# A BIAS PARITY QUESTION FOR STURMIAN WORDS 

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This article* is dedicated to Professor Solomon Marcus (1925-2016).


#### Abstract

We analyze a natural parity problem on the divisors associated to Sturmian words. We find a clear bias towards the odd divisors and obtain a sharp asymptotic estimate for the average of the difference odd-even function tamed by a mollifier, which proves various experimental results.


## 1. Introduction

Sturmian words are remarkable objects that lie at the frontier between order and chaos in the ample comprising world of binary sequences. They have interesting properties of which a few characterize them throughly. One of them says that a Sturmian word is a binary sequence that is not ultimately periodic and has minimal complexity, that is, it contains exactly $n+1$ distinct blocks of $n$ consecutive letters for each $n \geq 0$. Sturmian words where first introduced in 1940 by Morse and Hedlund [20] and since then, they have become a topic of intensive research [9], [10], [13], [14], [15], [16], [17], [18], [21], [22], [23].

Since an infinite word formed by just two letters is given by the sequence of natural numbers indicating the positions of just one of the two letters, a natural question is if this sequence has any special properties in the case of Sturmian words. Thus our object is to investigate the following parity problem that has an arithmetical flavor.

Suppose $w=w_{1} w_{2} \cdots$ is a binary word with letters from the alphabet $\mathcal{A}=\{a, b\}$. For each $n>0$, denote by $o_{w}(n)$ the number of divisors $j \mid n$ for which $w_{j}=b$ and the multiplicative-complement $n / j$ is odd, and similarly, let $e_{w}(n)$ be the number of divisors $j \mid n$ for which $w_{j}=b$ and its multiplicative-complement $n / j$ is even. Thus

$$
\begin{aligned}
& o_{w}(n):=\mid\left\{j \in \mathbb{N}: j \text { divides } n, w_{j}=b, n / j \text { odd }\right\} \mid \\
& e_{w}(n):=\mid\left\{j \in \mathbb{N}: j \text { divides } n, w_{j}=b, n / j \text { even }\right\} \mid
\end{aligned}
$$

The behavior of the parity functions is quite irregular, as can be seen in Figure 1 drawn for the Fibonacci word $w=a b a a b a b a a b a \ldots$, a generic, recursively generated Sturmian word [2], [19], [21, A003849].

We measure the deviation from the equilibrium of parity by the difference

$$
\begin{equation*}
D_{w}(n)=o_{w}(n)-e_{w}(n) \tag{1}
\end{equation*}
$$

The question is whether, for an arbitrary Sturmian word, there exists a bias in the distribution towards evens or odds and if the difference function is accurate enough to weight any possible dependence on the word $w$.

[^0]Question 1. For any arbitrary Sturmian word $w$, is there a particular tendency of $D_{w}(n)$ of being more positive than negative, or conversely, as $n$ increases towards infinity?


Figure 1. The Fibonacci parity functions $o_{w}(n)$ (left) and $e_{w}(n)$ (right) for $n \in[2,300]$.
At first look, while perhaps for low values of $n$, the function (1) that counts the deviation between the number of even or odd divisors may take somewhat disparate values, perhaps in the long run or at least on average, we should not expect any bias on the positive or negative values over the normal statistical variance. At least that happens in similar questions, such as the Lehmer problem [6] or on the parity of pairs of residue classes and their inverses ([3], [4], [5], [7, [8]). Although the balance is most likely tilted in the case of primes, where up to some limit, are more preponderant those of the form $4 k+3$ than those of the form $4 k+1$ (Chebyshev's bias, see Rubinstein and Sarnak [24]) or the more general Shanks-Rényi prime races problem (see Lamzouriz [12]).

In the case of the Fibonacci word, the distribution of $D_{w}(n)$ looks still quite random, but if we calculate its average, $\sum_{n=1}^{x} D_{w}(n)$, we observe a clear tendency of linear increase (see Figure 2). This shows a strong bias towards the odd divisors. Is there an explanation for such a strong discrepancy. We will see that the same behavior is characteristic and does not depend of the Fibonacci word $w$ and in Section 2 we will see that the appearance of exactly that slope of increase of the average of $D_{w}(n)$ is quite natural.

For any Sturmian word $w$ over the alphabet $A=\{a, b\}$, we denote by $\beta_{w}$ the limit proportion of the occurrence of letter $b$, that is,

$$
\beta_{w}:=\lim _{n \rightarrow \infty} \frac{|\{1 \leq j \leq n: w(j)=b\}|}{n} .
$$

The existence of $\beta_{w}$ is assured for any Sturmian word [1, Chapter 9]. For example, for the Fibonacci word, its precise value is $(3-\sqrt{5}) / 2$.

Our treatment of Question 1 conveniently tames the partial averages of the difference function by a mollifier. Thus, we wish to evaluate

$$
\begin{equation*}
M_{w}(x)=\sum_{n=1}^{x} D_{w}(n)\left(1-\frac{n}{x}\right) . \tag{2}
\end{equation*}
$$

In particular, we would like to answer to the following question: does $M_{w}(x)$ have constant sign for $x$ large enough, or does it have infinitely many changes of sign? As we shall see


Figure 2. The spread of $D_{w}(n)$ (left) and its partial averages $\sum_{n=1}^{x} D_{w}(n)$ for $x \in[2,1000]$ (right). In the images, $w$ is the Fibonacci word.
below, $M_{w}(x)$ is positive for $x$ large enough. We will actually prove a stronger statement, obtaining a sharp asymptotic formula for $M_{w}(x)$.

Theorem 1. Let $w$ be a Sturmian word. Then, for any $\delta>0$,

$$
\begin{equation*}
M_{w}(x)=\frac{\beta_{w} \log 2}{2} x+O_{\delta}\left(x^{\frac{1}{3}+\delta}\right) . \tag{3}
\end{equation*}
$$

Let us remark that since $\beta_{w}>0$, the asymptotic estimation 3 implies that $M_{w}(x)>0$ for all sufficiently large $x$, proving a strong bias towards the odd divisors for all Sturmian words. We also mention that the Fibonacci word, the slope of the main term of the estimate (3) is $\frac{(3-\sqrt{5}) \log 2}{4} \approx 0.264758$.

The paper is organized as follows: in Section 2 we discuss the analogue question on the parity of the divisors of positive integers instead of Sturmian words, in the next two sections we employ the techniques developed for the Dirichlet series associated to Sturmian words and then use them to prove Theorem 1 in the last section.

## 2. The parity problems for the sequence of positive integers

In this section we look on the parity problem for the infinite word $u=b b b \ldots$, whose letters are all equal, with $b$. The word $u$ is not a Sturmian word, but its analysis puts on perspective the more complex cases.

In the following, for simplicity, we will drop the subscript and write $o(n), e(n)$ and $D(n)$ instead of $o_{u}(n), e_{u}(n)$ and $D_{u}(n)$.

Like in the case of Sturmian words, the values of these parity functions are very irregular (see Figure 3) and the involvement of the primes is part of the motive.

For any $n \geq 1$, the parity functions $o(n), e(n)$ can easily be obtained if the decomposition in the prime factors of $n$ is known. Indeed, let $n=2^{\alpha} r$, with $\alpha>0, r$ odd and $r=p_{1}^{\alpha_{1}} \cdots p_{k}^{\alpha_{k}}$. Then, all the divisors of $n$ are the terms of the sum obtained after all multiplications are done in the formal product

$$
\left(1+2+\cdots+2^{\alpha}\right)\left(1+p_{1}+\cdots+p_{1}^{\alpha_{1}}\right) \cdots\left(1+p_{k}+\cdots+p_{k}^{\alpha_{k}}\right) .
$$

In particular, we see that the total number of divisors of $n$ is equal to $(\alpha+1)\left(\alpha_{1}+1\right) \cdots\left(\alpha_{k}+\right.$ $1)=(\alpha+1) d(r)$, where $d(\cdot)$ denotes the number of divisors function.


Figure 3. The values of $o(n)$ (left) and $e(n)$ (right) for $n \in[2,300]$.
Now, if $\alpha=0, n$ being odd has no even divisors, so it follows that $e(n)=0$ and $o(n)=o(r)=\left(\alpha_{1}+1\right) \cdots\left(\alpha_{r}+1\right)=d(r)$. If $\alpha \geq 1$, the number of odd divisors is still being equal to $d(r)$, while for each odd divisor of $n$ it corresponds $\alpha$ even divisors (those obtained by multiplying it by $2,2^{2}, \ldots, 2^{\alpha}$ ). Thus, we have the general formulas

$$
\left\{\begin{array}{l}
o(n)=d(r)  \tag{4}\\
e(n)= \\
e \cdot d(r)
\end{array}, \quad \text { for } n=2^{\alpha} r, \text { with } r\right. \text { odd. }
$$

Two special cases, in which $e(n)$ attains its minimum and its maximum values occur. The minimum of $e(n)$ appears often, since any odd $n$ has no even divisors. Thus $e(n)=0$ and $o(n)=d(n)$ for $n$ odd. In this case a local maximum, $o(n)=2^{k}$, is attained if $n=1 \cdot 3 \cdot 5 \cdots(2 k+1)$. At the other end, if $n$ is a power of 2 , then one is the only odd divisor of $n$, so $e(n)=\alpha$ and $o(n)=1$ if $n=2^{\alpha}$.


Figure 4. The spread of $D(n)$ and its partial averages in the interval $[2,1000]$.
Experimental results show that the partial averages of $D(n)$, that is, the sums $\sum_{n=1}^{x} D(n)$ tend to increase linearly with $x$ (see Figure 4 ) and the same behavior is apparent for the averages calculated on shorter intervals. Although, in rare situations, the increase may be bumpy in shorter intervals that contain special $n$ 's. For example, the larger jump in


Figure 5. Locally, in short intervals, in which some abnormal values of $D(n)$ occur, the partial averages may show a noisier effect. This happens, for example, in the interval of length 300 that starts at $10^{7}$.

Figure 5 is caused by the reach in divisors number $n=10000080=2^{4} \cdot 3^{2} \cdot 5 \cdot 17 \cdot 19 \cdot 43$, which implies $o(n)=48, e(n)=192$ and $D(n)=48-192=-144$.

Next, by (4) the formula for the difference function is

$$
\begin{equation*}
D(n)=o(n)-e(n)=(1-\alpha) d(r), \quad \text { for } n=2^{\alpha} r, \alpha \geq 0, r \text { odd. } \tag{5}
\end{equation*}
$$

Notice that if $n$ is even but not divisible by four, than $e(n)=o(n)=d(n)$, so $D(n)=0$.
In the following, we assume $x$ is sufficiently large and calculate the average of $D(n)$ over the positive integers $n \leq x$. Using formula (5), we have:

$$
\begin{equation*}
\sum_{n=1}^{x} D(n)=\sum_{\substack{r=1 \\ r \text { odd } 2^{\alpha} r \leq x}}^{x} \sum_{\alpha=0}^{\left[\frac{\log x}{\log 2}\right]}(1-\alpha) d(r) \tag{6}
\end{equation*}
$$

Let us denote the divisors sum over the odd integers by $I(t)$, that is

$$
I(t):=\sum_{\substack{r=1 \\ r \text { odd }}}^{t} d(r) .
$$

Then, (6) becomes

$$
\begin{equation*}
\sum_{n=1}^{x} D(n)=I(x)-I(x / 4)-2 I(x / 8)-\cdots-(1-\tau) I\left(x / 2^{\tau}\right)+R(x) \tag{7}
\end{equation*}
$$

where $\tau=\left[\frac{\log x}{\log 2}\right]$ and

$$
\begin{equation*}
R(x)=\tau I(y) \tag{8}
\end{equation*}
$$

for some $y \leq x / 2^{\tau+1}$, collects the remaining terms.
We need the following two lemmas.

Lemma 1. We have

$$
A_{o}(x):=\sum_{\substack{1 \leq n \leq x \\ n \text { odd }}}^{x} \frac{1}{n}=\frac{1}{2} \log x+\frac{\log 2}{2}+\frac{\gamma}{2}+O\left(\frac{1}{x}\right)
$$

where $\gamma$ is Euler's constant.
Proof. The harmonic sum up to $x$ is

$$
A(x)=\sum_{1 \leq n \leq x} \frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right)
$$

Then

$$
\begin{aligned}
A_{o}(x) & =A(x)-A(x / 2) / 2 \\
& =\log x+\gamma-\frac{1}{2} \log \frac{x}{2}-\frac{\gamma}{2}+O\left(\frac{1}{x}\right) \\
& =\frac{1}{2} \log x+\frac{\log 2}{2}+\frac{\gamma}{2}+O\left(\frac{1}{x}\right)
\end{aligned}
$$

what had to be proved.
The sum of odd divisors can be calculated by the well-known inclusion-exclusion Dirichlet method and this is the object of the next lemma.
Lemma 2. We have

$$
I(x)=\sum_{\substack{n=1 \\ n \text { odd }}}^{x} d(n)=\frac{1}{4} x \log x+x\left(\frac{\log 2}{2}+\frac{\gamma}{2}-\frac{1}{4}\right)+O(\sqrt{x})
$$

Proof. First we write the sum of the divisors as a double sum that counts lattice points under a hyperbola:

$$
I(x)=\sum_{\substack{n=1 \\ n \text { odd }}}^{x} d(n)=\sum_{\substack{1 \leq a b \leq x \\ a, b \text { odd }}}^{x} 1
$$

The contribution of the numerous smaller terms can be controlled efficiently counting them twice, in different order. Thus, we have

$$
\begin{aligned}
I(x) & =\sum_{\substack{1 \leq a \leq \sqrt{x} \\
a \text { odd }}} \sum_{\substack{1 \leq b \leq \frac{x}{a} \\
b \text { odd }}} 1+\sum_{\substack{1 \leq b \leq \sqrt{x} \\
b \text { odd }}} 1-\sum_{\substack{1 \leq a \leq \frac{x}{b} \\
a \text { odd }}} \sum_{\substack{1 \leq a \leq \sqrt{x} \\
a \text { odd }}} 1 \\
& =2 \sum_{\substack{1 \leq b \leq \sqrt{x} \\
b \\
a \text { odd }}}\left(\frac{x}{2 a}+O(1)\right)-\left(\frac{\sqrt{x}}{2}+O(1)\right)^{2} \\
& =x A_{o}(\sqrt{x})-\frac{x}{4}+O(\sqrt{x}) .
\end{aligned}
$$

Then, using Lemma 1, we find that

$$
\begin{aligned}
I(x) & =x\left(\frac{1}{2} \log \sqrt{x}+\frac{\log 2}{2}+\frac{\gamma}{2}+O\left(\frac{1}{\sqrt{x}}\right)\right)-\frac{x}{4}+O(\sqrt{x}) \\
& =\frac{1}{2} x \log \sqrt{x}+x\left(\frac{\log 2}{2}+\frac{\gamma}{2}-\frac{1}{4}\right)+O(\sqrt{x})
\end{aligned}
$$

which concludes the proof of the lemma.
Now we are ready to obtain the estimate of the average of the difference of parity functions.

Proposition 1. We have

$$
\sum_{1 \leq n \leq x} D(n)=\log 2 \cdot x+O(\sqrt{x})
$$

Proof. Replacing the corresponding terms of the sum in (7) and in the relation (8) by their estimate from Lemma 2, we have

$$
\begin{align*}
\sum_{n=1}^{x} D(n) & =I(x)-I(x / 4)-2 I(x / 8)-\cdots-(1-\tau) I\left(x / 2^{\tau}\right)+R(x)  \tag{9}\\
& =\frac{1}{4} x \log x S_{1}(\tau)+\frac{\log 2}{4} x S_{2}(\tau)+\frac{2 \log 2+2 \gamma-1}{4} x S_{1}(\tau)+O\left(\sqrt{x} S_{3}(\tau)\right)+R(x)
\end{align*}
$$

where we denoted

$$
\begin{aligned}
& S_{1}(\tau)=1-\frac{1}{2^{2}}-\frac{2}{2^{3}}-\cdots-\frac{\tau-1}{2^{\tau}} \\
& S_{2}(\tau)=\frac{1 \cdot 2}{2^{2}}+\frac{2 \cdot 3}{2^{3}}+\cdots+\frac{(\tau-1) \tau}{2^{\tau}} \\
& S_{3}(\tau)=1-\frac{1}{2^{2 / 2}}-\frac{2}{2^{3 / 2}}-\cdots-\frac{\tau-1}{2^{\tau / 2}}
\end{aligned}
$$

and $\tau=\left[\frac{\log x}{\log 2}\right]$. The sums $S_{1}(\tau), S_{2}(\tau), S_{3}(\tau)$ can be added and expressed in closed-form and then their sizes can be easily evaluated. Thus we find that all terms except the second from the right hand side of $(9)$ are no larger than $O(\sqrt{x})$. Then, since

$$
S_{2}(\tau)=4+O\left(\frac{\log x}{x}\right)
$$

the main term on the right hand side of $(9)$ is

$$
\frac{\log 2}{4} x S_{2}(\tau)=x \log 2+O(\log x)
$$

which concludes the proof.
Notice that the experiment drawn in Figure 4 is confirmed by the slope from the above proposition since $\log 2 \approx 0.69314$.

## 3. Infinite words and Dirichlet series

Let $A$ be a finite alphabet. Given a map $H: A \rightarrow \mathbb{C}$ and an infinite word $w: \mathbb{N} \rightarrow A$, we compose $H$ with $w$ and consider the associated Dirichlet series

$$
\begin{equation*}
F(H, w, s):=\sum_{n=1}^{\infty} \frac{H(w(n))}{n^{s}} \tag{10}
\end{equation*}
$$

which is absolutely convergent in the half-plane $\operatorname{Re} s>1$. The analytic function $F(H, w, s)$ captures some properties of the given word $w$. If $H$ is injective, the word is uniquely determined by the function $F(H, w, s)$. More precisely, the coefficients $H(w(n))$ can be
recovered from $F(H, w, s)$ via Perron type formulas. For any positive integer $n$, any $x$ in the interval $(n, n+1)$, and any real number $c>1$,

$$
\begin{equation*}
\sum_{j=1}^{n} H(w(j))=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \frac{F(H, w, s) x^{s}}{s} d s \tag{11}
\end{equation*}
$$

where the integral above is to be interpreted as the symmetric limit $\lim _{T \rightarrow \infty} \int_{c-i T}^{c+i T}$.
Working with the Dirichlet series $F(H, w, s)$ in the half-plane of convergence Re $s>1$ may reveal various properties of the given words $w$. More interesting are cases when the functions $F(H, w, s)$ have analytic or meromorphic continuation to larger half-planes. For example, for a word of the form $w=a a a a \ldots$, we have

$$
\begin{equation*}
F(H, w, s)=\sum_{n=1}^{\infty} \frac{H(a)}{n^{s}}=H(a) \zeta(s) \tag{12}
\end{equation*}
$$

a constant multiple of the Riemann zeta function $\zeta(s)$. In this case $F(H, w, s)$ has a meromorphic continuation to the entire complex plane, the only pole being a simple pole at $s=1$ (for $H(a)$ nonzero). Similarly, any Dirichlet $L$-function, and more generally any finite linear combination of Dirichlet $L$-functions $c_{1} L\left(s, \chi_{1}\right)+\cdots+c_{k} L\left(s, \chi_{k}\right)$ is of the form

$$
\begin{equation*}
c_{1} L\left(s, \chi_{1}\right)+\cdots+c_{k} L\left(s, \chi_{k}\right)=F(H, w, s) \tag{13}
\end{equation*}
$$

for some alphabet $A$, some map $H: A \rightarrow \mathbb{C}$, and some periodic word $w$. In all these cases $F(H, w, s)$ has analytic continuation to the entire complex plane, with a possible pole at $s=1$ if one or more of the characters $\chi_{1}, \ldots, \chi_{k}$ are principal.

Let now $A$ be an alphabet, $H: A \rightarrow \mathbb{C}$, and let $\lambda$ be a real number satisfying $0 \leq \lambda<1$. Suppose $w$ and $w^{\prime}$ are two words that coincide at enough many places so that for any $\varepsilon>0$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\left|\left\{j: 1 \leq j \leq n, w^{\prime}(j) \neq w j\right\}\right|}{n^{\lambda+\varepsilon}}=0 \tag{14}
\end{equation*}
$$

Then it is easy to see that the difference $F\left(H, w^{\prime}, s\right)-F(H, w, s)$ has analytic continuation to the half-plane $\operatorname{Re} s>\lambda$. Therefore in such cases $F\left(H, w^{\prime}, s\right)$ has analytic (respectively meromorphic) continuation to the half-plane Re $s>\lambda$ if and only if $F(H, w, s)$ has the same property.

## 4. Dirichlet series associated to Sturmian words

A one-parameter family of Dirichlet series whose coefficients are Sturmian words is studied by Kwon [11. For our purpose, we have proceed as follows. If $w$ is a Sturmian word, $F(H, w, s)$ has meromorphic continuation to the half-plane $\operatorname{Re} s>0$, the only pole being a simple pole at $s=1$. Therefore for any word $w^{\prime}$ which coincides with a Sturmian word $w$ at enough many positions so that (14) holds for some $0 \leq \lambda<1$, the corresponding Dirichlet series $F\left(H, w^{\prime}, s\right)$ has meromorphic continuation to the half-plane $\operatorname{Re} s>\lambda$, the only pole being a simple pole at $s=1$.

Let $w$ be a Sturmian word over the alphabet $A=\{a, b\}$, and assume for simplicity that $H(a)=0$ and $H(b)=1$. The proportion of positions $j$ up to $n$ for which $w(j)=b$ has a limit as $n$ tends to infinity. Let $\beta_{w}$ denote this limit,

$$
\begin{equation*}
\beta_{w}:=\lim _{n \rightarrow \infty} \frac{|\{1 \leq j \leq n: w(j)=b\}|}{n}=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq j \leq n} H(w(j)) \tag{15}
\end{equation*}
$$

Any two factors of $w$ having the same length contain about the same number of $a$ 's and $b$ 's. To be precise, for any positive integers $n_{1}, n_{2}$ and $L$,

$$
\begin{equation*}
\left|\sum_{n_{1} \leq j<n_{1}+L} H(w(j))-\sum_{n_{2} \leq j<n_{2}+L} H(w(j))\right| \leq 1 \tag{16}
\end{equation*}
$$

Therefore, for any positive integer $n$,

$$
\begin{equation*}
\left|\beta_{w} n-\sum_{1 \leq j \leq n} H(w(j))\right| \leq 1 \tag{17}
\end{equation*}
$$

For Res $>1$, we rewrite $F(H, w, s)$ as

$$
\begin{equation*}
F(H, w, s)=\sum_{n=1}^{\infty} \frac{H(w(n))}{n^{s}}=\sum_{n=1}^{\infty} H(w(n)) \int_{n}^{\infty} \frac{s}{t^{s+1}} d t \tag{18}
\end{equation*}
$$

We further have

$$
\begin{equation*}
F(H, w, s)=s \int_{1}^{\infty} \frac{\sum_{n \leq t} H(w(n))}{t^{s+1}} d t \tag{19}
\end{equation*}
$$

which may be rewritten as

$$
\begin{equation*}
F(H, w, s)=\frac{\beta_{w}}{s-1}+\beta_{w}-s \int_{1}^{\infty} \frac{\beta_{w} t-\sum_{n \leq t} H(w(n))}{t^{s+1}} d t . \tag{20}
\end{equation*}
$$

By (17), the numerator under the integral on the right side of (20) is $O(1)$. It follows that this integral represents an analytic function of $s$ in the entire half-plane $\operatorname{Re} s>0$. In conclusion, $F(H, w, s)$ has an analytic continuation to the half-plane $\operatorname{Re} s>0$, with the exception of a simple pole at $s=1$, with residue $\beta_{w}$.

In order to take advantage of the pole at $s=1$ in concrete applications, we need to have some knowledge on the growth of $|F(H, w, s)|$ for $s=\sigma+i t$ with $|t|$ large and $\sigma$ not too close to zero. For any fixed $\delta>0$, one has

$$
\begin{equation*}
|F(H, w, \sigma+i t)|=O_{\delta}\left(|t|^{1-\sigma+\delta}\right), \tag{21}
\end{equation*}
$$

uniformly for all $\delta \leq \sigma \leq 1+\delta$ and $|t| \geq 1$. For points on the vertical line $\sigma=1+\delta$, (21) follows directly from (10). For points on the vertical line $\sigma=\delta$, (21) follows from (20), taking into account that the numerator under the integral in (20) is $O(1)$. Then the convexity bound (21), for all $\delta \leq \sigma \leq 1+\delta$, follows from the general theory of Dirichlet series (see Titchmarsh [25).

## 5. Proof of Theorem 1

To prove the sharp asymptotic formula (3), we will make use of the properties of $F(H, w, s)$ discussed above.

Recall that the Dirichlet convolution of two arithmetical functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$ is the function $f * g: \mathbb{N} \rightarrow \mathbb{C}$ defined by

$$
\begin{equation*}
(f * g)(n)=\sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) . \tag{22}
\end{equation*}
$$

Let us observe that $D$ is the Dirichlet convolution of $H \circ w$ with the function $h$ given by $h(n)=(-1)^{n+1}$. Indeed,

$$
\begin{equation*}
((H \circ w) * h)(n)=\sum_{d \mid n} H(w(d)) h\left(\frac{n}{d}\right)=\sum_{\substack{d \mid n \\ w(d)=b}}(-1)^{\frac{n}{d}+1}=D_{w}(n) \tag{23}
\end{equation*}
$$

Dirichlet convolution corresponds to multiplication of the associated Dirichlet series. Therefore, in the half-plane of absolute convergence $\operatorname{Re} s>1$,

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{D_{w}(n)}{n^{s}}=F(H, w, s) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{s}} \tag{24}
\end{equation*}
$$

Here the sum on the right side of (24) equals

$$
\begin{equation*}
1-\frac{1}{2^{s}}+\frac{1}{3^{s}}-\frac{1}{4^{s}}+\cdots=\left(1-\frac{2}{2^{s}}\right) \sum_{n=1}^{\infty} \frac{1}{n^{s}} \tag{25}
\end{equation*}
$$

and we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{D_{w}(n)}{n^{s}}=\left(1-\frac{1}{2^{s-1}}\right) \zeta(s) F(H, w, s) \tag{26}
\end{equation*}
$$

Next, we use a variant of Perron's formula [26] in combination with (2) and (26) in order to express $M_{w}(x)$ as an integral over a vertical line. For any real numbers $x \geq 1$ and $c>1$,

$$
\begin{equation*}
M_{w}(x)=\sum_{n \leq x} D_{w}(n)\left(1-\frac{n}{x}\right)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \frac{\left(1-\frac{1}{2^{s-1}}\right) \zeta(s) F(H, w, s) x^{s}}{s(s+1)} d s \tag{27}
\end{equation*}
$$

The integrand on the right side of 27 is analytic on the entire half-plane $\operatorname{Re} s>0$, except for a pole at $s=1$. Notice that at $s=1$ both $\zeta(s)$ and $F(H, w, s)$ have simple poles, while $1-1 / 2^{s-1}$ has a simple zero. Therefore the integrand on the right side of (20) has a simple pole at $s=1$. The Taylor series expansion of $1-1 / 2^{s-1}$ about $s=1$ is

$$
\begin{equation*}
1-\frac{1}{2^{s-1}}=1-e^{-(s-1) \log 2}=(s-1) \log 2+\ldots \tag{28}
\end{equation*}
$$

Also, as we know,

$$
\begin{equation*}
\zeta(s)=\frac{1}{s-1}+\text { analytic } \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
F(H, w, s)=\frac{\beta_{w}}{s-1}+\text { analytic } \tag{30}
\end{equation*}
$$

Using (28), (29) and (30) it follows that the residue at $s=1$ of the integrand on the right side of (27) equals $\frac{\beta_{w} \log 2}{2} x$.

Let now $T>1$ be a parameter, whose precise value will be given later. We fix a small $\delta>0$, and then shift the line of integration on the right side of (27) to the left. In doing so, we encounter the pole at $s=1$. We choose the new contour as follows. We start vertically from $1+\delta-i \infty$ to $1+\delta-i T$, then move left to $\delta-i T$, then move vertically to $\delta+i T$, then go horizontally to $1+\delta+i T$, and then vertically to $1+\delta+i \infty$. Next, employing (21) in combination with known bounds for $\zeta(s)$ on the critical strip, we find that if we choose $T=x^{2 / 3}$, then the integral over the new contour is bounded as $O_{\delta}\left(x^{\frac{1}{3}+\delta}\right)$. Lastly, by the residue theorem we obtain the desired asymptotic formula, which completes the proof of the theorem.

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