

An Investigation on Partitions with Equal Products

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Abstract

An ordered triple (s, p, n) is called admissible if there exist two different multisets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ such that X and Y share the same sum s , the same product p , and the same size n . We first count the number of n such that (s, p, n) are admissible for a fixed s . We also fully characterize the values p such that (s, p, n) is admissible. Finally, we consider the situation where r different multisets are needed, instead of just two. This project is also related to John Conway's wizard puzzle from the 1960s.

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1 Introduction

A multiset $X = \{x_1, x_2, \dots, x_n\}$ of n positive integers is an n -partition of the sum $s = x_1 + x_2 + \dots + x_n$. Define the function $T\{x_1, x_2, \dots, x_n\} = (s, p, n)$, where $p = x_1 x_2 \cdots x_n$. Throughout this article, we will call s and p the sum and the product of the partition, respectively. Our main focus will be on ordered triples (s, p, n) for which there are at least two different n -partitions sharing the same sum s and the same product p . We call such ordered triples *admissible*. A positive integer s is *sum-admissible* if there exist integers p and n such that (s, p, n) is admissible; similarly, a positive integer p is *product-admissible* if there exist integers s and n such that (s, p, n) is admissible.

For each integer $r \geq 2$ and $n \geq 3$, let $s_r(n)$ be the smallest positive integer, if it exists, such that for all integers $s \geq s_r(n)$, there are at least r different n -partitions of s , namely $\{x_{i1}, x_{i2}, \dots, x_{in}\}$, where $i = 1, 2, \dots, r$, satisfying

- (a) $x_{ij} \neq x_{i'j'}$ if $(i, j) \neq (i', j')$, and
- (b) there exists $p \in \mathbb{N}$ such that for all $i = 1, 2, \dots, r$, $T\{x_{i1}, x_{i2}, \dots, x_{in}\} = (s, p, n)$.

If condition (a) is removed, then let $s_r^*(n)$ be the smallest positive integer, if it exists, such that for all integers $s \geq s_r^*(n)$, there are at least r different n -partitions of s satisfying only condition (b). The following theorem is proved by John B. Kelly in 1964.

Theorem 1.1 ([2]). *For every integer $n \geq 3$, $s_{n-1}(n)$ and $s_{n-1}^*(n)$ exist. Furthermore, $s_2(3) = 23$ and $s_2^*(3) = 19$.*

In the same paper, Kelly mentioned that the only known values of $s_{n-1}(n)$ and $s_{n-1}^*(n)$ were when $n = 3$, and all other values were unknown. He later showed that for $n \geq 3$ and for any positive integer r , there exist infinitely many integers s for which there are r mutually disjoint n -partitions of s such that the products of the partitions are all equal [3].

According to Kelly, his investigation into n -partitions of equal sum and product began with a conjecture, communicated orally, of T. S. Motzkin. Motzkin conjectured that for each sufficiently large s , there exists a positive integer p such that the triple $(s, p, 3)$ is admissible. Although Theorem 1.1 proves and generalizes Motzkin's conjecture, there are still many curious open questions related to this conjecture. In fact, just recently in 2015, Sadek and El-Sissi parameterized all admissible triples of the form $(s, p, 3)$ [4].

In this article, our study of admissible triples is threefold. First, in Section 2, we determine the value of the function

$$f(s) = |\{n \in \mathbb{N} : (s, p, n) \text{ is admissible for some } p \in \mathbb{N}\}|$$

for each positive integer s . Second, in Section 3 we provide a full characterization of product-admissible numbers. We further prove in Section 3 that if q is a prime and j is a positive integer, then q^j is product-admissible if and only if $j \geq 2q + 4$. Third, in Section 4, we provide an algorithm to effectively calculate the values of $s_r^*(n)$, and as a generalization of Kelly's results, we generate a list of values of $s_{n-1}^*(n)$ for $3 \leq n \leq 21$. Finally, we end our article in Section 5 with several conjectures regarding $s_r^*(n)$.

Before we move on to the next section, we would like to mention that the problem of integer partitions with equal products has connections with the “Conway’s wizard problem [1].” In the 1960’s, John Conway posed the following riddle.

Last night I sat behind two wizards on a bus and overheard the following:

- Blue Wizard: I have a positive integer number of children, whose ages are positive integers. The sum of their ages is the number of this bus, while the product is my own age.
- Red Wizard: How interesting! Perhaps if you told me your age and the number of your children, I could work out their individual ages?
- Blue Wizard: No, you could not.
- Red Wizard: Aha! At last, I know how old you are!

Apparently the Red Wizard had been trying to determine the Blue Wizard’s age for some time. Now, what was the number of the bus?

Solving this riddle is equivalent to finding a positive integer s such that there is a unique product p and an integer n to produce an admissible triple (s, p, n) .

2 The function $f(s)$

The following theorem is the main result of this section. For a fixed s , we count the number of positive integers n such that (s, p, n) is admissible for some product p .

Theorem 2.1. *When $1 \leq s \leq 11$, $f(s) = 0$, and when $s \geq 19$, $f(s) = s - 10$. Finally, $(f(s))_{s=12}^{18} = (1, 2, 4, 4, 6, 7, 7)$.*

In order to prove this theorem, we first introduce several lemmas regarding the function $F(s) = \{n \in \mathbb{N} : (s, p, n) \text{ is admissible for some } p \in \mathbb{N}\}$.

Lemma 2.2. *For each $s \in \mathbb{N}$, $\{1, 2, s - 7, s - 6, \dots, s - 1, s\} \cap F(s) = \emptyset$. In other words, if $n \in \{1, 2, s - 7, s - 6, \dots, s - 1, s\}$, then (s, p, n) is not admissible for any $p \in \mathbb{N}$.*

Proof. If $n = 1$, then the only 1-partition of s is $\{s\}$. If $n = 2$, then all the 2-partitions of s are of the form $\{r, s - r\}$, where $r \in \mathbb{N}$. Assume that there exist two different partitions $\{r, s - r\}$ and $\{r', s - r'\}$ satisfying $T\{r, s - r\} = T\{r', s - r'\} = (s, p, 2)$ for some $p \in \mathbb{N}$. Then $r(s - r) = r'(s - r')$, which implies $s(r - r') - (r^2 - r'^2) = (s - r - r')(r - r') = 0$. In other words, $r = r'$ or $r = s - r'$, contradicting that $\{r, s - r\} \neq \{r', s - r'\}$.

If $n = s - t$ for some $t = 0, 1, 2, \dots, 7$, we can now assume that $s - t \geq 3$ for a meaningful discussion. Here is a table of partitions of s into $s - t$ parts for each value t , together with their corresponding product p .

t	Partitions of s into $s - t$ parts	p
0	$\{1, 1, \dots, 1\}$	1
1	$\{1, 1, \dots, 1, 2\}$	2
2	$\{1, 1, \dots, 1, 3\}$	3
	$\{1, 1, \dots, 1, 2, 2\}$	4
3	$\{1, 1, \dots, 1, 4\}$	4
	$\{1, 1, \dots, 1, 2, 3\}$	6
	$\{1, 1, \dots, 1, 2, 2, 2\}$	8
4	$\{1, 1, \dots, 1, 5\}$	5
	$\{1, 1, \dots, 1, 2, 4\}$	8
	$\{1, 1, \dots, 1, 3, 3\}$	9
	$\{1, 1, \dots, 1, 2, 2, 3\}$	12
	$\{1, 1, \dots, 1, 2, 2, 2, 2\}$	16
5	$\{1, 1, \dots, 1, 6\}$	6
	$\{1, 1, \dots, 1, 2, 5\}$	10
	$\{1, 1, \dots, 1, 3, 4\}$	12
	$\{1, 1, \dots, 1, 2, 2, 4\}$	16
	$\{1, 1, \dots, 1, 2, 3, 3\}$	18
	$\{1, 1, \dots, 1, 2, 2, 2, 3\}$	24
	$\{1, 1, \dots, 1, 2, 2, 2, 2, 2\}$	32

t	Partitions of s into $s - t$ parts	p
6	$\{1, 1, \dots, 1, 7\}$	7
	$\{1, 1, \dots, 1, 2, 6\}$	12
	$\{1, 1, \dots, 1, 3, 5\}$	15
	$\{1, 1, \dots, 1, 4, 4\}$	16
	$\{1, 1, \dots, 1, 2, 2, 5\}$	20
	$\{1, 1, \dots, 1, 2, 3, 4\}$	24
	$\{1, 1, \dots, 1, 3, 3, 3\}$	27
	$\{1, 1, \dots, 1, 2, 2, 2, 4\}$	32
	$\{1, 1, \dots, 1, 2, 2, 3, 3\}$	36
	$\{1, 1, \dots, 1, 2, 2, 2, 2, 3\}$	48
$\{1, 1, \dots, 1, 2, 2, 2, 2, 2, 2\}$	64	
7	$\{1, 1, \dots, 1, 8\}$	8
	$\{1, 1, \dots, 1, 2, 7\}$	14
	$\{1, 1, \dots, 1, 3, 6\}$	18
	$\{1, 1, \dots, 1, 4, 5\}$	20
	$\{1, 1, \dots, 1, 2, 2, 6\}$	24
	$\{1, 1, \dots, 1, 2, 3, 5\}$	30
	$\{1, 1, \dots, 1, 2, 4, 4\}$	32
	$\{1, 1, \dots, 1, 3, 3, 4\}$	36
	$\{1, 1, \dots, 1, 2, 2, 2, 5\}$	40
	$\{1, 1, \dots, 1, 2, 2, 3, 4\}$	48
	$\{1, 1, \dots, 1, 2, 3, 3, 3\}$	54
	$\{1, 1, \dots, 1, 2, 2, 2, 2, 4\}$	64
	$\{1, 1, \dots, 1, 2, 2, 2, 3, 3\}$	72
	$\{1, 1, \dots, 1, 2, 2, 2, 2, 2, 3\}$	96
$\{1, 1, \dots, 1, 2, 2, 2, 2, 2, 2, 2\}$	128	

From this table, we can see that all products p are unique for each $n = s - t$. Therefore, $1, 2, s - 7, s - 6, \dots, s - 1, s \notin F(s)$. \square

Lemma 2.3. *Let $s \in \mathbb{N}$. If there exists a positive integer n such that $n \in F(s)$, then for all positive integers s' and n' satisfying $n' \leq s'$, we have $n + n' \in F(s + s')$.*

Proof. Suppose $n \in F(s)$. Then there exist at least two different multisets of n positive integers, $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$, satisfying $T\{x_1, x_2, \dots, x_n\} = T\{y_1, y_2, \dots, y_n\} = (s, p, n)$. For all positive integers s' and n' satisfying $n' \leq s'$, let $x_{n+1} = x_{n+2} = \dots = x_{n+n'-1} = y_{n+1} = y_{n+2} = \dots = y_{n+n'-1} = 1$ and $x_{n+n'} = y_{n+n'} = s' - (n' - 1)$. We can extend our multisets of n positive integers to $\{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+n'}\}$ and $\{y_1, y_2, \dots, y_n, y_{n+1}, \dots, y_{n+n'}\}$ such that

$$T\{x_1, x_2, \dots, x_n, \dots, x_{n+n'}\} = T\{y_1, y_2, \dots, y_n, \dots, y_{n+n'}\} = (s + s', p(s' - (n' - 1)), n + n').$$

This implies $n + n' \in F(s + s')$. \square

Lemma 2.4. (a) For $s = 11, 12, 15, 18$, $3 \notin F(s)$. Also, $4 \notin F(13)$.

(b) For $s = 13, 14, 16, 17$, $3 \in F(s)$. Also, $4 \in F(12)$.

Proof. Statement (a) is proved by exhaustion of all 3-partitions of 11, 12, 15, and 18, as well as all 4-partitions of 13. Statement (b) is due to the following observations.

$$\begin{aligned} T\{1, 6, 6\} &= T\{2, 2, 9\} = (13, 36, 3) \text{ implies } 3 \in F(13), \\ T\{1, 5, 8\} &= T\{2, 2, 10\} = (14, 40, 3) \text{ implies } 3 \in F(14), \\ T\{2, 5, 9\} &= T\{3, 3, 10\} = (16, 90, 3) \text{ implies } 3 \in F(16), \\ T\{3, 6, 8\} &= T\{4, 4, 9\} = (17, 144, 3) \text{ implies } 3 \in F(17), \\ \text{and } T\{1, 3, 4, 4\} &= T\{2, 2, 2, 6\} = (12, 48, 4) \text{ implies } 4 \in F(12). \end{aligned}$$

□

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. When $1 \leq s \leq 10$, for all positive integers $n \leq s$, $n \in \{1, 2, s-7, s-6, \dots, s-1, s\}$. By Lemma 2.2, $f(s) = 0$. When $s \geq 11$, we summarize the procedures in the following table.

Sum s	Values $n \leq s$																			
11	1	2	3	4	5	6	7	8	9	10	11									
12	1	2	3	④	5	6	7	8	9	10	11	12								
13	1	2	③	4	5	6	7	8	9	10	11	12	13							
14	1	2	③	4	5	6	7	8	9	10	11	12	13	14						
15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15					
16	1	2	③	4	5	6	7	8	9	10	11	12	13	14	15	16				
17	1	2	③	4	5	6	7	8	9	10	11	12	13	14	15	16	17			
18	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		
19	1	2	③	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
20	1	2	③	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

In this table, those crossed-out entries, i.e., ~~n~~ , indicate $n \notin F(s)$ by Lemma 2.4(a). Those circled entries, i.e., \textcircled{n} , indicate $n \in F(s)$ by Lemma 2.4(b). Those shaded entries indicate $n \in F(s)$ by Lemma 2.3. Since $s_2^*(3) = 19$ by Theorem 1.1, $3 \in F(s)$ for all $s \geq 19$. This fact is indicated by those boxed entries, i.e., \boxed{n} . Finally, those plain entries indicate $n \notin F(s)$ by Lemma 2.2. □

3 Product-admissible numbers

In Section 2, we fixed the sum in the triple (s, p, n) to study the function $f(s)$. We now turn our attention to fixing the product of the triple.

Theorem 3.1. Let q_1, q_2, \dots, q_k be primes, and let $j_1, j_2, \dots, j_k \in \mathbb{N}$. Then $p = q_1^{j_1} q_2^{j_2} \cdots q_k^{j_k}$ is product-admissible if and only if there exists a nonzero multivariate polynomial χ of k variables with integer coefficients such that

- $\chi(q_1, q_2, \dots, q_k) = 0$,
- $\chi_\ell(1, 1, \dots, 1) = 0$ for each $1 \leq \ell \leq k$, where χ_ℓ is the partial derivative of χ with respect to the ℓ -th variable,
- for each $1 \leq \ell \leq k$, the sum of the absolute values of the coefficients in χ_ℓ is at most $2j_\ell$, and
- $\chi(1, 1, \dots, 1) = 0$.

Proof. Let q_1, q_2, \dots, q_k be primes, and let $j_1, j_2, \dots, j_k \in \mathbb{N}$ be such that $p = q_1^{j_1} q_2^{j_2} \cdots q_k^{j_k}$ is product-admissible, i.e., there exists $n \in \mathbb{N}$ and at least two different multisets of n positive integers, $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$, satisfying $T\{x_1, x_2, \dots, x_n\} = T\{y_1, y_2, \dots, y_n\} = (s, p, n)$ for some $s \in \mathbb{N}$. Since $x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_n = q_1^{j_1} q_2^{j_2} \cdots q_k^{j_k}$, by the fundamental theorem of arithmetic, for each $1 \leq i \leq n$, we can let $x_i = q_1^{\alpha_{i1}} q_2^{\alpha_{i2}} \cdots q_k^{\alpha_{ik}}$ and $y_i = q_1^{\beta_{i1}} q_2^{\beta_{i2}} \cdots q_k^{\beta_{ik}}$, where $\alpha_{i\ell}, \beta_{i\ell} \in \mathbb{N} \cup \{0\}$ for all $1 \leq i \leq n$ and $1 \leq \ell \leq k$, and $\sum_{i=1}^n \alpha_{i\ell} = \sum_{i=1}^n \beta_{i\ell} = j_\ell$ for each $1 \leq \ell \leq k$.

For all (t_1, t_2, \dots, t_k) satisfying $0 \leq t_\ell \leq j_\ell$ for each $1 \leq \ell \leq k$, let a_{t_1, t_2, \dots, t_k} be the number of times $q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k}$ appears in $\{x_1, x_2, \dots, x_n\}$, and let b_{t_1, t_2, \dots, t_k} be the number of times $q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k}$ appears in $\{y_1, y_2, \dots, y_n\}$. For all $j \in \mathbb{N}$, let $[j] = \{0, 1, 2, \dots, j\}$. Then

- $$\sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} a_{t_1, t_2, \dots, t_k} q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k} = \sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} b_{t_1, t_2, \dots, t_k} q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k} = s,$$
- $$\sum_{\substack{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k] \\ \ell \leq k, \text{ and}}} a_{t_1, t_2, \dots, t_k} t_\ell = \sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} b_{t_1, t_2, \dots, t_k} t_\ell = j_\ell \text{ for each } 1 \leq \ell \leq k, \text{ and}$$
- $$\sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} a_{t_1, t_2, \dots, t_k} = \sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} b_{t_1, t_2, \dots, t_k} = n.$$

If we subtract the right hand side from the left, and relabel $c_{t_1, t_2, \dots, t_k} = a_{t_1, t_2, \dots, t_k} - b_{t_1, t_2, \dots, t_k}$ for each $(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]$, we get

- $$\sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} c_{t_1, t_2, \dots, t_k} q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k} = 0,$$
- $$\sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \cdots \times [j_k]} c_{t_1, t_2, \dots, t_k} t_\ell = 0 \text{ for each } 1 \leq \ell \leq k,$$

- $\sum_{\substack{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \dots \times [j_k] \\ 2j_\ell \text{ for each } 1 \leq \ell \leq k, \text{ and}}} |c_{t_1, t_2, \dots, t_k} t_\ell| \leq \sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \dots \times [j_k]} |a_{t_1, t_2, \dots, t_k} t_\ell| + |b_{t_1, t_2, \dots, t_k} t_\ell| =$
- $\sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \dots \times [j_k]} c_{t_1, t_2, \dots, t_k} = 0.$

This is equivalent to the existence of a multivariate polynomial χ of k variables with integer coefficients subject to the conditions in the statement of the theorem.

Conversely, if such a multivariate polynomial $\chi \in \mathbb{Z}[z_1, z_2, \dots, z_k]$ exists, denote the coefficient of $z_1^{t_1} z_2^{t_2} \dots z_k^{t_k}$ by a_{t_1, t_2, \dots, t_k} if it is positive, and denote the absolute value of the coefficient by b_{t_1, t_2, \dots, t_k} if it is negative. For each $(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \dots \times [j_k]$, let a_{t_1, t_2, \dots, t_k} and b_{t_1, t_2, \dots, t_k} be the number of times that $q_1^{t_1} q_2^{t_2} \dots q_k^{t_k}$ appears in the multisets X and Y respectively. Furthermore, for each $1 \leq \ell \leq k$, let $j'_\ell = j_\ell - \sum_{(t_1, t_2, \dots, t_k) \in [j_1] \times [j_2] \times \dots \times [j_k]} a_{t_1, t_2, \dots, t_k} t_\ell,$

and insert one copy of $q_1^{j'_1} q_2^{j'_2} \dots q_k^{j'_k}$ in both X and Y . From our constructions, it is apparent that $p = q_1^{j_1} q_2^{j_2} \dots q_k^{j_k}$ is product-admissible with X and Y being the two different multisets. \square

If we restrict to $k = 1$, then Theorem 3.1 implies that $p = q^j$ is product-admissible if and only if there exists a nonzero polynomial χ with integer coefficients such that

- $\chi(q) = 0,$
- $\chi'(1) = 0,$ where χ' is the derivative of $\chi,$
- the sum of the absolute values of the coefficients in χ' is at most $2j,$ and
- $\chi(1) = 0.$

In other words, there exists a nonzero polynomial ψ with integer coefficients such that

$$\chi(z) = (z - q)(z - 1)^2 \psi(z),$$

and the sum of the absolute values of the coefficients in χ' is at most $2j.$ This is a nice characterization, but we go one step further and prove the following theorem.

Theorem 3.2. *Let q be a prime and let $j \in \mathbb{N}.$ Then $p = q^j$ is product-admissible if and only if $j \geq 2q + 4.$*

Proof. If ψ is a constant polynomial such that $\psi(z) = 1,$ then $\chi(z) = z^3 - (q + 2)z^2 + (2q + 1)z - q.$ This implies the two multisets can be

$$\{q^3, \underbrace{q, q, \dots, q}_{2q+1 \text{ copies}}\} \text{ and } \{\underbrace{q^2, q^2, \dots, q^2}_{q+2 \text{ copies}}, \underbrace{1, 1, \dots, 1}_q\}.$$

At this moment, $(s, p, n) = (q^3 + 2q^2 + q, q^{2q+4}, 2q + 2).$ For all $j = 2q + 4 + j'$ for some $j' \in \mathbb{N},$ the two multisets can be

Inverting the matrices to the other side, we have

$$\begin{pmatrix} c_m \\ c_{m-1} \\ c_{m-2} \\ \vdots \\ c_1 \\ c_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Q_0 & & & & & & & & & & & \\ Q_1 & Q_0 & & & & & & & & & & \\ Q_2 & Q_1 & Q_0 & & & & & & & & & \\ Q_3 & Q_2 & Q_1 & Q_0 & & & & & & & & \\ Q_4 & Q_3 & Q_2 & Q_1 & Q_0 & & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ Q_{m+1} & \ddots & Q_4 & Q_3 & Q_2 & Q_1 & Q_0 & & & & & \\ Q_{m+2} & Q_{m+1} & \cdots & Q_4 & Q_3 & Q_2 & Q_1 & Q_0 & & & & \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{m+3} & & & & & & & & & & & \\ & \frac{1}{m+2} & & & & & & & & & & \\ & & \frac{1}{m+1} & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & \frac{1}{3} & & & & & & \\ & & & & & & \frac{1}{2} & & & & & \\ & & & & & & & 1 & & & & \end{pmatrix} \begin{pmatrix} d_{m+2} \\ d_{m+1} \\ d_m \\ \vdots \\ d_2 \\ d_1 \\ d_0 \end{pmatrix},$$

where $Q_\iota = \sum_{i=0}^\iota (\iota + 1 - i)q^i$ for all $\iota = 0, 1, 2, \dots, m + 2$.

From the second last row of the matrix multiplication, we have

$$0 = \sum_{\iota=0}^{m+1} \frac{Q_\iota}{\iota + 2} d_{\iota+1},$$

which implies

$$-d_1 = 2 \sum_{\iota=1}^{m+1} \frac{Q_\iota}{\iota + 2} d_{\iota+1}. \quad (2)$$

Note that for all $0 \leq \iota \leq m + 1$,

$$\begin{aligned} \frac{Q_\iota}{\iota + 2} - \frac{Q_{\iota-1}}{\iota + 1} &= \frac{1}{\iota + 2} \left(\sum_{i=0}^\iota (\iota + 1 - i)q^i - \left(1 + \frac{1}{\iota + 1}\right) \sum_{i=0}^{\iota-1} (\iota - i)q^i \right) \\ &= \frac{1}{\iota + 2} \sum_{i=0}^\iota \left(1 - \frac{\iota - i}{\iota + 1}\right) q^i > 0. \end{aligned}$$

Hence, $\frac{Q_\iota}{\iota + 2}$ decreases with ι . From the first row of the matrix multiplication, we note that

$d_{m+2} = c_m(m+3) \geq m+3$. Combining with inequality (1), equation (2) becomes

$$\begin{aligned}
-d_1 &\geq 2 \left(\frac{Q_{m+1}}{m+3}(m+3) + \frac{Q_m}{m+2}(-(2q+3)) \right) \\
&= 2 \left(\sum_{i=0}^{m+1} (m+2-i)q^i - \frac{1}{m+2} \left(2 \sum_{i=0}^m (m+1-i)q^{i+1} + 3 \sum_{i=0}^m (m+1-i)q^i \right) \right) \\
&= 2 \left(\sum_{i=0}^{m+1} (m+2-i)q^i - \frac{1}{m+2} \left(2 \sum_{i=1}^{m+1} (m+2-i)q^i + 3 \sum_{i=0}^{m+1} (m+1-i)q^i \right) \right) \\
&= 2 \left(\sum_{i=1}^{m+1} \left(m-3-i + \frac{3+5i}{m+2} \right) q^i + \frac{m^2+m+1}{m+2} \right).
\end{aligned}$$

It suffices to show that $-d_1 > 2q+4$, since this will contradict with inequality (1).

If $m=1$, then

$$-d_1 \geq \frac{2}{3}q^2 - \frac{2}{3}q + 2,$$

which is greater than $2q+4$ since $q \geq 5$. If $m=2$, then

$$-d_1 \geq q^3 + \frac{1}{2}q^2 + \frac{7}{2} > 2q + \frac{1}{2} + \frac{7}{2} = 2q + 4.$$

If $m=3$, then

$$-d_1 \geq \frac{6}{5}q^4 + \frac{6}{5}q^3 + \frac{6}{5}q^2 + \frac{6}{5}q + \frac{26}{5} > \frac{24}{5}q + \frac{26}{5} > 2q + 4.$$

Finally, if $m \geq 4$, then

$$\begin{aligned}
&-d_1 - (2q+4) \\
&\geq 2 \left(\frac{mq^{m+1} + (2m-3)q^m + (3m-6)q^{m-1} + (4m-9)q^{m-2}}{m+2} \right. \\
&\quad \left. + \sum_{i=2}^{m-3} \left(m-3-i + \frac{3+5i}{m+2} \right) q^i + \left(m-4 + \frac{8}{m+2} - 1 \right) q + \frac{m^2+m+1}{m+2} - 2 \right).
\end{aligned}$$

All coefficients of q^i and the constant term are positive, meaning $-d_1 > 2q+4$. \square

Corollary 3.3. *Let q be a prime and let $u \in \mathbb{N}$. Then $p = q^{2q+4}u$ is product-admissible.*

Proof. This is by noticing that $T\{q^3, \underbrace{q, q, \dots, q}_{2q+1 \text{ copies}}, u\} = T\{\underbrace{q^2, q^2, \dots, q^2}_{q+2 \text{ copies}}, \underbrace{1, 1, \dots, 1}_q, u\} = (q^3 + 2q^2 + q + u, q^{2q+4}u, 2q+3)$. \square

4 At least r partitions with the same product

In Sections 2 and 3, we focused on finding at least two different multisets $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$ such that $T\{x_1, x_2, \dots, x_n\} = T\{y_1, y_2, \dots, y_n\} = (s, p, n)$. In this section, we consider at least r multisets that correspond to the same triple (s, p, n) .

For all integers $r \geq 2$ and $n \geq 3$, recall from the introduction that $s_r^*(n)$ is the smallest positive integer such that for all integers $s \geq s_r^*(n)$, there are at least r different n -partitions of s , namely $X_i = \{x_{i1}, x_{i2}, \dots, x_{in}\}$ for $i = 1, 2, \dots, r$, satisfying

$$T(X_i) = (s, p, n)$$

for some $p \in \mathbb{N}$. As mentioned in Theorem 1.1, Kelly proved that $s_{n-1}^*(n) \in \mathbb{N}$ exists for all integers $n \geq 3$. He also stated that $s_2^*(3) = 19$, but $s_{n-1}^*(n)$ was unknown for $n \geq 4$.

To find the values of $s_{n-1}^*(n)$, we first define $s_r^0(n)$ as the smallest positive integer s such that there are at least r different n -partitions of s , namely $X_i = \{x_{i1}, x_{i2}, \dots, x_{in}\}$ for $i = 1, 2, \dots, r$, satisfying $T(X_i) = (s, p, n)$ for some $p \in \mathbb{N}$.

Theorem 4.1. *For all integers $r \geq 2$ and $n \geq 3$, $s_r^*(n+1) \leq s_r^0(n) + 1 \leq s_r^*(n) + 1$.*

Proof. Let $X_i = \{x_{i1}, x_{i2}, \dots, x_{in}\}$ for $i = 1, 2, \dots, r$ be r different partitions of $s = s_r^0(n)$ satisfying $T(X_i) = (s, p, n)$. For any $s' \geq s_r^0(n) + 1$, let $u = s' - s_r^0(n)$. Then $X'_i = \{x_{i1}, x_{i2}, \dots, x_{in}, u\}$ for $i = 1, 2, \dots, r$ are r different partitions of s' satisfying $T(X'_i) = (s', pu, n+1)$. Therefore, $s_r^*(n+1) \leq s_r^0(n) + 1$. The second inequality follows from the obvious fact that $s_r^0(n) \leq s_r^*(n)$. \square

Theorem 4.1 can be used as an algorithm to determine $s_r^*(n)$ by first computing $s_r^0(n-1)$, followed by checking all values $s \leq s_r^0(n-1) + 1$. To illustrate this process, we have computed $s_n^0(n)$ for $3 \leq n \leq 20$, listed in the following table. These results can be verified computationally.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$s_n^0(n)$	39	24	25	26	28	30	31	34	35	37	39	41	43	44	46	48	49	51

To determine $s_{n-1}^*(n)$ for $3 \leq n \leq 21$, we only need to check all values $s \leq s_{n-1}^0(n-1) + 1$. A longer list of $s_{n-1}^*(n)$ values can be found on the On-Line Encyclopedia of Integer Sequences as A317254 [5].

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$s_{n-1}^*(n)$	19	23	23	26	27	29	31	32	35	36	38	40	42	44	45	47	49	50	52

5 Concluding remarks and conjectures

Based on computational data for $6 \leq n \leq 60$, it can be observed that $s_{n-1}^*(n) = s_{n-1}^0(n-1) + 1$, which motivates the following conjecture.

Conjecture 5.1. *For all integers $n \geq 6$, $s_{n-1}^*(n) = s_{n-1}^0(n-1) + 1$.*

Computational data also leads us to the following conjectures. Note that in each of the following statements, $s_r^*(n) \leq s_r^*(n-1) + 1$ is given by Theorem 4.1.

Conjecture 5.2.

- (a) For all integers $n \geq 9$, $s_{n-2}^*(n) = s_{n-2}^*(n-1) + 1$.
- (b) For all integers $n \geq 7$, $s_{n-1}^*(n) = s_{n-1}^*(n-1) + 1$.
- (c) For all integers $n \geq 10$, $s_n^*(n) = s_n^*(n-1) + 1$.

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