LEAF-INDUCED SUBTREES OF LEAF-FIBONACCI TREES

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Abstract. In analogy to a concept of Fibonacci trees, we define the leaf-Fibonacci tree of size n and investigate its number of nonisomorphic leaf-induced subtrees. Denote by f_0 the one vertex tree and f_1 the tree that consists of a root with two leaves attached to it; the leaf-Fibonacci tree f_n of size $n \geq 2$ is the binary tree whose branches are f_{n-1} and f_{n-2}. We derive a nonlinear difference equation for the number $\mathrm{N}(f_n)$ of nonisomorphic leaf-induced subtrees (subtrees induced by leaves) of f_n, and also prove that $\mathrm{N}(f_n)$ is asymptotic to $1.00001887227319\ldots(1.48369689570172\ldots)^{\phi^n}$ ($\phi =$ golden ratio) as n grows to infinity.

1. Introduction

Fibonacci trees are an alternative approach to a binary search in computer science and information processing [10, p. 417]. The Fibonacci tree of order n is defined as the binary tree whose left branch is the Fibonacci tree of order $n-1$ and right branch is the Fibonacci tree of order $n-2$, while the Fibonacci tree of order 0 or 1 is the tree with only one vertex [10]. We show in Figure 1 the Fibonacci tree of order 5.

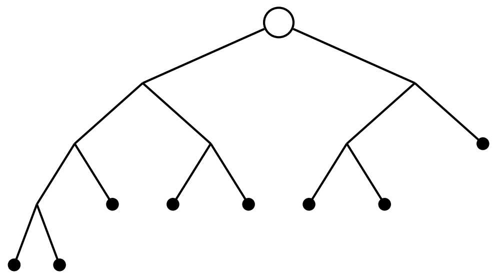

Figure 1. The Fibonacci tree of order 5.

Thus, the Fibonacci tree of order n has precisely F_{n+1} leaves (so $2F_{n+1} - 1$ vertices), where F_n denotes the n-th Fibonacci number:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n > 1\,.$$

Fibonacci trees are also a special case of so-called AVL ("Adel'son-Vel'skii and Landis"—named after the inventors) trees [1]; these trees have the defining property that for every

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internal vertex v, the heights (i.e., the greatest distance of a leaf from the root) of the left and right branches of the subtree rooted at v (consisting of v and all its descendants) differ by at most one. According to [1], AVL trees are the first data structure to be invented. Figure 2 shows an AVL tree of height 3. For more information on Fibonacci trees and their uses, we refer to [13, 8, 9, 7].

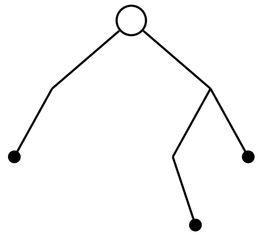

FIGURE 2. An AVL tree of height 3.

In analogy to the concept of Fibonacci trees from [10], we define the *leaf-Fibonacci tree of size (height)* n as follows:

- Denote by f_0 the tree with only one vertex and f_1 the tree that consists of a root with two leaves attached to it;
- For $n \geq 2$, connect the roots of the trees f_{n-1} and f_{n-2} to a new common vertex to obtain the tree f_n.

In other words, the leaf-Fibonacci tree f_n of size $n \geq 2$ is the binary tree whose branches are the leaf-Fibonacci trees f_{n-1} and f_{n-2}. Hence, f_n has precisely F_{n+2} leaves, where F_n is the n-th Fibonacci number ($F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \ldots$). Figure 3 shows the leaf-Fibonacci tree of size 5.

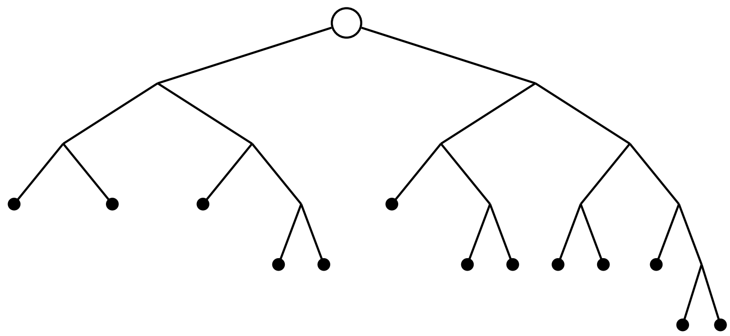

FIGURE 3. The leaf-Fibonacci tree of size 5.

In this note, we shall be interested in the number of nonisomorphic subtrees induced by leaves (henceforth, *leaf-induced subtrees*) of a leaf-Fibonacci tree of size n.

Let T be a rooted tree without vertices of outdegree 1 (also known as *topological* or *series-reduced* or *homeomorphically irreducible* trees [3, 2, 5, 12]). Every choice of k leaves of T induces another topological tree, which is obtained by extracting the minimal subtree of T that contains all the k leaves and suppressing (if any) all vertices of outdegree 1; see Figure 4 for an illustration. Every subtree obtained through this operation is sometimes

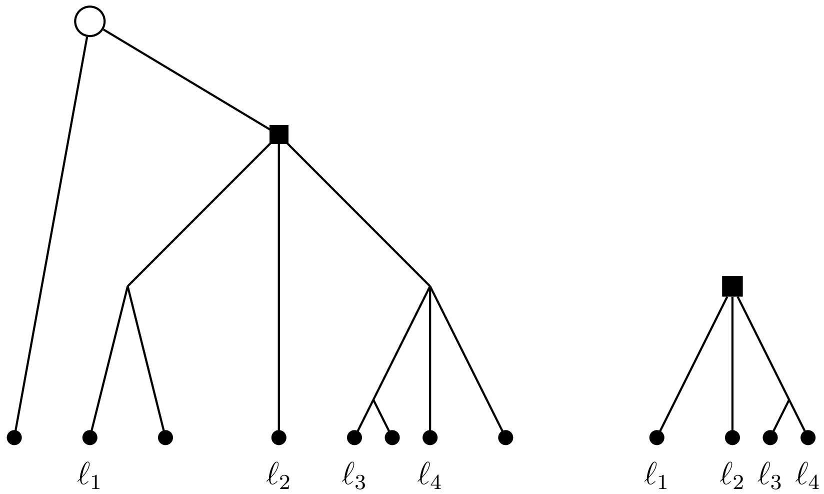

FIGURE 4. A topological tree (on the left) and the subtree induced by the leaves $\ell_1, \ell_2, \ell_3, \ell_4$ (on the right).

referred to as a *leaf-induced subtree* [4, 5]. We note the study of subtrees induced by leaves of binary trees finds a noteworthy relevance in phylogenetics—see Semple and Steel's book [11] which describes the mathematical foundations of phylogenetics.

Two rooted trees are said to be isomorphic if there is a graph isomorphism (preserving adjacency) between them that maps the root of one to the root of the other. It is important to note that the problem of enumerating leaf-induced subtrees becomes trivial if isomorphisms are not taken into account: in fact, it is clear that every topological tree with n leaves has exactly $2^n - 1$ leaf-induced subtrees.

We mention that nonisomorphic leaf-induced subtrees of a topological tree have been studied only very recently: Wagner and the present author [6] obtained exact and asymptotic enumeration results on the number of nonisomorphic leaf-induced subtrees of two classes of d-ary trees, namely so-called d-ary caterpillars and even d-ary trees. In [6], they also derived extremal results for the number of root containing leaf-induced subtrees of a topological tree.

We shall denote the number of nonisomorphic leaf-induced subtrees of the leaf-Fibonacci tree f_n by $\mathrm{N}(f_n)$. Our main results are a recurrence relation and an asymptotic formula for $\mathrm{N}(f_n)$. As it turns out, the plan to compute $\mathrm{N}(f_n)$ will be to consider the number of root containing leaf-induced subtrees of f_n.

In [14], Wagner alone studied the number of independent vertex subsets (set of vertices containing no pair of adjacent vertices) of a Fibonacci tree of order n with the notable difference that in his context, the Fibonacci tree of order 0 has no vertices. Wagner derived a system of recurrence relations for the number of independent vertex subsets of a Fibonacci tree of an arbitrary order n, and also proved that there are positive constants $A, B > 0$ such that the number of independent vertex subsets of a Fibonacci tree of order n is asymptotic to $A \cdot B^{F_n}$ as n grows to infinity. In the present study, we obtain a similar asymptotic formula for the number $\mathrm{N}(f_n)$ of nonisomorphic leaf-induced subtrees of the leaf-Fibonacci tree of size n: we demonstrate—not expectedly—that for some effectively computable constants $A_1, A_2 > 0$,

$$\mathrm{N}(f_n) \sim A_1 \cdot A_2^{F_n} \text{ as } n \to \infty\,.$$

2. Main results

We note from the recursive definition of the tree f_n that f_m is a leaf-induced subtree of f_n for every $m \leq n$. However, not every leaf-induced subtree of f_n is again a leaf-Fibonacci tree: in fact, by repeatedly removing leaves from f_n, one easily sees that f_n has leaf-induced subtrees of every number of leaves k between 1 and n.

As mentionned in the introduction, the plan to compute $\mathrm{N}(f_n)$ will be to consider the number of root containing leaf-induced subtrees of f_n.

Lemma 1. *All nonisomorphic leaf-induced subtrees with two or more leaves of f_n can be identified as containing the root of f_n.*

Proof. The tree f_0 has only one vertex which is also its leaf and root, so the statement holds vacuously for $n = 0$. The statement is trivial for $n = 1$ (f_1 is the only leaf-induced subtree in this case). Let $n > 1$ and consider a subset of $k > 1$ leaves of f_n. We argue by double induction on n and k:

- If all k leaves belong to f_{n-1} then by the induction hypothesis on n, the induced subtree with k leaves contains the root of f_{n-1}. Moreover, by the induction hypothesis on k, the tree f_{n-1} can be identified as containing the root of f_n (as f_{n-1} is clearly a leaf-induced subtree of f_n). Hence, the induced subtree with k leaves contains the root of f_n.
- If all k leaves belong to f_{n-2}, then we also deduce by the induction hypothesis that the induced subtree with k leaves is a root containing leaf-induced subtree of f_n.
- If k_1 leaves belong to f_{n-1} and $k - k_1$ leaves belong to f_{n-2}, then by the induction hypothesis, the induced subtrees with k_1 and $k - k_1$ leaves are root containing leaf-induced subtrees of f_{n-1} and f_{n-2}, respectively. Consequently, the root of the induced subtree with k leaves coincides with the root of f_n.

This completes the induction step as well as the proof of the lemma. $\square$

We then obtain the following proposition:

Proposition 2. *The number $N(f_n)$ of nonisomorphic leaf-induced subtrees of the leaf-Fibonacci tree f_n satisfies the following nonlinear recurrence relation:*

$$N(f_n) = 1 + \frac{1}{2}N(f_{n-2}) - \frac{1}{2}N(f_{n-2})^2 + N(f_{n-2}) \cdot N(f_{n-1}) \tag{1}$$

with initial values $N(f_0) = 1, N(f_1) = 2.$

Proof. It is obvious that $\mathrm{N}(f_0) = 1$ and $\mathrm{N}(f_1) = 2$. Let $n > 1$. By Lemma 1, $\mathrm{N}(f_n)$ is precisely one more the number of nonisomorphic root containing leaf-induced subtrees of f_n (the subtree with only one vertex has been included). Since all leaf-induced subtrees of the leaf-Fibonacci tree f_{n-2} are again leaf-induced subtrees of f_{n-1}, the nonisomorphic root containing leaf-induced subtrees of f_n can be categorised by two types of enumeration:

- Both branches of the induced subtree are leaf-induced subtrees of f_{n-2}. The total number of these possibilities is $\binom{1+\mathrm{N}(f_{n-2})}{2}$ as the induced subtrees have to be nonisomorphic.
- One of the branches of the induced subtree is a leaf-induced subtree of f_{n-2} while the other branch is a leaf-induced subtree of f_{n-1} but does not belong to the set of leaf-induced subtrees of f_{n-2}. The total number of these possibilities is $\mathrm{N}(f_{n-2})(\mathrm{N}(f_{n-1}) - \mathrm{N}(f_{n-2}))$.

Therefore, we obtain

$$\begin{aligned}\mathrm{N}(f_n) &= 1 + \binom{1+\mathrm{N}(f_{n-2})}{2} + \mathrm{N}(f_{n-2})(\mathrm{N}(f_{n-1}) - \mathrm{N}(f_{n-2})) \\ &= 1 + \frac{1}{2}\mathrm{N}(f_{n-2}) - \frac{1}{2}\mathrm{N}(f_{n-2})^2 + \mathrm{N}(f_{n-2}) \cdot \mathrm{N}(f_{n-1})\,,\end{aligned}$$

which completes the proof of the proposition. □

The sequence $(\mathrm{N}(f_n))_{n\geq 0}$ starts as

$\mathrm{N}(f_0) = 1,\ \mathrm{N}(f_1) = 2,\ \mathrm{N}(f_2) = 3,\ \mathrm{N}(f_3) = 6,\ \mathrm{N}(f_4) = 16,\ \mathrm{N}(f_5) = 82,\ \mathrm{N}(f_6) = 1193,$
$\mathrm{N}(f_7) = 94506,\ \mathrm{N}(f_8) = 112034631, \ldots$

We remark that recursion (1) cannot be solved explicitly. Therefore, finding an asymptotic formula should be in order. In the following theorem, we show—not expectedly—that $\mathrm{N}(f_n)$ grows doubly exponentially in n.

Theorem 3. *There are two positive constants* $K_1, K_2 > 0$ *(both solely depending on the first terms of* $(N(f_n))_{n\geq 0}$*) such that*

$$N(f_n) = (1 + o(1))K_1 \cdot K_2^{\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

as $n \to \infty$.

Proof. For ease of notation, set $A_n := \mathrm{N}(f_n)$. Then we have

$$A_n = 1 + \frac{1}{2}A_{n-2} - \frac{1}{2}A_{n-2}^2 + A_{n-2} \cdot A_{n-1}$$

with initial values $A_0 = 1, A_1 = 2$. Since the sequence $(A_n)_{n\geq 0}$ increases with n, it is not difficult to note that

$$A_n \geq \frac{1}{2}A_{n-1} \cdot A_{n-2}$$

for all $n \geq 2$. Also, since $A_n \geq A_2 = 3$ for all $n \geq 2$ and $1 + A_1/2 - A_1^2/2 = 0$, it is not difficult to see that

$$A_n \leq A_{n-1} \cdot A_{n-2}$$

for all $n \geq 3$. Thus, we have

$$\lim_{n\to\infty} \frac{A_{n-1}}{A_n} = 0\,, \tag{2}$$

which also implies that the sequence $(A_{n-1}/A_n)_{n\geq 1}$ is bounded for every $n \geq 1$. Let us use Q_n as a shorthand for $\log(A_n)$ and E_n as a shorthand for

$$\log\Big(1 + \frac{1}{2A_{n-1}} - \frac{A_{n-2}}{2A_{n-1}} + \frac{1}{A_{n-1}\cdot A_{n-2}}\Big)\,. \tag{3}$$

With these notations, we have

$$Q_n = Q_{n-1} + Q_{n-2} + E_n\,.$$

By setting $R_{n-1} := Q_{n-2}$, we obtain the following system (written in matrix form) of two linear difference equations:

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Q_{n-1} \\ R_{n-1} \end{pmatrix} + \begin{pmatrix} E_n \\ 0 \end{pmatrix}.$$

By iteration on n, one gets

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} Q_1 \\ Q_0 \end{pmatrix} + \sum_{i=2}^{n}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-i}\begin{pmatrix} E_i \\ 0 \end{pmatrix}$$

for all $n \geq 2$, as $R_1 = Q_0$. The eigenvalue decomposition gives us

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

with

$$\lambda_1 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1+\sqrt{5}}{2}\,.$$

It follows that

$$\begin{aligned}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m} &= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1^{m+1} - \lambda_2^{m+1} & \lambda_1\cdot\lambda_2^{m+1} - \lambda_1^{m+1}\cdot\lambda_2 \\ \lambda_1^m - \lambda_2^m & \lambda_1\cdot\lambda_2^m - \lambda_1^m\cdot\lambda_2 \end{pmatrix}\end{aligned}$$

for all integer values of m. Consequently, we have

$$\begin{pmatrix} Q_n \\ R_n \end{pmatrix} = \frac{\log(2)}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} \end{pmatrix} + \sum_{i=2}^{n}\frac{E_i}{\lambda_1 - \lambda_2}\begin{pmatrix} \lambda_1^{n-i+1} - \lambda_2^{n-i+1} \\ \lambda_1^{n-i} - \lambda_2^{n-i} \end{pmatrix}$$

for all $n \geq 2$ as $Q_0 = 0$ and $Q_1 = \log(2)$. Therefore, we obtain

$$Q_n = \frac{\log(2)}{\lambda_2 - \lambda_1}\big(\lambda_2^n - \lambda_1^n\big) + \frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i\big(\lambda_2^{n-i+1} - \lambda_1^{n-i+1}\big)$$

for all $n \geq 2$. Since the sequence $(E_n)_{n\geq 2}$ is bounded for every $n \geq 2$ (as $\lim_{n\to\infty} E_n = 0$ by virtue of (3) and (2)), we derive that

$$\sum_{i=2}^{n} |E_i| \cdot |\lambda_1|^{n-i+1} \leq \frac{|\lambda_1|^n - |\lambda_1|}{|\lambda_1| - 1} \cdot \sup_{2\leq m\leq n} |E_m|$$

for all $n \geq 2$. This implies that the quantity

$$\frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i \cdot \lambda_1^{n-i+1}$$

converges to a definite limit as $n \to \infty$ (note that $|\lambda_1| < 1$ and $\sup_{2\leq m\leq n} |E_m|$ is finite for every $n \geq 2$). On the other hand, we have

$$0 \leq \Big| \sum_{i=n+1}^{\infty} E_i \cdot \lambda_2^{n-i+1} \Big| \leq \frac{\lambda_2}{\lambda_2 - 1} \cdot \sup_{m\geq n+1} |E_m|$$

for all $n \geq 2$ (note that $\lambda_2 > 1$), which implies that

$$\frac{1}{\lambda_2 - \lambda_1}\sum_{i=n+1}^{\infty} E_i \cdot \lambda_2^{n-i+1} = \mathcal{O}\Big(\sup_{m\geq n+1} |E_m|\Big) = o(1)$$

as $n \to \infty$. Putting everything together, we arrive at

$$\begin{aligned} Q_n &= \frac{\lambda_2^n}{\lambda_2 - \lambda_1}\Big(\log(2) + \sum_{i=2}^{\infty} E_i \cdot \lambda_2^{-i+1}\Big) - \frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i \cdot \lambda_1^{n-i+1} \\ &\quad + \mathcal{O}\Big(\sup_{m\geq n+1} |E_m|\Big) + \mathcal{O}(\lambda_1^n) \\ &= \frac{\lambda_2^n}{\lambda_2 - \lambda_1}\Big(\log(2) + \sum_{i=2}^{\infty} E_i \cdot \lambda_2^{-i+1}\Big) - \frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i \cdot \lambda_1^{n-i+1} + o(1) \end{aligned}$$

as $n \to \infty$. We deduce that

$$\begin{aligned} A_n &= \Big(1 + \mathcal{O}\Big(\lambda_1^n + \sup_{m\geq n+1} |E_m|\Big)\Big) \\ &\qquad \cdot \exp\left(\frac{\lambda_2^n}{\lambda_2 - \lambda_1}\Big(\log(2) + \sum_{i=2}^{\infty} E_i \cdot \lambda_2^{-i+1}\Big) - \frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i \cdot \lambda_1^{n-i+1}\right) \\ &= (1 + o(1)) \cdot \exp\left(\frac{\lambda_2^n}{\lambda_2 - \lambda_1}\Big(\log(2) + \sum_{i=2}^{\infty} E_i \cdot \lambda_2^{-i+1}\Big) - \frac{1}{\lambda_2 - \lambda_1}\sum_{i=2}^{n} E_i \cdot \lambda_1^{n-i+1}\right) \end{aligned}$$

as $n \to \infty$. Call K_2 the quantity

$$\exp\left(\frac{1}{\lambda_2-\lambda_1}\Big(\log(2)+\sum_{i=2}^{\infty} E_i\cdot\lambda_2^{-i+1}\Big)\right),$$

and K_1 the quantity

$$\exp\left(-\frac{1}{\lambda_2-\lambda_1}\cdot\lim_{n\to\infty}\left(\sum_{i=2}^{n} E_i\cdot\lambda_1^{n-i+1}\right)\right).$$

Thus,

$$A_n=\mathrm{N}(f_n)=(1+o(1))K_1\cdot K_2^{\lambda_2^n}=(1+o(1))K_1\cdot K_2^{\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

as $n \to \infty$, where K_1 and K_2 can now be written as

$$K_2=\exp\left(\frac{1}{\sqrt{5}}\left(\log(2)+\sum_{i=2}^{\infty}\Big(\frac{1+\sqrt{5}}{2}\Big)^{-i+1}\right.\right.$$
$$\left.\left.\cdot\log\Big(1+\frac{1}{2\mathrm{N}(f_{i-1})}-\frac{\mathrm{N}(f_{i-2})}{2\mathrm{N}(f_{i-1})}+\frac{1}{\mathrm{N}(f_{i-1})\cdot\mathrm{N}(f_{i-2})}\Big)\right)\right)$$

and

$$K_1=\exp\left(-\frac{1}{\sqrt{5}}\cdot\lim_{n\to\infty}\left(\sum_{i=2}^{n}\Big(\frac{1-\sqrt{5}}{2}\Big)^{n-i+1}\right.\right.$$
$$\left.\left.\cdot\log\Big(1+\frac{1}{2\mathrm{N}(f_{i-1})}-\frac{\mathrm{N}(f_{i-2})}{2\mathrm{N}(f_{i-1})}+\frac{1}{\mathrm{N}(f_{i-1})\cdot\mathrm{N}(f_{i-2})}\Big)\right)\right).$$

By (numerically) evaluating K_1 and K_2, we obtain that the number of nonisomorphic leaf-induced subtrees of the leaf-Fibonacci tree f_n is asymptotically

$$1.00001887227319\cdots(1.48369689570172\ldots)^{\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

as $n \to \infty$. This completes the proof of the theorem. □

This asymptotic formula can also be written in terms of the Fibonacci number F_n: the number of leaves of f_n is given by

$$|f_n|=F_{n+2}=\frac{1}{\sqrt{5}}\left(\Big(\frac{1+\sqrt{5}}{2}\Big)^{2+n}-\Big(\frac{1-\sqrt{5}}{2}\Big)^{2+n}\right)$$

for every n; so we deduce that

$$\frac{10}{5+3\sqrt{5}}\cdot|f_n|\sim\Big(\frac{1+\sqrt{5}}{2}\Big)^n$$

as $n \to \infty$. This implies that

$$\begin{aligned} \mathrm{N}(f_n) &\sim K_1 \cdot K_2^{\frac{10}{5+3\sqrt{5}} \cdot |f_n|} \\ &= 1.00001887227319 \cdots (1.48369689570172 \dots)^{\frac{-5+3\sqrt{5}}{2} \cdot |f_n|} \end{aligned}$$

as $n \to \infty$.

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