A COMBINATORIAL CLASSIFICATION OF 2-REGULAR SIMPLE MODULES FOR NAKAYAMA ALGEBRAS

RENE MARCZINZIK, MARTIN RUBEY†, AND CHRISTIAN STUMP*

Abstract. We discuss combinatorial interpretations of several homological properties of Nakayama algebras in terms of Dyck path statistics. We thereby classify and enumerate various classes of Nakayama algebras. Most importantly, we classify their 2-regular simple modules, corresponding to exact structures on the category of projective modules. We also classify 1-regular simple modules, quasi-hereditary Nakayama algebras and Nakayama algebras of global dimension at most two. As it turns out, most classes are enumerated by well-known combinatorial sequences, such as Fibonacci, Riordan and Narayana numbers. We first obtain interpretations in terms of the Auslander-Reiten quiver of the algebra using homological algebra. Then, we apply suitable bijections to relate these to combinatorial statistics on Dyck paths.

1. Introduction

A *Nakayama algebra* is a finite-dimensional algebra over a field $\mathbb{F}$, all whose indecomposable projective and indecomposable injective modules are uniserial. The aim of this paper is to provide a dictionary between homological properties of Nakayama algebras and their modules, and combinatorial statistics on (possibly periodic) Dyck paths. Our main results concern 1- and 2-regular simple modules. By a result of Enomoto ([E18, Theorem 3.7]) the classification of 2-regular simple modules corresponds to the classification of exact structures on the category of projective modules.

Let Q be a finite quiver with path algebra $\mathbb{F}Q$, and let J denote the ideal generated by the arrows in Q. Then a two sided ideal I is called *admissible* if $J^m \subseteq I \subseteq J^2$ for some $m \geqslant 2$. In this article we assume that all Nakayama algebras are given by a connected quiver and admissible relations. Note that this assumption is no loss of generality for algebraically closed fields since every algebra is Morita equivalent to a quiver algebra in this case and all our notions are invariant under Morita equivalence. Using this language, Nakayama algebras are precisely the algebras $\mathbb{F}Q/I$, such that I is admissible and Q is either a linear quiver

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{n-2}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \end{tikzcd}$$

or a cyclic quiver

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[r] & \underset{1}{\bullet} \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \underset{n-2}{\bullet} \arrow[r] & \underset{n-1}{\bullet} \arrow[llllll, bend right] \end{tikzcd}$$

Date: November 15, 2018.

†Supported by the Austrian Science Fund (FWF): P 29275.
*Supported by DFG grants STU 563/2 "Coxeter-Catalan combinatorics" and STU 563/4-1 "Noncrossing phenomena in Algebra and Geometry".

Restriction	Statement	
no 1-regular simples	Cor. 3.19	(Riordan numbers)
no 2-regular simples	Cor. 3.20	(Dyck paths without 2-hills)
k 1-regular and ℓ 2-regular simples	Cor. 3.21	
ℓ simples of projective dimension 1	Cor. 3.16	(Narayana numbers)
ℓ simples of projective dimension 2	Cor. 3.17	(Dyck paths with ℓ big returns)
k simples of projective dimension 1 and ℓ simples of projective dimension 2	Cor. 3.18	
global dimension 2 and ℓ simples of projective dimension 2	Thm. 4.1	(subsets of cardinality 2ℓ)
global dimension 2 and restricted Gorenstein	Cor. 4.6	(Fibonacci numbers)
quasi-hereditary	Cor. 3.27	(balanced non-constant binary necklaces)
quasi-hereditary with a simple of dimension 2	Prop. 3.28	
quasi-hereditary without 1-regular simples	Cor. 3.33	(periodic Dyck paths without 1-rises)
quasi-hereditary without 2-regular simples	Cor. 3.34	(periodic Dyck paths without 2-hills)
quasi-hereditary with ℓ simples of projective dimension 1	Cor. 3.31	(periodic Dyck paths with ℓ peaks)
quasi-hereditary with ℓ simples of projective dimension 2	Cor. 3.32	(periodic Dyck paths with ℓ big returns)
global dimension 2 and ℓ simples of projective dimension 2	Thm. 4.1	(subsets of cardinality 2ℓ up to rotation by pairs)
global dimension 2 and restricted Gorenstein	Cor. 4.7	(cyclic compositions of non-singleton parts)

FIGURE 1. Enumerative results for LNakayama algebras and for CNakayama algebras.

For textbook introductions to Nakayama algebras we refer for example to [ARS97, AF92, SY11]. We write *LNakayama algebra* for a Nakayama algebra with linear quiver and *CNakayama algebra* for a Nakayama algebra with cyclic quiver. We moreover write *n-Nakayama algebra*, *n-LNakayama*, and *n-CNakayama* in the cases that the respective Nakayama algebra has n simple modules $S_0, \dots, S_{n-1}$. These are in one-to-one correspondence with the vertices of the quiver.

In Section 2 we provide identifications between $(n+1)$-LNakayama algebras and Dyck paths of semilength n (Proposition 2.8) and between n-CNakayama algebras and certain n-periodic Dyck paths (Proposition 2.9).

Section 3 contains the main results of this paper. These are descriptions of 1- and 2-regular simple modules for Nakayama algebras in terms of classical Dyck path statistics (Theorem 3.14 for LNakayama algebras and Theorem 3.29 for CNakayama algebras). In Section 4, we classify simple modules in Nakayama algebras of global dimension at most two (Theorem 4.1) and Nakayama algebras of global dimension at most two that satisfy the restricted Gorenstein condition (Theorem 4.5). As corollaries of these classification results, we also obtain explicit enumeration formulas in all considered situations as summarized in Figure 1.

The translation between Nakayama algebras and Dyck paths made it possible to search

- the Online Encyclopedia of Integer Sequences [OEIS] for counting formulas for the homological properties, and
- the combinatorial statistic finder FindStat [RS18] for combinatorial interpretations.

All major results, including the bijections involved, are based on these searches. In particular, results found by FindStat suggested the main bijection employed, which is a variant of the Billey-Jockusch-Stanley map and the Lalanne-Kreweras involution.

For the reader's convenience, we reference integer sequences in this article to the Online Encyclopedia of Integer Sequences [OEIS] and combinatorial bijections and statistics to FindStat [RS18]. We also provide all discussed homological properties for several small Nakayama algebras in Figure 2 for later reference. Experiments were carried out using the GAP-package QPA [QPA] and SageMath [Sage].

Kupisch series	1-reg.	2-reg.	pdim 1	pdim 2	gdim
[1]	-	-	-	-	0
[2, 1]	0	-	0	-	1
[2, 2, 1]	-	0	1	0	2
[3, 2, 1]	0,1	-	0,1	-	1
[2, 2, 2, 1]	-	-	2	1	3
[3, 2, 2, 1]	0	1	0,2	1	2
[2, 3, 2, 1]	2	0	1,2	0	2
[3, 3, 2, 1]	1	-	1,2	0	2
[4, 3, 2, 1]	0,1,2	-	0,1,2	-	1
[2, 2, 2, 2, 1]	-	-	3	2	4
[3, 2, 2, 2, 1]	0	-	0,3	2	3
[2, 3, 2, 2, 1]	-	0,2	1,3	0,2	2
[3, 3, 2, 2, 1]	1	-	1,3	2	3
[4, 3, 2, 2, 1]	0,1	2	0,1,3	2	2
[2, 2, 3, 2, 1]	3	-	2,3	1	3
[3, 2, 3, 2, 1]	0,3	1	0,2,3	1	2
[2, 3, 3, 2, 1]	2	-	2,3	1	3
[3, 3, 3, 2, 1]	-	-	2,3	1	3
[4, 3, 3, 2, 1]	0,2	-	0,2,3	1	2
[2, 4, 3, 2, 1]	2,3	0	1,2,3	0	2
[3, 4, 3, 2, 1]	1,3	-	1,2,3	0	2
[4, 4, 3, 2, 1]	1,2	-	1,2,3	0	2
[5, 4, 3, 2, 1]	0,1,2,3	-	0,1,2,3	-	1

Kupisch series	1-reg.	2-reg.	pdim 1	pdim 2	gdim
[3, 2]	-	1	0	1	2
[2, 3, 2]	-	-	1	0	3
[4, 3, 2]	1	2	0,1	2	2
[5, 4, 3]	0	-	0,1	2	2
[2, 2, 3, 2]	-	-	2	1	4
[2, 4, 3, 2]	2	-	1,2	0	3
[3, 2, 3, 2]	-	1,3	0,2	1,3	2
[3, 4, 3, 2]	1	-	1,2	0	3
[4, 3, 3, 2]	2	-	0,2	3	3
[5, 4, 3, 2]	1,2	3	0,1,2	3	2
[3, 5, 4, 3]	-	-	1,2	0	3
[6, 5, 4, 3]	0,2	-	0,1,2	3	2
[7, 6, 5, 4]	0,1	-	0,1,2	3	2
[2, 2, 2, 3, 2]	-	-	3	2	5
[2, 2, 4, 3, 2]	3	-	2,3	1	4
[2, 3, 2, 3, 2]	-	2	1,3	0,2	3
[2, 3, 4, 3, 2]	2	-	2,3	1	4
[2, 4, 3, 3, 2]	3	-	1,3	0	4
[2, 5, 4, 3, 2]	2,3	-	1,2,3	0	3
[3, 2, 3, 3, 2]	3	-	0,3	4	4
[3, 2, 4, 3, 2]	3	1,4	0,2,3	1,4	2
[3, 3, 4, 3, 2]	-	-	2,3	1	4
[3, 5, 4, 3, 2]	1,3	-	1,2,3	0	3
[4, 3, 3, 3, 2]	-	-	0,3	4	4
[4, 3, 4, 3, 2]	2	4	0,2,3	1,4	2
[4, 5, 4, 3, 2]	1,2	-	1,2,3	0	3
[5, 4, 3, 3, 2]	1,3	-	0,1,3	4	3
[5, 4, 4, 3, 2]	2,3	-	0,2,3	4	3
[6, 5, 4, 3, 2]	1,2,3	4	0,1,2,3	4	2
[3, 3, 5, 4, 3]	-	-	2,3	1	3
[3, 6, 5, 4, 3]	3	-	1,2,3	0	3
[4, 3, 5, 4, 3]	0,2	-	0,2,3	1	3
[4, 6, 5, 4, 3]	2	-	1,2,3	0	3
[6, 5, 4, 4, 3]	3	-	0,1,3	4	3
[7, 6, 5, 4, 3]	0,2,3	-	0,1,2,3	4	2
[4, 7, 6, 5, 4]	1	-	1,2,3	0	3
[8, 7, 6, 5, 4]	0,1,3	-	0,1,2,3	4	2
[9, 8, 7, 6, 5]	0,1,2	-	0,1,2,3	4	2

FIGURE 2. Some properties of small LNakayama algebras (left) and of small quasi-hereditary CNakayama algebras (right)

2. Preliminaries

Let A be an n-Nakayama algebra and let e_i denote the idempotent corresponding to vertex i in the corresponding quiver. The *Kupisch series* of A is the sequence $[c_0, c_1, \ldots, c_{n-1}]$, where $c_i \geqslant 1$ denotes the vector space dimension of the indecomposable projective module $e_i A$. This sequence uniquely determines the algebra up to isomorphism, see for example [AF92, Theorem 32.9]. For n-CNakayama algebras we extend the Kupisch series cyclically via $c_i = c_j$ for $i, j \in \mathbb{Z}$ with $i \equiv j \mod n$. Two Kupisch series give isomorphic CNakayama algebras if and only if they coincide up to cyclic rotation, corresponding to the cyclic rotation of the vertices of the quiver.

The following identification of Nakayama algebras and Kupisch series is classical and can be found, for example, in [AF92, Chapter 32]. First observe that the Kupisch series $[c_0, \ldots, c_{n-1}]$ of an n-Nakayama algebra A satisfies

$$\begin{aligned} c_{i+1} + 1 \geqslant c_i &\text{ for all } 0 \leqslant i < n, \\ c_i \geqslant 2 &\text{ for all } 0 \leqslant i < n-1. \end{aligned} \tag{2.1}$$

Moreover, A is an LNakayama algebra if and only if

$$c_{n-1} = 1. \tag{2.2}$$

A module over a quiver algebra has vector space dimension 1 if and only if it is simple, so the latter means that the projective module $e_{n-1}A$ is simple. Equivalently, the vertex $n-1$ in the quiver has no outgoing arrow. Together with Eq. (2.1) this forces $c_{n-2} = 2$ for LNakayama algebras. Otherwise, i.e., if

$$c_{n-1} \geqslant 2, \tag{2.3}$$

the Nakayama algebra A is a CNakayama algebra. Note that Eq. (2.1) forces $c_{n-1} \leqslant c_0 + 1$ in this case.

In total we obtain the following identification. Here and below, we use the term *necklace* for a sequence $[a_0, \dots, a_n]$ up to cyclic rotation and write $[a_0, \dots, a_n]_\circlearrowright$ in this case.

Proposition 2.4. *Sending an n-Nakayama algebra to its Kupisch series is a bijection between n-Nakayama algebras and* necklaces *satisfying Eq.* (2.1). *It moreover restricts to bijections between*

(1) n-LNakayama algebras and sequences *satisfying Eqs.* (2.1) *and* (2.2)*, and between*
(2) n-CNakayama algebras and necklaces *satisfying Eqs.* (2.1) *and* (2.3).

Remark 2.5. It is well known that a Nakayama algebra is selfinjective if and only if it is a CNakayama algebra with constant Kupisch series, see for example [SY11, Theorem 6.12 (Chapter IV)]. Over a selfinjective algebra every module is either projective or of infinite projective dimension.

Let A be an n-Nakayama algebra with Kupisch series $[c_0, \dots, c_{n-1}]$. The *coKupisch series* is the sequence $[d_0, \dots, d_{n-1}]$, where d_i is the vector space dimension of the indecomposable injective module $D(Ae_i)$ where $D := \mathrm{Hom}_{\mathbb{F}}(-, \mathbb{F})$ denotes the standard duality of a finite-dimensional algebra. Equivalently, d_i is the vector space dimension of the indecomposable projective left module Ae_i. For n-CNakayama algebras we extend the coKupisch series cyclically such that $d_i = d_j$ for $i, j \in \mathbb{Z}$ with $i \equiv j$ modulo n .

The Kupisch and coKupisch series are related by

$$d_i = \min\{k \mid k \geqslant c_{i-k}\}, \tag{2.6}$$

see [F68, Theorem 2.2]. In particular, this implies $\{c_0, \dots, c_{n-1}\} = \{d_0, \dots, d_{n-1}\}$ as multisets. A sequence is the coKupisch series of an n-Nakayama algebra if and only if the reverse sequence is a Kupisch series. Let A and B be n-Nakayama algebras such that the Kupisch series of A coincides with the reversed coKupisch series of B. Then also the coKupisch series of A coincides with the reversed Kupisch series of B. In particular, interchanging the Kupisch and the reversed coKupisch series is an involution on n-Nakayama algebras. It is given by mapping an n-Nakayama algebra to its opposite algebra.

2.1. **Nakayama algebras and Dyck paths.** The *Auslander-Reiten quiver* of a representation-finite quiver algebra is the quiver with vertices corresponding to the indecomposable modules of the algebra and arrows correspond to the irreducible maps between the indecomposable modules. We refer for example to [SY11, Chapter III] for a detailed introduction to Auslander-Reiten theory.

Nakayama algebras are representation-finite and it is well known that every indecomposable module of an n-Nakayama algebra A with Kupisch series $[c_0, \dots, c_{n-1}]$ is given up to isomorphism by $\mathbf{b}_{i,k} := e_iA/e_iJ^k$ where J denotes the Jacobson radical, $i \in \{0, 1, \dots, n-1\}$ and $k \in \{1, 2, \dots, c_i\}$. Note that $\dim \mathbf{b}_{i,k} = k$. We identify $\mathbf{b}_{i,c_i}$ with e_iA, which are exactly the indecomposable projective modules, and $\mathbf{b}_{i+1-d_i,d_i}$ with $D(Ae_i)$, which are exactly the indecomposable injective modules. The modules $S_i = \mathbf{b}_{i,1}$ are exactly the simple modules. For n-CNakayama

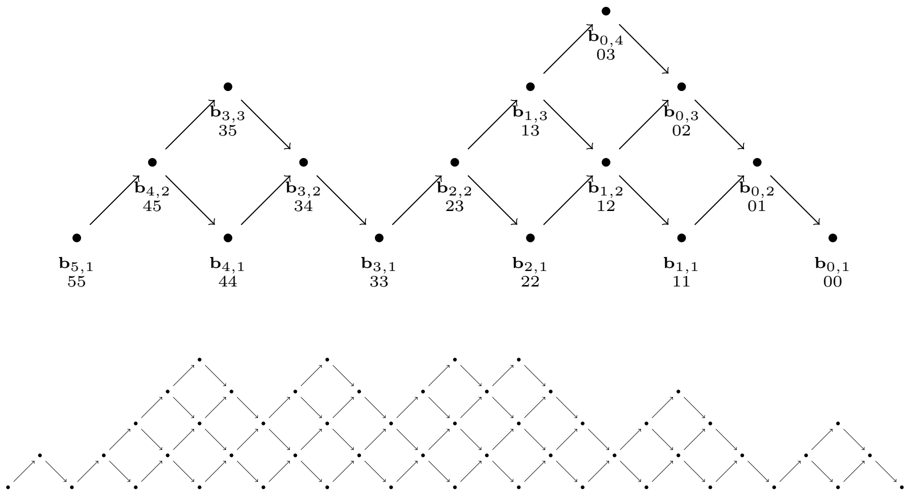
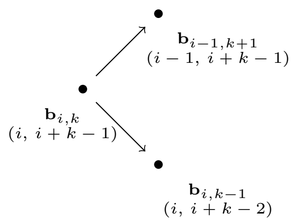


FIGURE 3. The Auslander-Reiten quiver of the Nakayama algebras with Kupisch series $[4, 3, 2, 3, 2, 1]$ and $[3, 2, 4, 3, 5, 5, 4, 5, 4, 5, 4, 3, 2, 2, 1]$ and with coKupisch series $[1, 2, 3, 4, 2, 3]$ and $[1, 2, 3, 2, 3, 4, 3, 4, 5, 5, 4, 5, 4, 5, 2]$. Modules without incoming arrow from the top left are projective, modules without outgoing arrow to the top right are injective.

algebras we regard the indices i of the modules $\mathbf{b}_{i,k}$ and S_i modulo n, so that they are defined for all $i \in \mathbb{Z}$.

The Auslander-Reiten quiver of an n-Nakayama algebra with Kupisch series given by $[c_0, \ldots, c_{n-1}]$ has vertices $\mathbf{b}_{i,k}$ with $0 \leqslant i < n$ and $1 \leqslant k \leqslant c_i$ and all possible arrows of the form



see, for example, [SY11, Theorem 8.7 (Chapter III)]. Note that exactly the maps $\mathbf{b}_{i,k} \to \mathbf{b}_{i-1,k+1}$ are injective, and exactly the maps $\mathbf{b}_{i,k} \to \mathbf{b}_{i,k-1}$ are surjective.

Proposition 2.7. *Let A be a Nakayama algebra with Kupisch series $[c_0, \ldots, c_{n-1}]$. The indecomposable module $\boldsymbol{b}_{i,m}$ is injective if and only if $c_{i-1} \leqslant m$. In particular, $\boldsymbol{b}_{i,c_i}$ is injective if and only if $c_{i-1} \leqslant c_i$ and dually $D(Ae_i)$ is projective if and only if $d_i \geqslant d_{i+1}$.*

Proof. See for example [AF92, Theorem 32.6]. □

We denote by $\tau(\mathbf{b}_{i,k}) := \mathbf{b}_{i+1,k}$ the *Auslander-Reiten translate* of a non-projective indecomposable module $\mathbf{b}_{i,k}$. In particular, $\tau(S_r) = S_{r+1}$ for non-projective S_r, see [ARS97, Proposition 2.11 (Chapter IV)].

As usual, we draw the Auslander-Reiten quiver such that all arrows go from left to right diagonally up or down. To refer to indecomposable modules in the Auslander-Reiten quiver of a Nakayama algebra it will be convenient to define the *coordinates* of $\mathbf{b}_{i,j}$ to be $(i, i + j - 1)$.

Given a Nakayama algebra with Kupisch series $[c_0, \ldots, c_{n-1}]$ and with coKupisch series $[d_0, \ldots, d_{n-1}]$, these coordinates have the property that the number of vertices with x-coordinate i is given by c_i and the number of vertices with y-coordinate j is given by d_j. Figure 3 shows two examples.

2.1.1. *LNakayama algebras and Dyck paths.* Sending an LNakayama algebra to the *top boundary* of its Auslander-Reiten quiver defines a bijection between LNakayama algebras and Dyck paths as follows. We choose a coordinate system for the $\mathbb{Z}^2$-grid by having the *horizontal step* $(0, 1)$ point left and the *vertical step* $(1, 0)$ point down. We identify a square in the $\mathbb{Z}^2$-grid with its top-left corner coordinates (i, j). We also denote the coordinates of a vertical step as the coordinates of its left end, and the coordinates of a horizontal step as the coordinates of its top end.

A *Dyck path* of semilength n is a path from $(0, 0)$ to (n, n) consisting of vertical and horizontal steps that never goes below the main diagonal $x = y$. Denote by $\mathcal{D}_n$ the collection of all Dyck paths of semilength n. In the following we use two *slightly shifted* variants of the area sequence associated with a Dyck path $D \in \mathcal{D}_n$: the *area sequence* $[c_0, c_1, \ldots, c_n]$ is obtained by setting c_k, for $0 \leqslant k \leqslant n$, to the number of lattice points with y-coordinate k in the region enclosed by the path and the main diagonal. Recall that we have identified a square with its top-left corner. For example, the area sequence of the Dyck path in Figure 4 is

$$[3, 2, 4, 3, 5, 5, 4, 5, 4, 5, 4, 3, 2, 2, 1].$$

Similarly, the *coarea sequence* $[d_0, \ldots, d_n]$ is obtained by setting d_k, for $0 \leqslant k \leqslant n$, to the number of lattice points with x-coordinate k in the region enclosed by the path and the main diagonal. In the example in Figure 4, the coarea sequence is

$$[1, 2, 3, 2, 3, 4, 3, 4, 5, 5, 4, 5, 4, 5, 2].$$

Sending a Dyck path $D \in \mathcal{D}_n$ to its area sequence is a bijection between $\mathcal{D}_n$ and sequences $[c_0, \ldots, c_n]$ satisfying exactly the same conditions as those for Kupisch series of $(n + 1)$-LNakayama algebras. Indeed, the pictorially indicated bijection between $(n+1)$-LNakayama algebras and Dyck paths of semilength n is formalized by sending an algebra to the unique Dyck path whose area sequence is given by the algebra's Kupisch series. This Dyck path can be interpreted as the top boundary of the Auslander-Reiten quiver of the LNakayama algebra. Observe that Eq. (2.6) implies that the coarea sequence of D also equals the coKupisch series of A.

We formulate this as the following proposition, and fix throughout this identification between LNakayama algebras and Dyck paths. This connection has already appeared in the literature, see for example [R13, p. 256] for (a variant of) this bijection.

Proposition 2.8. *Sending an LNakayama algebra to the top boundary of its Auslander-Reiten quiver defines a bijection between $(n + 1)$-LNakayama algebras and Dyck paths of semilength n.*

Proof. This follows immediately from Proposition 2.4(1). □

2.1.2. *CNakayama algebras and periodic Dyck paths.* Replacing the initial condition in Eq. (2.2) for LNakayama algebras with the initial condition in Eq. (2.3) for CNakayama algebras we obtain a description of these in terms of periodic Dyck paths.

A *balanced binary n-necklace* is a binary necklace consisting of n white and n black beads. In the above language, we represent a white bead by the letter v and the black bead by the letter h, so that a balanced binary n-necklace is a sequence

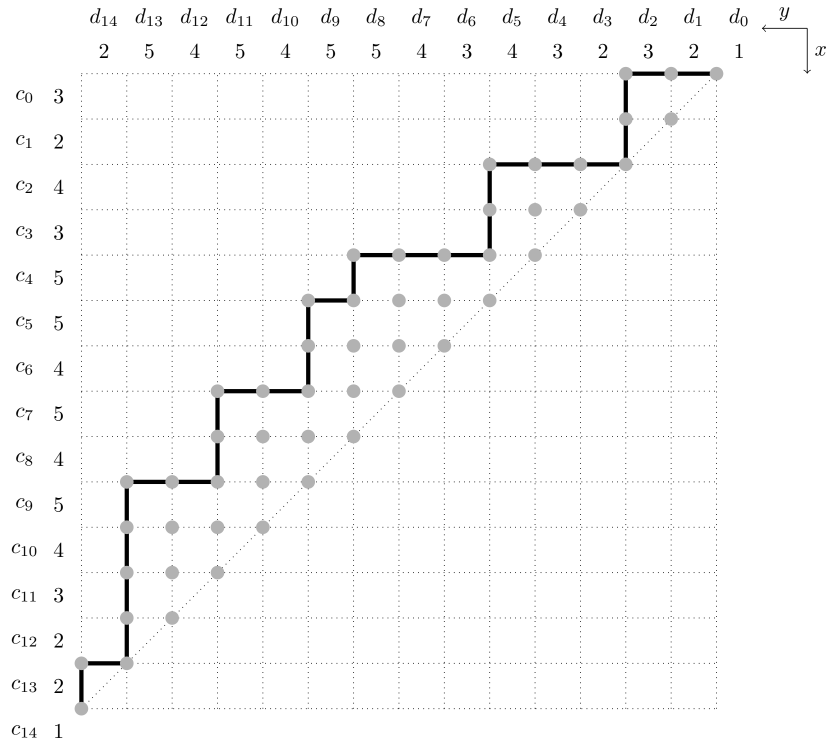


FIGURE 4. The Dyck path of semilength 14 corresponding to the Auslander-Reiten quiver in the bottom example in Figure 3.

of n letters v and h each, considered up to cyclic rotation. Formally, an *n-periodic Dyck path* is a balanced binary n-necklace together with an integer $c \geqslant 0$; we refer to this integer as its *global shift*. This corresponds to an actual path in the $\mathbb{Z}^2$-grid up to diagonal translations together with an explicit choice of a diagonal as follows. One draws a bi-infinite path given by the the balanced binary n-necklace where white beads represent vertical steps and black beads represent horizontal steps. This path is chosen so that it stays weakly but not strictly above the diagonal $y = x + c$, and two paths are identified if they coincide up to diagonal translation.

The *area sequence* of an n-periodic Dyck path is the necklace $[c_0, \dots, c_{n-1}]_{\circlearrowright}$, where c_k is the number of lattice points with y-coordinate k in the region enclosed by the path and the chosen diagonal. Note that, in contrast to the area sequence of an ordinary Dyck path of semilength n, the area sequence of an n-periodic Dyck path has length n rather than $n+1$. The *coarea sequence* is defined accordingly. Figure 5 shows the 5-periodic Dyck path $[h, v, v, h, v, h, h, h, v, v]_{\circlearrowright}$ with global shift 1 and area sequence $[4, 3, 3, 5, 4]_{\circlearrowright}$.

Similar to the case of ordinary Dyck paths, it is immediate from the definition that sending an n-periodic Dyck path to its area sequence (or, respectively, its reversed coarea sequence) is a bijection between n-periodic Dyck paths and necklaces $[c_0, \dots, c_{n-1}]_{\circlearrowright}$ satisfying exactly the same conditions as those for Kupisch series of n-CNakayama algebras.

Proposition 2.9. *Fix $c \geqslant 0$. Sending an n-CNakayama algebra with Kupisch series $[c_0, \dots, c_{n-1}]_{\circlearrowright}$ to the n-periodic Dyck path with area sequence $[c_0, \dots, c_{n-1}]_{\circlearrowright}$ is a bijection between n-CNakayama algebras whose Kupisch series have minimal entry $c + 2$ and n-periodic Dyck paths of global shift c.*

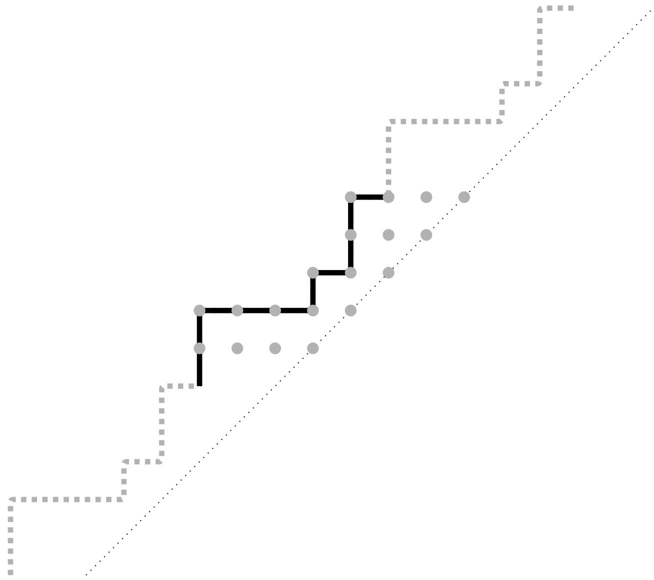


FIGURE 5. A 5-periodic Dyck path of global shift 1.

Proof. This follows immediately from Proposition 2.4(2). □

Corollary 2.10 ([BLL98, Exercise 3.1.10c])**.** *For any $c \geqslant 0$, the number of n-CNakayama algebras whose Kupisch series has minimal entry $c+2$ equals the number of balanced binary n-necklaces*[1]*. Explicitly, this number is*

$$\frac{1}{2n}\sum_{k\,|\,n}\phi(n/k)\binom{2k}{k},$$

where ϕ is Euler's totient, the number of integers relatively prime to the argument.

2.2. **Combinatorial statistics on Dyck paths.** It will be convenient to give names to certain special points in (periodic) Dyck paths. Note that, a priori, we cannot refer to individual steps in periodic Dyck paths or elements of the associated necklace, because they are only defined up to rotation. However, we can fix a (cyclic) labelling of the coordinates with $0,\dots,n-1$ as provided by the correspondence with the simple modules of the associated n-CNakayama algebra.

Definition 2.11. Let D be a Dyck path of semilength n, or a periodic Dyck path with period n.

A *peak*[2] at coordinates (i,j) is a horizontal step with x-coordinate i followed by a vertical step with y-coordinate j.

A *valley*[3] at coordinates (i,j) is a vertical step with y-coordinate j followed by a horizontal step with x-coordinate i.

A 1-*rise*[4] at coordinates (i,j) is a horizontal step with x-coordinate i and final y-coordinate j, which is neither preceded nor followed by a horizontal step.

A *double rise* at coordinates (i,j) is a segment of two horizontal steps whose midpoint has coordinates (i,j).

A *double fall* at coordinates (i,j) is a segment of two vertical steps whose midpoint has coordinates (i,j).

A *return*[5] at position i is a (necessarily vertical) step with final coordinates (i,i).

[1] www.oeis.org/A003239 [2] www.findstat.org/St000015 [3] www.findstat.org/St000053
[4] www.findstat.org/St000445 [5] www.findstat.org/St000011

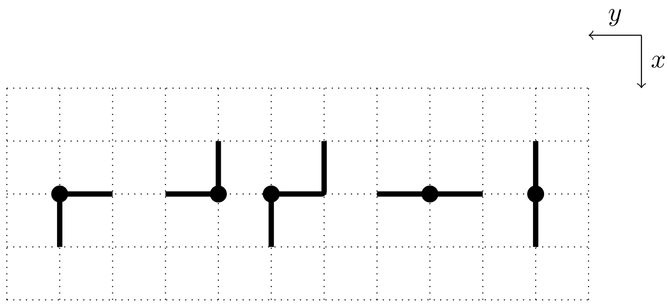


FIGURE 6. Coordinates of peaks, valleys, 1-rises, double rises, and double falls.

A *1-cut* at position i is an occurrence of a horizontal step with x-coordinate i and a vertical step with y-coordinate $i+1$.

A *k-hill*[6] at position i is a segment of k horizontal steps followed by k vertical steps, starting at (i,i).

A *rectangle* at coordinates $(i+1,j)$ is a valley at $(i+1,j)$, such that the next valley has x-coordinate strictly larger than $j+1$. In terms of area sequences, this is $c_{i+1}+1=c_i+c_{i+c_i}$, with $j=i+c_i-1$.

2.3. **Some homological properties of Nakayama algebras.**

In this section, we recall several known homological properties of Nakayama algebras that we need in later sections.

We quickly recall the definitions of projective dimension and Ext here and refer for example to [B91] for detailed information. Recall that the *projective cover* of a module M is by definition the unique surjective map (up to isomorphism) $P \to M$ such that P is projective of minimal vector space dimension. Dually the *injective envelope* of M is by definition the unique injective map $M \to I$ such that I is injective of minimal vector space dimension. One often just speaks of P as the projective cover for short and also as I being the injective envelope. We will often use that a module M is isomorphic to its projective cover P if and only if M has the same vector space dimension as its projective cover P. This follows immediately from the fact that a projective cover is a surjection and that a module homomorphism is an isomorphism if and only if it is surjective and both modules have the same vector space dimension. For a module M, the *first syzygy module* $\Omega^1(M)$ is by definition the kernel of the projective cover $P \to M$ of M. Inductively, one then defines for $n \geqslant 0$ the *n-th syzygy module* of M as $\Omega^n(M) := \Omega^1(\Omega^{n-1}(M))$ with $\Omega^0(M)=M$. The *projective dimension* $\mathrm{pd}(M)$ of M is defined as the smallest integer $n \geqslant 0$ such that $\Omega^n(M)$ is projective and as infinite in case no such n exists. For two A-modules M and N one defines $\mathrm{Ext}_A^1(M,N)$ as $\mathrm{Ext}_A^1(M,N) := D(\overline{\mathrm{Hom}}_A(N,\tau(M)))$, where $\tau(M)$ denotes the Auslander-Reiten translate of M and $\overline{\mathrm{Hom}}_A(X,Y)$ denotes the space of homomorphisms between two A-modules X and Y modulo the space of homomorphisms between X and Y that factor over an injective A-module. For $n \geqslant 1$, one then defines $\mathrm{Ext}_A^n(M,N) := \mathrm{Ext}_A^1(\Omega^{n-1}(M),N)$. We furthermore define $\mathrm{Ext}_A^0(M,N) := \mathrm{Hom}_A(M,N)$. Note that we choose here to present the definition of Ext in the probably shortest way possible (using the Auslander-Reiten formulas, see for example [SY11, Chapter III. theorem 6.3.]) and we refer for example to [B91, chapter 2.4.] for the classical definition. For the practical calculation of the

[6] www.findstat.org/St000674, www.findstat.org/St001139, www.findstat.org/St001141

projective cover, injective envelope and syzygies of modules in Nakayama algebras we refer the reader to the preliminaries of [M18].

Lemma 2.12. *Let A be a finite-dimensional algebra. Let S be a simple A-module and M an A-module with minimal projective resolution*

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0.$$

For $\ell \geqslant 0$, $\operatorname{Ext}_A^\ell(M, S) \neq 0$ if and only if there is a surjection $P_\ell \to S$. Dually, let

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

be a minimal injective coresolution of M. For $\ell \geqslant 0$, $\operatorname{Ext}_A^\ell(S, M) \neq 0$ if and only if there is an injection $S \to I_\ell$.

Proof. See for example [B91, Corollary 2.6.5]. □

3. 1-regular and 2-regular simple modules

In this section we provide characterizations of 1- and 2-regular simple modules of Nakayama algebras in terms of Kupisch series. We then exhibit bijections that transform these conditions into local properties of Dyck paths and periodic Dyck paths.

Definition 3.1. Let A be a finite-dimensional algebra and S a simple A-module. For $k \in \mathbb{N}$, the module S is *k-regular* if

(1) $\operatorname{pd}(S) = k$,
(2) $\operatorname{Ext}_A^i(S, A) = 0$ for $0 \leqslant i < k$, and
(3) $\dim \operatorname{Ext}_A^k(S, A) = 1$.

Recall that the condition $\dim \operatorname{Ext}_A^k(S, A) = 1$ is equivalent to the left A-module $\operatorname{Ext}_A^k(S, A)$ being simple, since modules over quiver algebras are simple if and only if they have vector space dimension equal to one.

The definition of k-regular simple modules is motivated by the notion of the restricted Gorenstein condition which is used in higher Auslander-Reiten theory, see for example [I07, Proposition 1.4 and Theorem 2.7]. We study the restricted Gorenstein condition in the special case of Nakayama algebras of global dimension at most two in Section 4. The simple module S_{n-1} for an n-LNakayama algebra is the unique simple projective module and thus S_{n-1} is never k-regular for $k \geqslant 1$. Thus it is no loss of generality to exclude the simple module S_{n-1} in our treatment of k-regular simple modules.

The most important case of k-regularity is 2-regularity, which was recently used by Enomoto [E18] to reduce the classification of exact structures on categories of finitely generated projective A-modules for Artin algebras A to the classification of 2-regular simple modules. We refer to [BU10] for the definitions and discussions of exact categories. Enomoto's result, restricted to finite-dimensional algebras, is as follows.

Theorem 3.2 ([E18, Theorem 3.7])**.** *Let A be a finite-dimensional algebra and let $\mathcal{E}$ be the category of finitely generated projective A-modules. Then there is a bijection between*

(1) exact structures on $\mathcal{E}$ and
(2) sets of isomorphism classes of 2-regular simple A-modules.

3.1. **Description in terms of Kupisch series.** For any k, a k-regular simple module S_i is non-projective by Definition 3.1(1). In the case of n-LNakayama algebras this means that $i < n-1$, whereas this is no restriction for CNakayama algebras since the latter do not have projective simple modules. Throughout this section, let A denote an n-Nakayama algebra with Kupisch series $[c_0, \dots, c_{n-1}]$ and coKupisch series $[d_0, \dots, d_{n-1}]$ and let S_i denote a simple A-module corresponding to the vertex i.

Theorem 3.3. *A simple non-projective module S_i is*

(1) 1*-regular*[7] *if and only if $c_i - c_{i+1} = d_{i+1} - d_i = 1$,*
(2) 2*-regular*[8] *if and only if*

$$c_i = d_{i+2} = 2 \quad \textit{and} \quad c_{i+1} - c_{i+2} = d_{i+1} - d_i = 1.$$

The first step towards this theorem is a description of the non-projective simple modules of projective dimensions one and two.

Proposition 3.4. *A simple non-projective module S_i has*

(1) $\mathrm{pd}(S_i) = 1$[9] *if and only if $c_{i+1} + 1 = c_i$, and*
(2) $\mathrm{pd}(S_i) = 2$[10] *if and only if $c_{i+1} + 1 = c_{i+c_i} + c_i$.*

Proof. We have that $\mathrm{pd}(S_i) = 1$ if and only if the module e_iJ in the short exact sequence

$$0 \to e_iJ \to e_iA \to S_i \to 0$$

is projective. This is the case if and only if e_iJ is isomorphic to its projective cover $e_{i+1}A$, which is equivalent to $c_i - c_{i+1} = 1$ by comparing vector space dimensions and using that $\dim(e_{i+1}A) = c_{i+1}$ and $\dim(e_iJ^k) = c_i - k$.

The beginning of a minimal projective resolution of S_i is given by splicing together the two short exact sequences

$$0 \to e_iJ \to e_iA \to S_i \to 0$$
$$0 \to e_{i+1}J^{c_i-1} \to e_{i+1}A \to e_iJ \to 0.$$

We have already seen in (1) that $\mathrm{pd}(S_i) \geqslant 2$ if and only if $c_i \leqslant c_{i+1}$. Moreover, $\mathrm{pd}(S_i) = 2$ if additionally $e_{i+1}J^{c_i-1}$ is projective. Now $e_{i+1}J^{c_i-1}$ being projective is equivalent to the condition that it is isomorphic to its projective cover $e_{i+c_i}A$, which in turn translates into the condition $c_{i+1} - (c_i - 1) = c_{i+c_i}$ by comparing vector space dimensions. ☐

Lemma 3.5. *For a simple non-projective module S_i, we have*

(1) $\mathrm{Hom}_A(S_i, A) = 0 \Leftrightarrow d_{i+1} = d_i + 1$,
(2) $\mathrm{Ext}^1_A(S_i, A) = 0 \Leftrightarrow c_i < c_{i+1} + 1$.

Proof. Note that $\mathrm{Hom}_A(S_i, A) = 0$ if and only if S_i does not appear in the socle of A, which is equivalent to the injective envelope $I(S_i) = D(Ae_i)$ of S_i being non-projective (here we use that the injective envelope of A is projective-injective for every Nakayama algebra, see for example [AF92, Theorem 32.2]). This translates into the condition $d_{i+1} > d_i$ by using Proposition 2.7.

For the second property, note that $\mathrm{Ext}^1_A(S_i, A) = 0$ if and only if $\mathrm{Ext}^1_A(S_i, e_rA) = 0$ for every indecomposable non-injective module e_rA. Thus, suppose that e_rA is non-injective, then

$$0 \to e_rA \to D(Ae_{r+c_r-1}) \to D(Ae_{r-1})$$

is the beginning of a minimal injective coresolution of a non-injective e_rA, see for example [M18, Preliminaries]. Lemma 2.12 entails that $\mathrm{Ext}^1_A(S_i, e_rA) \neq 0$ if and

[7] www.findstat.org/St001126 [8] www.findstat.org/St001125 [9] www.findstat.org/St001007
[10] www.findstat.org/St001011

only if there is an injection $S_i \to D(Ae_{r-1})$. Since $S_i = \mathbf{b}_{i,1}$ and $D(Ae_{r-1}) = \mathbf{b}_{r-d_{r-1},d_{r-1}}$, such an injection exists if and only if $i = r-1$. $\square$

Lemma 3.6. *We have the following two properties for a simple non-projective module S_i:*

(1) If $\operatorname{Hom}_A(S_i, A) = 0$ and $\operatorname{pd}(S_i) = 1$, then
$$\dim \operatorname{Ext}^1_A(S_i, A) = 1 \Leftrightarrow d_{i+1} = d_i + 1.$$
(2) If $\operatorname{Hom}_A(S_i, A) = \operatorname{Ext}^1_A(S_i, A) = 0$ and $\operatorname{pd}(S_i) = 2$, then
$$\dim \operatorname{Ext}^2_A(S_i, A) = 1 \Leftrightarrow d_{i+1} + 1 = d_i + d_{i+c_i}.$$

Proof. For the first property, we apply the left exact functor $\operatorname{Hom}_A(-, A)$ to the short exact sequence
$$0 \to e_iJ \to e_iA \to S_i \to 0$$
and use that $e_iJ \cong e_{i+1}A$ (since S_i is assumed to have projective dimension equal to one). We obtain the exact sequence
$$0 \to \operatorname{Hom}_A(S_i, A) \to \operatorname{Hom}_A(e_iA, A) \to \operatorname{Hom}_A(e_{i+1}A, A) \to \operatorname{Ext}^1_A(S_i, A) \to 0.$$
Comparing dimensions and using $\operatorname{Hom}_A(S_i, A) = 0$, we obtain the condition
$$\begin{aligned} 1 &= \dim \operatorname{Ext}^1_A(S_i, A) \\ &= \dim \operatorname{Hom}_A(S_i, A) + \dim \operatorname{Hom}_A(e_{i+1}A, A) - \dim \operatorname{Hom}_A(e_iA, A) \\ &= \dim(Ae_{i+1}) - \dim(Ae_i) \\ &= d_{i+1} - d_i. \end{aligned}$$

For the second property, we apply the left exact functor $\operatorname{Hom}_A(-, A)$ to the short exact sequence
$$0 \to e_iJ \to e_iA \to S_i \to 0,$$
and we obtain the exact sequence
$$0 \to \operatorname{Hom}_A(S_i, A) \to \operatorname{Hom}_A(e_iA, A) \to \operatorname{Hom}_A(e_iJ, A) \to \operatorname{Ext}^1_A(S_i, A) \to 0.$$
The condition $\operatorname{Ext}^1_A(S_i, A) = 0$, together with $\operatorname{Hom}_A(S_i, A) = 0$, is equivalent to
$$\operatorname{Hom}_A(e_iA, A) \cong \operatorname{Hom}_A(e_iJ, A),$$
which translates into the condition $\dim \operatorname{Hom}_A(e_iJ, A) = d_i$. Now we apply the functor $\operatorname{Hom}_A(-, A)$ to the short exact sequence
$$0 \to e_{i+1}J^{c_i-1} \to e_{i+1}A \to e_iJ \to 0,$$
where we use that $e_{i+1}J^{c_i-1} \cong e_{i+c_i}A$ is projective since S_i is assumed to have projective dimension equal to two. We obtain the exact sequence
$$0 \to \operatorname{Hom}_A(e_iJ, A) \to \operatorname{Hom}_A(e_{i+1}A, A) \to \operatorname{Hom}_A(e_{i+c_i}A, A) \to \operatorname{Ext}^1_A(e_iJ, A) \to 0.$$
Now note that $\operatorname{Ext}^1_A(e_iJ, A) \cong \operatorname{Ext}^1_A(\Omega^1(S_i), A) \cong \operatorname{Ext}^2_A(S_i, A)$. Comparing dimensions we obtain
$$\begin{aligned} 1 &= \dim \operatorname{Ext}^2_A(S_i, A) \\ &= \dim \operatorname{Hom}_A(e_iJ, A) + \dim \operatorname{Hom}_A(e_{i+c_i}A, A) - \dim \operatorname{Hom}_A(e_{i+1}A, A) \\ &= d_i + d_{i+c_i} - d_{i+1}. \end{aligned}$$
$\square$

Proof of Theorem 3.3. The description of 1-regular simple modules is a direct consequence of the respective first items in Proposition 3.4 and Lemmas 3.5 and 3.6. These lemmas also give that S_i is 2-regular if and only if
$$\begin{aligned} c_i < c_{i+1} + 1 &= c_i + c_{i+c_i}, \\ d_i + 2 = d_{i+1} + 1 &= d_i + d_{i+c_i}. \end{aligned} \tag{3.7}$$

We simplify these conditions as follows. The first condition implies that there are $c_{i+1}+1-c_i = c_{i+c_i} > 0$ horizontal steps with x-coordinate $i+1$ in the (possibly periodic) Dyck path corresponding to A. Thus there is a valley at $(i+1, i+c_i-1)$. The second condition implies $d_{i+c_i} = 2$. Therefore, the valley is on the main diagonal, which in turn implies that $c_i = 2$. Conversely,

$$\begin{aligned} c_i + c_{i+c_i} &= 2 + c_{i+2} = c_{i+1} + 1 \\ d_i + d_{i+c_i} &= d_i + 2 = d_{i+1} + 1. \end{aligned}$$

□

Example 3.8. Let A be the 5-LNakayama algebra with Kupisch series $[4, 3, 2, 2, 1]$ and coKupisch series $[1, 2, 3, 4, 2]$. By Proposition 3.4, the simple modules S_0, S_1 and S_3 have projective dimension 1 and the simple module S_2 has projective dimension 2. The simple module S_4 is projective. To see that S_0 and S_1 are 1-regular while S_3 is not, we compute

$$d_1 - d_0 = d_2 - d_1 = 1 \neq d_4 - d_3.$$

Moreover, S_2 is 2-regular because

$$c_2 = d_4 = 2 \quad \text{and} \quad c_3 - c_4 = d_3 - d_2 = 1.$$

3.2. **Description in terms of Dyck path statistics for LNakayama algebras.** Based on Theorem 3.3, we obtain combinatorial reformulations of 1- and 2-regularity in terms of Dyck paths. The first of these is a direct translation into the language of Dyck paths, the second uses a classical involution on Dyck paths to obtain a more local description, and the third uses another bijection which yields a completely local description in terms of two classical statistics.

Theorem 3.9. *Let A be an n-LNakayama algebra and let D be its corresponding Dyck path of semilength $n-1$. Let $\widehat{D}$ be the path obtained from D by adding a horizontal step from $(0,-1)$ to $(0,0)$ and a vertical step from $(n-1, n-1)$ to $(n, n-1)$. Then*

(1) S_i is 1-regular if and only if $\widehat{D}$ has a double rise with y-coordinate i and a double fall with x-coordinate $i+1$.

(2) S_i is 2-regular if and only if $\widehat{D}$ has a vertical step with final coordinates $(i+1, i+1)$, a double rise with y-coordinate i and a double fall with x-coordinate $i+2$.

Proof. This is immediate from Theorem 3.3. □

As announced, the second reformulation uses a classical involution which we now recall.

Definition 3.10. The *Lalanne-Kreweras involution*[11] LK on Dyck paths of semilength n is the following map:

(1) Draw the Dyck path D below the main diagonal, from the top right to the bottom left.

(2) Draw a vertical line emanating from the midpoint of each double horizontal step and a horizontal line emanating from the midpoint of each double vertical step.

(3) Mark the intersections of the i-th vertical and the i-th horizontal line for each i.

(4) Then $\mathrm{LK}(D)$ is the Dyck path of semilength n whose valleys are the marked points.

In Figure 7, the Lalanne-Kreweras involution of the black path yields the blue path. The vertical and horizontal lines drawn in the second step are coloured green. The black crosses should be ignored for now.

[11] www.findstat.org/Mp00120

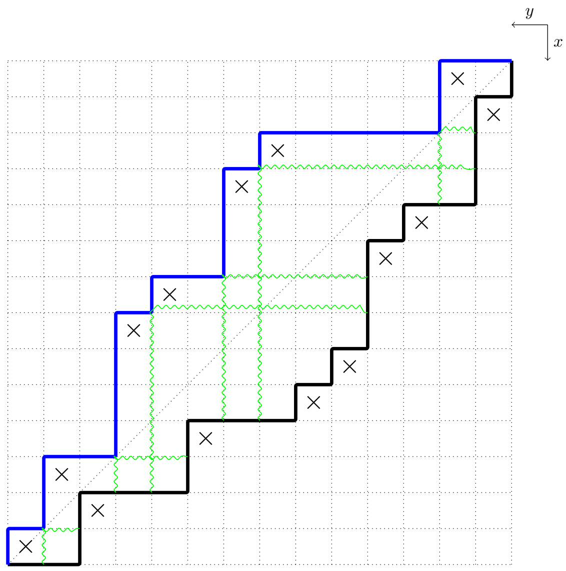


FIGURE 7. A Dyck path D (in black), its Lalanne-Kreweras involution (in blue) and the permutation (indicated by crosses) of the 321-avoiding permutation obtained by applying the Billey-Jockusch-Stanley bijection.

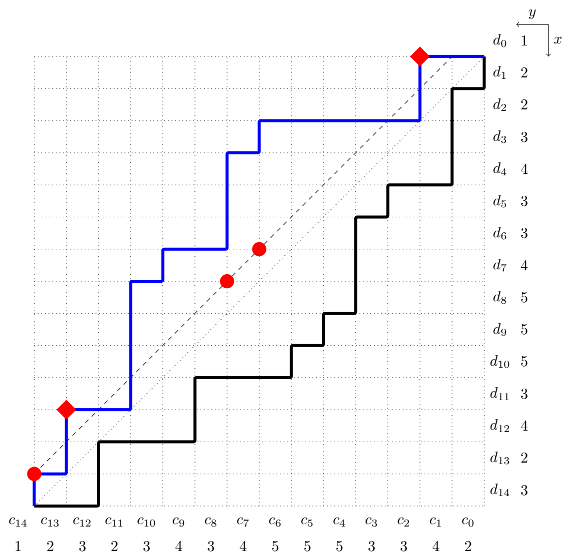


FIGURE 8. A complete example for Theorem 3.11. The Dyck path D, drawn below the main diagonal, is black, its Lalanne-Kreweras involution is blue. The area sequence for D is at the bottom, the coarea sequence for D at the right hand side. The 1-regular modules of the corresponding Nakayama algebra are S_6, S_7 and S_{13}. Corresponding 1-cuts are marked with a red circle. The 2-regular modules are S_0 and S_{11}. Corresponding 2-hills are marked with a red diamond.

Theorem 3.11. *Let A be an n-LNakayama algebra, let D be the Dyck path corresponding to A and let $E = \mathrm{LK}(D)$ be the image of D under the Lalanne-Kreweras involution. Then*

1. *S_i is 1-regular if and only if E has a 1-cut at position i.*
2. *S_i is 2-regular if and only if E has a 2-hill at position i.*

Proof. Suppose first that S_i is 1-regular. By Theorem 3.3(1), $c_i - c_{i+1} = d_{i+1} - d_i = 1$. Because of $c_i - c_{i+1} = 1$, there is a double horizontal step in the path below the diagonal, whose midpoint has y-coordinate $i+1$. Because of $d_{i+1} - d_i = 1$ there is a double vertical step whose midpoint has x-coordinate i. The corresponding vertical and horizontal lines (coloured green in Figure 7) intersect at the diagonal $y = x + 1$, dashed in Figure 8.

By the definition of the Lalanne-Kreweras involution, the Dyck path $\mathrm{LK}(D)$ has a valley at the end of every green line. Therefore, for each vertical line there is a peak of $\mathrm{LK}(D)$ with the same y-coordinate as the line, and for each horizontal line there is a peak at the same x-coordinate as the line. Specifically, for two green lines intersecting at $(i, i+1)$, there is a peak corresponding to the vertical line with y-coordinate $i+1$, and a peak corresponding to the horizontal line with x-coordinate i, that is, a 1-cut.

Conversely, if there are two such peaks, the two corresponding vertical and horizontal lines intersect at the diagonal $y = x + 1$, implying that S_i is 1-regular.

Let us now show that 2-regular modules correspond to hills of size 2. It is clear that a hill of size 2 in $\mathrm{LK}(D)$ forces the conditions on D in Theorem 3.3(2).

Conversely, the conditions $c_i = d_{i+2} = 2$ imply that the Dyck path D below the diagonal has a return to the diagonal with x- and y-coordinate $i+1$.

Let us ignore for the moment the degenerate cases where D begins or ends with a single vertical followed by a single horizontal step. Then, the horizontal line emanating from the midpoint of the double vertical step forced by $d_{i+1} = d_i + 1$ must be matched with the vertical line emanating from the midpoint of the last double horizontal step before - to the right and above - the return to the diagonal. Thus, the intersection of these two lines is on the diagonal of D.

Similarly, the vertical line emanating from the midpoint of the double horizontal step forced by $c_{i+1} = c_{i+2} + 1$ must be matched with the horizontal line emanating from the first double vertical step after the return to the diagonal, and their intersection is on the diagonal. Finally, we observe that the distance between these two intersections is 2. $\square$

In the following we describe a bijection on Dyck paths that yields an even simpler description of the 1- and 2-regular simple modules. The main ingredient is the Billey-Jockusch-Stanley bijection, which is closely related to the Lalanne-Kreweras involution:

Definition 3.12. The *Billey-Jockusch-Stanley bijection*[12] BJS sends a Dyck path D of semilength n to a 321-avoiding permutation π of the numbers $\{1, \dots, n\}$ as follows:

(1) Draw the Dyck path D below the main diagonal.
(2) Put crosses into the cells corresponding to the valleys of D.
(3) Then, working from left to right, for each column not yet containing a cross we put a cross into the lowest cell whose row does not yet contain a cross.

This yields the permutation matrix of the reverse complement of π.

Note that one can equivalently fill in the crosses in step (3) from right to left, putting crosses into the top-most available cell.

[12] www.findstat.org/Mp00129

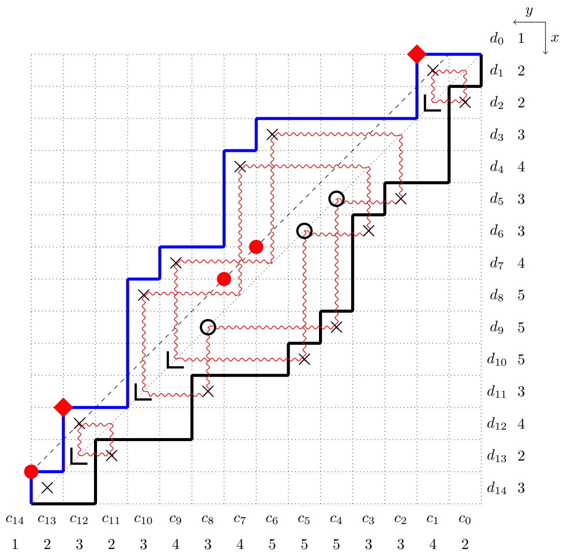


FIGURE 9. The cycle diagram of the permutation associated with the Dyck path, together with the points indicating the 1- and 2-regular modules. The configurations along the diagonal specifying the composition are indicated with black 'L'-shapes and circles.

In Figure 7, the black crosses indicate the permutation matrix of $\mathrm{BJS}(D)$. As visible there, we have the following relation between the Lalanne-Kreweras involution and the Billey-Jockusch-Stanley bijection:

Proposition 3.13. *The peaks of* $\mathrm{LK}(D)$ *are at the positions of the crosses of the permutation matrix of* $\mathrm{BJS}(D)$ *strictly above the main diagonal.*

Our final reformulation uses a slightly more involved bijection, but has the advantage of describing 1- and 2-regular statistics in a completely local way.

Theorem 3.14. *Let A be an n-LNakayama algebra and let D be corresponding Dyck path. Then there is an explicit bijection ϕ, such that*

(1) S_i is 1-regular if and only if $\phi(D)$ has a 1-rise with x-coordinate i.
(2) S_i is 2-regular if and only if $\phi(D)$ has a 2-hill at position i.

Proof. Taking into account Theorem 3.11, it suffices to provide a bijection ψ on Dyck paths that preserves hills of size 2 and maps 1-cuts to 1-rises.

Let E be a Dyck path of semilength n. We first construct Elizalde's 'cycle diagram' of the 321-avoiding permutation associated with E by the Billey-Jockusch-Stanley bijection. For the Dyck path $E = \mathrm{LK}(D)$ in Figure 7, this is carried out in Figure 9.

We then record the sequence of configurations of lines emanating from the main diagonal of the cycle diagram, and construct a (weak) composition α as follows. Points on the main diagonal with both lines being in the upper left of the diagram (as drawn in Figure 9) correspond to 1-cuts, and are ignored. If the horizontal line is in the upper left, and the vertical line in the lower right of the diagram, the point is also ignored.

Of the remaining points, those who have their horizontal line in the lower right and their vertical line in the upper left of the diagram, serve as delimiters of the composition. In the figure, these are indicated by black 'L'-shapes. The i-th part of the composition, α_i, is the number of points between the i-th and the $(i+1)$-st

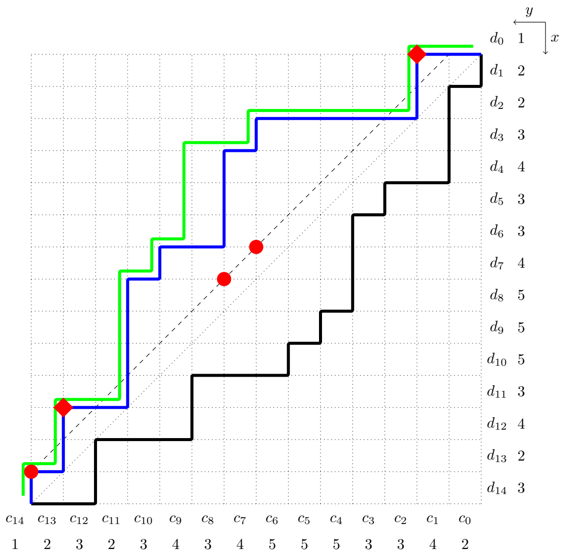


FIGURE 10. A complete example for Theorem 3.14. The Dyck path D is black, its Lalanne-Kreweras involution is blue. The area sequence for D is at the bottom, the coarea sequence at the right hand side. The 1-regular modules of the corresponding Nakayama algebra are S_6, S_7 and S_{13}. Corresponding 1-cuts of $\mathrm{LK}(D)$ are marked with a red circle. The 2-regular modules are S_0 and S_{11}. Corresponding 2-hills of $\mathrm{LK}(D)$ are marked with a red diamond. The final path, $\phi(D)$, in green, has 1-rises at x-coordinates 6, 7 and 13, and 2-hills at 0 and 11. It is slightly shifted to improve visibility.

delimiter, with both lines in the lower right of the diagram. In the figure, these points are indicated by black circles. Thus, the composition corresponding to the configuration in Figure 9 is $\alpha = (0, 3, 0, 0)$.

Finally, $\psi(E)$ is the unique Dyck path that has peaks at the same x-coordinates as E, 1-rises at the x-coordinates of the 1-cuts of E, and the lengths of the remaining rises given by adding 2 to each part of α. □

Theorem 3.15. *Let A be an n-LNakayama algebra and let D be corresponding Dyck path. Then D has a rectangle at $(i+1, j)$ if and only if* $\mathrm{LK}(D)$ *has a return at position $j+1$, which is not a 1-hill.*

Proof. Suppose that D has a rectangle at $(i+1, j)$. Thus, drawing D below the main diagonal, there is a valley at $(j, i+1)$, and the next valley has y-coordinate strictly larger than $j+1$. In particular, $\mathrm{BJS}(D)$ has no crosses in the region below and to the right of $(j+1, j+1)$. Because $\mathrm{BJS}(D)$ is a permutation, by the pigeonhole principle, it has no crosses in the region above and to the left of $(j+1, j+1)$ either. Thus, $\mathrm{LK}(D)$ must have a return with x-coordinate $j+1$. This cannot be a 1-hill, because there is a cross in the cell with x-coordinate j, corresponding to the valley $(j, i+1)$ and a 1-hill would correspond to a cross in the cell with coordinates (j, j). □

We conclude with some corollaries enumerating LNakayama algebras with certain homological restrictions.

Corollary 3.16. *The number of $(n+1)$-LNakayama algebras with exactly ℓ simple modules of projective dimension 1 and the number of $(n+1)$-LNakayama algebras with exactly ℓ simple modules of projective dimension at least 2 equal the Narayana numbers*[13]*, counting Dyck paths of semilength n with exactly ℓ peaks. Explicitly, this number is*

$$\frac{1}{n}\binom{n}{\ell-1}\binom{n}{\ell}.$$

Proof. This is a direct consequence of Proposition 3.4(1). □

Corollary 3.17. *The number of $(n+1)$-LNakayama algebras with exactly ℓ simple modules of projective dimension 2 is the number of Dyck paths of semilength n with exactly ℓ returns which are not 1-hills. Explicitly, this number is*

$$\sum_{k=0}^{n-2\ell}\frac{\ell}{k+\ell}\binom{2(k+\ell)}{k}\binom{n-k-\ell}{\ell}.$$

Corollary 3.18. *Let $a_{n,k,\ell}$ be the number of n-LNakayama algebras with k simple modules of projective dimension 1 and ℓ simple modules of projective dimension 2 and let*

$$\begin{aligned}N(x,q,t) &= \sum_{n,k,\ell} a_{n,k,\ell}x^n q^k t^\ell\\ &= x + qx^2 + (q^2+qt)x^3 + (q^3+3q^2t+qt)x^4 + \cdots\end{aligned}$$

Then

$$\begin{aligned}\Big(x^2(q-t)q(t-1) - x(2qt-2q+t)+t-1\Big)N(x,q,t)^2\\ + \Big((qt-2q+t)x-t+2\Big)N(x,q,t)-1=0.\end{aligned}$$

Corollary 3.19. *The number of $(n+1)$-LNakayama algebras without 1-regular simple modules equals the Riordan number*[14]*, counting Dyck paths of semilength n without 1-rises. Explicitly, this number is*

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{n-k-1}{k-1}.$$

Corollary 3.20. *For $n\geqslant 1$, the number of $(n+1)$-LNakayama algebras A without 2-regular simple modules (that is, such that the category of finitely generated projective modules has a unique exact structure) equals the number of Dyck paths without 2-hills*[15]*. Explicitly, this number is*

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{k+1}{n-k+1}\binom{2n-3k}{n-k}.$$

Proof. The formula for the number of such Dyck paths was provided by Ira Gessel [MO18]. □

Corollary 3.21. *Let $a_{n,k,\ell}$ be the number of n-LNakayama algebras with k simple 1-regular and ℓ simple 2-regular modules and let*

$$\begin{aligned}N(x,q,t) &= \sum_{n,k,\ell} a_{n,k,\ell}x^n q^k t^\ell\\ &= x + qx^2 + (q^2+t)x^3 + (q^3+2qt+q+1)x^4 + \cdots\end{aligned}$$

[13] www.oeis.org/A001263 [14] www.oeis.org/A005043 [15] www.oeis.org/A114487

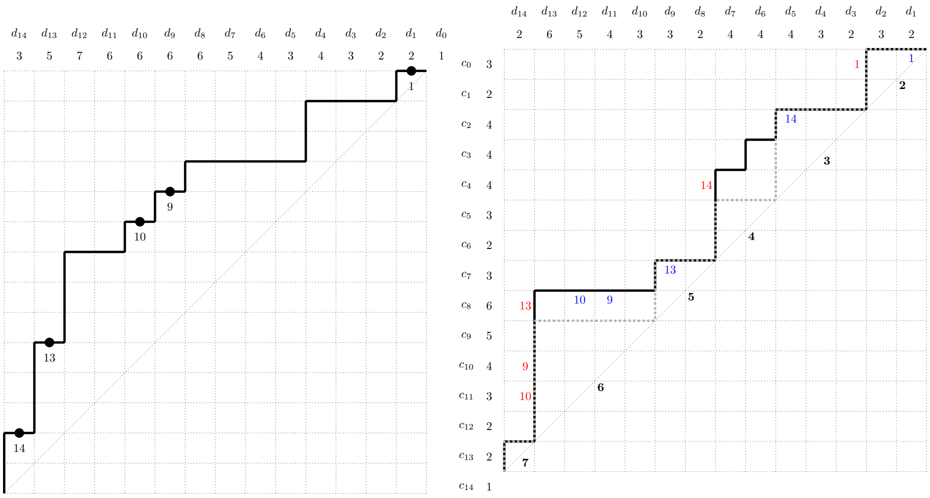


FIGURE 11. A Dyck path of semilength 14 and its image under the zeta map.

Then

$$\Big(x^3(t-1)^2 + x^2(t-1)(q-1) - x(t-1+q-1) + 1\Big)N(x,q,t)^2 + \Big(2x^2(t-1) + x(q-1) - 1\Big)N(x,q,t) + x = 0.$$

3.2.1. *Describing* 1*-regular simple modules using the zeta map.* We finish this section with an alternative approach to Theorem 3.14(1) using the *zeta map*. We refer to [Hag08, page 50] for the history of this map and its original context. Let D be a Dyck path of semilength n with coarea sequence $(d_0, \dots, d_n)$. We obtain $\zeta(D)$, adapted to our convention as follows:

- First, let a_k be the number of indices i with $d_i = k$ and build an intermediate Dyck path (the *bounce path*) consisting of a_2 horizontal steps, followed by a_2 vertical steps, followed by a_3 horizontal steps and a_3 vertical steps, and so on.
- Then, we fill the rectangular regions between two consecutive peaks of the bounce path. Observe that the rectangle between the k-th and the $(k+1)$-st peak must be filled by a_{k+1} vertical steps and a_{k+2} horizontal steps. We do so by scanning the coarea sequence $(d_0, \dots, d_n)$ and drawing a vertical or a horizontal step whenever we encounter a $k+1$ or a $k+2$, respectively.

In the example in Figure 11, a Dyck path D (on the left) with coarea sequence

$$[1, 2, 2, 3, 4, 3, 4, 5, 6, 6, 6, 6, 7, 5, 3]$$

and its image $\zeta(D)$ (on the right) under the zeta map is shown. In dotted grey, the intermediate bounce path is shown.

For a given Dyck path with coarea sequence $(d_0, \dots, d_n)$, the definition of the zeta map yields a labelling of the vertical steps and of the horizontal steps of $\zeta(D)$ with the indices $\{1, \dots, n\}$ by associating to $1 \leqslant j \leqslant n$ the vertical and the horizontal step drawn using the entry d_j. In the example in Figure 11, the vertical steps are labelled from top to bottom by the permutation $[1, 2, 3, 5, 14, 4, 6, 7, 13, 8, 9, 10, 11, 12]$ as are the horizontal steps from right to left. In symbols and in terms of the inverse permutation, the vertical step labelled j has initial x-coordinate

$$k(j) = \#\big\{0 \leqslant i \leqslant n \ :\ d_i < d_j\big\} + \#\big\{0 \leqslant i < j \ :\ d_i = d_j\big\} - 1$$

and the horizontal step labelled by j has final y-coordinate $k(j)+1$. In the example, the $k(j)$ is given by $[0, 1, 2, 5, 3, 6, 7, 9, 10, 11, 12, 13, 8, 4]$.

We then have the following alternative to Theorem 3.14(1).

Theorem 3.22. *Let D be a Dyck path of semilength n with coarea sequence $(d_0, \dots, d_n)$ and let A be the Nakayama algebra corresponding to $\zeta(D)$. Then D has*

- *a peak with y-coordinate j if and only if the simple A-module $S_{k(j)}$ has projective dimension 1, and*
- *a 1-rise with y-coordinate j if and only if $S_{k(j)}$ is 1-regular.*

We extract the crucial observation towards the proof in the following lemma.

Lemma 3.23. *Let D be a Dyck path of semilength n and let $(c_0, \dots, c_n)$ and $(d_0, \dots, d_n)$ be the area and, respectively, the coarea sequence of $\zeta(D)$. Then,*

- *for any $1 \leqslant j \leqslant n$, the path D has a peak with y-coordinate j if and only if $c_{k(j)} - c_{k(j)+1} = 1$, and*
- *for any $2 \leqslant j \leqslant n$, the path D has a valley with y-coordinate $j-1$ if and only if $d_{k(j)+1} - d_{k(j)} = 1$.*

Proof. Both properties are immediate consequences of the definition of the zeta map. □

Proof of Theorem 3.22. Let $2 \leqslant j \leqslant n$. Then D has a 1-rise with y-coordinate j if and only if it has both a peak with y-coordinate j and also a valley with y-coordinate $j-1$. The statement now follows from Proposition 3.4(1) and Theorem 3.3(1). The boundary case $j = 1$ follows from the observation that $k(1) = 0$, implying $d_{k(j)} = 1$ and $d_{k(j)+1} = 2$. □

In the example in Figure 11, the 1-rises in D are marked and labelled by letters A – E. For each 1-rise, the corresponding horizontal and the corresponding vertical step is marked with the given letter inside $\zeta(D)$ in blue and in red, respectively. This means that the Nakayama algebra for $\zeta(D)$ has 1-regular simple module

$$\{S_{k(1)}, S_{k(9)}, S_{k(10)}, S_{k(13)}, S_{k(14)}\} = \{S_0, S_{10}, S_{11}, S_8, S_4\}$$

and simple modules

$$\{S_{k(4)}, S_{k(8)}, S_{k(12)}\} = \{S_5, S_9, S_{13}\}$$

of projective dimension 1 that are not 1-regular.

3.3. **Description in terms of Dyck path statistics for CNakayama algebras.** To extend Theorem 3.11 to CNakayama algebras, we introduce an analogue of the Lalanne-Kreweras involution for certain periodic Dyck paths.

Let us first specify a map LK^0 on the set $\mathcal{D}_n^0$ of periodic Dyck paths with global shift 0 and non-constant area sequence. Given a path D in this set we essentially use Definition 3.10 to construct $\mathrm{LK}^0(D)$. For item (3) of this definition, we fix any return of D to the diagonal, and stipulate that we mark the intersection of the i-th vertical line *after this return* with the i-th horizontal line *after this return*.

Let us emphasize however, that $\mathrm{LK}^0(D)$ is not necessarily in $\mathcal{D}_n^0$. To circumvent this defect, let $\mathcal{D}_n^{\mathrm{r}}$ be the set of periodic Dyck paths that have a rectangle as defined in Section 2.2. We will see below that the image of LK^0 is exactly $\mathcal{D}_n^{\mathrm{r}}$. Moreover, the CNakayama algebras corresponding to $\mathcal{D}_n^{\mathrm{r}}$ are precisely the quasi-hereditary CNakayama algebras, as we will explain below.

Let us now describe the inverse of LK^0 explicitly. Again, we essentially use Definition 3.10 to construct the image of a path D in $\mathcal{D}_n^{\mathrm{r}}$. However, since D may not have any returns to the diagonal, we have to make item (3) of the definition precise in a different way. Specifically, fix any index i such that D has a rectangle with x-coordinate $i+1$. Drawing D below the main diagonal, we then stipulate that the 'first' horizontal line has x-coordinate $i+c_i$, and the 'first' vertical line has y-coordinate $i+c_i$. In particular, the image of D has a return at $i+c_i$.

Theorem 3.24. *Let $\mathcal{D}^{\mathrm{r}}_n$ be the set of n-periodic Dyck paths with a rectangle. Then the map LK^0 is a bijection between $\mathcal{D}^0_n$ and $\mathcal{D}^{\mathrm{r}}_n$, with inverse LK^{r}. Moreover, LK^0 is an involution on $\mathcal{D}^0_n \cap \mathcal{D}^{\mathrm{r}}_n$.*

Essentially, this theorem allows us to extend LK^0 to a map on the union $\mathcal{D}^0_n \cup \mathcal{D}^{\mathrm{r}}_n$. Of course, whenever two maps agree on the intersection of their domains, one can regard them as a single map. However, in the case at hand this is particularly interesting, because the definitions of LK^0 and its inverse LK^{r} are so similar.

Definition 3.25. The *generalized Lalanne-Kreweras involution* LK is the map

$$\mathrm{LK} : \mathcal{D}^0_n \cup \mathcal{D}^{\mathrm{r}}_n \to \mathcal{D}^0_n \cup \mathcal{D}^{\mathrm{r}}_n$$
$$D \mapsto \begin{cases} \mathrm{LK}^0(D) & \text{if } D \in \mathcal{D}^0_n \\ \mathrm{LK}^{\mathrm{r}}(D) & \text{if } D \in \mathcal{D}^{\mathrm{r}}_n. \end{cases}$$

This is well-defined, because LK^0 is an involution on $\mathcal{D}^0_n \cap \mathcal{D}^{\mathrm{r}}_n$.

Proof of Theorem 3.24. Let us first show that $\mathrm{LK}^0(D)$ is in $\mathcal{D}^{\mathrm{r}}$. To see this, fix any return of D which is followed by a double vertical step in the path drawn below the main diagonal. Then the valley corresponding to the midpoint of this double vertical step in $\mathrm{LK}^0(D)$ is, by construction, at the left hand side of a rectangle.

To verify that LK^{r} is the inverse map, it suffices to note that, by construction, all rectangles correspond to returns followed by a double vertical step.

It remains to prove that LK^0 is an involution on $\mathcal{D}^0_n \cap \mathcal{D}^{\mathrm{r}}_n$. To this end, regard the (classical) Dyck path $\tilde{E}$ obtained from restricting the image $E = \mathrm{LK}^0(D)$ to a period beginning with a return to the diagonal. Then, by definition of the classical Lalanne-Kreweras involution and LK^0, the positions of the valleys of $\mathrm{LK}(\tilde{E})$ coincide with the positions of the valleys of $\mathrm{LK}^0(E)$ in the same region - there are only additional peaks in $\mathrm{LK}(\tilde{E})$ at the beginning and the end of the period. Since the valleys determine the periodic Dyck path, $\mathrm{LK}^0(E) = D$. □

Let us now turn to a description of quasi-hereditary Nakayama algebras. We briefly recall the general definition, and then give an alternative description in the case of Nakayama algebras. Let A be a quiver algebra and let $e := (e_1, e_2, \ldots, e_n)$ denote an ordered complete set of primitive orthogonal idempotents of A, where complete means that $\mathbf{1}_A = \sum_{k=1}^{n} e_k$. For $i \in \{1, \ldots, n\}$, set $\epsilon_i := e_i + e_{i+1} + \cdots + e_n$, and also set $\epsilon_{n+1} := 0$. Moreover, define the *right standard modules* $\Delta(i) := e_iA/e_iA\epsilon_{i+1}A$ and dually the *left standard modules* $\Delta(i)^{op}$ as the right standard modules of the opposite algebra of A. Define the *right costandard modules* then as $\nabla(i) := D(\Delta(i)^{op})$. An algebra A is then called *quasi-hereditary* in case there is an ordering $e := (e_1, e_2, \ldots, e_n)$ such that $\mathrm{End}_A(\Delta(i))$ is a division algebra for all i and $\mathrm{Ext}^2_A(\Delta(i), \nabla(j)) = 0$ for all i, j. Note that we used here one of the many characterizations of quasi-hereditary algebras and we refer [DK94, Theorem A.2.6.] for many more equivalent characterizations. It is well known that any quiver algebra with an acyclic quiver is quasi-hereditary and thus every LNakayama algebra is quasi-hereditary. Not all CNakayama algebras are quasi-hereditary, but there is an easy homological characterization as the next proposition shows.

Proposition 3.26. *A CNakayama algebra is quasi-hereditary if and only if it has a simple module of projective dimension 2.*

Proof. See [UY90, Proposition 3.1]. □

We remark that the more general class of standardly stratified Nakayama algebras has been recently classified in [MM18].

Thus, by Proposition 3.4, the CNakayama algebras corresponding to $\mathcal{D}_n^{\mathrm{r}}$ are precisely those which are quasi-hereditary. The new description and Corollary 2.10 yields their number.

Corollary 3.27. *For any $c \geqslant 0$, there is an explicit bijection between quasi-hereditary n-CNakayama algebras and n-CNakayama algebras whose Kupisch series is non-constant and has minimal entry $c+2$. In particular, the number of quasi-hereditary n-CNakayama algebras is*

$$\frac{1}{2n}\sum_{k\,|\,n}\phi(n/k)\binom{2k}{k}-1.$$

As an aside, we compute the size of $\mathcal{D}_n^0 \cap \mathcal{D}_n^{\mathrm{r}}$.

Proposition 3.28. *The number of quasi-hereditary n-CNakayama algebras whose Kupisch series have minimal entry 2 equals*

$$\frac{1}{n}\sum_{k\,|\,n}\phi(n/k)\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}\binom{2k-2m-2}{k-2}.$$

Proof. Let us call an area sequence $[c_0,\dots,c_{n-1}]_{\circlearrowleft}$ *primitive*, if (without loss of generality) $c_{n-1} = 2$ and $c_i > 2$ for $i \neq n-1$. Note that the concatenation of primitive area sequences has no rectangle if and only if none of the factors has a rectangle. Thus, it is sufficient to count primitive area sequences without rectangle, and apply the cycle construction (eg. [BLL98, eq.1.4(18)]).

The number of primitive area sequences of length n without rectangles equals the number of 321-avoiding permutations without fixed points, counted by the Fine numbers[16]. This can be seen by interpreting $[c_0-1,\dots,c_{n-1}-1]$ as the area sequence of a Dyck path, and applying the Billey-Jockusch-Stanley bijection. Fixed points in the resulting permutation then correspond to rectangles. □

Theorem 3.29. *Let A be an n-CNakayama algebra, and let D be the corresponding periodic Dyck path. Suppose that $D \in \mathcal{D}_n^0 \cup \mathcal{D}_n^{\mathrm{r}}$. Then*

(1) S_i is 1-regular if and only if $\mathrm{LK}(D)$ has a 1-cut at position i,
(2) S_i is 2-regular if and only if $\mathrm{LK}(D)$ has a 2-hill at position i.

Note that a CNakayama algebra A has no 2-regular simple modules if the corresponding periodic Dyck path D is not in $\mathcal{D}_n^0 \cap \mathcal{D}_n^{\mathrm{r}}$. Thus, the theorem above actually covers all CNakayama algebras.

Proof. The proof of Theorem 3.11 applies verbatim. □

Remark 3.30. There is an alternative way to extend the map $\mathrm{LK} = \mathrm{LK}^0 = \mathrm{LK}^{\mathrm{r}}$ on $\mathcal{D}_n^0 \cap \mathcal{D}_n^{\mathrm{r}}$ as follows. For a given periodic Dyck path D with area sequence $[a_0,\dots,a_{n-1}]_{\circlearrowleft}$ with global shift c, let $\tilde{D}$ be the corresponding periodic Dyck path with global shift 0 and area sequence $[a_0-c,\dots,a_{n-1}-c]_{\circlearrowleft}$. One may now define an involution on periodic Dyck paths with global shift c for which the associated path with global shift 0 lies inside $\mathcal{D}_n^0 \cap \mathcal{D}_n^{\mathrm{r}}$ by mapping this associated path via the involution LK and then adding back the global shift. Observe that this map preserves the global shift, but does not coincide with LK outside of $\mathcal{D}_n^0 \cap \mathcal{D}_n^{\mathrm{r}}$.

[16] www.oeis.org/A000957

We conclude with some corollaries enumerating CNakayama algebras with certain homological restrictions. All of these are obtained using the bijection between quasi-hereditary CNakayama algebras and periodic Dyck paths with global shift 0.

Corollary 3.31. *The number of quasi-hereditary n-CNakayama algebras with exactly ℓ simple modules of projective dimension 1 equals the number of n-periodic Dyck paths with exactly ℓ peaks. Explicitly, this number is*

$$\frac{1}{n} \sum_{k \mid \gcd(\ell,n)} \phi(k) \binom{n/k-1}{\ell/k-1} \binom{n/k}{\ell/k}.$$

Corollary 3.32. *The number of quasi-hereditary n-CNakayama algebras with exactly $\ell > 0$ simple modules of projective dimension 2 equals the number of n-periodic Dyck paths with exactly ℓ returns which are not 1-hills. Explicitly, this number is*

$$\sum_{k \mid \gcd(\ell,n)} \frac{\phi(k)}{k} \sum_{m=0}^{(n-2\ell)/k} \frac{1}{m+\ell/k} \binom{2(m+\ell/k)}{m} \binom{(n-\ell)/k-m-1}{\ell/k-1}.$$

Corollary 3.33. *The number of quasi-hereditary n-CNakayama algebras without 1-regular simple modules equals the number of n-periodic Dyck paths with global shift 0 without 1-rises. Explicitly, this number is*

$$\frac{1}{n} \sum_{k \mid n} \phi(k) \sum_{m=0}^{n/k-1} \binom{n/k}{m} \binom{n/k-m-1}{m-1}.$$

Corollary 3.34. *The number of quasi-hereditary n-CNakayama algebras without 2-regular simple modules equals the number of n-periodic Dyck paths with global shift 0 without 2-hills. Explicitly, this number is*

$$\sum_{k \mid n} \frac{\phi(k)}{k} \sum_{m=0}^{\lfloor \frac{n}{2k} \rfloor} (-1)^m \frac{1}{n/k-m} \binom{2n/k-3m-1}{n/k-m-1} - 1.$$

4. Nakayama algebras of global dimension one and two

The *global dimension* $\mathrm{gd}(A)$ of an algebra A is the maximal projective dimension of a simple module, see for example [ARS97, Proposition I.5.1]. In this section we consider Nakayama algebras of global dimension at most two.

Theorem 4.1. *An n-Nakayama algebra A has global dimension 1 if and only if it has Kupisch series $[n, \dots, 1]$ corresponding to the unique Dyck path without valleys. Any other n-LNakayama algebra has global dimension 2 if and only if for all i such that S_i is non-projective, we have*

$$c_{i+1} + 1 \in \{c_i, c_{i+c_i} + c_i\},$$

i.e., if and only if all valleys of the corresponding (possibly periodic) Dyck path are rectangles.

If A is an n-LNakayama algebra or the (possibly periodic) Dyck path D corresponding to A is in $\mathcal{D}_n^0 \cup \mathcal{D}_n^{\mathrm{r}}$, the Nakayama algebra has global dimension 2 if and only if $\mathrm{LK}(D)$ *has height 2.*

Moreover, $(n+1)$-LNakayama algebras of global dimension 2 with exactly ℓ simple modules of projective dimension 2 are in bijection with subsets of $\{1, \dots, n\}$ of cardinality 2ℓ, counted by $\binom{n}{2\ell}$.

n-CNakayama algebras of global dimension 2 with exactly ℓ simple modules of projective dimension 2 are in bijection with subsets of $\{0, \dots, n-1\}$ of cardinality

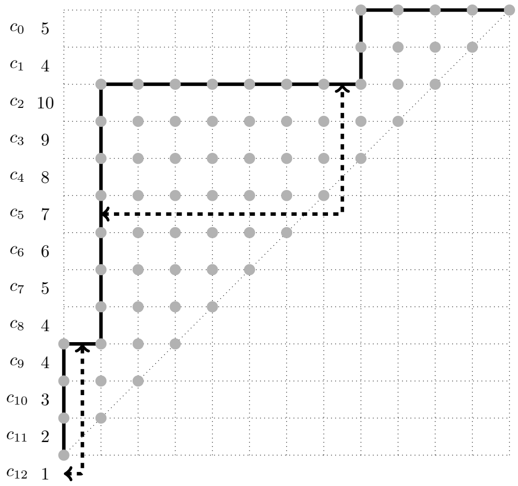


FIGURE 12.

2ℓ up to rotation by pairs[17]. *Explicitly, this number is*

$$\frac{2}{n} \sum_{k \,|\, \gcd(\ell,n)} \phi(k) \binom{n/k}{2\ell/k}.$$

Proof. The global dimension of a Nakayama algebra equals the maximal projective dimension of a simple module S_i. Thus, the characterization in terms of the Kupisch series is an immediate consequence of Proposition 3.4. The reformulation in terms of rectangles is immediate from the definition.

It remains to describe the claimed bijections. Let A be an $(n+1)$-LNakayama algebra whose simple modules of projective dimension 2 are $S_{i_1}, \dots, S_{i_\ell}$, corresponding to rectangles of D with x-coordinates $1 < i_1 + 1 < \cdots < i_\ell + 1 \leqslant n$. Drawing D below the main diagonal, BJS(D) puts crosses into the cells of the corresponding valleys, with top left coordinates $(j_1, i_1 + 1), \dots, (j_\ell, i_\ell + 1)$. Then, working from right to left, BJS puts crosses into the cells on the main diagonal, with top-left coordinates $(0, 1), \dots, (i_1 - 1, i_1)$.

Because there is a rectangle with x-coordinate i_1+1, the next valley at (j_2, i_2+1) has y-coordinate strictly larger than j_1+1. Thus, there are no crosses corresponding to valleys of D with y-coordinates $i_1+2, \dots, j_1+1$. Therefore, BJS puts crosses into the cells on the super diagonal with top-left coordinates $(i_1, i_1+2), \dots, (j_1-1, j_1+1)$.

The process then continues by putting crosses into the cells on the main diagonal again, with top-left coordinates $(j_1 + 1, j_1 + 2), \dots, (i_2 - 1, i_2)$, and so on. It is not hard to see that any Dyck path of height 2 can be obtained this way.

Similarly, one finds that mapping D to the set $i_1 + 1 < j_1 + 1 < i_2 + 1 < \cdots < i_\ell + 1 < j_\ell + 1$ is a bijection with subsets of $\{1, \dots, n\}$ of size 2ℓ. $\square$

Example 4.2. The 13-LNakayama algebra with Kupisch series

$$[5, 4, 10, 9, 8, 7, 6, 5, 4, 4, 3, 2, 1]$$

has global dimension 2 and its simple modules S_i have projective dimension 2 exactly for indices $i \in \{1, 8\}$, where we compute

$$c_2 + 1 - c_1 = 7 = c_5 = c_{1+c_1}, \quad c_9 + 1 - c_8 = 1 = c_{12} = c_{8+c_8}.$$

The corresponding Dyck path is shown in Figure 12, and is sent to the set $\{2, 5, 9, 12\}$. To see how to recover the path from this set $\{j_1 = 2, j_2 = 5, j_3 = 9, j_4 = 12\}$, observe

[17] www.oeis.org/A052823

that we obtain that $c_{i+1} + 1 = c_i$ for all i except

$$i \in \{j_1 - 1, j_3 - 1\} = \{1, 8\},$$

and

$$c_1 = j_2 - (j_1 - 1) = 4, \quad c_8 = j_4 - (j_3 - 1) = 4.$$

This in turn uniquely determines the Kupisch series as given.

Combining Theorem 3.3(2) with Theorem 4.1, we thus obtain the following description of 2-regular simple modules of Nakayama algebras of global dimension 2.

Corollary 4.3. *Let A be an n-Nakayama algebra of global dimension 2.*

(1) if A is a CNakayama algebra, S_i is 2-regular if and only if $c_i = 2$.
(2) if A is an LNakayama algebra, S_{n-2} and S_{n-1} are never 2-regular, and S_i is 2-regular for $i < n-2$ if and only if $c_i = 2$.

Proof. Suppose that $c_i = 2$. It follows that the (possibly periodic) Dyck path D corresponding to A has a valley at $(i+1, i+1)$ and in particular $d_{i+2} = 2$. By Theorem 4.1, $c_{i+1} = 1$ or $c_{i+1} = c_{i+2} + 1$. In the former case, A is an LNakayama algebra and $i = n-1$. It remains to show that in the latter case, we also have $d_{i+1} = d_i + 1$, as required by Theorem 3.3(2). Suppose on the contrary that $d_{i+1} \leqslant d_i$, so that D has a valley at $(j+1, i)$ for some j. This valley cannot belong to a rectangle with x-coordinate $j+1$, because then the next valley should have y-coordinate strictly larger than $i+1$. Thus, by Theorem 4.1, $c_{j+1} + 1 = c_j$, contradicting the assumption that there is a valley at $(j+1, i)$. □

Observe that the conclusion in the previous corollary does not hold in general for Nakayama algebras of higher global dimension as the next example shows.

Example 4.4. Let A be the LNakayama algebra with Kupisch series $[2, 2, 2, 1]$. Then A has global dimension 3 and does not have any simple 2-regular modules. The LNakayama algebra with Kupisch series $[2, 3, 2, 2, 2, 1]$ also has global dimension 3, and S_0 is its only simple 2-regular module.

Let us now use our preceding results to classify LNakayama algebras of global dimension at most two that satisfy the restricted Gorenstein condition. An algebra A with finite global dimension $k \geqslant 0$ satisfies the *restricted Gorenstein condition* if $k = 0$ or if every simple left and right module with projective dimension k is k-regular, see for example [I07, Definition 1.3].

We say that a Dyck path is a *bounce path* if it is of the form $h^{a_1} v^{a_1} \dots h^{a_\ell} v^{a_\ell}$ for positive integers $a_1, \dots, a_\ell$. Observe that this is the case if and only if all its valleys are of the form (i, i). Moreover, an LNakayama algebra has a associated Dyck path that is a bounce path if and only if its Kupisch series is of the form

$$[a_1 + 1, \dots, 2,\ a_2 + 1, \dots, 2,\ \dots,\ a_\ell + 1, \dots, 2, 1].$$

Theorem 4.5. *Let A be an n-Nakayama algebra and let D be the associated Dyck path. Then A has global dimension at most 2 and satisfies the restricted Gorenstein condition if and only if D is a bounce path and has no 1-hill after position 0 and before position $n-1$.*

Similarly, let A be an n-CNakayama algebra. Then A has global dimension at most 2 and satisfies the restricted Gorenstein condition if and only if D is a bounce path and has no 1-hills.

Proof. Suppose that D is a bounce path without 1-hills after position 0 and before position $n-1$. Then all valleys of D belong to rectangles, so by Theorem 4.1, the global dimension of A is at most 2. Since all rectangles of D are returns, the corresponding simple modules are all 2-regular by Theorem 3.9(2).

Conversely, suppose that A has global dimension at most 2, but D has a rectangle with x-coordinate $i+1$ which is not a return. Then $c_i > 2$, so S_i is not 2-regular.

To conclude that A satisfies the restricted Gorenstein condition, it remains to recall that the simple left modules of A are the simple right modules of the opposite algebra, corresponding to the reversed Dyck path. The above reasoning applies verbatim. □

Corollary 4.6. *The number of $(n+1)$-LNakayama algebras of global dimension at most 2 that satisfy the restricted Gorenstein condition equals the Fibonacci number*[18] *$F(n+1)$, counting subsets of $\{1,2,\dots,n-1\}$ that contain no consecutive integers. Explicitly, this number is given by the recurrence*

$$F(n+2) = F(n) + F(n+1)$$

with initial conditions $F(1) = F(2) = 1$.

Proof. A bounce path of semilength n can be identified with the subset of $\{1,\dots,n-1\}$ given by the positions of its valleys. Under this identification, a 1-hill at a position between 1 and $n-2$ corresponds to two consecutive numbers in the given subset, which implies the claim. □

The analogous result for CNakayama algebras is as follows.

Corollary 4.7. *The number of n-CNakayama algebras of global dimension 2 that satisfy the restricted Gorenstein condition equals the number of cyclic compositions of n into parts of size at least 2*[19]*. Explicitly, this number is*

$$\frac{1}{n}\sum_{k\,|\,n}\phi(n/k)\big(F(k-1)+F(k+1)\big)-1$$

where F is the Fibonacci number defined above.

We remark that by [I07, Proposition 1.4.], the class of Nakayama algebras with global dimension at most 2 satisfying the restricted Gorenstein condition coincides with the class of Nakayama algebras of global dimension at most 2 that are 2-Gorenstein. For the general enumeration of 2-Gorenstein LNakayama algebras we refer to the recent article [RS17].

References

[AF92] Anderson, Frank W.; Fuller, Kent R., *Rings and Categories of Modules.* Graduate Texts in Mathematics, Volume 13, Springer-Verlag, 1992.

[ARS97] Auslander, Maurice; Reiten, Idun; Smalø, Sverre, *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997. xiv+425 pp.

[AB69] Auslander, Maurice; Brider, Mark, *Stable module theory.* Memoirs of the American Mathematical Society, number 94, 1969.

[B91] Benson, David J., *Representations and cohomology I: Basic representation theory of finite groups and associative algebras.* Cambridge Studies in Advanced Mathematics, Volume 30, Cambridge University Press, 1991.

[BLL98] Bergeron, François; Labelle, Gilbert; Leroux, Pierre, *Combinatorial species and tree-like structures.* Encyclopedia of Mathematics and its Applications, Volume 67, Cambridge University Press, 1998.

[BU10] Buehler, Theo, *Exact categories.* Expositiones Mathematicae Volume 28, Issue 1, 2010, Pages 1-69.

[CPS88] Cline, Edward; Parshall, Brian; Scott, Leonard, *Finite dimensional algebras and highest weight categories.* Journal für die reine und angewandte Mathematik (1988) Volume: 391, page 85-99.

[DK94] Drozd, Yurj; Kirichenko, Vladimir, *Finite Dimensional Algebras.* Springer-Verlag 1994.

[E18] Enomoto, Haruhisa, *Classifications of exact structures and Cohen-Macaulay-finite algebras.* Advances in Mathematics Volume 335, 7 September 2018, Pages 838-877.

[18] www.oeis.org/A000045 [19] www.oeis.org/A032190

[F68] Fuller, Kent R., *Generalized Uniserial Rings and their Kupisch Series*. Math. Zeitschr. Volume 106, pages 248-260, 1968.

[Hag08] Haglund, James, *The q, t-Catalan numbers and the space of diagonal harmonics*. University Lecture Series, Amer. Math. Soc. Volume 41, 2008.

[I07] Iyama, Osama, *Auslander-Reiten theory revisited*. Skowronski, Andrzej (ed.), Trends in representation theory of algebras and related topics. Proceedings of the 12th international conference on representations of algebras and workshop (ICRA XII), Torun, Poland, August 15-24, 2007. Zuerich: European Mathematical Society (EMS)

[K97] Koenig, Steffen, *A criterion for quasi-hereditary, and an abstract straightening formula*. Inventiones mathematicae September 1997, Volume 127, Issue 3, pp 481-488.

[La99] Lam, Tsit-Yuen, *Lectures on Modules and Rings*. Graduate Texts in Mathematics, Volume 189, Springer-Verlag 1999.

[MM18] Madsen, Dag Oskar; Marczinzik, René, *On bounds of homological dimensions in Nakayama algebras*. Proc. Amer. Math. Soc. Ser. B 5 (2018), 40-49.

[M18] Marczinzik, René, *Upper bounds for the dominant dimension of Nakayama and related algebras*. Journal of Algebra Volume 496, 15 February 2018, Pages 216-241.

[MO18] https://mathoverflow.net/questions/302936.

[QPA] The QPA-team, QPA - Quivers, path algebras and representations - a GAP package, Version 1.25; 2016 (https://folk.ntnu.no/oyvinso/QPA/).

[R13] Ringel, Claus Michael, *The Gorenstein projective modules for the Nakayama algebras. I*. Journal of Algebra Volume 385, 1 July 2013, Pages 241-261.

[RS17] Rubey, Martin; Stump, Christian, *Double deficiencies of Dyck paths via the Billey-Jockusch-Stanley bijection*. J. Integer Seq. 20(9), 2017.

[RS18] Rubey, Martin; Stump, Christian; et al, *FindStat - The combinatorial statistics database*. http://www.FindStat.org (2018).

[Sage] The Sage Developers, *SageMath, the Sage Mathematics Software System*. http://www.sagemath.org (2018).

[SY11] Skowronski, Andrzej; Yamagata, Kunio, *Frobenius Algebras I: Basic Representation Theory*. EMS Textbooks in Mathematics, 2011.

[OEIS] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org.

[UY90] Uematsu, Morio; Yamagata, Kunio, *On serial quasi-hereditary rings*. Hokkaido Math. J. Volume 19, Number 1 (1990), 165-174.

(R. Marczinzik) Institute of algebra and number theory, University of Stuttgart, Germany
E-mail address: marczire@mathematik.uni-stuttgart.de

(M. Rubey) Fakultät für Mathematik und Geoinformation, TU Wien, Austria
E-mail address: martin.rubey@tuwien.ac.at

(C. Stump) Fakultt fr Mathematik, Ruhr-Universität Bochum, Germany
E-mail address: christian.stump@rub.de