Algebraic structures on typed decorated rooted trees

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Abstract

Typed decorated trees are used by Bruned, Hairer and Zambotti to give a description of a renormalisation process on stochastic PDEs. We here study the algebraic structures on these objects: multiple prelie algebras and related operads (generalizing a result by Chapoton and Livernet), noncommutative and cocommutative Hopf algebras (generalizing Grossman and Larson's construction), commutative and noncocommutative Hopf algebras (generalizing Connes and Kreimer's construction), bialgebras in cointeraction (generalizing Calaque, Ebrahimi-Fard and Manchon's result). We also define families of morphisms and in particular we prove that any Connes-Kreimer Hopf algebra of typed and decorated trees is isomorphic to a Connes-Kreimer Hopf algebra of non typed and decorated trees (the set of decorations of vertices being bigger), trough a contraction process.

Keywords. typed tree; combinatorial Hopf algebras; prelie algebras; operads.

AMS classification. 05C05, 16T30, 18D50, 17D25.

Introduction

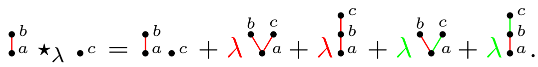
Bruned, Hairer and Zambotti used in [3] typed trees in an essential way to give a systematic description of a canonical renormalisation procedure of stochastic PDEs. Typed trees are rooted trees which edges are decorated by elements of a fixed set $\mathcal{T}$ of types. They also appear in a context of low dimension topology in [14] (there, described as nested parentheses) and for the description of combinatorial species in [1]. We here study several algebraic structures on these trees, generalizing results of Connes and Kreimer, Chapoton and Livernet, Grossman and Larson, Calaque, Ebrahimi-Fard and Manchon.

We first define grafting products of trees, similar to the prelie product of [5]. For any type t, we obtain a prelie product $\bullet_t$ on the space $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ of $\mathcal{T}$-typed trees which vertices are decorated by elements of a set $\mathcal{D}$: for example, if | and | are two types, if a, b, $c \in \mathcal{D}$, then:

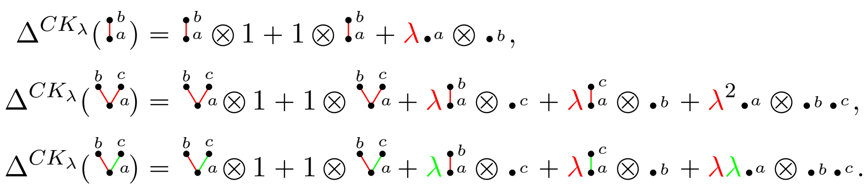
Then $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, equipped with all these products, is a $\mathcal{T}$-multiple prelie algebra (Definition 3), and we prove in Corollary 9 that it is the free $\mathcal{T}$-multiple prelie algebra generated by $\mathcal{D}$, generalizing the result of [6]. Consequently, we obtain a combinatorial description of the operad of $\mathcal{T}$-multiple prelie algebras in terms of $\mathcal{T}$-typed trees with indexed vertices (Theorem 11): for example,

We also give a desription of the Koszul dual operad and of its free algebras in Propositions 12 and 13, generalizing a result of [5].

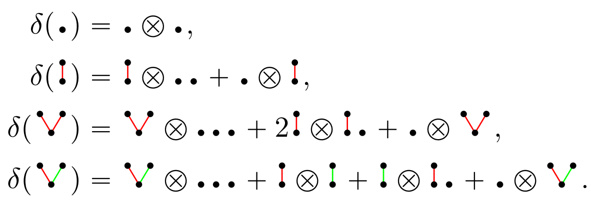
For any family $\lambda = (\lambda_t)_{t\in\mathcal{T}}$ with a finite support, the product $\bullet_\lambda = \sum \lambda_t \bullet_t$ is prelie: using the Guin-Oudom construction [16, 15], we obtain a Hopf algebraic structure $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}} = (S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), \star_\lambda, \Delta)$ on the symmetric algebra generated by $\mathcal{T}$-typed and $\mathcal{D}$-decorated trees, that is to say on the space of $\mathcal{T}$-typed and $\mathcal{D}$-decorated forests. The coproduct Δ is given by partitions of forests into two forests and the $\star_\lambda$ product is given by grafting. For example:



In the non-typed case, we get back the Grossman-Larson Hopf algebra of trees [9]. Dually, we obtain Hopf algebras $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$, generalizing the Connes-Kreimer Hopf algebra [7] of rooted trees. For example:



This Hopf algebra satisfies a universal property in Hochschild cohomology, as the Connes-Kreimer's one. We describe it in the simpler case where $\mathcal{T}$ is finite (Theorem 22). We finally give a second coproduct δ on $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$, such that $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ is a Hopf algebra in the category of $(S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), m, \delta)$-right comodules, generalizing the result of [4]. This coproduct δ is given by a contraction-extraction process. For example, in the non-decorated case:



We are also interested in morphisms between these objects. Playing linearly with types, we prove that if λ and μ are both nonzero, then the prelie algebras $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ and $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\mu)$ are isomorphic (Corollary 18). Consequently, if λ and μ are both nonzero, the Hopf algebras $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}}$ and $\mathcal{H}^{GL_\mu}_{\mathcal{D},\mathcal{T}}$ are isomorphic; dually, the Hopf algebras $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ and $\mathcal{H}^{CK_\mu}_{\mathcal{D},\mathcal{T}}$ are isomorphic (Corollary 24). Using Livernet's rigidity theorem [11] and a nonassociative permutative coproduct defined in Proposition 14, we prove that if $\lambda \neq 0$, then $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is, as a prelie algebra, freely generated by a family of typed trees $\mathcal{D}' = \mathcal{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$ satisfying a condition on the type of edges outgoing the root (Corollary 15). As a consequence, the Hopf algebra $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ of typed and decorated trees is isomorphic to a Connes-Kreimer Hopf algebra of non typed and decorated trees $\mathcal{H}^{CK}_{\mathcal{D}'}$, and an explicit isomorphism is described with the help of contraction in Proposition 26.

This paper is organized as follows: the first section gives the basic definition of typed rooted trees and enumeration results, when the number of types and decorations are finite. The second section is about the $\mathcal{T}$-multiple prelie algebra structures on these trees and the underlying operads. The freeness of the prelie structures on typed decorated trees and its consequences are studied in the third section. In the last section, the dual Hopf algebras $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}}$ and $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ are defined and studied.

Notations 1.
1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, prelie algebras...) in this text will be taken over $\mathbb{K}$.
2. For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$.
3. For any set $\mathcal{T}$, we denote by $\mathbb{K}^{\mathcal{T}}$ the set of family $\lambda = (\lambda_t)_{t\in\mathcal{T}}$ of elements of $\mathbb{K}$ indexed by $\mathcal{T}$, and we denote by $\mathbb{K}^{(\mathcal{T})}$ the set of elements $\lambda \in \mathbb{K}^{\mathcal{T}}$ with a finite support. Note that if $\mathcal{T}$ is finite, then $\mathbb{K}^{\mathcal{T}} = \mathbb{K}^{(\mathcal{T})}$.

1 Typed decorated trees

1.1 Definition

Definition 1. *Let $\mathcal{D}$ and $\mathcal{T}$ be two nonempty sets.*

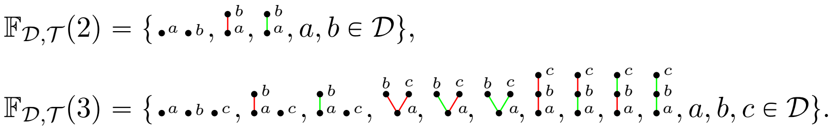
1. *A $\mathcal{D}$-decorated $\mathcal{T}$-typed forest is a triple $(F, \mathrm{dec}, \mathrm{type})$, where:*
   - *F is a rooted forest. The set of its vertices is denoted by $V(F)$ and the set of its edges by $E(F)$.*
   - *$\mathrm{dec} : V(F) \longrightarrow \mathcal{D}$ is a map.*
   - *$\mathrm{type} : E(F) \longrightarrow \mathcal{T}$ is a map.*

   If the underlying rooted forest of F is connected, we shall say that F is a $\mathcal{D}$-decorated $\mathcal{T}$-typed tree.
2. *For any finite set A, we denote by $\mathbb{T}_{\mathcal{T}}(A)$ the set of A-decorated $\mathcal{T}$-typed trees T such that $V(T) = A$ and $\mathrm{dec} = Id_A$, and by $\mathbb{F}_{\mathcal{T}}(A)$ the set of A-decorated $\mathcal{T}$-typed forests F such that $V(F) = A$ and $\mathrm{dec} = Id_A$.*
3. *For any $n \geqslant 0$, we denote by $\mathbb{T}_{\mathcal{D},\mathcal{T}}(n)$ the set of isoclasses of $\mathcal{D}$-decorated $\mathcal{T}$-typed trees T such that $|V(T)| = n$ and by $\mathbb{F}_{\mathcal{D},\mathcal{T}}(n)$ the set of $\mathcal{D}$-decorated $\mathcal{T}$-typed forests F such that $|V(F)| = n$. We also put:*

$$\mathbb{T}_{\mathcal{D},\mathcal{T}} = \bigsqcup_{n\geqslant 0} T_{\mathcal{D},\mathcal{T}}(n), \qquad \mathbb{F}_{\mathcal{D},\mathcal{T}} = \bigsqcup_{n\geqslant 0} F_{\mathcal{D},\mathcal{T}}(n).$$

Example 1. We shall represent the types of the edges by different colors and the decorations of the vertices by letters alongside them. If $\mathcal{T}$ contains two elements, represented by | and |, then:

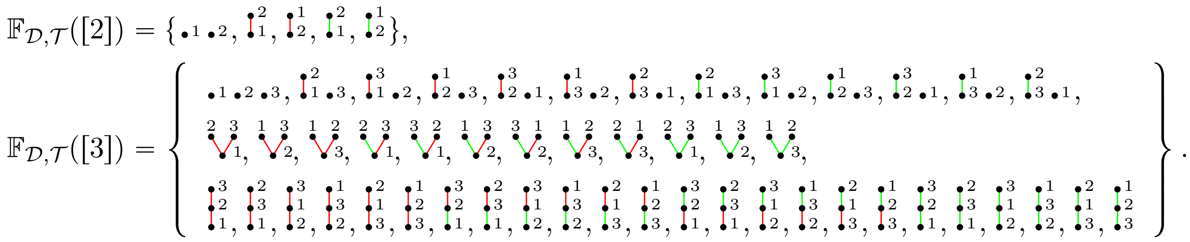
$$\mathbb{F}_{\mathcal{D},\mathcal{T}}(1) = \{\bullet_d, d \in \mathcal{D}\},$$



Note that for any a, b, $c \in \mathcal{D}$:

Moreover:

$$\mathbb{F}_{\mathcal{D},\mathcal{T}}([1]) = \{\bullet_1\},$$



Remark 1. If $|\mathbb{T}| = 1$, all the edges of elements of $\mathbb{F}_{\mathcal{D},\mathcal{T}}$ have the same type: we work with $\mathcal{D}$-decorated rooted forests. In this case, we shall omit $\mathcal{T}$ in the indices describing the forests, trees, spaces we are considering.

1.2 Enumeration

We assume here that $\mathcal{D}$ and $\mathcal{T}$ are finite, of respective cardinality D and T. For all $n \geqslant 0$, we put:

$$t_{D,T}(n) = |\mathbb{T}_{\mathcal{T},\mathcal{D}}(n)|, \qquad f_{D,T}(n) = |\mathbb{F}_{\mathcal{T},\mathcal{D}}(n)|,$$

$$T_{D,T}(X) = \sum_{n=1}^{\infty} t_{D,T}(n)X^n, \qquad F_{D,T}(X) = \sum_{n=0}^{\infty} f_{D,T}(n)X^n.$$

As any element of $\mathbb{F}_{\mathcal{T},\mathcal{D}}$ can be uniquely decomposed as the disjoint union of its connected components, which are elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, we obtain:

$$F_{D,T}(X) = \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)}}. \tag{1}$$

We put $\mathcal{T} = \{t_1, \ldots, t_T\}$. For any $d \in \mathcal{D}$, we consider:

$$B_d : \left\{ \begin{array}{rcl} (\mathbb{F}_{\mathcal{D},\mathcal{T}})^T & \longrightarrow & \mathbb{T}_{\mathcal{D},\mathcal{T}} \\ (F_1, \ldots, F_T) & \longrightarrow & B_d(F_1, \ldots, F_T), \end{array} \right.$$

where $B_d(F_1, \ldots, F_T)$ is the tree obtained by grafting the forests $F_1, \ldots, F_n$ on a common root decorated by d; the edges from this root to the roots of F_i are of type t_i for any $1 \leqslant i \leqslant T$. Then B_d is injective, homogeneous of degree 1, and moreover $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ is the disjoint union of the $B_d((\mathbb{F}_{\mathcal{D},\mathcal{T}})^T)$, $d \in \mathcal{D}$. Hence:

$$T_{D,T}(X) = DX(F_{D,T})^T = DX \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)T}}. \tag{2}$$

Note that (2) allows to compute $t_{D,T}(n)$ by induction on n, and (1) allows to deduce $f_{D,T}(n)$.

Lemma 2. *For any $n \in \mathbb{N}$,*
$$t_{D,T}(n) = \frac{t_{TD,1}(n)}{T}.$$

Proof. By induction on n. If $n = 1$, $t_{D,T}(1) = D$ and $t_{TD,1} = TD$, which gives the result. Let us assume the result at all ranks $k < n$. Then $t_{D,T}(n)T$ is the coefficient of X^n in:

$$TDX \prod_{k=1}^{n-1} \frac{1}{(1-X^k)^{t_{D,T}(k)T}} = TDX \prod_{k=1}^{n-1} \frac{1}{(1-X^k)^{t_{TD,1}(k)}},$$

which is precisely $t_{TD,1}(n)$. □

Example 2. We obtain:

$$\begin{aligned}
t_{D,T}(1) &= D,\\
t_{D,T}(2) &= D^2t,\\
t_{D,T}(3) &= \frac{D^2T(3D+1)}{2},\\
t_{D,T}(4) &= \frac{D^2T(8S^2T^2+3DT+1)}{3},\\
t_{D,T}(5) &= \frac{D^2T(125D^3T^3+54D^2T^2+31DT+6)}{24},\\
t_{D,T}(6) &= \frac{D^2T(162D^4T^4+80D^3T^3+45D^2T^2+10DT+3)}{15},\\
t_{D,T}(7) &= \frac{D^2T(16807D^5T^5+9375D^4T^4+5395D^3T^3+2025D^2T^2+838DT+120)}{720}
\end{aligned}$$

Specializing, we find the following sequences of the OEIS [18]:

$T \backslash D$	1	2	3	4
1	*A0081*	*A038055*	*A038059*	*A136793*
2	*A00151*	*A136794*		
3	*A006964*			
4	*A052763*			
5	*A052788*			
6	*A246235*			
7	*A246236*			
8	*A246237*			
9	*A246238*			
10	*A246239*			

2 Multiple prelie algebras

We here fix a nonempty set $\mathcal{T}$ of types of edges.

2.1 Definition

Definition 3. *A $\mathcal{T}$-multiple prelie algebra is a family $(V, (\bullet_t)_{t\in\mathcal{T}})$, where V is a vector space and for all $t \in \mathcal{T}$, $\bullet_t$ is a bilinear product on V such that:*

$$\forall t, t' \in \mathcal{T},\ \forall x, y, z \in V, \qquad x \bullet_{t'} (y \bullet_t z) - (x \bullet_{t'} y) \bullet_t z = x \bullet_t (z \bullet_{t'} y) - (x \bullet_t z) \bullet_{t'} y.$$

For any $t \in \mathcal{T}$, $(V, \bullet_t)$ is a prelie algebra. More generally, for any family $\lambda = (\lambda_t)_{t\in\mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, putting $\bullet_\lambda = \sum \lambda_t \bullet_t$, $(V, \bullet_\lambda)$ is a prelie algebra.

Proposition 4. *Let $\mathcal{D}$ be any set; we denote by $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ the vector space generated by $\mathbb{T}_{\mathcal{D},\mathcal{T}}$. For any $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, $v \in V(T)$ and $t \in \mathcal{T}$, we denote by $T \bullet_t^{(v)} T'$ the $\mathcal{D}$-decorated $\mathcal{T}$-typed tree obtained by grafting T' on v, the created edge being of type t. We then define a product $\bullet_t$ on $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ by:*

$$\forall T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \qquad T \bullet_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T'.$$

Then $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is a $\mathcal{T}$-multiple prelie algebra.

Proof. Let T, T', T'' be elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $t', t'' \in \mathcal{T}$.

$$\begin{aligned}
&(T \bullet_{t'} T') \bullet_{t''} T'' - T \bullet_{t'} (T' \bullet_{t''} T'') \\
&= \sum_{v \in Vert(T), v' \in Vert(T) \sqcup Vert(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - \sum_{v \in Vert(T), v' \in Vert(T')} T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\
&= \sum_{v \in Vert(T), v' \in Vert(T)} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' \\
&+ \sum_{v \in Vert(T), v' \in Vert(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\
&= \sum_{v \in Vert(T), v' \in Vert(T)} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' \\
&= \sum_{v \in Vert(T), v' \in Vert(T)} (T \bullet_{t''}^{(v')} T'') \bullet_{t'}^{(v)} T' \\
&= (T \bullet_{t''} T'') \bullet_{t'} T' - T \bullet_{t''} (T'' \bullet_{t'} T').
\end{aligned}$$

So $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ is indeed a $\mathcal{T}$-multiple prelie algebra. □

Example 3. If a,b, $c \in \mathcal{D}$ and $|$, $|$ $\in \mathcal{T}$:

2.2 Guin-Oudom extension of the prelie products

Notations 2. Let V be a vector space. We denote

$$V^{\oplus \mathcal{T}} = \bigoplus_{t \in \mathcal{T}} V \delta_t.$$

Lemma 5. *If for any $t \in \mathcal{T}$, $\bullet_t$ is a bilinear product on a vector space V, we define $\bullet : (V^{\oplus \mathcal{T}})^{\otimes 2} \longrightarrow V^{\oplus \mathcal{T}}$ by:*

$$x\delta t \bullet x'\delta_{t'} = (x \bullet_{t'} y)\delta_t.$$

Then $(V, (\bullet_t)_{t \in \mathcal{T}})$ is a $\mathcal{T}$-multiple prelie algebra if, and only if, $(V^{\oplus \mathcal{T}}, \bullet)$ is a prelie algebra.

Proof. Let $x, x', x'' \in V$, $t, t', t'' \in \mathcal{T}$. Then:

$$x\delta_t \bullet (x'\delta_{t'} \bullet x''\delta_{t''}) - (x\delta_t \bullet x'\delta_{t'}) \bullet x''\delta_{t''} = \big((x \bullet_{t'} x') \bullet_{t''} x'' - x \bullet_{t'} (x' \bullet_{t''} x'')\big)\, \delta_t,$$

which implies the result. □

Notations 3. The symmetric algebra $S(V)$ is given its usual coproduct Δ, making it a bialgebra:

$$\forall x \in V, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x.$$

We shall use Sweedler's notation: for any $w \in S(V)$, $\Delta(w) = \sum w^{(1)} \otimes w^{(2)}$.

Theorem 6. *Let V be a $\mathcal{T}$-multiple prelie algebra. One can define a product*

$$\bullet : S(V) \otimes S(V^{\otimes \mathcal{T}}) \longrightarrow S(V)$$

in the following way: for any $u, v \in S(V)$, $w \in S(V^{\oplus \mathcal{T}})$, $x \in V$, $t \in \mathcal{T}$,

$$\begin{aligned}
1 \bullet w &= \varepsilon(w),\\
u \bullet 1 &= u,\\
uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}),\\
u \bullet w(x\delta_t) &= (u \bullet w) \bullet_t x - x \bullet (w \bullet_t x),
\end{aligned}$$

where $\bullet_t$ is extended to $S(V) \otimes V$ and $S(V^{\oplus \mathcal{T}}) \otimes V$ by:

$$\forall x_1, \ldots, x_k, x \in V,\ t_1, \ldots, t_k \in \mathcal{T}, \qquad x_1 \ldots x_k \bullet_t x = \sum_{i=1}^{k} x_1 \ldots (x_i \bullet_t x) \ldots x_k,$$

$$(x_1\delta_{t_1}) \ldots (x_k\delta_{t_k}) \bullet_t x = \sum_{i=1}^{k} (x_1\delta_{t_1}) \ldots ((x_i \bullet_t x)\delta_{t_i}) \ldots (x_k\delta_{t_k}).$$

Proof. Unicity. The last formula allows to compute $x \bullet w$ for any $x \in V$ and $w \in S(V^{\oplus \mathcal{T}})$ by induction on the length of w; the other ones allow to compute $u \bullet w$ for any $u \in S(V)$ by induction on the length on u. So this product $\bullet$ is unique.

Existence. Let us use the Guin-Oudom construction [15, 16] on the prelie algebra $V^{\otimes \mathcal{T}}$. We obtain a product $\bullet$ defined on $S(\mathfrak{g}^{\oplus \mathcal{T}})$ such that for any $u, v, w \in S(\mathfrak{g}^{\oplus \mathcal{T}})$, $x \in V^{\oplus \mathcal{T}}$:

$$\begin{aligned}
1 \bullet w &= \varepsilon(w),\\
u \bullet 1 &= u,\\
uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}),\\
u \bullet wx &= (u \bullet w) \bullet x - x \bullet (w \bullet x).
\end{aligned}$$

Let $f : \mathcal{T} \longrightarrow \mathbb{K}$ be any nonzero map. We consider the surjective algebra morphism $F : S(V^{\oplus \mathcal{T}}) \longrightarrow S(V)$, sending $x\delta_t$ to $f(t)x$ for any $x \in V$, $t \in \mathcal{T}$. Its kernel is generated by the elements $X_{t,t'}x = (f(t')\delta_t - f(t)\delta_{t'})x$, where $x \in V$ and $t, t' \in \mathcal{T}$. We denote by J the vector space generated by the elements $X_{t,t'}x$. Let us prove that for any $w \in S(V^{\oplus \mathcal{T}})$, $J \bullet w \subseteq J$ by induction on the length n of w. If $n = 0$, we can assume that $w = 1$ and this is obvious. If $n = 1$, we can assume that $w = x'\delta_{t''}$. Then:

$$X_{t,t'}x \bullet w = (f(t')\delta_t - f(t)\delta_{t'})x \bullet_{t''} x' = X_{t,'t'}x \bullet_{t''} x' \in J.'$$

Let us assume the result at rank $n-1$. We can assume that $w = w'x'\delta_t$, the length of w' being $n-1$. For any $x \in J$:

$$x \bullet w = (x \bullet w') \bullet x' - x \bullet (w' \bullet x').$$

The length of w' and $w' \bullet x'$ is $n-1$, so $x \bullet w'$ and $x \bullet (w' \bullet x')$ belong to J. From the case $n = 1$, $(x \bullet w') \bullet x' \in J$, so $x \bullet w \in J$.

For any $x \in J$, $u, v \in S(V^{\oplus \mathcal{T}})$:

$$xu \bullet v = \underbrace{x \bullet v^{(1)}}_{\in J}(u \bullet v^{(2)}) \in Ker(F).$$

This proves that $Ker(F) \bullet S(V^{\oplus \mathcal{T}}) \subseteq Ker(F)$. Hence, $\bullet$ induces a product also denoted by $\bullet$, defined from $S(V) \otimes S(V^{\otimes \mathcal{T}})$ to $S(V)$. It is not difficult to show that it does not depend on the choice of f and satisfies the required properties. $\square$

Definition 7. *Let $d \in \mathcal{D}$, $T_1, \ldots, T_k \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, $t_1, \ldots, t_k \in \mathcal{T}$. We denote by*

$$B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}\right)$$

the $\mathcal{T}$-typed $\mathcal{D}$-decorated tree obtained by grafting $T_1, \ldots, T_k$ on a common root decorated by d, the edge between this root and the root of T_i being of type t_i for any i. This defines a map $B_d : S\left(Vect(\mathbb{T}_{\mathcal{D},\mathcal{T}})^{\oplus\mathcal{T}}\right) \longrightarrow S(Vect(\mathbb{T}_{\mathcal{D},\mathcal{T}}))$.

Lemma 8. *For any $d \in \mathcal{D}$, $T_1, \ldots, T_k \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, $t_1, \ldots, t_k \in \mathcal{T}$:*

$$B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}\right) = \bullet_d \bullet \prod_{i\in[k]} T_i\delta_{t_i}.$$

Proof. We write $F = \prod_{i\in[k]} T_i\delta_{t_i}$. We proceed by induction on k. If $k = 0$, then $F = 1$ and $\bullet_d \bullet 1 = \bullet_d = B_d(1)$. let us assume the result at rank $k-1$, with $k \geqslant 1$. We can write $F = F'T\delta_t$, with $length(F') = k-1$, $T = T_k$ and $t = t_k$. Then:

$$\begin{aligned}
\bullet_d \bullet F &= (\bullet_d \bullet F') \bullet T\delta_t - \bullet_d \bullet (F' \bullet T\delta_t)\\
&= B_d(F') \bullet_t T - B_d(F' \bullet_t T)\\
&= B_d(F'T\delta_t) + B_d(F' \bullet_t T) - B_d(F' \bullet_t T)\\
&= B_d(F).
\end{aligned}$$

So the result holds for all $k \geqslant 0$. □

Corollary 9. *Let A be a $\mathcal{T}$-multiple prelie algebra and, for any $d \in \mathcal{D}$, $a_d \in A$. There exists a unique $\mathcal{T}$-multiple algebra morphism $\phi : \mathfrak{g}_{\mathcal{D},\mathcal{T}} \longrightarrow A$, such that for any $d \in \mathcal{D}$, $\phi(\bullet_d) = a_d$. In other words, $\mathfrak{g}_{\mathcal{T},\mathcal{D}}$ is the free $\mathcal{T}$-multiple prelie algebra generated by $\mathcal{D}$.*

Proof. Unicity. Using the Guin-Oudom product and lemma 8, ϕ is the unique linear map inductively defined by:

$$\phi\left(B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}\right)\right) = a_d \bullet \prod_{i\in[k]} \phi(T_i)\delta_{t_i}.$$

Existence. Let $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $t \in \mathcal{T}$. Let us prove that $\phi(T \bullet_t T') = \phi(T) \bullet_t \phi(T')$ by induction on $n = |T|$. If $n = 1$, we assume that $T = \bullet_d$. Then $T \bullet_t T' = B_d(T'\delta_t)$, so:

$$\phi(T \bullet_t T') = a_d \bullet (\phi(T')\delta_t = a_d \bullet_t \phi(T') = \phi(T) \bullet_t \phi(T').$$

Let us assume the result at all ranks $< |T|$. We put:

$$T = B_d\left(\prod_{i=1}^{k} T_i\delta_{t_i}\right).$$

By definition of the prelie product of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ in terms of graftings:

$$
\begin{aligned}
T\bullet T' &= B_d\left(\prod_{i=1}^k T_i\delta_{t_i}T'\delta_t\right)+\sum_{j=1}^k B_d\left(\prod_{i\neq j}T_i\delta_{t_i}(T_j\bullet_t T')\delta_{t_j}\right),\\
\phi(T\bullet T') &= a_d\bullet\prod_{i=1}^k\phi(T_i)\delta_{t_i}\phi(T')\delta_t+\sum_{j=1}^k a_d\bullet\prod_{i\neq j}\phi(T_i)\delta_{t_i}(\phi(T_j\bullet_t T'))\delta_{t_j}\\
&= a_d\bullet\prod_{i=1}^k\phi(T_i)\delta_{t_i}\phi(T')\delta_t+\sum_{j=1}^k a_d\bullet\prod_{i\neq j}\phi(T_i)\delta_{t_i}(\phi(T_j)\bullet_t\phi(T'))\delta_{t_j}\\
&= a_d\bullet\prod_{i=1}^k\phi(T_i)\delta_{t_i}\phi(T')\delta_t+a_d\bullet\left(\left(\prod_{i=1}^k\phi(T_i)\delta_{t_i}\right)\bullet\phi(T')\delta_t\right)\\
&= \left(a_d\bullet\prod_{i=1}^k\phi(T_i)\delta_{t_i}\right)\bullet\phi(T')\delta_t\\
&= \phi(T)\bullet_t\phi(T').
\end{aligned}
$$

So ϕ is a $\mathcal{T}$-multiple prelie algebra morphism. □

Remark 2. In other words, $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ is the free $\mathcal{T}$-multiple prelie algebra generated by $\mathcal{D}$.

2.3 Operad of typed trees

We now describe an operad of typed trees, in the category of species. We refer to [2, 12, 13] for notations and definitions on operads.

Notations 4. If $T\in\mathbb{T}_{\mathcal{T}}(A)$ and $a\in T$:

1. The subtrees formed by the connected components of the set of vertices, descendants of a (a excluded) are denoted by $T_1^{(a)},\ldots,T_{n_a}^{(a)}$. The type of the edge from a to the root of $T_i^{(a)}$ is denoted by t_i.
2. The tree formed by the vertices of T which are not in $T_1^{(a)},\ldots,T_{n_a}^{(a)}$, at the exception of a, is denoted by $T_0^{(a)}$.

Proposition 10. *For any nonempty finite set A, we denote by $\mathcal{P}_{\mathcal{T}}(A)$ the vector space generated by $\mathbb{T}_{\mathcal{T}}(A)$. We define a composition $\circ$ on $\mathcal{P}_{\mathcal{T}}$ in the following way: for any $T\in\mathbb{T}_{\mathcal{T}}(A)$, $T'\in\mathbb{T}_{\mathcal{T}}(B)$ and $a\in A$,*

$$T\circ_a T'=\sum_{v_1,\ldots,v_{n_a}\in V(T')}(\ldots((T_0^{(a)}\bullet_\lambda^{(t_0)}T')\bullet_{v_1}^{(t_1)}T_1^{(a)})\ldots)\bullet_{v_{n_a}}^{(t_{n_a}^{(a)})}T_{n_a}^{(a)}.$$

With this composition, $\mathcal{P}_{\mathcal{T}}$ is an operad in the category of species.

Proof. Note that the tree $(\ldots((T_0^{(a)}\bullet_\lambda^{(t_0)}T')\bullet_{v_1}^{(t_1)}T_1^{(a)})\ldots)\bullet_{v_{n_a}}^{(t_{n_a}^{(a)})}T_{n_a}$, which is shortly denote by $T\bullet_\lambda^{(v)}T'$, is obtained in the following process:

1. Delete the branches $T_1^{(a)},\ldots,T_{n_a}^{(a)}$ coming from a in T. One obtains a tree T'', and a is a leaf of T''.
2. Identify $a\in V(T'')$ with the root of T'.
3. Graft $T_1^{(a)}$ on v_1,…, $T_{n_a}^{(a)}$ on v_{n_a}.

This obviously does not depend on the choice of the indexation of $T_1^{(a)}, \ldots, T_{n_a}^{(a)}$.

Let $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$, $T'' \in \mathbb{T}_{\mathcal{T}}(C)$.

- If $a', a'' \in A$, with $a' \neq a''$, then:

$$\begin{aligned}(T \circ_{a'} T') \circ_{a''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a'}^{(v')} T') \bullet_{a''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a''}^{(v'')} T'') \bullet_{a'}^{(v')} T' \\ &= (T \circ_{a''} T'') \circ_{a'} T'.\end{aligned}$$

- If $a' \in A$ and $b'' \in B$, then:

$$\begin{aligned}(T \circ_{a'} T') \circ_{b''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} (T \bullet_{a'}^{(v')} T') \bullet_{b''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} T \bullet_{a'}^{(v')} (T' \bullet_{b''}^{(v'')} T'') \\ &= T \circ_{a'} (T' \circ_{b''} T'').\end{aligned}$$

Moreover, $\bullet^a \bullet_\lambda T = T$ for any tree T, and if $a \in V(T)$, $T \bullet_\lambda \bullet^a T$. So $\mathcal{P}_{\mathbb{T}}$ is indeed an operad in the category of species. □

Consequently, the family $(\mathcal{P}_{\mathcal{T}}(n))_{n \geqslant 0}$ is a "classical" operad, which we denote by $\mathcal{P}_{\mathcal{T}}$.

Example 4.

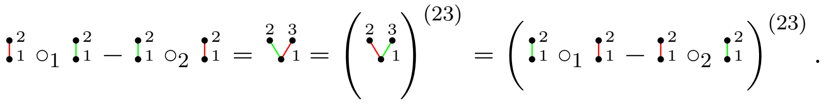
In the non-typed case, this theorem is proved in [6]:

Theorem 11. *The operad of $\mathcal{T}$-multiple prelie algebras is isomorphic to $\mathcal{P}_{\mathcal{T}}$, via the isomorphism Φ sending, for any $t \in \mathcal{T}$, $\bullet_t$ to the tree* *, where the edge is of type t.*

Proof. The operad of $\mathcal{T}$-multiple prelie algebras is generated by the binary elements $\bullet_t$, $t \in \mathcal{T}$, with the relations

$$\forall t, t' \in \mathcal{T}, \qquad \bullet_{t'} \circ_2 \bullet_t - \bullet_t \circ_1 \bullet_{t'} = (\bullet_t \circ_2 \bullet_{t'} - \bullet_{t'} \circ_1 \bullet_t)^{(23)}.$$

Firstly, if t and t' are elements of $\mathcal{T}$, symbolized by | and |, by the preceding example:



So the morphism ϕ exists. Let us prove that it is surjective: let $T \in \mathbb{T}_{\mathcal{T}}(n)$, we show that it belongs to $Im(\Phi)$ by induction on n. It is obvious if $n = 1$ or $n = 2$. Let us assume the result at all ranks $< n$. Up to a reindexation we assume that:

$$T = B_1(T_1 \delta_{t_1} \ldots T_k \delta_{t_k}),$$

where for any $1 \leqslant i < j \leqslant k$, if $x \in V(T_i)$ and $y \in V(T_j)$, then $x < y$. We denote by T_i' the standardization of T_i. By the induction hypothesis on n, $T_i' \in Im(\Phi)$ for all i. We proceed by induction on k. The type t_k will be represented in red. If $k = 1$, then:

$T =$  $\circ_2 T_1 \in Im(\Phi)$.

Let us assume the result at rank $k-1$. We put $T' = B_1(T_1\delta_{t_1}\dots T_{k-1}\delta_{t_{k-1}})$. By the induction hypothesis on n, $T' \in Im(\Phi)$. Then:

$$\overset{\bullet 2}{\underset{\bullet 1}{\vert}} \circ_1 T' = T + x,$$

where x is a sum of trees with n vertices, such that the fertility of the root is $k-1$. Hence, $x \in Im(\Phi)$, so $T \in Im(\Phi)$.

Let $\mathcal{D}$ be a set. The morphism ϕ implies that the free $\mathcal{P}_{\mathbb{T}}$-algebra generated by $\mathcal{D}$, that is to say $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, inherits a $\mathcal{T}$-multiple prelie algebra structure defined by:

$$\forall x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}, \qquad x \circ y = \overset{\bullet 2}{\underset{\bullet 1}{\vert}} \cdot (x \otimes y),$$

where $\cdot$ is the $\mathcal{P}_{\mathcal{T}}$-algebra structure of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$. For any trees T, T' in $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, by definition of the operadic composition of $\mathcal{P}_{\mathcal{T}}$:

$$T \circ_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T',$$

so $\circ_t = \bullet_t$ for any t. As $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, (\bullet_t)_{t\in\mathcal{T}})$ is the free $\mathcal{T}$-multiple prelie algebra generated by $\mathcal{D}$, Φ is an isomorphism. □

Remark 3. Let us assume that $\mathcal{T}$ is finite, of cardinality T. Then the components of $\mathcal{P}_{\mathcal{T}}$ are finite-dimensional. As the number of rooted trees which vertices are the elements of $[n]$ is n^{n-1}, for any $n \geqslant 0$ the dimension of $\mathcal{P}_{\mathcal{T}}(n)$ is $T^{n-1}n^{n-1}$, and the formal series of $\mathcal{P}_{\mathcal{T}}$ is:

$$f_T(X) = \sum_{n\geqslant 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}(n))}{n!} X^n = \sum_{n\geqslant 1} \frac{(Tn)^{n-1}}{n!} X^n = \frac{f_1(TX)}{T}.$$

2.4 Koszul dual operad

If $\mathcal{T}$ is finite, then $\mathcal{P}_{\mathcal{T}}$ is a quadratic operad. Its Koszul dual can be directly computed:

Proposition 12. *The Koszul dual operad $\mathcal{P}_{\mathcal{T}}^!$ of $\mathcal{P}_{\mathcal{T}}$ is generated by $\diamond_t$, $t \in \mathcal{T}$, with the relations:*

$$\forall t, t' \in \mathcal{T}, \qquad \diamond_{t'} \circ_1 \diamond_t = \diamond_t \circ_2 \diamond_{t'}, \qquad \diamond_{t'} \circ_1 \diamond_t = (\diamond_t \circ_1 \diamond_{t'})^{(23)}.$$

The algebras on $\mathcal{P}_{\mathcal{T}}^!$ are called $\mathcal{T}$-multiple permutative algebras: such an algebra A is given bilinear products $\diamond_t$, $t \in \mathcal{T}$, such that:

$$\forall x, y, z \in A, \qquad (x \diamond_t y) \diamond_{t'} z = x \diamond_t (y \diamond_{t'} z),$$
$$(x \diamond_t y) \diamond_{t'} z = (x \diamond_{t'} z) \diamond_t y.$$

In particular, for any t, $\diamond_t$ is a permutative product.

Of course, the definition of $\mathcal{T}$-multiple permutative algebras makes sense even if $\mathcal{T}$ is infinite. Permutative algebras are introduced in [5]. If A is a $\mathcal{T}$-multiple permutative algebra, then for any $(\lambda_t)_{t\in\mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, $\diamond_a = \sum \lambda_t \diamond_t$ is a a permutative product on A.

Proposition 13. *Let V be a vector space. Then $V \otimes S(V^{\oplus\mathcal{T}})$ is given a $\mathcal{T}$-multiple permutative algebra structure:*

$$\forall t \in \mathcal{T},\ v, v' \in V, w, w' \in S(V^{\oplus\mathcal{T}}), \qquad (v \otimes w) \diamond_t (v' \otimes w') = v \otimes ww'(v'\delta_t).$$

This $\mathcal{T}$-multiple permutative algebra is denoted by $P_{\mathcal{T}}(V)$. For any $\mathcal{T}$-multiple permutative algebra V and any linear map $\phi : V \longrightarrow A$, there exists a unique morphism $\Phi : P_{\mathcal{T}}(V) \longrightarrow A$ such that for any $v \in V$, $\Phi(v \otimes 1) = \phi(v)$.

Proof. Let $t,\, t' \in \mathcal{T},\, v,\, v,\, v'' \in V,\, w,\, w',\, w'' \in S(V^{\oplus \mathcal{T}})$.

$$\begin{aligned}
&(v \otimes w \diamond_t v' \otimes w') \diamond_{t'} v'' \otimes w'' \\
&= v \otimes w \diamond_t (v' \otimes w' \diamond_{t'} v'' \otimes w'') \\
&= (v \otimes w \diamond_t v' \diamond_{t'} v'' \otimes w'') \otimes w' \\
&= v \otimes w w' w'' (v' \delta_t)(v'' \delta_{t'}),
\end{aligned}$$

so $P_{\mathcal{T}}(V)$ is $\mathcal{T}$-multiple permutative.

Existence of Φ. Let $t_1, \ldots, t_k \in \mathcal{T},\, v, v_1, \ldots, v_k \in V$. We inductively define $\Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_k \delta_{t_k}))$ by:

$$\begin{aligned}
\Phi(v \otimes 1) &= \phi(v), \\
\Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_k \delta_{t_k})) &= \Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_{k-1} \delta_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \text{ if } k \geqslant 1.
\end{aligned}$$

Let us prove that this does not depend on the order chosen on the factors $v_i \delta_{t_i}$ by induction on k. If $k = 0$ or 1, there is nothing to prove. Otherwise, if $i < k$:

$$\begin{aligned}
&\Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_{i-1} \delta_{t_{i-1}})(v_{i+1} \delta_{t_{i+1}}) \ldots (v_k \delta_{t_k})) \diamond_{t_i} \phi(v_i) \\
&= (\Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_{i-1} \delta_{t_{i-1}})(v_{i+1} \delta_{t_{i+1}}) \ldots (v_{k-1} \delta_{t_{k-1}})) \diamond_{t_k} \phi(v_k)) \diamond_{t_i} \phi(v_i) \\
&= (\Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_{i-1} \delta_{t_{i-1}})(v_{i+1} \delta_{t_{i+1}}) \ldots (v_{k-1} \delta_{t_{k-1}})) \diamond_{t_i} \phi(v_i)) \diamond_{t_k} \phi(v_k) \\
&= \Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_{k-1} \delta_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \\
&= \Phi(v \otimes (v_1 \delta_{t_1}) \ldots (v_k \delta_{t_k})).
\end{aligned}$$

So Φ is well-defined. Let us prove that Φ is a $\mathcal{T}$-multiple permutative algebra morphism. Let $v, v' \in V,\, w,\, w' = (v_1 \delta_{t_1}) \ldots (v_k \delta_{t_k}) \in S(V^{\oplus \mathcal{T}})$, and $t \in \mathcal{T}$. Let us prove that $\Phi(v \otimes w \diamond_t v' \otimes w'') = \Phi(v \otimes w) \diamond_t \Phi(v' \otimes w')$ by induction on k. If $k = 0$:

$$\begin{aligned}
\Phi(v \otimes w \diamond_t v' \otimes 1) &= \Phi(v \otimes w(v' \delta_t)) \\
&= \Phi(v \otimes w) \diamond_t \phi(v') \\
&= \Phi(v \otimes w) \diamond_t \Phi(v' \otimes 1).
\end{aligned}$$

Otherwise, we put $w'' = (v_1 \delta_{t_1}) \ldots (v_{k-1} \delta_{t_{k-1}})$. Then:

$$\begin{aligned}
\Phi(v \otimes w \diamond_t v' \otimes w') &= \Phi(v \otimes w w'' (v' \delta_t)(v_k \delta_{t_k})) \\
&= \Phi(v \otimes w w'' (v' \delta_t)) \diamond_{t_k} \phi(v_k) \\
&= \Phi(v \otimes w \diamond_t v' \otimes w'') \diamond_{t_k} \phi(v_k) \\
&= (\Phi(v \otimes w) \diamond_t \Phi(v' \otimes w'')) \diamond_{t_k} \phi(v_k) \\
&= \Phi(v \otimes w) \diamond_t (\Phi(v' \otimes w'') \diamond_{t_k} \phi(v_k)) \\
&= \Phi(v \otimes w') \diamond_t \Phi(v' \otimes w').
\end{aligned}$$

So Φ is a $\mathcal{T}$-multiple permutative algebra morphism.

Unicity. For any $v, v_1, \ldots, v_k \in V,\, t_1, \ldots, t_k \in \mathcal{T}$:

$$v \otimes (v_1 \delta_{t_1}) \ldots (v_k \delta_{t_k}) = (v \otimes (v_1 \delta_{t_1}) \ldots (v_{k-1} \delta_{t_{k-1}})) \diamond_{t_k} v_k.$$

It is then easy to prove that $P_{\mathcal{T}}(V)$ is generated by $V \otimes 1$ as a $\mathcal{T}$-multiple permutative algebra. Consequently, Φ is unique. □

Remark 4. We proved that $P_{\mathcal{T}}(V)$ is freely generated by V, identified with $V \otimes 1$. Consequently, $\mathcal{P}^!_{\mathcal{T}}(n)$ has the same dimension as the multilinear component of $V \otimes S(V^{\oplus \mathcal{T}})$ with $V = Vect(X_1, \ldots, X_n)$, that is to say:

$$Vect(X_i \otimes (X_1 \delta_{t_1}) \ldots (X_{i-1} \delta_{t_{i-1}})(X_{i+1} \delta_{t_{i+1}}) \ldots (X_n \delta_{t_n}),\; 1 \leqslant i \leqslant n, t_j \in \mathcal{T}),$$

so:

$$\dim(\mathcal{P}^!_{\mathcal{T}}(n)) = nT^{n-1}.$$

The formal series of $\mathcal{P}^!_{\mathcal{T}}$ is:

$$f^!_T(X) = \sum_{n\geqslant 1} \frac{\dim(\mathcal{P}^!_{\mathcal{T}}(n))}{n!} X^n = \sum_{n\geqslant 1} \frac{T^{n-1}}{(n-1)!} X^n = X\exp(TX) = \frac{f^!_1(TX)}{T}.$$

It is possible to prove that $\mathcal{P}^!_{\mathcal{T}}$ is a Koszul operad (and, hence, $\mathcal{P}_{\mathcal{T}}$ too) using the rewriting method of [12].

3 Structure of the prelie products

3.1 A nonassociative permutative coproduct

Proposition 14. *For all $t \in \mathcal{T}$, we define a coproduct $\rho_t : \mathfrak{g}_{\mathcal{D},\mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}^{\otimes 2}$ by:*

$$\forall T = B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}\right) \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \qquad \rho_t(T) = \sum_{j\in[k]} B_d\left(\prod_{i\in[k],\, i\neq j} T_i\delta_{t_i}\right) \otimes T_j\delta_{t,t_j}.$$

Then:

1. *For all $t, t' \in \mathcal{T}$, $(\rho_t \otimes Id) \circ \rho_{t'} = ((\rho_{t'} \otimes Id) \circ \rho_t)^{(23)}$.*
2. *For any $x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}$, $t, t' \in \mathcal{T}$, with Sweedler's notations $\rho_t(x) = \sum x^{(1)_t} \otimes x^{(2)_t}$,*

$$\rho_t(x \bullet_{t'} y) = \delta_{t,t'} x \otimes y + \sum x^{(1)_t} \bullet_{t'} y \otimes x^{(2)_t} + \sum x^{(1)_t} \otimes x^{(2)_t} \bullet_{t'} y.$$

3. *For any $\mu = (\mu_t)_{t\in\mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, we put:*

$$\rho_\mu = \sum_{t\in\mathcal{T}} \mu_t \rho_t : \mathfrak{g}_{\mathcal{D},\mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}^{\otimes 2}.$$

This makes sense, as any tree in $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ does not vanish only under a finite number of ρ_t. Then ρ_μ is a nonassociative permutative (NAP) coproduct; for any $x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}$, by the second point, using Sweeder's notation for ρ_μ:

$$\rho_\mu(x \bullet_\lambda y) = \left(\sum_{t\in\mathcal{T}} \lambda_t\mu_t\right) x \otimes y + \sum x^{(1)_\mu} \bullet_\lambda \otimes x^{(2)_\mu} + \sum x^{(1)_\mu} \otimes x^{(2)_\mu} \bullet_b .$$

In particular, if $\sum_{t\in\mathcal{T}} \lambda_t\mu_t = 1$, $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda, \rho_\mu)$ is a NAP prelie bialgebra in the sense of [11].

Proof. 1. For any tree T:

$$(\rho_t \otimes Id) \circ \rho_{t'}(T) = \sum_{p,q\in[k],p\neq q} B_d\left(\prod_{i\in[k],i\neq p,q} T_i\delta_{t_i}\right) \otimes T_p\delta_{t_p,t} \otimes T_q\delta_{t_q,t'},$$

which implies the result.

2. For any tree T, T':

$$\begin{aligned}
\rho_t(T \bullet_{t'} T') &= \rho_t\left(B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}T'\delta_{t'}\right) + \sum_{i\in[k]} B_d\left(\prod_{j\in[k],j\neq i} T_j\delta_{t_j}(T_i \bullet_{t'} T')\delta_{t_i}\right)\right)\\
&= B_d\left(\prod_{i\in[k]} T_i\delta_{t_i}\right)\otimes T'\delta_{t,t'} + \sum_{i\in[k]} B_d\left(\prod_{j\in[k],j\neq i} T_j\delta_{t_j}T'\delta_{t'}\right)\otimes T_i\delta_{t_i,t}\\
&+ \sum_{i\in[k]} B_d\left(\prod_{j\in[k],j\neq i} T_j\delta_{t_j}\right)\otimes (T_i \bullet_{t'} T')\delta_{t_i,t'}\\
&+ \sum_{i\neq j\in[k]} B_d\left(\prod_{p\in[k],p\neq i,j} T_p\delta_{t_p}(T_j \bullet_{t'} T')\delta_{t_j}\right)\otimes T_i\delta_{t_i,t}\\
&= T\otimes T'\delta_{t,t'} + \sum_{i\in[k]} B_d\left(\prod_{j\in[k],j\neq i} T_j\delta_{t_j}\right)\bullet_{t'} T'\otimes T_i\delta_{t_i,t}\\
&+ \sum_{i\in[k]} B_d\left(\prod_{j\in[k],j\neq i} T_j\delta_{t_j}\right)\otimes T_i \bullet_{t'} T'\delta_{t_i,t}\\
&= T\otimes T'\delta_{t,t'} + T^{(1)_t} \bullet_{t'} T' \otimes T^{(2)_t} + T^{(1)_t}\otimes T^{(2)_t} \bullet_{t'} T'.
\end{aligned}$$

3. Obtained by summation. □

Corollary 15. *If $\lambda \in \mathbb{K}^{(\mathcal{T})}$ is nonzero, let us choose $t_0 \in \mathcal{T}$ such that $\lambda_{t_0} \neq 0$. The prelie algebra $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is freely generated by the set $\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$ of $\mathcal{T}$-typed $\mathcal{D}$-decorated trees T such that there is no edge outgoing the root of T of type t_0.*

Proof. For any tree T, we denote by α_T the number of edges outgoing the root of T of type T_0. Our aim is to prove that $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is freely generated by the trees T such that $\alpha_T = 0$. We define a family of scalar b by:

$$\forall t \in \mathcal{T}, \qquad \mu_t = \begin{cases} 0 \text{ if } t \neq t_0,\\ \dfrac{1}{\lambda_{t_0}} \text{ if } t = t_0.\end{cases}$$

Note that $\rho_\mu = \dfrac{1}{\lambda_{t_0}}\rho_{t_0}$. By proposition 14, $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda, \rho_\mu)$ is a NAP prelie bialgebra, so by Livernet's rigidity theorem [11], it is freely generated by $Ker(\rho_\mu) = Ker(\rho_{t_0})$. Obviously, if $\alpha_T = 0$, $T \in Ker(\rho_{t_0})$. Let us consider $x = \sum_{T\in\mathbb{T}_{\mathcal{D},\mathcal{T}}} x_T T \in Ker(\rho_{t_0})$. We consider the map:

$$\Upsilon : \begin{cases} \mathfrak{g}_{\mathcal{D},\mathcal{T}} \otimes \mathfrak{g}_{\mathcal{D},\mathcal{T}} & \longrightarrow & \mathfrak{g}_{\mathcal{D},\mathcal{T}}\\ T\otimes T' & \longrightarrow & T \bullet^{\mathrm{root}(T)}_{t_0} T'.\end{cases}$$

By definition of ρ_{t_0}, for any tree T, $\Upsilon \circ \rho_{t_0}(T) = \alpha_T T$. Consequently:

$$0 = \Upsilon \circ \rho_{t_0}(x) = \sum_{T\in\mathbb{T}_{\mathcal{D},\mathcal{T}}} x_T \alpha_T T.$$

So if $\alpha_T \neq 0$, $x_T = 0$, and x is a linear span of trees such that $\alpha_T = 0$: the set of trees T such that $\alpha_T = 0$ is a basis of $Ker(\rho_{t_0})$. □

If $|\mathcal{D}| = D$ and $|\mathcal{T}| = T$, the number of elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ of degree n is denoted by $t'_{D,T}(n)$; it does not depend on t_0. By direct computations:

$$\begin{aligned}
t'_{D,T}(1) &= D,\\
t'_{D,T}(2) &= D^2(T-1),\\
t'_{D,T}(3) &= \frac{D^2(T-1)(3DT-D+1)}{2},\\
t'_{D,T}(4) &= \frac{D^2(T-1)(16D^2T^2-8D^2T+D^2+6DT-3D+2)}{6}.
\end{aligned}$$

In the particular case $D = 1$, $T = 2$, one recovers sequence A005750 of the OEIS.

3.2 Prelie algebra morphisms

Notations 5. Let $\mathcal{T}$ and $\mathcal{T}'$ be two sets of types. We denote by $\mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ the space of matrices $M = (m_{t,t'})_{(t,t')\in\mathcal{T}\times\mathcal{T}'}$, such that for any $t' \in \mathcal{T}'$, $(m_{t,t'})_{t\in\mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$. If $\mathcal{T} = \mathcal{T}'$, we shall simply write $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$. If $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ and $M' \in M_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, then:

$$MM' = \left(\sum_{t'\in\mathcal{T}'} m_{t,t'}m'_{t',t''}\right)_{(t,t'')\in\mathcal{T}\times\mathcal{T}''} \in \mathcal{M}_{\mathcal{T},\mathcal{T}''}(\mathbb{K}).$$

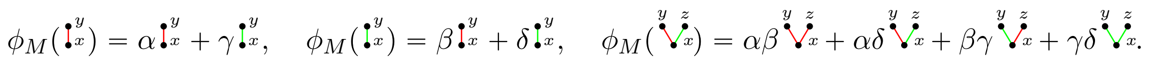
If $\lambda \in \mathbb{K}^{(\mathcal{T}')}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, then:

$$M\lambda = \left(\sum_{t'\in\mathcal{T}'} m_{t,t'}\lambda_{t'}\right)_{t\in\mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}, \qquad M^\top\mu = \left(\sum_{t\in\mathcal{T}} m_{t,t'}\mu_t\right)_{t'\in\mathcal{T}'} \in \mathbb{K}^{\mathcal{T}'}.$$

In particular, $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$ is an algebra, acting on $\mathbb{K}^{(\mathcal{T})}$ on the left and on $\mathbb{K}^{\mathcal{T}}$ on the right.

Definition 16. *Let $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. We define a map $\Phi_M : \mathcal{H}_{\mathcal{D},\mathcal{T}'} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}$, sending $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$ to the forest obtained by replacing* type(e) *by $\sum_{t\in\mathcal{T}} m_{t,\mathrm{type}(e)}t$ for any $e \in E(F)$, F being considered as linear in any of its edges. The restriction of Φ_M to $\mathfrak{g}_{\mathcal{D},\mathcal{T}'}$ is denoted by $\phi_M : \mathfrak{g}_{\mathcal{D},\mathcal{T}'} \longrightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}$.*

Example 5. If $\mathcal{T}$ contains two elements, the first one being represented in red and the second one in green, if $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, for any $x, y, z \in \mathcal{D}$:



Remark 5. For any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, $\Phi_M \circ \Phi_{M'} = \Phi_{MM'}$.

Proposition 17. *Let $\lambda \in \mathbb{K}^{(\mathcal{T})}$, $\mu \in \mathbb{K}^{\mathcal{T}}$ and $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. Then ϕ_M is a prelie morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$ and a NAP coalgebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \rho_{M^\top\mu})$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \rho_\mu)$.*

Proof. Let $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$. For any $t \in \mathcal{T}$, for any $v \in V(T)$:

$$\phi_M(T \bullet_t^{(v)} T') = \sum_{t'\in\mathcal{T}} m_{t',t}\phi_M(T) \bullet_{t'} \phi_M(T'),$$

so:

$$\phi_M(T \bullet_\lambda T') = \sum_{t,t'\in\mathcal{T}} m_{t',t}\lambda_t\phi_M(T) \bullet_{t'} \phi_M(T') = \phi_M(T) \bullet_{M\lambda} \phi_M(T').$$

We proved that ϕ_M is a prelie algebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$.

For any $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$:

$$\rho_t \circ \phi_M(T) = \sum_{t' \in \mathcal{T}} m_{t,t'}(\phi_M \otimes \phi_M) \circ \rho_{t'}(T),$$

so:

$$\rho_\mu \circ \phi_M(T) = \sum_{t,t' \in \mathcal{T}} m_{t,t'}\mu_t(\phi_M \otimes \phi_M) \circ \rho_{t'}(T) = (\phi_M \otimes \phi_M) \circ \rho_{M^\top \mu}(T).$$

So $\phi_M : (\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \rho_\mu)$ is a NAP coalgebra morphism. □

Corollary 18. *For any $\lambda \in\in \mathbb{K}^{(\mathcal{T})}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, such that $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, for any $t_0 \in \mathcal{T}$, the NAP prelie bialgebras $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda, \rho_\mu)$ and $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{t_0}, \rho_{t_0})$ are isomorphic.*

Proof. Let us denote by $\lambda^{(0)}$ the element of $\mathbb{K}^{(\mathcal{T})}$ defined by:

$$\lambda_t^{(0)} = \delta_{t,t_0}.$$

Note that for any $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, $\phi_M : (\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{\lambda^{(0)}}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda^{(0)}}, \rho_\mu)$ is an isomorphism. In particular, for a well-chosen M, $M\lambda^{(0)} = \lambda$; we can assume that $\lambda = \lambda^{(0)}$ without loss of generality. Then, by hypothesis, $\mu_{t_0} = 1$. We define a matrix $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$ in the following way:

$$m_{t,t'} = \begin{cases} \delta_{t,t_0} \text{ if } t' = t_0, \\ \delta_{t,t'} - \mu_{t'}\delta_{t,t_0} \text{ otherwise.} \end{cases}$$

Then M is invertible. Moreover, $M\lambda^{(0)} = \lambda^{(0)}$ and $M^\top \mu = \lambda^{(0)}$. So ϕ_M is an isomorphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{\lambda^{(0)}}, \rho_{\lambda^{(0)}})$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda, \rho_\mu)$. □

Proposition 19. *Let $\lambda \in \mathbb{K}^{(\mathcal{T})}$, and $t_0 \in \mathcal{T}$. We define a prelie algebra morphism ψ_{t_0} : $(\mathfrak{g}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}, \bullet) \longrightarrow (\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$, sending $\bullet_T$ to T for any $T \in \mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$. Then ψ_{t_0} is a prelie algebra isomorphism if, and only if, $\lambda_{t_0} \neq 0$.*

Proof. If $\lambda_{t_0} \neq 0$, then by corollary 15, $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is freely generated by $\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$, so ψ_{t_0} is an isomorphism. If $\lambda_{t_0} = 0$, then it is not difficult to show that any tree T with two vertices, with its unique edge of type t_0, does not belong to $Im(\psi_{t_0})$. □

4 Hopf algebraic structures

We here fix a family $\lambda \in \mathbb{K}^{(\mathcal{T})}$.

4.1 Enveloping algebra of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$

Using again the Guin-Oudom construction, we obtain the enveloping algebra of $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$. We first identify the symmetric coalgebra $S(\mathfrak{g}_{\mathcal{D},\mathcal{T}})$ with the vector space generated by $\mathbb{F}_{\mathcal{D},\mathcal{T}}$, which we denote by $\mathcal{H}_{\mathcal{D},\mathcal{T}}$. Its product m is given by disjoint union of forests, its coproduct by:

$$\forall T_1, \ldots, T_k \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \qquad \Delta(T_1 \ldots T_n) = \sum_{I \subseteq [n]} \prod_{i \in I} T_i \otimes \prod_{i \notin I} T_i.$$

We denote by $\bullet_\lambda$ the Guin-Oudom extension of $\bullet_\lambda$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ and $\star_\lambda$ the associated associative product.

Theorem 20. *For any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, $T_1, \ldots, T_n \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$:*

$$F \bullet_\lambda T_1 \ldots T_n = \sum_{\substack{v_1,\ldots,v_n \in V(F),\\ t_1,\ldots,t_n \in \mathcal{T}}} \left(\prod_{i\in[n]} \lambda_{t_i}\right) (\ldots (F \bullet_{t_1}^{(v_1)} T_1) \ldots) \bullet_{t_n}^{(v_n)} T_n,$$

$$F \star_\lambda T_1 \ldots T_n = \sum_{I \subseteq [n]} \left(F \bullet_\lambda \prod_{i\in I} T_i\right) \prod_{i \notin I} T_i.$$

The Hopf algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, \star_\lambda, \Delta)$ is denoted by $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}}$. Moreover, for any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, for any $\lambda \in \mathbb{K}^{(\mathcal{T}')}$, Φ_M is a Hopf algebra morphism from $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}'}$ to $\mathcal{H}^{GL_{M\lambda}}_{\mathcal{D},\mathcal{T}}$. The extension of ψ_{t_0} as a Hopf algebra morphism from $\mathcal{H}^{GL}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}$ to $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}}$ is denoted by Ψ_{t_0}; it is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

In particular, if $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the Grossman-Larson Hopf algebra [9].

4.2 Dual construction

Proposition 21. *Let $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.*

1. *A cut c of T is a nonempty subset of $E(T)$; it is said to be admissible if any path in the tree from the root to a leaf meets at most one edge in c. The set of admissible cuts of T is denoted by $Adm(T)$.*
2. *If c is admissible, one of the connected components of $T\backslash c$ contains the root of c: we denote it by $R^c(T)$. The product of the other connected components of $T\backslash c$ is denoted by $P^c(T)$.*

Let $\lambda \in \mathbb{K}^{\mathcal{T}}$. We define a multiplicative coproduct Δ^{CK_λ} on the algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m)$ by:

$$\forall T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \qquad \Delta^{CK_\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in Adm(T)} \left(\prod_{e\in c} \lambda_{\mathrm{type}(e)}\right) R^c(T) \otimes P^c(T).$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \Delta^{CK_\lambda})$ is a Hopf algebra, which we denote by $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$.

Proof. We first assume that $\lambda \in \mathbb{K}^{(\mathcal{T})}$. Let us define a nondegenerate pairing $\langle -, - \rangle$ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by:

$$\forall F, F' \in \mathbb{F}_{\mathcal{D},\mathcal{T}}, \qquad \langle F, F' \rangle = \delta_{F,F'} s_F,$$

where s_F is the number of symmetries of F. Let us consider three forests F, F', F''. We put:

$$F = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\lambda_t}, \qquad F' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a'_T}, \qquad F'' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a''_T}.$$

Then:

$$\begin{aligned}
\langle \Delta(F), F' \otimes F'' \rangle &= \sum_{a=b+c} \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} \frac{\lambda_t!}{\mu_t! c_T!} \langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\mu_t}, F' \rangle \langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{c_T}, F'' \rangle \\
&= \sum_{a=b+c} \delta_{b,a'} \delta_{c,a''} \frac{\lambda_t!}{a'_T! a''_T!} s_{F'} s_{F''} \\
&= \delta_{a,a'+a''} \frac{\lambda_t!}{a'_T! a''_T!} a'_T! a''_T! s_T^{a'_T + a''_T} \\
&= \delta_{a,a'+a''} \lambda_t! s_T^{\lambda_t} \\
&= \delta_{F,F'F''} s_F \\
&= \langle F, F'F'' \rangle.
\end{aligned}$$

Therefore:

$$\forall x, y, z \in \mathcal{H}_{\mathcal{D},\mathcal{T}}, \qquad \langle \Delta(x), y \otimes z \rangle = \langle x, yz \rangle.$$

Let F, G be two forests and T be a tree. Observe that if F is a forest with at least two trees, then $F \star_\lambda G$ does not contain any tree, so $\langle F \star_\lambda G, T \rangle = 0$. If $F = 1$, then $\langle F \star_\lambda G, T \rangle \neq 0$ if, and only if, $G = T$; moreover, $\langle 1 \star_\lambda T, T \rangle = 1$. If F is a tree, then:

$$\langle F \star_\lambda G, T \rangle = \langle F \bullet_\lambda G, T \rangle.$$

Moreover, if $F = B_d(F')$ and $G = T_1 \dots T_k$:

$$F \bullet_\lambda G = \sum_{I \subseteq [k]} \sum_{(t_i) \in \mathcal{T}^k} \left(\prod_{i \in [k]} \lambda_{t_i} \right) B_d \left(\prod_{i \in I} T_i \delta_{t_i} F' \bullet \prod_{i \notin I} T_i \delta_{t_i} \right),$$

where $\bullet$ is the prelie product on $\mathfrak{g}_{\mathcal{D},\mathcal{T}}^{\mathcal{T}}$ induced by the $\mathcal{T}$-multiplie prelie structure. Consequently, we can inductively define a coproduct $\Delta^{CK_\lambda} : \mathcal{H}_{\mathcal{D},\mathcal{T}} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}} \otimes \mathcal{H}_{\mathcal{D},\mathcal{T}}$, multiplicative for the product m, such that, if we denote for any tree T, $\overline{\Delta}_{CK}(T) = \Delta(T) - 1 \otimes T$, for any tree $T = B_d(T_1 \delta_{t_1} \dots T_k \delta_{t_k})$:

$$\overline{\Delta}_\lambda^{CK}(T) = (B_d \otimes Id) \left(\prod_{i \in [k]} (\overline{\Delta}_\lambda^{CK}(T_i) \delta_{t_i} \otimes 1 + \lambda_{t_i} 1 \otimes T_i) \right). \tag{3}$$

Then, for any $x, y, z \in \mathcal{H}_{\mathcal{D},\mathcal{T}}$:

$$\langle x \star_\lambda y, z \rangle = \langle x \otimes y, \Delta^{CK_\lambda}(z) \rangle.$$

A quite easy induction on the number of vertices of trees proves that this coproduct is indeed the one we define in the statement of the proposition. As $\langle -, - \rangle$ is nondegenerate, $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \Delta^{CK_\lambda})$ is a Hopf algebra, dual to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{GL_\lambda}$.

In the general case, for any $x \in \mathcal{H}_{\mathcal{D},\mathcal{T}}$, there exists a finite subset $\mathcal{T}'$ of $\mathcal{T}$ such that $x \in \mathcal{H}_{\mathcal{D},\mathcal{T}'}$. Putting $\lambda' = \lambda_{|\mathcal{T}'}$, $\lambda' \in \mathbb{K}^{\mathcal{T}'} = \mathbb{K}^{(\mathcal{T}')}$, so:

$$(\Delta^{CK_\lambda} \otimes Id) \circ \Delta^{CK_\lambda}(x) = (\Delta^{CK_{\lambda'}} \otimes Id) \circ \Delta^{CK_{\lambda'}}(x) = (Id \otimes \Delta^{CK_{\lambda'}}) \circ \Delta^{CK_{\lambda'}}(x) = (Id \otimes \Delta^{CK_\lambda}) \circ \Delta^{CK_\lambda}(x).$$

Hence, Δ_λ is coassociative, and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\lambda}$ is a Hopf algebra. □

Example 6. Let us fix a subset $\mathcal{T}'$ of $\mathcal{T}$ and choose $(\lambda_t)_{t \in \mathcal{T}}$ such that:

$$\lambda_t = \begin{cases} 1 \text{ if } t \in \mathcal{T}', \\ 0 \text{ otherwise.} \end{cases}$$

For any tree $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, let us denote by $Adm_{\mathcal{T}'}(T)$ the set of admissible cuts c of T such that the type of any edge in c belongs to $\mathcal{T}'$. Then:

$$\Delta^{CK_\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in Adm_{\mathcal{T}'}(T)} R^c(T) \otimes P^c(T).$$

Remark 6.
1. If $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the usual Connes-Kreimer Hopf algebra of $\mathcal{D}$-decorated rooted trees, which we denote by $\mathcal{H}_{\mathcal{D}}^{CK}$, and its duality with the Grossman-Larson Hopf algebra [7, 10, 17].

2. If $\mathcal{T}$ and $\mathcal{D}$ are finite, for any $\lambda \in \mathbb{K}^{\mathcal{T}}$, both $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{GL_\lambda}$ are graded Hopf algebra (by the number of vertices), and their homogeneous components are finite-dimensional. Via the pairing $\langle -, - \rangle$, each one is the graded dual of the other.

4.3 Hochschild cohomology of coalgebras

For the sake of simplicity, we assume that the set of types $\mathcal{T}$ is finite and we put $\mathcal{T} = \{t_1, \ldots, t_N\}$.

Let (C, Δ) be a coalgebra and let (M, δ_L, δ_R) be a C-bicomodule. One defines a complex, dual to the Hochschild complex for algebras, in the following way:

1. For any $n \geqslant 0$, $H_n = \mathcal{L}(M, C^{\otimes n})$.
2. For any $L \in H_n$:

$$b_n(L) = (Id \otimes L) \circ \delta_L + \sum_{i=1}^{n} (-1)^i (Id^{\otimes(i-1)} \otimes \Delta \otimes Id^{\otimes(n-i)}) \circ L + (-1)^{n+1} (L \otimes Id) \circ \delta_R.$$

In particular, one-cocycles are maps $L : M \longrightarrow C$ such that:

$$\Delta \circ L = (Id \otimes L) \circ \delta_L + (L \otimes Id) \circ \delta_R.$$

We shall consider in particular the bicomodule (M, δ_L, δ_R) such that:

$$\forall x \in C, \qquad \begin{cases} \delta_L(x) = 1 \otimes x, \\ \delta_R(x) = \Delta(x). \end{cases}$$

If C is a bialgebra, then $M^{\otimes N}$ is also a bicomodule:

$$\forall x_t \in C, \qquad \begin{cases} \delta_L \left(\bigotimes_{1 \leqslant i \leqslant N} x_i \right) = 1 \otimes \bigotimes_{1 \leqslant i \leqslant N} x_i, \\ \delta_R \left(\bigotimes_{1 \leqslant i \leqslant N} x_i \right) = \bigotimes_{1 \leqslant i \leqslant N} x_i^{(1)} \otimes \prod_{1 \leqslant i \leqslant N} x_i^{(2)}. \end{cases}$$

We denote by $\underline{1} = (1)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, and we take $C = \mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_{\underline{1}}}$. One can identify $S(Vect(\mathbb{T}_{\mathcal{D},\mathcal{T}})^{\oplus \mathcal{T}})$ and $C^{\otimes N}$, $x\delta_{T_i}$ being identified with $1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ for any $x \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $1 \leqslant i \leqslant N$. Then for any d, $B_d : C^{\otimes N} \longrightarrow C$ is a 1-cocycle. Moreover, there is a universal property, proved in the same way as for the Connes-Kreimer's one [7]:

Theorem 22. *Let B be a commutative bialgebra and, for any $d \in \mathcal{D}$, let $L_d : C^{\otimes N} \longrightarrow C$ be a 1-cocycle:*

$$\forall d \in \mathcal{D},\ \forall x_t \in B, \qquad \Delta \circ L_d \left(\bigotimes_{1 \leqslant i \leqslant N} x_i \right) = 1 \otimes \bigotimes_{1 \leqslant i \leqslant N} x_i + L_d \left(\bigotimes_{1 \leqslant i \leqslant N} x_i^{(1)} \right) \otimes \prod_{1 \leqslant i \leqslant N} x_i^{(2)}.$$

There exists a unique bialgebra morphism $\phi : \mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_{\underline{1}}} \longrightarrow C$ such that for any $d \in \mathcal{D}$, $\phi \circ L_d = B_d \circ \phi^{\otimes N}$.

4.4 Hopf algebra morphisms

Our aim is, firstly, to construct Hopf algebras morphisms between $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\mu}$; secondly, to construct Hopf algebra isomorphisms between $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\lambda}$ and $\mathcal{H}_{\mathcal{D}'}^{CK}$ for a well-chosen $\mathcal{D}'$.

Proposition 23. *Let $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, $\lambda \in \mathbb{K}^{\mathcal{T}}$. Then $\Phi_M : \mathcal{H}_{\mathcal{D},\mathcal{T}'}^{CK_{M^\top \lambda}} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}^{CK_\lambda}$ is a Hopf algebra morphism.*

Proof. Φ_M is a obviously an algebra morphism. Let $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.

$$\begin{aligned}\Delta_\lambda \circ \Phi_M(T) &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T)\\ &+ \sum_{c \in Adm(T)} \prod_{e \in c} \left(\sum_{t \in \mathcal{T}} m_{t,\mathrm{type}(e)} \lambda_t \right) \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T))\\ &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T)\\ &+ \sum_{c \in Adm(T)} \prod_{e \in c} (M^\top \lambda)_{\mathrm{type}(e)} \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T))\\ &= (\Phi_M \otimes \Phi_M) \circ \Delta_{M^\top a}(T).\end{aligned}$$

So Φ_M is a coalgebra morphism from $\mathcal{H}^{CK_{M^\top \lambda}}_{\mathcal{D},\mathcal{T}'}$ to $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$. □

Corollary 24. *Let $\lambda, \mu \in \mathbb{K}^{\mathcal{T}}$, both nonzero. Then $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ and $\mathcal{H}^{CK_\mu}_{\mathcal{D},\mathcal{T}}$ are isomorphic Hopf algebras.*

Proof. There exists $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, such that $M^\top \lambda = \mu$. Then Φ_M is an isomorphism between $\mathcal{H}^{CK_\mu}_{\mathcal{D},\mathcal{T}}$ and $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$. □

Definition 25. *Let us fix $t_0 \in \mathcal{T}$. For any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, we shall say that $\{T_1, \ldots, T_k\} \lhd_{t_0} F$ if the following conditions hold:*

- *$\{T_1, \ldots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, by restriction.*
- *For any $i \in [k]$, $T_i \in \mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$.*

If $\{T_1, \ldots, T_k\} \lhd_{t_0} F$, we denote by $F/\{T_1, \ldots, T_k\}$ the forest obtained by contracting T_i to a single vertex for any $i \in [k]$, decorating this vertex by T_i, and forgetting the type of the remaining edges. Then $F/\{T_1, \ldots, T_k\}$ is a $\mathcal{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$-decorated forest.

Proposition 26. *Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, $t_0 \in \mathcal{T}$. Let us consider the map:*

$$\Psi^*_{t_0} : \left\{ \begin{array}{rcl} \mathcal{H}_{\mathcal{D},\mathcal{T}} & \longrightarrow & \mathcal{H}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}} \\ F \in \mathbb{F}_{\mathcal{D},\mathcal{T}} & \longrightarrow & \displaystyle\sum_{\{T_1,\ldots,T_k\} \lhd_{t_0} F} \left(\prod_{e \in E(F) \setminus \sqcup E(T_i)} \lambda_{\mathrm{type}(e)} \right) F/\{T_1, \ldots, T_k\}. \end{array} \right.$$

Then $\Psi^*_{t_0}$ is a Hopf algebra morphism from $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ to $\mathcal{H}^{CK}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}$. It is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

Proof. First case. We first assume that $\mathcal{D}$ and $\mathcal{T}$ are finite. In this case, $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ is the graded dual of $\mathcal{H}^{GL_\lambda}_{\mathcal{D},\mathcal{T}}$, with the Hopf pairing $\langle -, - \rangle$; grading $\mathcal{H}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}$ by the number of vertices of the decorations, $\mathcal{H}^{CK}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}$ is the graded dual of $\mathcal{H}^{GL}_{\mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}}$. Moreover, $\Psi^*_{t_0}$ is the transpose of Ψ_{t_0} of proposition 19, so is a Hopf algebra morphism. If $\lambda_{t_0} \neq 0$, Ψ_{t_0} is an isomorphism, so $\Psi^*_{t_0}$ also is.

General case. Let $x, y \in \mathcal{H}_{\mathcal{D},\mathcal{T}}$. There exist finite $\mathcal{D}', \mathcal{T}'$, such that $x, y \in \mathcal{H}_{\mathcal{D}',\mathcal{T}'}$; we can assume that $t_0 \in \mathcal{T}'$. We denote by $\lambda' = \lambda_{|\mathcal{T}'}$. Then, by the preceding case, denoting by Ψ'_{t_0} the restriction of $\Psi^*_{t_0}$ to $\mathcal{H}_{\mathcal{D}',\mathcal{T}'}$:

$$\begin{aligned}\Psi^*_{t_0}(xy) = \Psi'_{t_0}(xy) = \Psi'_{t_0}(x)\Psi'_{t_0}(y) = \Psi^*_{t_0}(x)\Psi^*_{t_0}(y),\\ \Delta^{CK_\lambda} \circ \Psi^*_{t_0}(x) = \Delta^{CK_{\lambda'}} \circ \Psi'_{t_0}(x) = (\Psi'_{t_0} \otimes \Psi'_{t_0}) \circ \Delta^{CK_{\lambda'}}(x) = (\Psi^*_{t_0} \otimes \Psi^*_{t_0}) \circ \Delta^{CK_\lambda}(x),\end{aligned}$$

so Ψ is a Hopf algebra morphism.

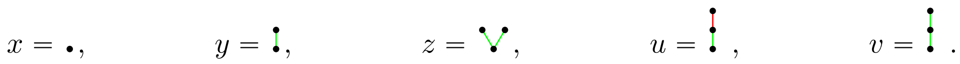
Let us assume that $\lambda_{t_0} \neq 0$. If $\Psi^*_{t_0}(x) = 0$, then $\Psi'_{t_0}(x) = 0$. As $a'_{t_0} \neq 0$, by the first case, $x = 0$, so $\Psi^*_{t_0}$ is injective. Moreover, there exists $z \in \mathcal{H}_{\mathcal{D}',\mathcal{T}'}$, such that $\Psi'_{t_0}(z) = y$; so $\Psi^*_{t_0}(z) = y$, and $\Psi^*_{t_0}$ is surjective.

Let us assume that $\lambda_{t_0} = 0$. Let T be a tree with two vertices, such that its unique edge is of type t_0. As $T \notin \mathbb{T}^{(t_0)}_{\mathcal{D},\mathcal{T}}$, $\Phi_{t_0}(T)$ has a unique term, given by the partition $X = \{\{x_1\},\{x_2\}\}$, where x_1 and x_2 are the vertices of T. Hence:

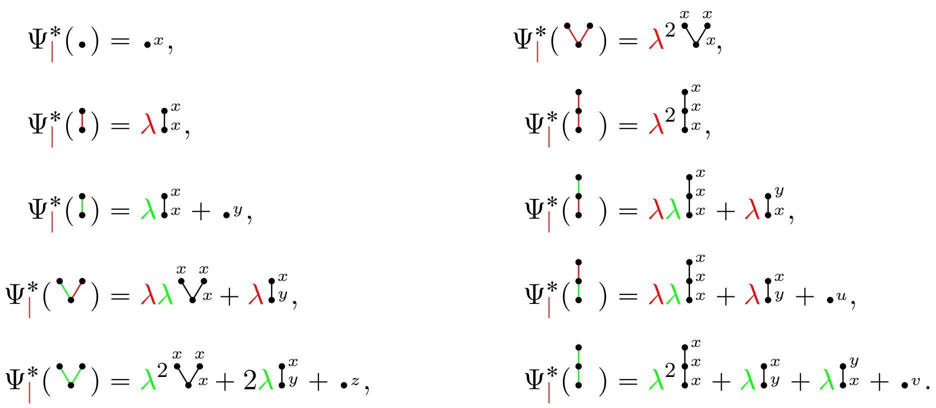
$$\Psi^*_{t_0}(T) = \lambda_{t_0} T' = 0,$$

so $\Psi^*_{t_0}$ is not injective. $\square$

Example 7. Here, $\mathcal{T}$ contains two elements, | and |. In order to simplify, we omit the decorations of vertices. We put:



Applying $\Psi^*_{|}$:



Remark 7. Although it is not indicated, Ψ_{t_0} and $\Psi^*_{t_0}$ depend on λ.

4.5 Bialgebras in cointeraction

By [8], for any $\lambda \in \mathbb{K}^{(\mathcal{T})}$, the operad morphism θ_a : **Prelie** $\longrightarrow \mathcal{P}_\mathcal{T}$, which send $\bullet$ to $\bullet_\lambda$, where **Prelie** is the operad of prelie algebras, induces a pair of cointeracting bialgebras for any finite set $\mathcal{D}$. By construction, the first bialgebra of the pair is $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$. Let us describe the second one.

Definition 27. *Let $F \in \mathbb{F}_{\mathcal{T},\mathcal{D}}$. We shall say that $\{T_1,\ldots,T_k\} \lhd F$ if:*

1. *$\{T_1,\ldots,T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, by restriction.*
2. *For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.*

If $\{T_1,\ldots,T_k\} \lhd F$ and $\mathrm{dec} : [k] \longrightarrow \mathcal{D}$, we denote by $(F/\{T_1,\ldots,T_k\},\mathrm{dec})$ the forest obtained by contracting T_i to a single vertex, and decorating this vertex by $\mathrm{dec}(i)$, for all $i \in [k]$. This is an element of $\mathbb{F}_{\mathcal{D},\mathcal{T}}$.

Proposition 28. *If $\mathcal{D}$ is finite, $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ is the free commutative algebra generated by pairs (T,d), where $T \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$ and $d \in \mathcal{D}$. The coproduct is given, for any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, $d \in \mathcal{D}$, by:*

$$\delta(F,d) = \sum_{\{T_1,\ldots,T_k\} \lhd F} \sum_{\mathrm{dec}:[k] \longrightarrow \mathcal{D}} ((F/\{T_1,\ldots,T_k\},\mathrm{dec}),d) \otimes (T_1,\mathrm{dec}(1))\ldots(T_k,\mathrm{dec}(k)).$$

Then $(\mathcal{H}'_{\mathcal{D},\mathcal{T}}, m, \delta)$ *is a bialgebra, and* $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ *is a coalgebra in the category of* $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$*-comodules via the coaction given, for any* $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$*, by:*

$$\overline{\delta}(T) = \sum_{\{T_1,\ldots,T_k\} \lhd T} \sum_{\mathrm{dec}:[k] \longrightarrow \mathcal{D}} ((T/\{T_1,\ldots,T_k\}, \mathrm{dec}) \otimes (T_1, \mathrm{dec}(1)) \ldots (T_k, \mathrm{dec}(k)).$$

Corollary 29. *Let us assume that* $\mathcal{D}$ *is given a semigroup law denoted by* $+$*. If* $F \in \mathbb{F}_{\mathcal{T},\mathcal{D}}$*, and* $\{T_1, \ldots, T_k\} \lhd F$*, then naturally* $T_i \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$ *for any* i *and the* $\mathcal{T}$*-typed forest* $F/\{T_1, \ldots, T_k\}$ *is given a* $\mathcal{D}$*-decoration, decorating the vertex obtained in the contradiction of* T_i *by the sum of the decorations of the vertices of* T_i*. Then* $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ *is given a second coproduct* δ *such that for any* $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$*:*

$$\delta(F) = \sum_{\{T_1,\ldots,T_k\} \lhd F} F/\{T_1, \ldots, T_k\} \otimes T_1 \ldots T_k.$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \delta)$ *is a bialgebra and* $\mathcal{H}^{CK_\lambda}_{\mathcal{D},\mathcal{T}}$ *is a coalgebra in the category of* $\mathcal{H}_{\mathcal{D},\mathcal{T}}$*-comodules via the coaction* δ*.*

Proof. We denote by I the ideal of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ generated by pairs (T, d) such that $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $d \in \mathcal{D}$, with:

$$d \neq \sum_{v \in V(T)} \mathrm{dec}(v).$$

The quotient $\mathcal{H}'_{\mathcal{D},\mathcal{T}}/I$ is identified with $\mathcal{H}_{\mathcal{D},\mathcal{T}}$, trough the surjective algebra morphism:

$$\varpi : \left\{ \begin{array}{rcl} \mathcal{H}'_{\mathcal{D},\mathcal{T}} & \longrightarrow & \mathcal{H}_{\mathcal{D},\mathcal{T}} \\ (F, d) \in \mathbb{F}_{\mathcal{D},\mathcal{T}} \times \mathcal{D} & \longrightarrow & \begin{cases} F \text{ if } d = \displaystyle\sum_{v \in V(F)} \mathrm{dec}(v), \\ 0 \text{ otherwise.} \end{cases} \end{array} \right.$$

Let us prove that I is a coideal. Let $T \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$, $d \in \mathcal{D}$, $\{T_1, \ldots, T_k\} \lhd F$, $\mathrm{dec} : [k] \longrightarrow \mathcal{D}$ such that $((T/\{\mathcal{T}_1, \ldots, T_k\}, \mathrm{dec}), d) \notin I$ and for any i, $(T_i, \mathrm{dec}(i)) \notin I$. Then:

$$\forall i \in [k], \qquad \sum_{v \in V(T_i)} \mathrm{dec}(v) = \mathrm{dec}(i), \qquad \sum_{i=1}^{k} \mathrm{dec}(i) = d.$$

Hence:

$$\sum_{v \in V(T)} \mathrm{dec}(v) = \sum_{i=1}^{k} \sum_{v \in V(T_i)} \mathrm{dec}(v) = \sum_{i=1}^{k} \mathrm{dec}(i) = d,$$

so $(T, d) \notin I$. Consequently, if $T \in I$, then $((T/\{\mathcal{T}_1, \ldots, T_k\}, \mathrm{dec}), d) \in I$ or at least one of the $(T_i, \mathrm{dec}(i))$ belongs to I. Hence:

$$\delta(I) \subseteq I \otimes \mathcal{H}'_{\mathcal{D},\mathcal{T}} + \otimes \mathcal{H}'_{\mathcal{D},\mathcal{T}} \otimes I.$$

So I is a coideal. The coproduct induced on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by the morphism ϖ is precisely the one given in the setting of this Corollary. □

In particular, if $\mathcal{D}$ is reduced to a single element, denoted by $*$, if we give it its unique semigroup structure $(* + * = *)$, We obtain again the result of [4].

References

[1] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, Cambridge, 1998, Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[2] Murray R. Bremner and Vladimir Dotsenko, *Algebraic operads*, CRC Press, Boca Raton, FL, 2016, An algorithmic companion.

[3] Yvain Bruned, Martin Hairer, and Lorenzo Zambotti, *Algebraic renormalisation of regularity structures*, arXiv:1610.08468, 2016.

[4] Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon, *Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series*, Adv. in Appl. Math. **47** (2011), no. 2, 282–308.

[5] Frédéric Chapoton, *Un endofoncteur de la catégorie des opérades*, Dialgebras and related operads, Lecture Notes in Math., vol. 1763, Springer, Berlin, 2001, pp. 105–110.

[6] Frédéric Chapoton and Muriel Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices (2001), no. 8, 395–408.

[7] Alain Connes and Dirk Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. **199** (1998), no. 1, 203–242.

[8] Loïc Foissy, *Algebraic structures associated to operads*, arXiv:1702.05344, 2017.

[9] Robert Grossman and Richard G. Larson, *Hopf-algebraic structure of combinatorial objects and differential operators*, Israel J. Math. **72** (1990), no. 1-2, 109–117, Hopf algebras.

[10] Michael E. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3795–3811.

[11] Muriel Livernet, *A rigidity theorem for pre-Lie algebras*, J. Pure Appl. Algebra **207** (2006), no. 1, 1–18.

[12] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012.

[13] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002.

[14] Richard J. Mathar, *Topologically distinct sets of non-intersecting circles in the plane*, arXiv:1603.00077, 2016.

[15] Jean-Michel Oudom and Daniel Guin, *Sur l'algèbre enveloppante d'une algèbre pré-Lie*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 5, 331–336.

[16] ———, *On the Lie enveloping algebra of a pre-Lie algebra*, J. K-Theory **2** (2008), no. 1, 147–167.

[17] Florin Panaite, *Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees*, Lett. Math. Phys. **51** (2000), no. 3, 211–219.

[18] N. J. A Sloane, *On-line encyclopedia of integer sequences*, avalaible at http://www.research.att.com/∼njas/sequences/Seis.html.