intersect no perimeter tiles. (This need not be true for arbitrary permutations w, because it relies on the convexity of $X(w_0^{(n)})$.) This allows us to advance the result of Corollary 3.8 in this setting.

Note that in the following theorem, and in some subsequent results, we require $n > 2$. This is because $X(21)$ is, itself, a rhombus, and its sole rhombic tiling is just a single rhombus. That single tile is, simultaneously, a perimeter tile of all four types, and would be overcounted.

Theorem 4.5. For any $n > 2$,

$$\min\left\{\mathsf{perim}(T) : T \in T(w_0^{(n)})\right\} = 3.$$

Moreover, there exists $T \in T(w_0^{(n)})$ with $\mathsf{perim}(T) = 3$, in which only two of the three perimeter tiles are side-perimeter tiles.

Proof. From Corollary 3.8, we know that T has at least one left-perimeter tile and at least one right-perimeter tile. The question, then, is whether these could be the only perimeter tiles in T. As remarked above, there must be at least one perimeter tile contained entirely among any n consecutive boundary edges of $X(w_0)$.

The polygon $X(w_0)$ has $2n$ edges, and so the only way to position a left-perimeter tile and a right-perimeter tile so that there are $n-2$ edges between them in each direction is if the left-perimeter tile has depth d and the right-perimeter tile has depth $n-d$. This would mean that the edge labels of the left-perimeter tile are $\{d, d+1\}$, and the edge labels of the right-perimeter tile are $\{w(n-d), w(n-d+1)\} = \{d+1, d\}$. This contradicts Corollary 2.12. Therefore, $\mathsf{perim}(T) \geq 3$.

As shown in Figure 7, there are tilings for which $\mathsf{perim}(T) = 3$, and it is possible for only two of the required three perimeter tiles in T to be side-perimeter tiles. $\square$

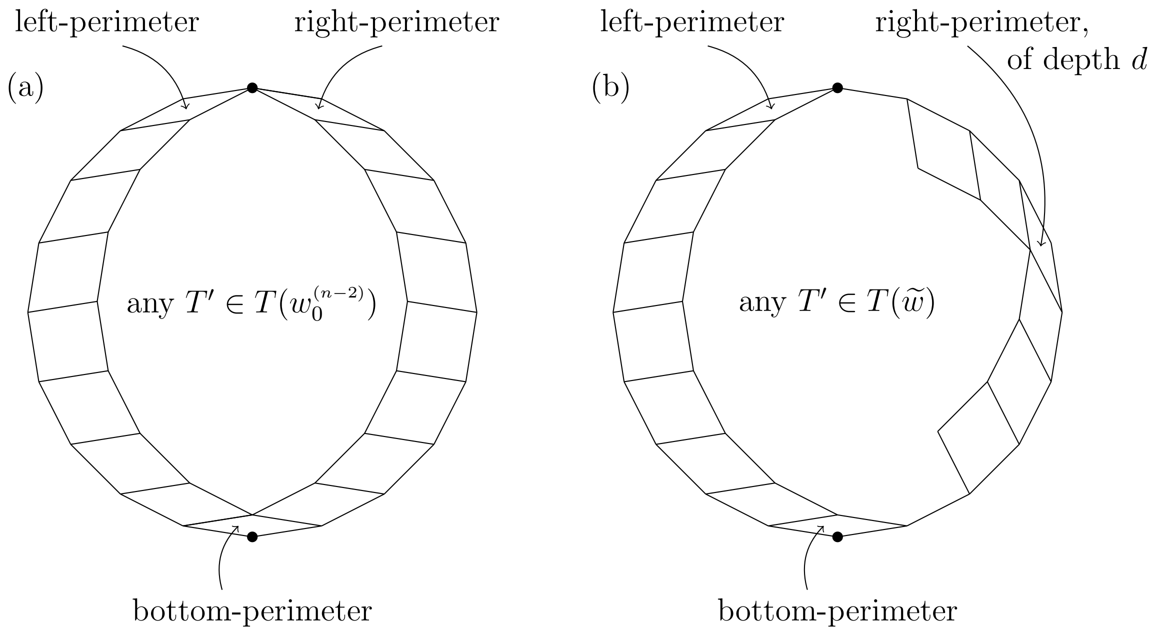


FIGURE 7. Two families of elements in $T(w_0^{(n)})$ minimizing the number (three, as shown in Theorem 4.5) of perimeter tiles. For the second family, (b), in which the right-perimeter tile has generic depth d, the permutation $\widetilde{w}$ is $(n-1)(n-d-1)(n-2)(n-3)\cdots(n-d+1)(n-d-2)\cdots 2(n-d)1 \in S_{n-1}$.

In analogy to Theorem 4.5, we now give a sharp upper bound to the total number of perimeter tiles that can appear in rhombic tilings of $X(w_0)$. This will improve upon Proposition 3.7.

Theorem 4.6. For any $n \geq 2$,

$$\max\left\{\mathsf{perim}(T) : T \in T(w_0^{(n)})\right\} = 2\left\lfloor\frac{n-1}{2}\right\rfloor + 1, \tag{1}$$

and at most $n-1$ of those perimeter tiles can be side-perimeter tiles.

Proof. Notice that the quantity on the righthand side of Equation (1) is equal to n when n is odd, and $n-1$ when n is even. We already know, from Proposition 3.7, that there are at most n perimeter tiles.

Consider the case when n is even. For there to be exactly n perimeter tiles in $T \in T(w_0^{(n)})$, the entire boundary of $X(w_0^{(n)})$ would belong to perimeter tiles. This would mean that the edge labels of the perimeter tiles, read in counterclockwise order from the top vertex, are either

$$\{1,2\}, \{3,4\}, \ldots, \{n-1,n\}, \{1,2\}, \{3,4\}, \ldots, \{n-1,n\},$$

or

$$\{2,3\}, \{4,5\}, \ldots, \{n-2,n-1\}, \{n,1\}, \{2,3\}, \{4,5\}, \ldots, \{n-2,n-1\}, \{n,1\},$$

depending on whether or not there is a top-perimeter tile. Each option would violate Corollary 2.12, so $\mathsf{perim}(T) \leq n-1$.

If n is odd, and there are n perimeter tiles, then the entire boundary of $X(w_0^{(n)})$ must belong to perimeter tiles. In particular, the parity of n means that one of those tiles has edge labels $\{1, n\}$, and is either a top- or a bottom-perimeter tile. Therefore there can be at most $n-1$ side-perimeter tiles.

Therefore, the number of perimeter tiles in T is bounded by the quantity in Equation (1), and it remains to show that this bound is achievable. We do so in Figure 8. □

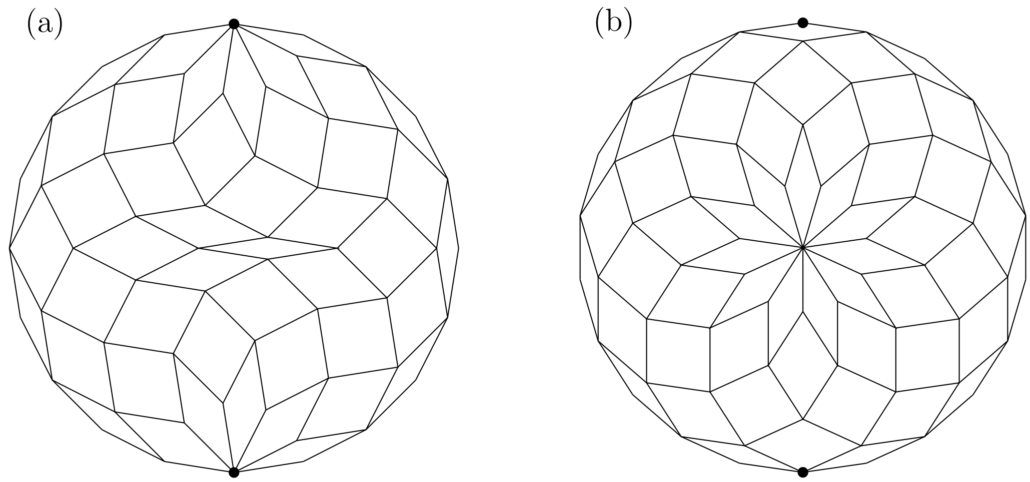


FIGURE 8. Rhombic tilings of $X(w_0^{(n)})$ maximizing the number of perimeter tiles. The figures are drawn for $n = 10$ (tiling (a)) and $n = 11$ (tiling (b)), and can easily be generalized. As shown in Theorem 4.6, this maximum is $n-1 = 9$ for the tiling in (a), and $n = 11$ for the tiling in (b).

In light of these bounds, we can make the following conclusions.

Corollary 4.7. For any $n \geq 2$, there exist commutation classes $C, C' \in C(w_0^{(n)})$ with the following properties.

(a) All elements of C have the same leftmost letter, and all have the same rightmost letter, and exactly one copy of either s_1 or s_{n-1} appears; and

(b) If n is even, then the set of leftmost letters that appear in elements of C' is $\mathcal{L}$, and the set of rightmost letters that appear in elements of C' is $\mathcal{R}$, where $\{\mathcal{L}, \mathcal{R}\} = \{\mathcal{O}, \mathcal{E}\}$ for

$$\mathcal{O} = \{s_1, s_3, s_5, \ldots\} \cap \{s_1, s_2, \ldots, s_{n-1}\} \text{ and}$$
$$\mathcal{E} = \{s_2, s_4, s_6, \ldots\} \cap \{s_1, s_2, \ldots, s_{n-1}\}.$$

If n is odd, then the set of leftmost letters that appear in elements of C' and the set of rightmost letters that appear in elements of C' are either both $\mathcal{O}$ or both $\mathcal{E}$. If they are $\mathcal{O}$ (respectively, $\mathcal{E}$), then elements of C' contain exactly one copy of s_{n-1} (respectively, s_1).

Proof. (a) This follows from Corollary 3.5 and Theorem 4.5.

(b) This follows from Corollary 3.5 and Theorem 4.6.

□

As before, we demonstrate Corollary 4.7 with a (reasonably sized) example.

Example 4.8. The following commutation classes in $C(4321)$ demonstrate the previous corollary.

(a) $\{s_1s_2s_3s_1s_2s_1, s_1s_2s_1s_3s_2s_1\}$
(b) $\{s_2s_3s_1s_2s_1s_3, s_2s_1s_3s_2s_1s_3, s_2s_3s_1s_2s_3s_1, s_2s_1s_3s_2s_3s_1\}$

5. Perimeter tiles for the longest element: average cases

As in the previous section, fix an integer $n > 1$, and write w_0 for $w_0^{(n)}$ when clear from context. Theorems 4.5 and 4.6 give a range for the number of perimeter tiles (and, specifically, side-perimeter tiles) that can appear in a rhombic tiling of $X(w_0)$. In this section, we address the complementary question: what is the average number of perimeter tiles that appears in a rhombic tiling of $X(w_0)$? Consider the set $T(w_0)$ with the uniform probability distribution. Viewing $\mathsf{perim}(T)$ as a random variable on rhombic tilings $T \in T(w_0)$ that counts the perimeter tiles, answering that question amounts to understanding

$$\mathbb{E}[\mathsf{perim}(T)].$$

We will also compute

$$\mathbb{E}[\mathsf{L\text{-}perim}(T)],\ \mathbb{E}[\mathsf{R\text{-}perim}(T)],\ \mathbb{E}[\mathsf{T\text{-}perim}(T)],\ \text{and } \mathbb{E}[\mathsf{B\text{-}perim}(T)],$$

where these statistics identify left-, right-, top-, and bottom-perimeter tiles in T, respectively. This will answer questions like: what is the average number of letters that can appear as leftmost letters in reduced words within a single commutation class of $R(w_0)$? What is the probability that the reduced decompositions in a commutation class of $R(w_0)$ contain exactly one copy of s_1?

In order to answer these questions, we will borrow terminology from [10].

Definition 5.1. Let $w \in \mathfrak{S}_n$ be a permutation and $T \in T(w)$ a rhombic tiling of $X(w)$.

(a) An n-edge path in T from the top vertex of $X(w)$ to the bottom vertex is a *border*.
(b) For $i \in [n]$, the tiles in T whose edge labels include i form a *strip* joining the edges labeled i on the leftside and rightside boundaries of $X(w)$.

In [10], Elnitsky recognizes that deleting the strip with $i = n$ in a rhombic tiling of $X(w_0^{(n)})$ will produce a rhombic tiling of $X(w_0^{(n-1)})$ in which that deleted strip appears as a border. This allows rhombic tilings of convex $2n$-gons to be generated recursively.

Corollary 5.2 ([10, Lemma 3.3 and Corollary 3.4])**.** For any $n > 2$, rhombic tilings of $X(w_0^{(n)})$ are in bijection with borders in rhombic tilings of $X(w_0^{(n-1)})$, and hence

$$\left|T(w_0^{(n)})\right| = \sum_{T \in T(w_0^{(n-1)})} \#\text{ borders in } T.$$

Our first step toward calculating the average number of perimeter tiles appearing in a rhombic tiling of $X(w_0)$ will be to calculate how many times a given perimeter tile appears among all such rhombic tilings. The symmetry of $X(w_0)$ — in particular, Proposition 4.3 — will imply that we need only calculate this value for a particular chosen perimeter tile. We will choose the top-perimeter tile.

Lemma 5.3. For any $n > 2$, the rhombic tilings of $X(w_0^{(n)})$ that have top-perimeter tiles are in bijection with borders in $U \in T(w_0^{(n-1)})$ whose topmost edge is along the leftside boundary of $X(w_0^{(n-1)})$.

Proof. This follows from Corollary 5.2. □

In the following two examples, we demonstrate how to enumerate top-perimeter tiles among all rhombic tiling of $X(w_0^{(n)})$ for $n \in \{4, 5\}$. The case $n = 4$ can be confirmed by Figure 6. The case $n = 5$ can also be confirmed, but, because $|T(54321)| = 62$, we omit that confirmation here.

Example 5.4.

(a) Among the eight rhombic tilings of $X(4321)$, there are three that contain top-perimeter tiles. As shown in Figure 9, these can be enumerated by the borders in rhombic tilings of $X(321)$ whose topmost edge is along the leftside boundary of the hexagon:
$$2 + 1 = 3.$$

(b) Among the sixty-two rhombic tilings of $X(54321)$, there are twenty that contain top-perimeter tiles. As shown in Figure 9, these can be enumerated by borders in rhombic tilings of $X(4321)$ whose topmost edge is along the leftside boundary of the octagon:
$$1 + 1 + 2 + 3 + 4 + 4 + 3 + 2 = 20.$$

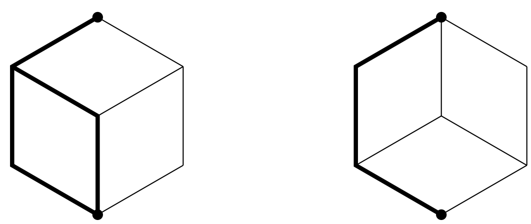


FIGURE 9. The elements of $T(321)$. Thick lines indicate the borders whose topmost edge is along the leftside boundary of $X(321)$.

As can be noticed in Figures 9 and 10, Lemma 5.3 is not necessarily the cleanest way to enumerate top-perimeter tiles in $X(w_0^{(n)})$. In particular, although the lemma refers to $X(w_0^{(n-1)})$, the full $2(n-1)$-gon is never used. Instead, only a $2(n-2)$-gon positioned in the lower left of $X(w_0^{(n-1)})$ is being utilized.

Definition 5.5. Given a permutation w and a tiling $T \in T(w)$, a *subtiling* S of T is obtained by removing, iteratively, a nonnegative number of right-perimeter tiles from T.

If S is a subtiling of $T \in T(w)$, then $S \in T(v)$ for a permutation v that is less than or equal to w in the weak Bruhat order. It is possible that this v might not satisfy Requirement 2.4(b), but this is not a problem because we will only reference v for the purposes of this recursive counting process.

Example 5.6. The tiling $T \in T(42153)$ depicted in Figure 3(a) has nine subtilings. These are depicted in Figure 11.

We are now able to recast Lemma 5.3 without a restriction on the borders.

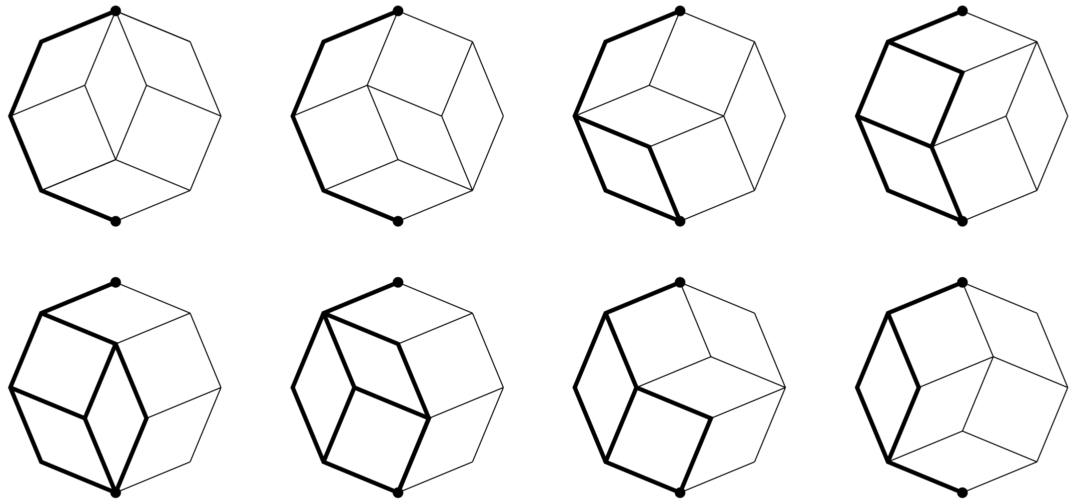


FIGURE 10. The elements of $T(4321)$. Thick lines indicate the borders whose topmost edge is along the leftside boundary of $X(4321)$.

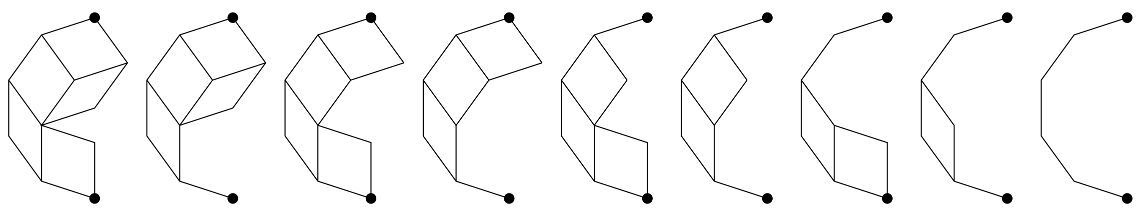


FIGURE 11. The nine subtilings of Figure 3(a).

Proposition 5.7. For any $n > 2$, the rhombic tilings of $X(w_0^{(n)})$ that have top-perimeter tiles are in bijection with borders in subtilings S of tilings $V \in T(w_0^{(n-2)})$, counted with multiplicity. Put another way,

$$\sum_{T \in T(w_0^{(n)})} \mathsf{T\text{-}perim}(T) = \sum_{V \in T(w_0^{(n-2)})} \sum_{\substack{\text{subtilings} \\ S \text{ of } V}} \#\text{borders in } S. \tag{2}$$

Proof. Suppose that $T \in T(w_0^{(n)})$ contains a top-perimeter tile. From Lemma 5.3, this T corresponds to a border in a tiling $U \in T(w_0^{(n-1)})$. Within this U, consider the maximal (in terms of number of tiles) subtiling S, where $S \in T(w)$ for some w, in which the topmost edges along the leftside and rightside boundaries of the polygon $X(w)$ coincide. In other words, $w(1) = 1$. The borders described in Lemma 5.3 are exactly the borders of subtilings S obtained in this way.

By requiring that a border in $U \in T(w_0^{(n-1)})$ have topmost edge along the leftside boundary of $X(w_0^{(n-1)})$, we are in fact requiring that this border fit within an Elnitsky polygon for the permutation of $\{2, 3, \ldots, n-1\}$ turning $23\cdots(n-1)$ into $(n-1)(n-2)\cdots 32$. (This is consistent with the observation in the previous paragraph that $w(1) = 1$.) In other words, with a straightforward relabeling, this is equivalent to requiring that this border fit within a tiling of $X(w_0^{(n-2)})$. $\square$

Example 5.8. There are two rhombic tilings of $X(321)$, and each has four subtilings. Among those four subtilings for each element of $T(321)$, the total number of borders is 10. Thus, by

Proposition 5.7, the total number of top-perimeter tiles among rhombic tilings of $X(54321)$ is $10 + 10 = 20$, confirming Example 5.4(b).

Using Proposition 5.7, we can compute the frequency of top-perimeter appear among all rhombic tilings of $X(w_0^{(n)})$, and Proposition 4.3 lets us characterize the data more generally. These values appear in Table 1 for small values of n.

n	number of perimeter tiles of each (fixed depth) type among all $T(w_0^{(n)})$
2	1
3	1
4	3
5	20
6	268

TABLE 1. The total number of perimeter tiles of each type, among all $T(w_0^{(n)})$, for $2 \leq n \leq 6$.

We can now use these results to compute the average number of perimeter tiles appearing in any rhombic tiling of $X(w_0^{(n)})$, in terms of the number of rhombic tilings. We phrase these results in the terminology of Definition 4.1.

Theorem 5.9. For any $n > 2$ and $T \in T(w_0^{(n)})$ chosen with uniform probability,

$$\mathbb{E}[\mathsf{perim}(T)] = 2n \left(\sum_{V \in T(w_0^{(n-2)})} \sum_{\substack{\text{subtilings} \\ S \text{ of } V}} \#\text{borders in } S \right) \cdot \frac{1}{|T(w_0^{(n)})|}.$$

Moreover, this statistic can be refined to identify top- or bottom-perimeter tiles:

$$\begin{aligned} \mathbb{E}[\mathsf{T\text{-}perim}(T)] &= \mathbb{E}[\mathsf{B\text{-}perim}(T)] \\ &= \left(\sum_{V \in T(w_0^{(n-2)})} \sum_{\substack{\text{subtilings} \\ S \text{ of } V}} \#\text{borders in } S \right) \cdot \frac{1}{|T(w_0^{(n)})|}, \end{aligned} \tag{3}$$

and similarly left- or right-perimeter tiles:

$$\begin{aligned} \mathbb{E}[\mathsf{L\text{-}perim}(T)] &= \mathbb{E}[\mathsf{R\text{-}perim}(T)] \\ &= (n-1) \left(\sum_{V \in T(w_0^{(n-2)})} \sum_{\substack{\text{subtilings} \\ S \text{ of } V}} \#\text{borders in } S \right) \cdot \frac{1}{|T(w_0^{(n)})|}. \end{aligned} \tag{4}$$

Proof. Proposition 5.7 gives the expectation $\mathbb{E}[\mathsf{T\text{-}perim}(T)]$. It also gives $\mathbb{E}[\mathsf{B\text{-}perim}(T)]$, by Proposition 4.3. The calculations for $\mathbb{E}[\mathsf{perim}(T)]$, $\mathbb{E}[\mathsf{L\text{-}perim}(T)]$, and $\mathbb{E}[\mathsf{R\text{-}perim}(T)]$ follow from that same symmetry result via the rotations discussed in Lemma 4.2: there are $2n$ positions at which a perimeter tile can occur in $X(w_0^{(n)})$, while there are $n-1$ positions (depths) at which a left-perimeter (or right-perimeter) tile can occur. □

To give more balance to the numerators and denominators in Theorem 5.9, recall Corollary 5.2, which characterizes $|T(w_0^{(n)})|$ in terms of borders in elements of $T(w_0^{(n-1)})$.

Observe the Coxeter-theoretic significance of Theorem 5.9.

Corollary 5.10. On average, the number of leftmost (equivalently, rightmost) letters appearing among the reduced decompositions of a commutation class of $R(w_0^{(n)})$ is given in Equation (4). Similarly, the number of commutation classes of $R(w_0^{(n)})$ whose reduced decompositions contain exactly one s_1 (equivalently, s_{n-1}) is given in Equation (2), which also appears as the numerator in the expectation calculated in Equation (3).

Data related to Theorem 5.9 and Corollary 5.10 appear in Table 2, for small values of n.

n	$\sum \mathsf{perim}(T)$	$\sum \mathsf{T\text{-}perim}(T) = \sum \mathsf{B\text{-}perim}(T)$	$\sum \mathsf{L\text{-}perim}(T) = \sum \mathsf{R\text{-}perim}(T)$	$\lvert T(w_0^{(n)})\rvert$
2	1	1	1	1
3	6	1	2	2
4	24	3	9	8
5	200	20	80	62
6	3216	268	1340	908

TABLE 2. The total number of perimeter tiles, overall and by type, among all $T(w_0^{(n)})$, and $|T(w_0^{(n)})|$ itself, for $2 \le n \le 6$. All sums are taken over $T \in T(w)$.

The columns of Table 2 are entries A320944, A320945, A320946, and A006245, respectively, in [20].

6. Isoperimetric results for rhombic tilings

The typical isoperimetric problem is an optimization question leveraging perimeter against area. As such, it can be phrased in two ways.

- For fixed A, what is the smallest perimeter among all regions with area A?
- For fixed P, what is the largest area among all regions with perimeter P?

Recall that in the contour-based setting, Theorem 1.1 says that this optimization problem is solved by the circle.

Definition 6.1. Let Ω be a region bounded by a simple closed curve. Let $P(\Omega)$ be the length of that closed curve and $A(\Omega)$ the area of its enclosure Ω. The *isoperimetric quotient* of Ω is

$$Q(\Omega) = \frac{4\pi A(\Omega)}{P(\Omega)^2}. \tag{5}$$

If one were to "score" regions according to their isoperimetric quotients (equivalently, in political science literature, their Polsby-Popper scores [18]), then plump convex regions would score higher than spindly ones. A demonstration of this appears in Figure 12.

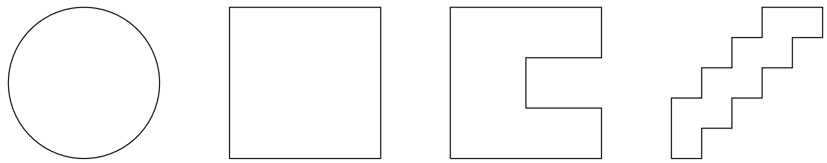


FIGURE 12. Four regions whose contour-based isoperimetric quotients (1, $\pi/4$, $2\pi/15$, $9\pi/100$) are in decreasing order from left to right.

In the context of the rhombic tilings discussed in this paper, recall that there are natural interpretations of "area" and "perimeter," as discussed in Definitions 3.1 and 4.1. Because the number of rhombi in any $T \in T(w)$ is equal to the length $\ell(w)$ of the permutation, the area as defined above depends only on w, and is independent of the choice of $T \in T(w)$.

Definition 6.2. Let the *area* of a permutation w be $\ell(w)$, the number of tiles in any $T \in T(w)$.

This sets up a framework for analyzing tiling-based isoperimetric properties. For example, one might ask, for a given permutation w, what are the minimal (and, for that matter, possible) perimeters obtainable among $T \in T(w)$, and which T achieve them. One might also fix an area A and consider the w and T achieving

$$\min\{\mathsf{perim}(T) : \ell(w) = A \text{ and } T \in T(w)\}.$$

Lemma 6.3. For any $A > 1$, there are permutations of area A and strong perimeter 2 (that is, every rhombic tiling of Elnitsky's polygon has exactly two perimeter tiles):

$$23 \cdots A(A+1)1 \quad \text{and} \quad (A+1)12 \cdots A.$$

Proof. Let $w = 23 \cdots A(A+1)1$ and $w' = (A+1)12 \cdots A$, both elements of $\mathfrak{S}_{A+1}$. First note that $\ell(w) = \ell(w') = A$, so both permutations have area A. Moreover, their Elnitsky polygons each have a single rhombic tiling, and it contains exactly two perimeter tiles, as depicted in Figure 13. $\square$

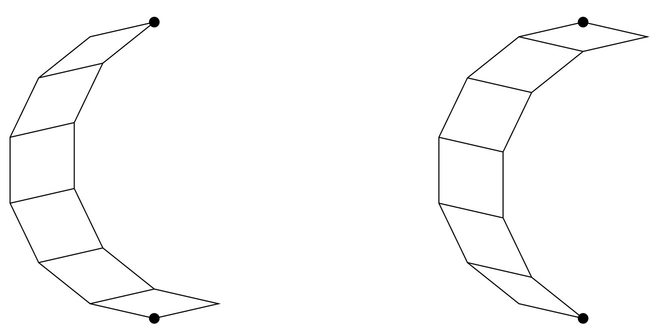


FIGURE 13. The only rhombic tilings of $X(2345671)$ and $X(7123456)$, each of which has exactly two perimeter tiles.

Note that in each of the tilings shown in Figure 13, there is a tile that is both a side-perimeter tile and a top- or bottom-perimeter tile.

As discussed after Corollary 3.8, if the area of a permutation is greater than 1, then the strong perimeter of any of its tilings is greater than 1 as well. Thus, apart from trivial cases, the strong perimeter of any tiling is always at least 2. In other words, the permutations in Lemma 6.3 illustrate minimal strong perimeters.

Combining Theorem 4.5 and Lemma 6.3 demonstrates an important distinction between the contour-based isoperimetric problem and the analogous problem in this discrete setting.

Corollary 6.4. Among all fixed-area tilings of Elnitsky polygons, the tilings that minimize strong perimeter are not the tilings of $X(w_0)$.

Lemma 6.3 gives two classes of permutations whose tilings achieve minimal strong perimeter. Moreover, for each permutation described in that lemma, the *only* rhombic tiling(s) of Elnitsky's polygon has exactly two (strong) perimeter tiles. This begs the question: what other permutations have this property? In fact, in light of the Coxeter-theoretic significance of perimeter tiles discussed in Corollary 3.5, there are two variants of this question that are particularly interesting.

Question 6.5.

(a) What are the permutations w for which all rhombic tilings of $X(w)$ have exactly two perimeter tiles?
(b) What are the permutations w for which all rhombic tilings of $X(w)$ have exactly two side-perimeter tiles?

The permutations described in Lemma 6.3 satisfy both parts of Question 6.5, and we now characterize all permutations that satisfy either question. We begin with a corollary of Elnitsky's work, describing how right-perimeter tiles relate to convex portions of the rightside boundary of $X(w)$.

Lemma 6.6. Suppose that w is a permutation and $w(d) > w(d+1)$. Then there exists a rhombic tiling of $X(w)$ with a right-perimeter tile of depth d.

We can also guarantee the possibility of a top-perimeter tile when $w(1) \neq 1$, and a bottom-perimeter tile when $w(n) \neq n$. Recall that these inequalities are built into Requirement 2.4(b).

Lemma 6.7. Let $w \in \mathfrak{S}_n$. If $w(1) \neq 1$, then there exists a rhombic tiling of $X(w)$ with a top-perimeter tile. If $w(n) \neq n$, then there exists a rhombic tiling of $X(w)$ with a bottom-perimeter tile.

Proof. Suppose $w(1) \neq 1$. Define the permutation $\widetilde{w} \in \mathfrak{S}_{n-1}$ as

$$\widetilde{w}(i) := \begin{cases} w(i+1) & w(i+1) < w(1), \text{ and} \\ w(i+1) - 1 & w(i+1) > w(1). \end{cases}$$

Construct a tiling $T \in T(w)$ using tiles with edge labels $\{i, w(1)\}$ along the leftside boundary of $X(w)$, for $i < w(1)$, and tiling the rest of $X(w)$ as in any rhombic tiling of $X(\widetilde{w})$. As constructed, this T contains a top-perimeter tile. An example of this procedure in which $w \in \mathfrak{S}_7$ and $w(1) = 5$ is depicted in Figure 14.

The existence of bottom-perimeter tiles when $w(n) \neq n$ can be proved analogously. □

We now turn to Question 6.5(a), whose answer will bring Lemma 6.3 to mind.

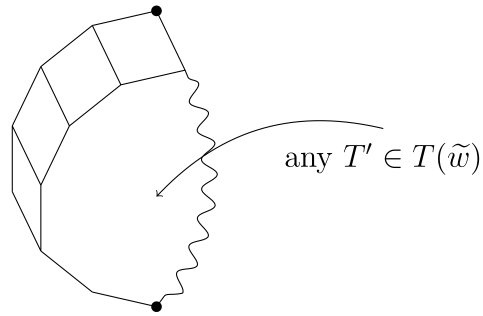


FIGURE 14. Constructing $T \in T(w)$ with a top-perimeter tile when $w(1) \neq 1$, as described in the proof of Lemma 6.7.

Theorem 6.8. For any $n > 2$, a permutation $w \in \mathfrak{S}_n$ has strong perimeter 2 (that is, is such that all rhombic tilings of $X(w)$ have exactly two perimeter tiles) if and only if w is

$$n123\cdots(n-1) \quad \text{ or } \quad 234\cdots n1.$$

Proof. First note that by Lemma 6.3, the permutations $n123\cdots(n-1)$ and $234\cdots n1$ have the desired perimeter property.

Now suppose that all rhombic tilings of $X(w)$ have exactly two perimeter tiles. By Requirement 2.4(b), we have $w(1) \neq 1$ and $w(n) \neq n$. By Lemma 6.7, there exists $T \in T(w)$ with a top-perimeter tile. Therefore, by Corollary 3.8, if T is to have exactly two perimeter tiles, then this top-perimeter tile must also be a left- or right-perimeter tile. To be a left-perimeter tile (call this "Case 1"), we would have to have $w(1) = 2$, whereas to be a right-perimeter tile ("Case 2"), we would have to have $w(2) = 1$. In each Case, that top-perimeter tile is consequently forced in any rhombic tiling of $X(w)$.

By a similar argument, we find that any rhombic tiling of $X(w)$ contains a bottom-perimeter tile. Recall, again, Lemma 3.8. In Case 1, then, this bottom-perimeter tile must also be a right-perimeter tile and so $w(n-1) = n$. On the other hand, in Case 2, the bottom-perimeter tile would also be a left-perimeter tile, and so $w(n) = n-1$.

In each of these Cases, we have identified two (distinct, because $n > 2$) perimeter tiles that exist in every element of $T(w)$.

To satisfy the requirements of our hypothesis about the strong perimeter of w, Lemma 6.6 implies that there can be no $w(d) > w(d+1)$ besides $d = n-1$ in Case 1, and $d = 1$ in Case 2. Thus $w = 234\cdots n1$ in Case 1, and $w = n123\cdots(n-1)$ in Case 2. $\square$

We now see that Question 6.5(a) was highly restrictive — there only two permutations in each $\mathfrak{S}_n$ with the desired property, and the Elnitsky polygon for each of those permutations only has one rhombic tiling.

In contrast to the result of Theorem 6.8, Question 6.5(b) seeks to address a more Coxeter-focused version of the isoperimetric problem in the context of these rhombic tilings. More specifically, this question seeks to know when there is only one choice of leftmost letter and only one choice of rightmost letter within any commutation class of reduced decompositions of $w \in \mathfrak{S}_n$, without further restrictions on the number of s_1 or s_{n-1} letters that might appear in those reduced decompositions.

Theorem 6.9. For any $n > 2$, a permutation $w \in \mathfrak{S}_n$ is such that all rhombic tilings of $X(w)$ have exactly two side-perimeter tiles if and only if w has the form

$$(k+1)(k+2)\cdots n12\cdots k \qquad \text{or} \qquad (k+1)(k+2)\cdots nk12\cdots(k-1)$$

for some $k \in [n-1]$.

Proof. First note that by Corollary 3.5, any permutation of the type listed in the theorem statement does indeed have the desired property:

- $T \in T((k+1)(k+2)\cdots n12\cdots k)$ can (in fact, must) only have a right-perimeter tile of depth $n-k$, and no other. Similarly, it can (in fact, must) only have a left-perimeter tile of depth k. The edge labels of the right-perimeter tile are $\{1, n\}$, and the edge labels of that left-perimeter tile are $\{k, k+1\}$.
- $T \in T((k+1)(k+2)\cdots nk12\cdots(k-1))$ can only have right-perimeter tiles of depths $n-k$ or $n-k+1$. Because these depths differ by 1, they cannot both be right-perimeter tiles in T. Similarly, T can only have left-perimeter tiles of depths $k-1$ or k. Because these depths differ by 1, they cannot both be left-perimeter tiles in T. The edge labels of the right-perimeter tile are $\{k, n\}$ or $\{1, k\}$, respectively, and the edge labels of the left-perimeter tile are $\{k-1, k\}$ or $\{k, k+1\}$, respectively.

Suppose that all rhombic tilings of $X(w)$ have exactly two side-perimeter tiles, and recall that $w(1) \neq 1$ and $w(n) \neq n$ by Requirement 2.4(b). By Corollary 3.8, one of these side-perimeter tiles must be a left-perimeter tile, one a right-perimeter tile, and since $n > 2$ these tiles must be distinct. Iteratively applying Lemma 6.6 shows that if $w(i) > w(i+1)$ and $w(j) > w(j+1)$ for $|i-j| > 1$, then there exists a rhombic tiling of $X(w)$ with right-perimeter tiles of depths i and j. Thus, to have exactly one right-perimeter tile in any tiling, there can be no such $\{i, j\}$. So w has at most two descents, and if there are two descents then they must be consecutive.

Suppose w has exactly one descent. Then, because $w(1) \neq 1$ and $w(n) \neq n$, the permutation w must be such that $w(1) < w(2) < \cdots < w(m) = n$ and $w(m+1) = 1 < w(m+2) < \cdots < w(n)$ for some $m \in [n-1]$. Moreover, it must be that $w(n) + 1 = w(1)$, and hence

$$w = (k+1)(k+2)\cdots n12\cdots k.$$

Now suppose that w has exactly two descents. Then, again, because $w(1) \neq 1$ and $w(n) \neq n$, the permutation w must must the form $w(1) < w(2) < \cdots < w(m) = n$, $w(m+1) = k$, and $w(m+2) = 1 < w(m+3) < \cdots < w(n)$ for some $m \in [n-2]$ and $k \in [2, n-1]$. To avoid two left-perimeter tiles in the same $T \in T(w)$, Corollary 3.5 tells us that these same sorts of conclusions must hold for the inverse permutation w^{-1}. This means that $w^{-1}(k+1) = 1$ and $w^{-1}(k-1) = n$, from which we obtain

$$w = (k+1)(k+2)\cdots nk12\cdots(k-1),$$

as desired. Note that when w has this form, the choice of left-perimeter tile (edge labels $\{k-1, k\}$ or $\{k, k+1\}$) and the choice of right-perimeter tile (edge labels $\{1, k\}$ or $\{k, n\}$) are independent of each other, as long as $2 < k < n-1$. $\square$

Although it was not requested in Questions 6.5(a) or (b), the results of Theorems 6.8 and 6.9 let us describe the elements of $T(w)$ in those settings. This relies on a much stronger result from [27].

Proposition 6.10 ([27, Theorem 6.4])**.** There is an element of $T(w)$ containing a sub-$2k$-gon with edge labels $\{i_1 < i_2 < \cdots < i_k\}$ if and only if $i_k \cdots i_1$ forms a $k \cdots 21$-pattern in w; that is, if and only if $w^{-1}(i_k) < \cdots < w^{-1}(i_2) < w^{-1}(i_1)$.

The proposition implies, among other things, that Elnitsky's polygon for a 321-avoiding permutation will have no subhexagons, which means that it has exactly one rhombic tiling (recovering a result of [3]). This accounts for all of the permutations from Theorems 6.8 and 6.9 except for the second category of the latter theorem. Permutations in that class have $(n-k)(k-1)$, a positive number, 321-patterns, and so there are multiple rhombic tilings of their Elnitsky polygons. Example polygons for the permutations described in Theorem 6.9, and what is forced in their tilings, are depicted in Figure 15.

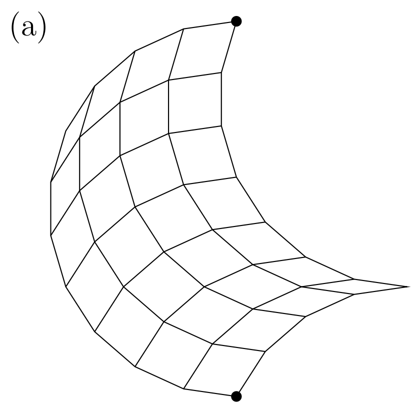


Figure 15. Examples of $X(w)$ for permutations w satisfying the statement of Theorem 6.9. Both permutations have $n = 11$ and $k = 4$, with (a) having the first form in the theorem statement and (b) having the second. The polygon in (a) has exactly one rhombic tiling because the permutation has no 321-pattern. The polygon in (b) has many rhombic tilings because the permutation has twenty-one 321-patterns, but there can only ever be one left-perimeter tile and one right-perimeter tile in any particular rhombic tiling. The possible left- and right-perimeter tiles (which can be chosen independently of each other in this case) are indicated with dashed and dotted edges in the figure.

7. Perimeter tiles in domino tilings: an introduction

A highly active area of mathematical research (in combinatorics and statistical mechanics, in particular) examines properties and enumerations of domino tilings. See, for example, [7, 9, 15, 21], among many others.

Definition 7.1. A *domino* is a 1×2 rectangle. A *domino tiling* of a region R is a tiling of R by dominoes.

Of course, not all regions R can be tiled by dominoes. Among other properties, R must have sides of integer lengths, only (internal or external) right angles along its boundary, and even area.

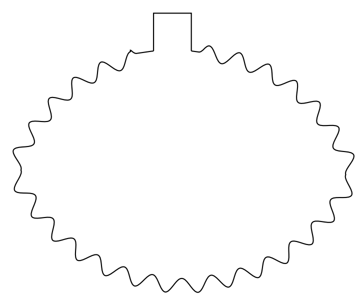
In light of the tiling-based isoperimetric analysis of Sections 4-6, it is interesting to consider what such a perspective would have to say about domino tilings. Recall Definition 3.2 (which designated a tile as a "(strong) perimeter" tile based on the number of edges it shared with the boundary of the region) and the paragraphs preceding it. Because dominoes and the regions that they tile already reside in a metric space, we have some choice about how to define (strong) perimeter dominoes.

Definition 7.2. Consider a domino tiling of a region R and a domino d in that tiling. If the domino d shares ...

- a length-2 edge with the boundary of R, then d is a *longside-perimeter domino*;
- a contiguous length-2 section of its boundary with the boundary of R, then d is a *nonspecific-perimeter domino*.

Again, we distinguish these from *weak perimeter dominoes*, which share a length-1 section of boundary with the boundary of the region, and yet are not nonspecific-perimeter dominoes.

Longside-perimeter dominoes are a special case of nonspecific-perimeter dominoes. The distinction between these definitions will only matter if the region R contains a protruding 1×1 square, such as in the figure below.



As cited above, there is a substantial literature on domino tilings, and the range of questions, regions, machinery, restrictions, and so on, is quite broad. In this section here, we focus our attention on just a small family of regions, with the goal of drawing attention to what could be quite an interesting area of research. For the regions we will consider, distinguishing between the options in Definition 7.2 will have no impact. As such, we will not stipulate one or the other option, and will simply refer to *perimeter* dominoes. Of course, for more general regions, this distinction could indeed matter, and the study of each type of perimeter tile, as well as their relationships to each other, would be an interesting avenue of research.

Definition 7.3. Throughout this section, let $R_{m,n}$ denote the $m \times n$ rectangle. Because will use it frequently, set $R_n := R_{2,n}$. For a region R, let $D(R)$ be the set of domino tilings of R.

Not only is the region R_n of manageable size and complexity, but it also has attractive enumerative properties and lends itself well to recursive procedures. For example, it is an easy exercise to show that domino tilings of R_n are enumerated by the Fibonacci number $f(n+1)$ [20, A000045].

We begin our analysis of perimeter tiles in R_n with an analogue to Theorems 4.5 and 4.6. Note that as in previous sections, we restrict to $n > 1$ to avoid overcounting the one domino in the one tiling of R_1. We will recycle the function

$$\mathsf{perim}$$

from Definition 4.1 to count perimeter dominoes in a tiling $D \in D(R_n)$.

Proposition 7.4. For any $n \geq 2$,

$$\min\{\mathsf{perim}(D) : D \in D(R_n)\} = 2 \qquad \text{and} \qquad \max\{\mathsf{perim}(D) : D \in D(R_n)\} = n.$$

Proof. In any of the four corners of the region R_n, either vertical or horizontal placement of the domino occupying that corner will produce a perimeter tile. The two corners along an edge of length two may be covered by the same (perimeter) domino, and there is no way to cover more than two corners by a single domino. Therefore there must be at least two perimeter dominoes in any tiling. Moreover, if all dominoes are oriented so as to be parallel to the length-2 edge of R_n, then there will be exactly two perimeter dominoes in the tiling.

If, in contrast, all dominoes are oriented so as to be perpendicular to the length-2 edge of R_n, with the first domino having the opposite orientation if n is odd, then all $2n/2 = n$ dominoes will be perimeter tiles. □

Examples of the tilings described in the proof of Proposition 7.4 are shown in Figure 16.

FIGURE 16. Domino tilings of R_9 having (a) the least and (b) the greatest number of perimeter dominoes. Perimeter dominoes are shaded.

In analogy to the work of Section 5, we now study the total number of perimeter dominoes appearing among all domino tilings of R_n. We will do so by constructing recurrence relations and generating functions.

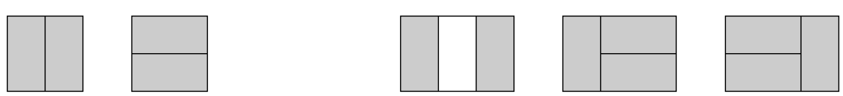
Definition 7.5. For $n > 2$, let P_n be the number of perimeter dominoes appearing among all elements of $D(R_n)$.

Proposition 7.6. For any $n > 2$,

$$P_{n+1} = P_n + P_{n-1} + 2f(n),$$

where $f(n)$ is the nth Fibonacci number, and $P_2 = 4$ and $P_3 = 8$.

Proof. To compute P_2 and P_3, we look at the two elements of $D(P_2)$ and the three elements of $D(P_3)$. The perimeter dominoes in each of those tilings have been shaded in the figures below.



For the sake of clarity in this argument, we will draw all rectangles R_n so that the side of length n is horizontal. It will help us to consider the sequence $\{a_n\}$, which we define to count perimeter dominoes that are not a leftmost vertical domino, appearing among all elements of $D(R_n)$. Thus, for example, we see in the figures above that $a_2 = 3$ and $a_3 = 6$.

There are two possible ways to tile the leftmost column of R_n: by a vertical domino or by two horizontal dominoes. In each case, the placed dominoes are perimeter tiles. From these options, we see that

$$\begin{aligned} P_{n+1} &= \Big(1 \cdot |D(R_n)| + a_n\Big) + \Big(2 \cdot |D(R_{n-1})| + a_{n-1}\Big) \\ &= 1 \cdot f(n+1) + a_n + 2 \cdot f(n) + a_{n-1}. \end{aligned}$$

By the same breakdown of cases, we have

$$a_{n+1} = 0 \cdot f(n+1) + a_n + 2 \cdot f(n) + a_{n-1}.$$

Therefore $P_{n+1} = a_{n+1} + f(n+1)$, and so

$$\begin{aligned} P_n + P_{n-1} &= a_n + f(n) + a_{n-1} + f(n-1) \\ &= a_{n+1} - 2f(n) + f(n+1) \\ &= P_{n+1} - 2f(n), \end{aligned}$$

completing the proof. □

The values P_n, along with $|D(R_n)|$, are presented in Table 3 for $n \in [2, 9]$. The sequence $\{P_n\}$ is entry A320947 of [20].

n	P_n	$\lvert D(R_n)\rvert = f(n+1)$
2	4	2
3	8	3
4	16	5
5	30	8
6	56	13
7	102	21
8	184	34
9	328	55

TABLE 3. The total number of perimeter dominoes, along with the number of domino tilings, of a $2 \times n$ rectangle, for $n \in [2, 9]$.

Corollary 7.7.

$$\sum_{n \geq 2} P_n x^n = \frac{4x^2 - 4x^4 - 2x^5}{(1 - x - x^2)^2}.$$

We close this section with a brief look at isoperimetric properties in the setting of $D(R_n)$.

Lemma 7.8. If $m, n > 2$, then

$$\min\{\mathsf{perim}(D) : D \in D(R_{m,n})\} \geq 4$$

and

$$\max\{\mathsf{perim}(D) : D \in D(R_{m,n})\} \leq m + n - 2 < \frac{mn}{2}.$$

Proof. Because m and n are both greater than 2, no single domino can cover two corners of the rectangle $R_{m,n}$. Therefore $\min\{\mathsf{perim}(D) : D \in D(R_{m,n})\} \geq 4$.

As in the proof of Proposition 3.7, we note that at most $(2m + 2n - 4)/2 = m + n - 2$ perimeter dominoes can be used to cover the $2m + 2n - 4$ squares along the boundary of $R_{m,n}$. Because $m, n > 2$, this value is strictly less than $mn/2$. □

Note the implications of Lemma 7.8 for isoperimetric-type questions about rectangles.

Corollary 7.9. Among all rectangles of a fixed area, the dimensions that can achieve both the least and the greatest number of (non-weak) perimeter dominoes is the $2 \times n$ rectangle.

Contrast this with the contour-based isoperimetric theorem, Theorem 1.1: in the context of domino tilings, it is the long skinny rectangles that achieve the extreme strong perimeters, while plump rectangles achieve neither end of the spectrum. This demonstrates, once again, the different behavior of isoperimetric questions in the contour-based context, versus the tiling-based context presented here.

8. Directions for further research

The work and results presented here suggest many areas of future research.

Regarding the rhombic tilings of Elnitsky polygons, it would be interesting to have analogues of the work in Sections 4 and 5 for arbitrary permutations. The Coxeter-theoretic significance of such results inspires questions in other directions, too. What do these properties tell us about structure in Coxeter graphs, or in the contracted graphs of [2]? What do they imply for enumerative and structural features of the Bruhat order? Elnitsky's work extends to Coxeter groups of types B and D, as well. What can we say about perimeter tiles in those settings?

From Proposition 7.6, we get a sense of the average number of perimeter dominoes in a randomly chosen domino tiling of a $2 \times n$ rectangle. One could look at analogues of Proposition 7.6 and Corollary 7.7 for arbitrary rectangles. One could similarly examine the circumstances of perimeter dominoes — either longside- or nonspecific- — that appear in domino tilings of non-rectangular regions, including regions with nontrivial topology, like the "holey square" of [29].

In this work, we have only looked at two types of tilings, but, of course, there are infinitely many tiling-style problems, many of which are known to have compelling mathematics. There are also scenarios where one is studying a problem that might be rephrased as a tiling-based isoperimetric question. What do our methods have to say in those applications? Do strong perimeter tiles bear extra significance there? Are weak perimeter tiles mathematically noteworthy, too? Can these tiles (either type) be enumerated, either directly or probabilistically?

As the reader can appreciate, there are many directions for this research to take, and many research communities that make take an interest in its pursuit.

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