

About the ordinances of the vectors of the n -dimensional Boolean cube in accordance with their weights

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Abstract

The problem "Given a Boolean function f of n variables by its truth table vector. Find (if exists) a vector $\alpha \in \{0, 1\}^n$ of maximal (or minimal) weight, such that $f(\alpha) = 1$." arises in computing the algebraic degree of Boolean functions or vectorial Boolean functions called S-boxes. The solutions to this problem have useful generalizations and applications. To find effective solutions we examine the ways of ordering the vectors of the Boolean cube in accordance with their weights. The notion " k -th layer" of the n -dimensional Boolean cube is involved in the definition and examination of the "weight order" relation. It is compared with the known relation "precedes". We enumerate the maximum chains for both relations. An algorithm that generates the vectors of the n -dimensional Boolean cube in accordance with their weights is developed. The lexicographic order is chosen as a second criterion for an ordinance of the vectors of equal weights. The algorithm arranges the vectors in a unique way called a weight-lexicographic order. It is represented by the serial numbers of the vectors, instead of the vectors itself. Its time and space complexities are $\Theta(2^n)$, i.e., of linear type with respect to the size of the output. The obtained results are summarized and added as a new sequence (A294648) in the OEIS.

Keywords: Boolean cube, binary vector, serial number, lexicographic order, weight order, maximum chains enumerating, weight-lexicographic order generating, power set generating, ranking

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1 Introduction

The binary vectors (binary words, binary sequences, bit strings, etc.) play an important role in all areas of Discrete mathematics and Computer science. The set of all n -dimensional binary vectors $\{0,1\}^n$ is often called an n -dimensional *Boolean cube* (*hypercube*). The most natural order of its vectors is the lexicographic order, which is a total order. There are other important orders—for example, in a Gray code (various types, considered exhaustively in the survey of C. Savage [20]), or in accordance with the relation "precedence" which is a partial order. Here we consider a similar one—order in accordance with the weights of the binary vectors. Our study of this order is motivated by searching for efficient solutions to the following problem: "Given a Boolean function f of n variables by its Truth Table vector, denoted by $TT(f)$. Find (if exists) a vector $\alpha \in \{0,1\}^n$ of maximal (or minimal) weight, such that $f(\alpha) = 1$." This problem arises in computing the algebraic degree of Boolean functions or vectorial Boolean functions (called S-boxes) [4, 5]. The solutions to this problem have useful generalizations and applications that are commented here. The most natural way to solve this problem is to perform an exhaustive (linear) search: for any vector $\beta \in \{0,1\}^n$ it checks whether $f(\beta) = 1$ and selects the one with a maximal (resp. minimal) weight. Since the values of $TT(f)$ correspond to the lexicographic order of the vectors of the n -dimensional Boolean cube, such solution needs $\Theta(2^n)$ checks. However, if the search checks the values of $TT(f)$ in accordance with the vectors' weights, the search will finish after finding the first vector $\beta \in \{0,1\}^n$, such that $f(\beta) = 1$. Once the desired order of the vectors has been obtained, this approach needs $O(2^n)$ operations. This order can be obtained by an algorithm that: (1) computes the vectors' weights and (2) sorts the vectors in accordance with their weights. So it needs at least $\Theta(n \cdot 2^n)$ operations.

The ways for ordering the vectors of the Boolean cube in accordance with their weights are examined here. The necessary basic notions concerning the Boolean cube and their properties are given in Section 2. In Section 3 the key notion "k-th layer" of the Boolean cube is involved in discussing the ways for an ordinance of the vectors of the Boolean cube in accordance with their weights. The corresponding relation "precedes by weight" is defined, investigated and compared with the known relation "precedes". The maximum chains for the Partially Ordered Sets (POSets) determined by both relations are enumerated (the corresponding notes are appended to the sequences A051459 and A000142 in the OEIS [21]). A special way for an ordinance of the vectors in weight order (WO) is defined in the second part

of Section 3. It is represented by the serial numbers of the vectors instead of the vectors themselves. Theorem 6 shows that the lexicographic order is the second criterion for an ordinance of the vectors of equal weights in their WO. So a unique order is obtained and it is called a *Weight-Lexicographic Order* (WLO). An algorithm that generates the sequence of the serial numbers of all n -dimensional binary vectors in WLO is created. It is represented in Section 4. The essential parts of its code (in the C programming language) and some results of its performance are given. These results are summarized and added as a new sequence (A294648) in the OEIS [21]. The correctness of the algorithm is shown and its time and space complexities are evaluated—they are both $\Theta(2^n)$. Some concluding remarks are given in the last section.

2 Basic notions and properties

After studying numerous books and textbooks, we could not find a whole topic (part or chapter), devoted to the n -dimensional Boolean cube. The notions and the assertions given in this part can be found in most of the sources cited here. However, they are scattered in different parts and have different names and notations. So we represent the necessary basic notions about the Boolean cube and its properties following [3].

Here \mathbb{N} denotes the set of natural numbers. We consider that $0 \in \mathbb{N}$ and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ is the set of positive natural numbers.

Usually, the *n -dimensional Boolean cube* is defined as $\{0, 1\}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \{0, 1\}, \text{ for } i = 1, 2, \dots, n\}$, i.e., the set of all n -dimensional binary vectors. So their number is $|\{0, 1\}^n| = |\{0, 1\}|^n = 2^n$. The following alternative, inductive and constructive definition is more useful for us further.

Definition 1. 1) The set $\{0, 1\} = \{(0), (1)\}$ is called *one-dimensional Boolean cube* and its elements (0) and (1) are called *one-dimensional binary vectors*.

2) Let $\{0, 1\}^{n-1} = \{\alpha_0, \alpha_1, \dots, \alpha_{2^{n-1}-1}\}$ be the $(n-1)$ -dimensional Boolean cube and $\alpha_0, \alpha_1, \dots, \alpha_{2^{n-1}-1}$ be its $(n-1)$ -dimensional binary vectors.

3) The *n -dimensional Boolean cube* $\{0, 1\}^n$ is built by taking the vectors of $\{0, 1\}^{n-1}$ twice: firstly, each vector of $\{0, 1\}^{n-1}$ is prefixed by zero, and thereafter each vector of $\{0, 1\}^{n-1}$ is prefixed by one, i.e.,

$$\{0, 1\}^n = \{(0, \alpha_0), (0, \alpha_1), \dots, (0, \alpha_{2^{n-1}-1}), (1, \alpha_0), (1, \alpha_1), \dots, (1, \alpha_{2^{n-1}-1})\}.$$

Figure 1 shows how the vectors of the n -dimensional Boolean cube are obtained following the definition.

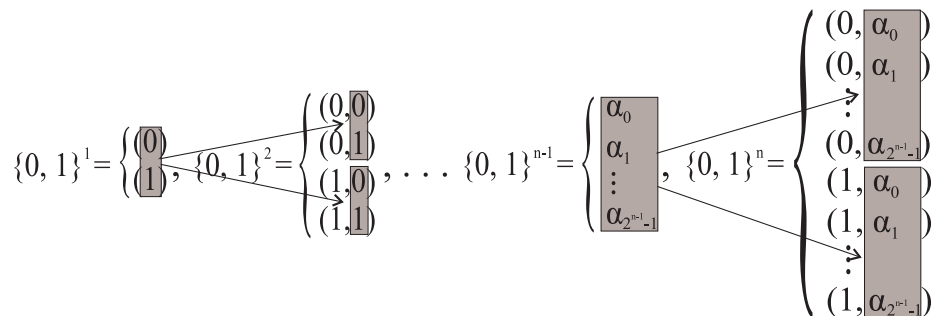


Figure 1: Building of $\{0, 1\}^n$ in accordance with Definition 1

Definition 2. Let $\alpha = (a_1, a_2, \dots, a_n) \in \{0, 1\}^n$ be an arbitrary vector. The natural number $\#\alpha = \sum_{i=1}^n a_i \cdot 2^{n-i}$ is called a *serial number* of the vector α . In other words $\#\alpha$ is the natural number whose n -digit binary representation is $a_1 a_2 \dots a_n$.

This notion and some of the following ones are illustrated in Figure 2. Furthermore, they are shown in Example 1.

Definition 3. Let $\alpha = (a_1, a_2, \dots, a_n) \in \{0, 1\}^n$ be an arbitrary vector. A *weight* (or *Hamming weight*) of α is the natural number $wt(\alpha)$, equal to the number of non-zero coordinates of α , i.e., $wt(\alpha) = \sum_{i=1}^n a_i$.

Definition 4. For arbitrary vectors $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n) \in \{0, 1\}^n$ the relation *lexicographic precedence* $R_{\leq} \subseteq \{0, 1\}^n \times \{0, 1\}^n$ is defined as follows: $(\alpha, \beta) \in R_{\leq}$, if $\alpha = \beta$ or $\exists i, 0 \leq i < n$, such that $a_1 = b_1, a_2 = b_2, \dots, a_i = b_i$, but $a_{i+1} < b_{i+1}$. When $(\alpha, \beta) \in R_{\leq}$ we say that α *lexicographically precedes* β and write $\alpha \leq \beta$.

It is easy to verify that the relation R_{\leq} is reflexive, antisymmetric and transitive. So R_{\leq} is a *partial order* in the cube. Furthermore, each pair of vectors $\alpha, \beta \in \{0, 1\}^n$ are *comparable* with respect to R_{\leq} , i.e., either $\alpha \leq \beta$, or $\beta \leq \alpha$ holds—this property is called a *totality*. So R_{\leq} is a *total order* in $\{0, 1\}^n$. This means that its vectors can be *ordered* (or *sorted*) *lexicographically* in a unique way in the sequence $\alpha_0, \alpha_1, \dots, \alpha_k, \dots, \alpha_{2^n-1}$, such that $\alpha_l \leq \alpha_k$, for all $l < k$, and $\alpha_k \leq \alpha_r$, for all $k < r$, and for any $k = 0, 1, \dots, n$. This order is called also a *standard order*.

Theorem 1. *The vectors of $\{0, 1\}^n$, obtained in accordance with Definition 1 are in lexicographic order, for any $n \in \mathbb{N}^+$.*

Proof. Following Definition 1, the proof of the theorem by induction is easy.

1) For $k = 1$ the assertion is obvious.

2) Suppose that for arbitrary integer $k > 1$ the vectors of $\{0, 1\}^k$ are in lexicographic order.

3) We consider the vectors of $\{0, 1\}^{k+1}$. The first half of them are obtained from $\{0, 1\}^k$ by adding zero in the beginning of each of them. In accordance with the inductive suggestion, they are in lexicographic order. The second half of them are obtained by adding one in the beginning of each vector of $\{0, 1\}^k$ and so they are in lexicographic order too. Finally, since each vector from the first half begins with zero, it precedes lexicographically each vector from the second half (because it begins with one). Therefore the vectors of $\{0, 1\}^{k+1}$ are in lexicographic order.

So the theorem holds for any $n \in \mathbb{N}^+$. □

Theorem 2. *Let the vectors of $\{0, 1\}^n$ be in lexicographic order. Then:*

1) *The serial numbers of the vectors form the sequence of natural numbers: $0, 1, \dots, 2^n - 1$. So, $\alpha \leq \beta$ if and only if $\#\alpha \leq \#\beta$.*

2) *The weights of the vectors in the second half of $\{0, 1\}^n$ are obtained by adding 1 to the weights of corresponding vectors from the first half of the cube.*

The proof of this theorem is analogous to the proof of Theorem 1. The theorem states the bijection between the vectors in lexicographic order and their serial numbers. It also shows the relation between the vectors in lexicographic order and their weights. Its assertion is illustrated in Figure 2.

Definition 5. Let $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$ be arbitrary vectors of $\{0, 1\}^n$. A *Hamming distance* between α and β is the natural number $d(\alpha, \beta)$ equal to the number of coordinates in which α and β differ. If $d(\alpha, \beta) = 1$, then α and β are called *adjacent*, or more precisely *adjacent in i -th coordinate*, if they differ in this coordinate only. If $d(\alpha, \beta) = n$, the vectors α and β are called *opposite* to each other.

The *graph* of the n -dimensional boolean cube is defined as $H_n = (V_n, E_n)$, where $V_n = \{0, 1\}^n$ (i.e., the vectors of the cube are vertices of H_n) and $E_n = \{\{\alpha, \beta\} \mid \alpha, \beta \in \{0, 1\}^n : d(\alpha, \beta) = 1\}$ (i.e., each pair adjacent vectors are connected by an edge). The graphs H_1, \dots, H_4 are shown in Figure

# α	the vectors of $\{0,1\}^n$ in lexicographic order	$wt(\alpha)$
0	(0,0,...,0,0,0)	0
1	(0,0,...,0,0,1)	1
2	(0,0,...,0,1,0)	1
3	(0,0,...,0,1,1)	2
4	(0,0,...,1,0,0)	1
5	(0,0,...,1,0,1)	2
6	(0,0,...,1,1,0)	2
7	(0,0,...,1,1,1)	3
\vdots		\vdots
$2^{n-1}-2$	(0,1,...,1,1,0)	n-2
$2^{n-1}-1$	(0,1,...,1,1,1)	n-1
2^{n-1}	(1,0,...,0,0,0)	1
$2^{n-1}+1$	(1,0,...,0,0,1)	2
\vdots		\vdots
2^n-2	(1,1,...,1,1,0)	n-1
2^n-1	(1,1,...,1,1,1)	n

Diagrammatic annotations: A large left curly bracket groups the first 2^{n-1} rows, labeled $+2^{n-1}$. A smaller left curly bracket groups the first 4 rows, labeled $+2^2$. A right curly bracket groups the first 4 rows, labeled $+1$. A large right curly bracket groups the last 4 rows, labeled $+1$.

Figure 2: Illustration of the statement of Theorem 2

3. The geometric reasons that determine the name *cube* (or more precisely *hypercube*) can be seen in the figure.

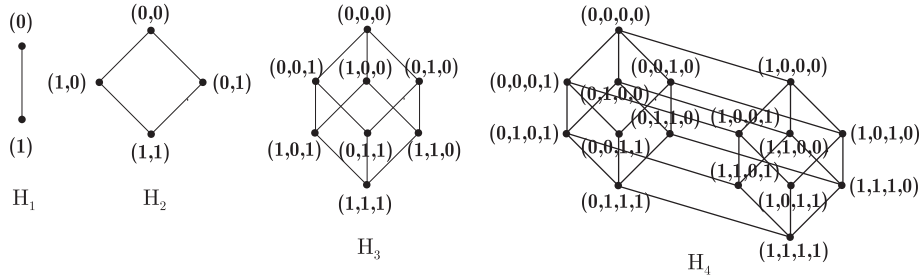


Figure 3: The graphs H_1, \dots, H_4

Besides the lexicographic order and the order of the vectors of $\{0,1\}^n$ in a Gray code, another important order is given by the following relation.

Definition 6. The *precedence* relation is denoted by R_{\preceq} and it is defined as follows: for arbitrary vectors $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n) \in \{0,1\}^n$, $(\alpha, \beta) \in R_{\preceq}$ if $a_i \leq b_i, \forall i = 1, 2, \dots, n$. When $(\alpha, \beta) \in R_{\preceq}$ we say

that α precedes β and write $\alpha \preceq \beta$. When $\alpha \preceq \beta$ or $\beta \preceq \alpha$ the vectors α and β are called *comparable*, and otherwise—*incomparable*.

It is easy to verify that R_{\preceq} is reflexive, antisymmetric and transitive. So R_{\preceq} is a *partial order* in $\{0, 1\}^n$. In other words $\{0, 1\}^n$ is a *partially ordered set* (POSet) with respect to R_{\preceq} and it is denoted by $(\{0, 1\}^n, R_{\preceq})$ or simply by $(\{0, 1\}^n, \preceq)$. Since not all pairs $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n) \in \{0, 1\}^n$ are comparable (for example, all vectors of equal weights are incomparable), R_{\preceq} is not a total order, unlike the lexicographic order.

The vector $\alpha \in \{0, 1\}^n$ is called a *minimal element* of the POSet $(\{0, 1\}^n, R_{\preceq})$, if $\alpha \preceq \beta$, for any $\beta \in \{0, 1\}^n$. Analogously, the vector δ is called a *maximal element* of $\{0, 1\}^n$ with respect to R_{\preceq} , if $\gamma \preceq \delta$, for any $\gamma \in \{0, 1\}^n$. So, the *zero vector* $(0, 0, \dots, 0)$ (i.e., the all zeros vector of n -coordinates, denoted by $\tilde{0}_n$ further) and the *unit vector* $(1, 1, \dots, 1)$ (i.e., the all ones vector of n -coordinates, denoted by $\tilde{1}_n$ further) are the minimal and the maximal element of the POSet $(\{0, 1\}^n, \preceq)$, correspondingly. If any pair of vectors of the subset $C \subset \{0, 1\}^n$ are comparable, they can be ordered in a unique way in a *chain*, for example $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}, \dots, \alpha_{i_m}$, such that $\alpha_{i_l} \preceq \alpha_{i_k}$, for $l < k$, and $\alpha_{i_k} \preceq \alpha_{i_r}$, for $k < r$, and for $k = 1, 2, \dots, m$. So C is a totally ordered set. A chain that is not a proper subset of any other chain is a *maximal chain*. For example, $(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1)$ is a maximal chain in $\{0, 1\}^3$, whereas $(0, 1, 0, 0), (0, 1, 0, 1), (1, 1, 0, 1), (1, 1, 1, 1)$ is not a maximal chain in $\{0, 1\}^4$ —see Figure 3. The maximal chain should contain the minimal and the maximal element of the POSet with respect to the corresponding relation. Each chain of the greatest possible size is called a *maximum* (or *longest*) chain.

Definition 7. Let $U = \{x_1, x_2, \dots, x_n\}$ be a given set, $n \in \mathbb{N}^+$, and $X \subseteq U$. The vector $\alpha = (a_1, a_2, \dots, a_n) \in \{0, 1\}^n$, defined as:

$$a_i = \begin{cases} 0, & \text{if } x_i \notin X, \\ 1, & \text{if } x_i \in X, \end{cases}$$

for $i = 1, 2, \dots, n$, is called a *characteristic vector* of the set X .

Example 1. Let $U = \{a, b, c, d, e, f\}$, $X = \{b, c, e\}$ and $Y = \{c, a, f, d\}$. Since $|U| = 6$, $\alpha = (0, 1, 1, 0, 1, 0) \in \{0, 1\}^6$ is the characteristic vector of X , and $\beta = (1, 0, 1, 1, 0, 1)$ —the characteristic vector of Y . The vectors $\gamma = \tilde{0}_6$ and $\delta = \tilde{1}_6$ are the characteristic vectors of $\emptyset \subseteq U$ and $U \subseteq U$, correspondingly. Furthermore:

- $\#\alpha = 26$, $\#\beta = 45$, $\#\gamma = 0$, $\#\delta = 2^6 - 1 = 65$;
- $wt(\alpha) = 3$, $wt(\beta) = 4$, $wt(\gamma) = 0$, $wt(\delta) = 6$;
- $d(\alpha, \gamma) = 3$, $d(\alpha, \beta) = 5$, $d(\beta, \delta) = 2$, etc.;
- $\gamma \leq \alpha \leq \beta \leq \delta$, in accordance with Definition 4 and Theorem 2;
- $\gamma \preceq \alpha$, $\gamma \preceq \beta$, $\alpha \preceq \delta$, $\beta \preceq \delta$, etc., but α and β are incomparable with respect to R_{\preceq} .

Theorem 3. Let U be an n -element set, $n \in \mathbb{N}^+$, and $\mathcal{P}(U)$ be the power set of U . Let $f : \mathcal{P}(U) \rightarrow \{0, 1\}^n$ be a function defined as follows: $f(X) = \alpha$, where $\alpha \in \{0, 1\}^n$ is the characteristic vector of X , for any $X \in \mathcal{P}(U)$. Then f is a bijection.

The proof of the theorem is easy and we omit it. The theorem states that the vectors of the n -dimensional Boolean cube are bijectively related to the subsets of a given n -dimensional set by the notion of "characteristic vector". Furthermore, the function f from Theorem 3 bijectively relates (maps) the bitwise operations on the binary vectors to the operations on the subsets of a given n -element set U as follows: \vee (disjunction) and \cup (union); \wedge (conjunction) and \cap (intersection); $\bar{}$ (negation) and $\bar{}$ (complement); \oplus (sum modulo 2, XOR) and Δ (symmetric difference), correspondingly. These properties are generalized in the following theorem [3, 7, 8, 12, 13].

Theorem 4. Let U be an n -element set, $n \in \mathbb{N}^+$. Then the Boolean algebras $(\mathcal{P}(U), \cup, \cap, \bar{}, \emptyset, U)$ and $(\{0, 1\}^n, \vee, \wedge, \bar{}, \tilde{0}_n, \tilde{1}_n)$ are isomorphic.

The bijection f from Theorem 3 concerns the relations R_{\subseteq} (defined on a given universal set U , $|U| = n \in \mathbb{N}^+$) and R_{\preceq} (defined on $\{0, 1\}^n$). For arbitrary $A, B \subseteq U$, having characteristic vectors $\alpha, \beta \in \{0, 1\}^n$, correspondingly, it is easy to prove that:

$$A \subseteq B \Leftrightarrow \alpha \preceq \beta, \text{ i.e., } (A, B) \in R_{\subseteq} \Leftrightarrow (f(A), f(B)) \in R_{\preceq}.$$

Thus f is an isomorphism between the POSets $(\mathcal{P}(U), R_{\subseteq})$ and $(\{0, 1\}^n, R_{\preceq})$ that preserves the relations and the orders corresponding to them. This property is illustrated in Figure 6 by the graphs of the corresponding relations, for $n = 3$.

These important *structural properties* are used for:

- Computer representations of sets by binary vectors or arrays and performance of the basic operations on them—see [1, 2, 6, 9–14, 16–19, 22], etc. The serial numbers of the vectors of $\{0, 1\}^n$ are used for *ranking* the subsets of U by the notion characteristic vector and this is the most natural ranking function. We note that in this way, the lexicographic order of the vectors (i.e. the natural numbers $0, 1, \dots, 2^n - 1$) corresponds to the reverse lexicographic order of the subsets of U .
- Generating all subsets of a given n -element set in a definite order. This topic is considered exhaustively in [9, 19], other good expositions are [14–16, 20, 22], etc.
- Generating the k -elements subsets (combinations) of a given n -element set in a definite order. Such algorithms are considered in [9, 14–16, 19, 20, 22].
- Ranking and unranking of combinatorial structures. Such algorithms are discussed in [9, 15, 19, 22], etc.

The following exposition is related to all these applications.

3 Ordinances of the vectors of the Boolean cube in accordance with their weights

We start with the following key notion.

Definition 8. For an arbitrary $k \in \mathbb{N}, k \leq n$, the set of all n -dimensional binary vectors of weight k is called a k -th layer of the n -dimensional Boolean cube. We denote it by $L_{n,k} = \{\alpha \mid \alpha \in \{0, 1\}^n : wt(\alpha) = k\}$.

3.1 The weight-order relation

Figure 3 illustrates the notion of layer from Definition 8. All vectors in the same horizontal level in the figure form the corresponding layer of the cube. Since k coordinates can be chosen (and filled in with ones) among n coordinates in $\binom{n}{k}$ ways, hence $|L_{n,k}| = \binom{n}{k}$, for $k = 0, 1, \dots, n$. These numbers (i.e., binomial coefficients) form the n -th row of Pascal's triangle and it is well-known that $\sum_{k=0}^n \binom{n}{k} = 2^n = |\{0, 1\}^n|$. Obviously, the family of all layers $L_n = \{L_{n,0}, L_{n,1}, \dots, L_{n,n}\}$ is a *partition* of the n -dimensional Boolean cube into layers. Moreover, the *sequence of layers* $L_{n,0}, L_{n,1}, \dots, L_{n,n}$ is an order of the vectors of $\{0, 1\}^n$ in accordance with their weights. This means

that when $\alpha, \beta \in \{0, 1\}^n$ and $wt(\alpha) < wt(\beta)$, then α precedes β in the sequence of layers, and when $wt(\alpha) = wt(\beta) = k$, then $\alpha, \beta \in L_{n,k}$ and there is no precedence between them. More precisely, the corresponding relation $R_{<wt}$ can be defined as follows: for arbitrary $\alpha, \beta \in \{0, 1\}^n$, $(\alpha, \beta) \in R_{<wt}$ if $wt(\alpha) < wt(\beta)$. We want $R_{<wt}$ to be reflexive and so we set $(\alpha, \alpha) \in R_{<wt}$. When $(\alpha, \beta) \in R_{<wt}$ we say that " α precedes by weight β " and write also $\alpha <_{wt} \beta$. It is easy to verify that $R_{<wt}$ is a partial order in $\{0, 1\}^n$ and we refer to it as a *Weight-Order* (WO) further.

The vectors of $L_{n,k}$ can be rearranged in $\binom{n}{k}!$ ways, for $k = 0, 1, \dots, n$. Thus we obtain $\prod_{k=0}^n \binom{n}{k}!$ ways for WO of the vectors of $\{0, 1\}^n$. The product values obtained for $n = 1, 2, 3, 4, \dots$ are $1, 2, 36, 414720, \dots$, correspondingly. They form the sequence A051459 in the OEIS [21], which is defined (by Yuval Dekel, Nov 15 2003) very shortly as "Number of orderings of the subsets of a set with n elements that are compatible with the subsets' sizes; i.e., if A, B are two subsets with $A \leq B$ then $Card(A) \leq Card(B)$ ". This description corresponds to the assertion of Theorem 3 and to the notion WO, since the vectors in the layer $L_{n,k}$ are characteristic vectors of all k -element subsets of an n -element set, for $k = 0, 1, \dots, n$. In addition, we conclude that $\prod_{k=0}^n \binom{n}{k}!$ is the number of:

- all maximum chains in the POSet $(\{0, 1\}^n, R_{<wt})$;
- all possible topological orders (or sorts) of the directed acyclic graph defined by the same POSet.

The corresponding comments were added to A051459.

Let us consider the connection between the relations $R_{<wt}$ and R_{\preceq} . We note that $\alpha \preceq \beta$ always implies $\alpha <_{wt} \beta$. However, $\alpha <_{wt} \beta$ does not imply $\alpha \preceq \beta$ in the general case. Here is a simple example that confirms this assertion—if $\alpha = (1, 0, 0, 0)$ and $\beta = (0, 1, 1, 0)$, then $\alpha <_{wt} \beta$, whereas α and β are incomparable with respect to the relation " \preceq ". Therefore $R_{\preceq} \subset R_{<wt}$.

Now we shall enumerate the maximum chains in the POSet $(\{0, 1\}^n, R_{\preceq})$. The following assertion will help for this goal.

Lemma 1. *Let α be an arbitrary vector of the layer $L_{n,k}$, for some integer $k, 0 < k < n$. Then α has k adjacent vectors in the layer $L_{n,k-1}$ and also $n - k$ adjacent vectors in the layer $L_{n,k+1}$.*

Proof. We consider an arbitrary vector $\beta \in L_{n,k}$ and we assume that β contains units in the coordinates i_1, i_2, \dots, i_k , where $1 \leq i_1 \leq \dots \leq i_k \leq n$. The set of all vectors adjacent to β is partitioned into two subsets. The

first one contains all vectors α , such that $\alpha \preceq \beta$, i.e., exactly one of the coordinates i_1, i_2, \dots, i_k is inverted to zero and all remaining coordinates are the same. So they are elements of $L_{n,k-1}$ and there are k such vectors. The second subset contains all vectors γ , such that $\beta \preceq \gamma$, i.e., all coordinates i_1, i_2, \dots, i_k are ones and exactly one of the remaining $n - k$ coordinates is inverted to one. Analogously, all these vectors are elements of $L_{n,k-1}$ and their number is $n - k$. \square

Theorem 5. *The number of maximum chains in the POSet $(\{0, 1\}^n, R_{\preceq})$ is equal to $n!$, for any $n \in \mathbb{N}^+$.*

Proof. Obviously, the length of any maximum chain is equal to the number of layers in $\{0, 1\}^n$, which is $n + 1$. Let $\tilde{0}_n, \alpha_1, \dots, \alpha_k, \dots, \alpha_{n-1}, \tilde{1}_n$ be a maximum chain. Starting from the vector $\tilde{0}_n$ and following Lemma 1, there are n possible ways to choose the vector $\alpha_1 \in L_{n,1}$ which is adjacent to $\tilde{0}_n$. There are $n - 1$ possible ways to choose a vector $\alpha_2 \in L_{n,2}$ which is adjacent to α_1 , etc. There are k ways to choose a vector α_k which is adjacent to α_{k-1} , etc. Finally, the last vector $\tilde{1}_n$ can be chosen in a unique way. Applying the multiplication rule we obtain that $n \cdot (n - 1) \dots (n - k) \dots 2 \cdot 1 = n!$ maximum chains can be obtained. \square

The values of $n!$, for $n = 0, 1, 2, \dots$ form the sequence A000142 (called Factorial numbers) in the OEIS [21]. Among its numerous comments, only one corresponds to the assertion of Theorem 5. It was done on Feb 05 2006 by Rick L. Shepherd as follows: "The number of chains of maximal length in the power set of 1, 2, ..., n ordered by the subset relation." Beside the assertion of Theorem 5, we added one more comment to the sequence A000142—it contains *the number of all shortest paths* (obtained by Breadth First Search, for example) between the nodes $\tilde{0}_n$ and $\tilde{1}_n$ in the graph H_n .

3.2 The weight-lexicographic order relation

For the problem formulated in Section 1 the WO of the vectors of $\{0, 1\}^n$ is sufficient. However, we need the serial numbers of the vectors in the sequence of layers instead of the vectors themselves. So, we shall represent the WO by a sequence with the serial numbers of the vectors in the layers, in accordance with Theorem 2. For that purpose, for an arbitrary layer $L_{n,k} = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$ of $\{0, 1\}^n$, we denote by $l_{n,k} = \#\alpha_0, \#\alpha_1, \dots, \#\alpha_m$ the *sequence of serial numbers*, corresponding to the vectors of $L_{n,k}$. If $l_n = l_{n,0}, l_{n,1}, \dots, l_{n,n}$ denotes the *sequence of all serial numbers*, corresponding to the vectors in the sequence of layers $L_{n,0}, L_{n,1}, \dots, L_{n,n}$, then

l_n represents a WO of the vectors of $\{0, 1\}^n$. Briefly, we refer to l_n as a *WO sequence* of $\{0, 1\}^n$. One of all possible $\prod_{k=0}^n \binom{n}{k}!$ WO sequences deserves a special attention. For its consideration, we need the following operation on a sequence of integers.

Definition 9. Let $n, m \in \mathbb{N}^+$ and $s = a_1, a_2, \dots, a_n$ be a sequence of integers. We define the operation *addition of the natural number m to the sequence s* as follows: $s + m = a_1 + m, a_2 + m, \dots, a_n + m$.

This operation can be seen in Figure 2. Following the idea in this figure and Definition 1, we define the special WO sequence l_n inductively.

Definition 10. 1) The WO sequence of the one-dimensional Boolean cube is $l_1 = 0, 1$.

2) Let $l_{n-1} = l_{n-1,0}, l_{n-1,1}, \dots, l_{n-1,n-1}$ be the WO sequence of the $(n-1)$ -dimensional Boolean cube.

3) The WO sequence of the n -dimensional Boolean cube $l_n = l_{n,0}, l_{n,1}, \dots, l_{n,n}$ is defined as follows:

- $l_{n,0} = 0$ and it corresponds to the layer $L_{n,0} = \{\tilde{0}_n\}$;
- $l_{n,n} = 2^n - 1$ and it corresponds to the layer $L_{n,n} = \{\tilde{1}_n\}$;
- $l_{n,k} = l_{n-1,k}, l_{n-1,k-1} + 2^{n-1}$, for $k = 1, 2, \dots, n-1$. Here $l_{n,k}$ is a concatenation of two sequences: the sequence $l_{n-1,k}$ is taken (or copied) firstly, and the sequence $l_{n-1,k-1} + 2^{n-1}$ follows after it. The sequence $l_{n,k}$ corresponds to the layer $L_{n,k}$.

The corresponding recursive definition of l_n is:

$$\text{If } n = 1, \quad l_1 = 0, 1.$$

$$\text{If } n > 1, \quad l_n = l_{n,0}, \dots, l_{n,k}, \dots, l_{n,n}, \text{ where:}$$

$$l_{n,k} = \begin{cases} 0, & \text{if } k = 0, \\ 2^n - 1, & \text{if } k = n, \\ l_{n-1,k}, l_{n-1,k-1} + 2^{n-1}, & \text{for } 0 < k < n. \end{cases}$$

Figure 4 and Figure 5 illustrate how the WO sequences l_2 and l_3 are obtained in accordance with Definition 10.

The last two definitions resemble the definition of Pascal's triangle. As we noted, the length of $l_{n,k} = \binom{n}{k} = |L_{n,k}|$, for $k = 0, 1, \dots, n$. Instead of the rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ used in Pascal's triangle, we use a similar rule $l_{n,k} = l_{n-1,k}, l_{n-1,k-1} + 2^{n-1}$. The next theorem clarifies it.

Theorem 6. Let $l_n = l_{n,0}, l_{n,1}, \dots, l_{n,n}$ be the WO sequence, obtained in accordance with Definition 10, for an arbitrary $n \in \mathbb{N}^+$. Then, the serial numbers in the sequence $l_{n,k}$ determine a lexicographic order of the vectors of the corresponding layer $L_{n,k}$, for $k = 0, 1, \dots, n$.

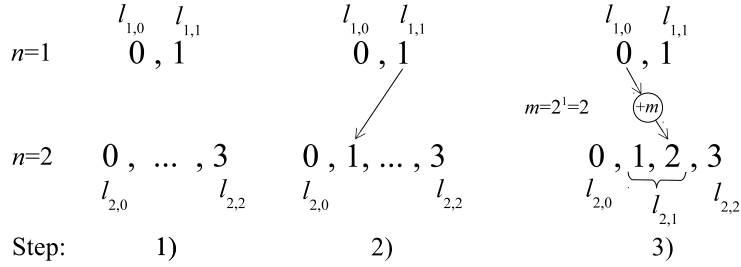


Figure 4: The WO sequence l_2 , obtained from l_1

Proof. We prove the theorem by induction on m , $m \in \mathbb{N}^+$, following Definition 10.

1) For $m = 1$ the assertion is obvious.

2) Suppose that the theorem holds, for an arbitrary integer $m > 1$, and the sequence $l_m = l_{m,0}, l_{m,1}, \dots, l_{m,m}$ is obtained in accordance with Definition 10.

3) Let $l_{m+1} = l_{m+1,0}, l_{m+1,1}, \dots, l_{m+1,m+1}$ be the sequence, obtained in accordance with Definition 10. For $l_{m+1,0} = 0$ and $l_{m+1,m+1} = 2^{m+1} - 1$, the corresponding layers $L_{m+1,0} = \{\tilde{0}_{m+1}\}$ and $L_{m+1,m+1} = \{\tilde{1}_{m+1}\}$ are in lexicographic order. Furthermore, $l_{m+1,0}$ and $l_{m+1,m+1}$ are in their right places in l_{m+1} . Let $l_{m+1,k}$ be one of the rest of the subsequences in l_{m+1} , for an arbitrary integer k , $1 \leq k \leq m$. In accordance with Definition 10, $l_{m+1,k}$ is a concatenation of two subsequences: $l_{m,k}$ and $l_{m,k-1} + 2^m$, placed in that order. So, the layer $L_{m+1,k}$ corresponding to $l_{m+1,k}$ is partitioned into two groups. The first one consists of all vectors of $L_{m+1,k}$, that begin with zero. Hence their serial numbers coincide with these in the sequence $l_{m,k}$. It corresponds to the layer $L_{m,k}$, whose vectors are in lexicographic order, in accordance with the inductive suggestion. So the vectors in the first group are also in lexicographic order. The second group includes all vectors of $L_{m+1,k}$ that begin with one. So their serial numbers are obtained by an addition of the integer 2^m to the serial numbers of the sequence $l_{m,k-1}$. Following the inductive suggestion, the vectors of the corresponding layer $L_{m,k-1}$ are in lexicographic order and therefore the vectors in the second group are also in lexicographic order. Moreover, each vector from the first group precedes lexicographically each vector from the second group. Therefore, the sequence $l_{m+1,k}$ determines a lexicographic order in the corresponding layer $L_{m+1,k}$. This conclusion holds for any integer k , $1 \leq k \leq m$, and so the theorem is proven. \square

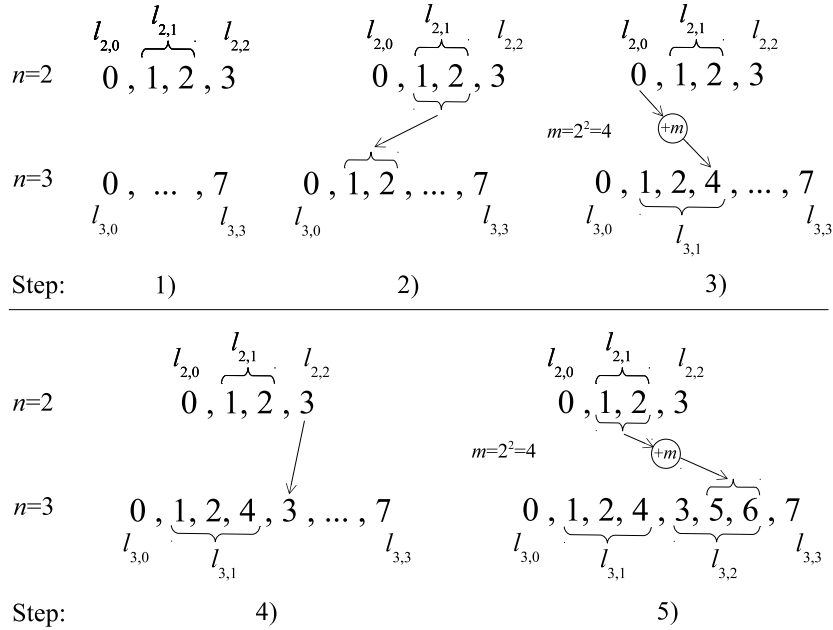


Figure 5: The WO sequence l_3 , obtained from l_2

Theorem 6 states that Definition 10 determines a *second criterion* for ordering in the WO of the Boolean cube—this is the *lexicographic order*. Since it is a unique total order for each subsequence $l_{n,k}$, $0 \leq k \leq n$, a unique total weight order for the sequence l_n is obtained. We call it a *Weight-Lexicographic Order* (WLO). It is represented by the corresponding WLO sequence l_n .

4 The WLO algorithm

We developed an algorithm called *WLO algorithm* that computes the sequence l_n for a given input $n \in \mathbb{N}^+$. The algorithm uses an array for the binomial coefficients from Pascal’s triangle (i.e., the lengths of the subsequences), and one more array where the beginning of each subsequence is computed and stored. The values in these two arrays are computed firstly. The code of the corresponding function is given below in the C programming language.

Listing 1: Filling in both additional arrays

```
typedef unsigned int uint;
```

```

// ... definitions of constants, variables and arrays
// the array P_t stands for Pascal's triangle
// the array ss_beg stands for subsequence beginning
void fill_in_both_triangles (int n) {
    P_t[0][0]= 1;
    for (int r= 1; r<=n; r++) { // r stands for row
        P_t[r][0]= 1; P_t[r][1]= r;
        ss_beg[r][0]= 0; ss_beg[r][1]= 1;
        for (int c= 2; c<r; c++) { // c stands for column
            P_t[r][c]= P_t[r-1][c-1] + P_t[r-1][c];
            ss_beg[r][c]= ss_beg[r][c-1] + P_t[r][c-1];
        }
        P_t[r][r]= 1;
        ss_beg[r][r]= ss_beg[r][r-1] + P_t[r][r-1];
    }
}

```

The WLO algorithm is based on Definition 10. Starting from l_1 it computes consecutively the sequences l_2, l_3, \dots, l_n in the array `seqs` as follows.

Listing 2: Computing the WLO sequence l_n

```

void fill_in_seqs (int n) {
    seqs[1][0]= 0; seqs[1][1]= 1; // initialization for n=1
    uint m= 2; // to be added to a subsequence, m=2^1
    for (int r= 2; r<=n; r++) {
        seqs[r][0]= 0;
        uint k=1; // a second index for the array seqs
        for (int c=1; c<=r; c++) {
            uint seq_len= P_t[r-1][c]; // Preparing for the
            uint subseqbeg= ss_beg[r-1][c]; // first step.
            for (uint j=0; j<seq_len; j++) // I step-copying of
                seqs[r][k++]= seqs[r-1][subseqbeg+j]; // a subseq.
            seq_len= P_t[r-1][c-1]; // Preparing for the
            subseqbeg= ss_beg[r-1][c-1]; // second step.
            for (uint j=0; j<seq_len; j++) // II step-addition
                // the number m to a subsequence.
                seqs[r][k++]= seqs[r-1][subseqbeg+j] + m;
        }
        m *= 2;
    }
}

```

Some results obtained by the algorithm, for $n = 1, 2, \dots, 5$, are given in Table 1.

More results can be seen in the OEIS [21], sequence A294648.

Figure 6 summarizes some of discussed results and illustrates:

- the bijection between subsequences of l_3 and the layers of $\{0, 1\}^3$;

Table 1: Results from the WLO algorithm, for $n = 1, 2, \dots, 5$

n	l_n
1	0, 1
2	0, 1, 2, 3
3	0, 1, 2, 4, 3, 5, 6, 7
4	0, 1, 2, 4, 8, 3, 5, 6, 9, 10, 12, 7, 11, 13, 14, 15
5	0, 1, 2, 4, 8, 16, 3, 5, 6, 9, 10, 12, 17, 18, 20, 24, 7, 11, 13, 14, 19, 21, 22, 25, ...

- the bijection f between the vectors of $\{0, 1\}^3$ and the subsets of $\{a, b, c\}$ (see Theorem 3);
- the isomorphism f between the POSets $(\{0, 1\}^3, \preceq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$ by the graphs of the corresponding relations.

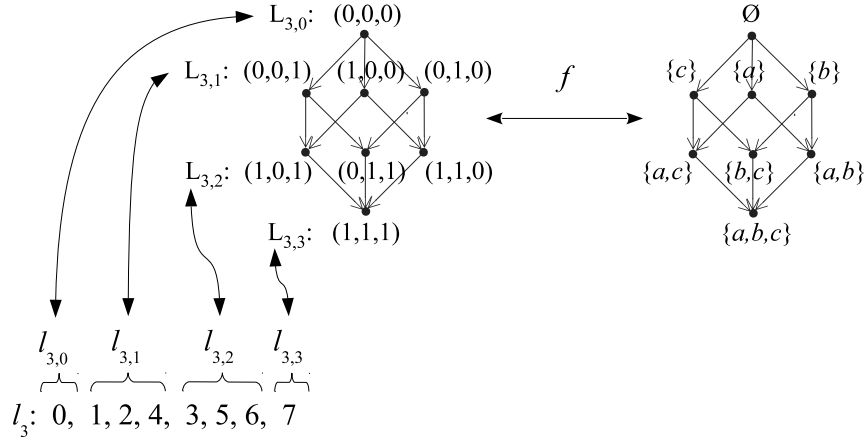


Figure 6: Illustration of the bijections between the sequence l_3 , the layers of $\{0, 1\}^3$ and the subsets of $\{a, b, c\}$, as well as the isomorphism between the POSets $(\{0, 1\}^3, \preceq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$

Like we said, the WLO algorithm is based on Definition 10 and it follows its steps. This fact, Theorem 6 and the notes after it determine its correctness. Let us consider the time complexity of the algorithm. The time for filling in both additional arrays (for Pascal's triangle and for the beginning of each subsequence) is $\Theta(n^2)$. The function `fill_in_seqs` in Listing 2 runs as follows. On the k -th step, $2 \leq k \leq n$, it copies generally $2^{k-1} - 1$ values from l_{k-1} to l_k , and also it adds the constant 2^{k-1} to $2^{k-1} - 1$ members of l_{k-1} and stores them in l_k . So, it performs $\Theta(2^k)$ assignments and $\Theta(2^{k-1})$

summations, i.e., $\Theta(2^k)$ operations generally on the k -th step. Therefore, the time complexity of the algorithm is

$$\sum_{k=2}^n \Theta(2^k) = \Theta\left(\sum_{k=2}^n 2^k\right) = \Theta(2^{n+1}) = \Theta(2^n).$$

So, the time complexity of the WLO algorithm is of an exponential type with respect to the size of the input n . It can not be better since it produces an output of exponential size. What is more important is that the algorithm has a linear time complexity with respect to the size of the output.

Let us consider the space complexity of the WLO algorithm. For clarity, in Listing 2 we use a two-dimensional array of size $2^n \times 2^n$ and hence, the space complexity is $\Theta(2^{2n})$. We recall that the existence of l_k is sufficient to obtain l_{k+1} . So, instead of the square array we can use:

- Two one-dimensional static arrays of size 2^n —for the existing sequence l_k and for the new sequence l_{k+1} . After we obtain l_{k+1} , we change the role of the arrays to obtain the next sequence l_{k+2} , and so on.
- One-dimensional arrays of size 2^k which are created dynamically in the k -th step, for $k = 1, 2, \dots, n$.

In both cases the space complexity of the WLO algorithm reduces to $\Theta(2^n)$.

5 Conclusions

Trying to find an efficient solution to the problem formulated in Section 1 we considered the weight order of the vectors of the Boolean cube. We examined this order more generally and we obtained solutions to two enumeration problems, concerning the POSets $(\{0, 1\}^n, R_{<wt})$ and $(\{0, 1\}^n, R_{\leq})$. The corresponding notes were added to the sequences A051459 and A000142 in the OEIS [21]. We defined one special WO—the WLO and proved that it is a unique total order. Based on this, we developed the WLO algorithm that generates all n -dimensional binary vectors in accordance with their weights, where the lexicographical order is chosen as a second criterion. The vectors of the cube in WLO have very compact representation by their serial numbers. Following the obtained results, a new sequence (A294648) was appended to the OEIS [21].

Our next goal is to perform tests and to evaluate the efficiency of the proposed solution to the initial problem. Moreover, the bijection between the n -dimensional Boolean cube and the power set of a given n -element set

(Theorem 3) means that the WLO algorithm can have more general applications. For example, it can be used in solving problems related to representing and generating the power set of a given set, or some of its subsets (for example, k -element subsets, or combinations), etc., as it is shown in Figure 6. In such cases, if the elements of a given set are in lexicographic order, the WLO sequence of their characteristic vectors implies a cardinality order of the subsets. However, the corresponding subsets of equal cardinalities will be in a reverse lexicographic order.

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