Rota—Baxter operators on a sum of fields

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Abstract

We count the number of all Rota—Baxter operators on a finite direct sum $A = F \oplus F \oplus \ldots \oplus F$ of fields and count all of them up to conjugation with an automorphism. We also study Rota—Baxter operators on A corresponding to a decomposition of A into a direct vector space sum of two subalgebras. We show that every algebra structure induced on A by a Rota—Baxter of nonzero weight is isomorphic to A.

Keywords: Rota—Baxter operator, (un)labeled rooted tree, 2-coloring, subtree acyclic digraph, transitive digraph.

1 Introduction

Given an algebra A and a scalar $\lambda \in F$, where F is a ground field, a linear operator $R\colon A \to A$ is called a Rota—Baxter operator (RB-operator, for short) on A of weight λ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \quad (1)$$

holds for any $x, y \in A$. The algebra A is called Rota—Baxter algebra (RB-algebra).

G. Baxter in 1960 introduced the notion of Rota—Baxter operator [3] as natural generalization of by parts integration formula. In 1960–1970s such operators were studied by G.-C.Rota [19], P. Cartier [10], J. Miller [17], F. Atkinson [2] and others.

In 1980s, the deep connection between constant solutions of the classical Yang—Baxter equation from mathematical physics and RB-operators on a semisimple finite-dimensional Lie algebra was discovered by A. Belavin and V. Drinfel'd [4] and M. Semenov-Tyan-Shanskii [20].

About different connections of Rota—Baxter operators with symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra see in the monograph [14] written by L. Guo in 2012.

In the paper, we study Rota—Baxter operators on a finite direct sum $A = F \oplus F \oplus \ldots \oplus F$ of n copies of a field F. We continue investigations fulfilled by S. de Bragança in 1975 [6] and by H. An and C. Bai in 2008 [1]. Since all RB-operators on A of weight zero are trivial [12], i.e., equal to 0, we study only RB-operators on A of nonzero weight λ.

In §2, we formulate some preliminaries about RB-operators, including splitting RB-operators which are projections on a subalgebra A_1 parallel to another one A_2 provided the direct vector space sum decomposition $A = A_1 \dot{+} A_2$.

In §3, we show that RB-operators on A of nonzero weight λ are in bijection with 2-colored transitive subtree acyclic digraphs (subtree acyclic digraphs were defined by F. Harary et al. in 1992 [15]) or equivalently with labeled rooted trees on $n+1$ vertices with 2-colored non-root vertices. For the last, we apply the result of R. Castelo and A. Siebes [11]. Thus, the number of all RB-operators on A of nonzero weight λ equals $2^n(n+1)^{n-1}$. With the help of the bijection, we show that splitting RB-operators on A of nonzero weight λ are in one-to-one correspondence with labeled rooted trees on $n+1$ vertices with properly 2-colored non-root vertices. We also study the number of all RB-operators and all splitting RB-operators on A up to conjugation with an automorphism of A.

In 2012, D. Burde et al. initiated to study so called post-Lie algebra structures [7]. One of the questions arisen in the area [7, 8, 9] is the following one: starting with a semisimple Lie algebra endowed RB-operator of weight 1 what kind of Lie algebras we will get under the new Lie bracket $[R(x),y]+[x,R(y)]+[x,y]$? Such problems could be stated not only for Lie algebras but also for associative or commutative ones. In §4, we show that every algebra structure induced on a finite direct sum A of fields by a Rota—Baxter operator of nonzero weight is isomorphic to A itself.

2 Preliminaries

Trivial RB-operators of weight λ are zero operator and $-\lambda\mathrm{id}$.

Statement 1 [14]. Given an RB-operator R of weight λ,

a) the operator $-R-\lambda\mathrm{id}$ is an RB-operator of weight λ,

b) the operator $\lambda^{-1}R$ is an RB-operator of weight 1, provided $\lambda\neq 0$.

Given an algebra A, let us define a map ϕ on the set of all RB-operators on A as $\phi(R)=-R-\lambda(R)\mathrm{id}$. It is clear that ϕ^2 coincides with the identity map.

Statement 2 [5]. Given an algebra A, an RB-operator R on A of weight λ, and $\psi\in\mathrm{Aut}(A)$, the operator $R^{(\psi)}=\psi^{-1}R\psi$ is an RB-operator on A of weight λ.

Statement 3 [14]. Let an algebra A to split as a vector space into the direct sum of two subalgebras A_1 and A_2. An operator R defined as

$$R(a_1+a_2)=-\lambda a_2,\quad a_1\in A_1,\ a_2\in A_2, \tag{2}$$

is RB-operator on A of weight λ.

Let us call an RB-operator from Statement 3 as *splitting* RB-operator with subalgebras A_1,A_2. Note that the set of all splitting RB-operators on an algebra A is in bijection with all decompositions A into a direct sum of two subalgebras A_1,A_2.

Remark 1. Given an algebra A, let R be a splitting RB-operator on A of weight λ with subalgebras A_1,A_2. Hence, $\phi(R)$ is an RB-operator of weight λ and

$$\phi(R)(a_1+a_2)=-\lambda a_1,\quad a_1\in A_1,\ a_2\in A_2.$$

So $\phi(R)$ is splitting RB-operator with the same subalgebras A_1,A_2.

Lemma 1 [5]. Let A be a unital algebra, R be an RB-operator on A of nonzero weight λ. If $R(1)\in F$, then R is splitting.

We call an RB-operator R satisfying the conditions of Lemma 1 as *inner-splitting* one.

Lemma 2 [12]. Let $A = A_1 \oplus A_2$ be an algebra, R be an RB-operator on A of weight λ. Then the induced linear map $P\colon A_1 \to A_1$ defined by the formula $P(x_1+x_2) = \mathrm{Pr}_{A_1}(R(x_1))$, $x_1 \in A_1$, $x_2 \in A_2$, is an RB-operator on A_1 of weight λ.

3 RB-operators on a sum of fields

Statement 4 [1, 6, 12]. Let $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$ be a direct sum of copies of a field F. A linear operator $R(e_i) = \sum\limits_{k=1}^{n} r_{ik}e_k$, $r_{ik} \in F$, is an RB-operator on A of weight 1 if and only if the following conditions are satisfied:

(SF1) $r_{ii} = 0$ and $r_{ik} \in \{0,1\}$ or $r_{ii} = -1$ and $r_{ik} \in \{0,-1\}$ for all $k \neq i$;

(SF2) if $r_{ik} = r_{ki} = 0$ for $i \neq k$, then $r_{il}r_{kl} = 0$ for all $l \notin \{i,k\}$;

(SF3) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{ki} = 0$ and $r_{kl} = 0$ or $r_{il} = r_{ik}$ for all $l \notin \{i,k\}$.

Example [2, 17]. The following operator is an RB-operator on A of weight 1:

$$R(e_i) = \sum_{l=i+1}^{s} e_l,\ 1 \leq i < s, \quad R(e_s) = 0, \quad R(e_i) = -\sum_{l=i}^{n} e_l,\ s+1 \leq i \leq n.$$

Remark 2. It follows from (SF3) that $r_{ik}r_{ki} = 0$ for all $i \neq k$. In [1], the statement of Statement 4 was formulated with this equality and (SF1) but without (SF2) and the general version of (SF3). That's why the formulation in [1] seems to be not complete.

Remark 3. The sum of fields in Statement 4 can be infinite.

In advance, we will identify an RB-operator on A with its matrix.

Let us calculate the number of different RB-operators of nonzero weight λ on $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$. By Statement 1a, we may assume that $\lambda = 1$. For $n = 1$, we have only two RB-operators $\{0, -\mathrm{id}\}$. For $n = 2$ we have 12 cases [1]:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Here we identify an RB-operator with its matrix $R \in M_2(F)$ by the rule $R(e_i) = \sum\limits_{k=1}^{n} r_{ik}e_k$.

For $n = 3$, we have $8 \cdot 16 = 128$ variants [1]:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 2c+1 & 2c+1 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 2c+1 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 2c+1 & c \end{pmatrix},$$
$$\begin{pmatrix} a & 0 & 0 \\ 2b+1 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 2b+1 & b & 0 \\ 2c+1 & 2c+1 & c \end{pmatrix}, \begin{pmatrix} a & 2a+1 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 2a+1 & 0 \\ 0 & b & 0 \\ 2c+1 & 2c+1 & c \end{pmatrix},$$

$$\begin{pmatrix} a & 2a+1 & 2a+1 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 2a+1 & 2a+1 \\ 0 & b & 0 \\ 0 & 2c+1 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 2b+1 & b & 2b+1 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 2b+1 \\ 0 & 0 & c \end{pmatrix},$$

$$\begin{pmatrix} a & 0 & 0 \\ 2b+1 & b & 2b+1 \\ 2c+1 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 2a+1 & 2a+1 \\ 0 & b & 2b+1 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 2a+1 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 & 2a+1 \\ 2b+1 & b & 2b+1 \\ 0 & 0 & c \end{pmatrix}$$

for $a = r_{11}, b = r_{22}, c = r_{33} \in \{0, -1\}$.

For $n = 4$, computer can help to state that there are exactly 2000 RB-operators of weight 1 on A. Thus, we get the first four terms from the sequence A097629 [18].

Theorem 1. Let $A = Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$ be a direct sum of copies of a field F. The number of different RB-operators on A of nonzero weight λ equals $2^n(n+1)^{n-1}$.

PROOF. Let R be an RB-operator on A of weight λ. We may assume that $\lambda = 1$. We follow the previous notations. We have 2^n variants to choose the values of the elements r_{ii}, $i = 1, \ldots, n$. The choice of any of them, say r_{ii}, influences only on the possible signs of all elements r_{ik}, $k \neq i$. So, we may put $r_{ii} = 0$ for all i and fix the factor 2^n for the answer.

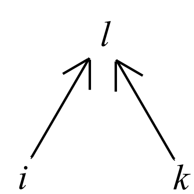
Now, we want to construct a directed graph G on n vertices by any matrix $R = (r_{ij})_{i,j=1}^n$ with chosen $r_{ii} = 0$. We consider the matrix R as the adjacency matrix of a directed graph G. Let us interpretate conditions (SF2) and (SF3) in terms of digraphs. Firstly, we rewrite (SF3) as two conditions:

(SF3a) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{ki} = 0$;

(SF3b) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{kl} = 0$ or $r_{il} = r_{ik}$ for all $l \notin \{i, k\}$.

The condition (SF3a) says that if we have an edge between two vertices $i \neq k$, then the direction of such edge is well-defined, so, it is a correctness of getting a digraph by the matrix R. In graph theory, the condition (SF3b) is called *transitivity*, i.e., if have edges $(i, k) \in E$ and $(k, l) \in E$, then we have an edge $(i, l) \in E$.

Secondly, we read the condition (SF2) in terms of digraphs in such way: there are no in G induced subgraphs isomorphic to H with $V(H) = \{i, k, l\}$ and $E(H) = \{(i, l), (k, l)\}$ (see Pict. 1). In [11] the subgraph H was called *immorality*, thus, a digraph without immoralities is called *moral* digraph [16].



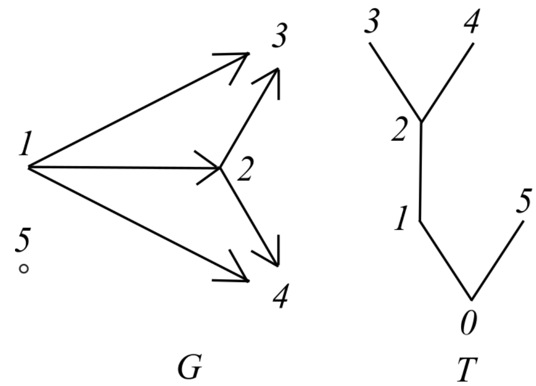
PICTURE 1. The forbidden induced subgraph H on three vertices $\{i, k, l\}$ due to (SF2)

We may reformulate our problem of counting the number N of different RB-operators on A of nonzero weight λ in such way: What is the number of all transitive moral transitive digraphs on n vertices? In terms of [11], the last is the same as the number of all moral TDAGs on n vertices, here TDAG is the abbreviation for Transitive Directed Acyclic Graph (we are interested on transitive digraphs which are surely acyclic). In the graph-theoretic context, moral DAGs are known as subtree acyclic digraphs [15]. Thus,

$$N/2^n = \#\{\text{moral TDAGs on } n \text{ vertices}\}$$
$$= \#\{\text{transitive subtree acyclic digraphs on } n \text{ vertices}\}. \quad (3)$$

In [11], the authors constructed a bijection between the set of moral TDAGs on n vertices and the set of labeled rooted trees on $n+1$ vertices as follows (see Pict. 2). Define the function $f(i)$ for a vertex i by induction. For a source i (i.e., such a vertex i that there are no edges (j,i) in a digraph), we put $f(i)=0$. For a not-source vertex j, we may find the unique source i such that there exists a directed path p from i to j. So, we define $f(j)$ as the length of p. Now, we construct a labeled rooted tree $T=(U,F)$ by a moral TDAG $G=G(V,E)$:

$$U = V\cup\{0\},\quad F=\{(0,i)\mid f(i)=0\}\cup\{(i,j)\mid (i,j)\in E,\ f(i)=f(j)-1\}.$$



PICTURE 2. The corresponding graph G and tree T to the RB-operator $R(e_1)=e_2+e_3+e_4$, $R(e_2)=-e_2-e_3-e_4$, $R(e_3)=-e_3$, $R(e_4)=0$, $R(e_5)=-e_5$.

Applying the above constructed correspondence, the number of moral TDAGs on n vertices equals $(n+1)^{n-1}$ by the Cayley theorem, and so $N=2^n(n+1)^{n-1}$. Theorem is proved.

Below we will apply the easy fact that $\mathrm{Aut}(A)\cong S_n$. It could be derived, e.g., from the Molin—Wedderburn—Artin theory, in particular from the uniqueness up to a rearrangement of summands of decomposition of a semisimple finite-dimensional associative algebra into a finite direct sum of simple ones.

Corollary 1 [6]. Let $A=Fe_1\oplus Fe_2\oplus\ldots\oplus Fe_n$ be a direct sum of copies of a field F and R be an RB-operator on A of nonzero weight 1. There exists an automorphism ψ of A such that the matrix of the operator $R^{(\psi)}$ in the basis $e_1,\ldots,e_n$ is an upper-triangular matrix with entries $r_{ij}\in\{0,\pm1\}$ and $r_{ii}\in\{0,-1\}$.

Proof. As we did in the proof of Theorem 1, we define by R a labeled rooted tree T. Define $t = \max\{f(i) \mid i \in V(T)\}$ and $k_j = \#\{i \mid f(i) = j\}$. We may reorder indexes $1, 2, \ldots, n$ by action of a permutation from $S_n \cong \mathrm{Aut}(A)$ in a way such that

$$f(1) = \ldots = f(k_0) = 0,$$
$$f(k_0+1) = \ldots = f(k_0+k_1) = 1,$$
$$\ldots$$
$$f(n-k_t+1) = \ldots = f(n) = t.$$

Due to the definition of T, we get the upper-triangular matrix. The restrictions on the values of elements immediately follow from Statement 4.

Corollary 2. There is a bijection between the set of RB-operators of nonzero weight λ on $Fe_1 \oplus Fe_2 \oplus \ldots \oplus Fe_n$ and

a) the set of 2-colored subtree acyclic digraphs on n vertices;

b) the set of labeled rooted trees on $n+1$ vertices with 2-colored non-root vertices.

Now, we want to compute the number r_n of RB-operators of nonzero weight λ on $A = Fe_1 \oplus \ldots \oplus Fe_n$ which lie in different orbits under the action of the automorphism group $\mathrm{Aut}(A) \cong S_n$. The group $\mathrm{Aut}(A)$ acts on the set of RB-operators of weight λ in the way described in Statement 2, $\psi\colon R \to R^{(\psi)} = \psi^{-1}R\psi$.

In a light of Corollary 2b, we may interpretate the number r_n as the number of **unlabeled** rooted trees on $n+1$ vertices with 2-colored non-root vertices. It is exactly the sequence A000151 [18], the first eight values are 2, 7, 26, 107, 458, 2058, 9498, 44947 etc. Let us fix that in advance we will use two colors: white and black, white color corresponds to the case $r_{ii} = 0$ and black color corresponds to $r_{ii} = -\lambda$. Considering the rooted tree T with $n+1$ vertices, we may assume that the root is colored in the third color, say grey.

Note that the map ϕ acts on a labeled (or unlabeled) rooted tree T on $n+1$ vertices with 2-colored non-root vertices as follows. The ϕ interchanges a color in every non-root vertex.

Let us describe splitting RB-operators of nonzero weight λ on A.

Theorem 2. An RB-operator R of nonzero weight λ on $A = Fe_1 \oplus \ldots \oplus Fe_n$ is splitting if and only if the corresponding (labeled) rooted tree $T = T(R)$ on $n+1$ vertices is properly colored.

Proof. Wuthout loss of generality, we put $\lambda = 1$. For simplicity, let us consider the graph $T' = T \setminus \{\mathrm{root}\}$, which is a forest in general case.

Let us prove the statement by induction on n. For $n = 1$, we have either $R = 0$ (the only non-root vertex is white) or $R = -\lambda\mathrm{id}$ (the only non-root vertex is black), both RB-operators are splitting with subalgebras F and (0).

Suppose that we have proved Theorem 2 for all natural numbers less than n. Let a graph T' with n vertices be disconnected, denote by $T_1, \ldots, T_k$ the connected components of T'. So, $A = A_1 \oplus \ldots A_k$ for $A_s = \mathrm{Span}\{e_j \mid j \in V(T_s)\}$. Define R_s as the induced RB-operator $R|_{A_s}$ (see Lemma 2). By the definition, R is splitting if and only if $A =$

$\ker(R)\dot{+}\ker(R+\mathrm{id})$ or equivalently $A_s = \ker(R_s)\dot{+}\ker(R_s+\mathrm{id})$, $s=1,\ldots,k$. By the induction hypothesis, we have such decomposition for every s if and only if the coloring of T_s is proper.

Now consider the case when T' is connected. We may assume that e_1 corresponds to the vertex 1, the only source in G, and $\{2,\ldots,k\}$ is the set of all vertices of G with the value of $f(x)$ equal to 1. We also define T_s for $s=2,\ldots,k$ as the connected component of $T'\setminus\{1\}$ which contains the vertex s. Note that R induces the RB-operator of weight λ on the subalgebra $A_s = \mathrm{Span}\{e_j \mid j\in V(T_s)\}$ for all s by Lemma 2.

The condition of R to be splitting is equivalent to the condition

$$\mathrm{rank}\,(R)+\mathrm{rank}\,(R+\mathrm{id}) = n. \tag{4}$$

Analysing the e_1-coordinate, we have

$$n = \mathrm{rank}\,(R)+\mathrm{rank}\,(R+\mathrm{id}) \geq 1+\mathrm{rank}\,(R')+\mathrm{rank}\,(R'+\mathrm{id})$$

for R', the induced RB-operator on the subalgebra $\mathrm{Span}\{e_j \mid j\geq 2\}$. Thus, $\mathrm{rank}\,(R')+\mathrm{rank}\,(R'+\mathrm{id}) = n-1$, i.e. R' is splitting or equivalently $R|_{A_s}$ is spplitting for every $s=2,\ldots,k$. By the induction hypothesis, the graph $T'\setminus\{1\}$ is properly 2-colored. It remains to prove that the vertices $2,\ldots,k$ are colored in the same color and the vertex 1 is colored in another one.

Up to the action of ϕ, which preserves the splitting structure of an RB-operator (see Remark 1), we may assume that the vertex 1 is colored in white. Since we know that $\mathrm{rank}\,(R+\mathrm{id}) = \mathrm{rank}\,(R'+\mathrm{id})+1$, we have to state the equality $\mathrm{rank}\,(R) = \mathrm{rank}\,(R')$. So, the condition (4) is fulfilled if and only if the first row $(0,1,1,\ldots,1)$ of the matrix R is linearly expressed via other rows. By the definition of the matrix R, the vertices $2,\ldots,k$ have to be colored in black. Theorem is proved.

Corollary 3. An RB-operator R of nonzero weight λ on $A = Fe_1\oplus\ldots\oplus Fe_n$ is inner-splitting if and only if in $T = T(R)$ all vertices with even value of f are colored in one color and all vertices with odd value of f are colored in another color.

Proof. Up to ϕ, we may assume that $R(1)=0$. Thus, any vertex with the value of $f(x)$ equal to 0 has to be colored in white. By Theorem 2, $T' = T\setminus\{\mathrm{root}\}$ is properly 2-colored, so, all vertices with the value of $f(x)$ equal to 1 are colored in black, all vertices with the value of $f(x)$ equal to 2 are colored in white and so on.

Now, we collect all our knowledges about all RB-operators (in Table 1) and all nonisomorphic RB-operators (in Table 2) of nonzero weight on a sum of fields $A = Fe_1\oplus Fe_2\oplus\ldots\oplus Fe_n$.

We have noticed that the first values of number of splitting RB-operators coincides with the sequence A007830 [18] (in labeled case) and coincides with the sequence A000106 [18] (in unlabeled case). Actually it should be proven for all n.

Remark 4. Counting rooted trees on $n+1$ vertices with properly 2-colored non-root vertices is not the same as counting properly 2-colored forests on n vertices.

TABLE 1. Number of RB-operators of nonzero weight on a sum of n fields

Class of RB-operators	Description	formula and OEIS [18]	first 5 values
all	labeled rooted trees on $n+1$ vertices with 2-colored non-root vertices	$2^n(n+1)^{n-1}$ A097629	$2, 12, 128, 2000, 41472$
splitting	labeled rooted trees on $n+1$ vertices with properly 2-colored non-root vertices	$2(n+2)^{n-1}$?! A007830 ?!	$2, 8, 50, 432, 4802$
inner-splitting	labeled rooted trees on $n+1$ vertices (twice)	$2(n+1)^{n-1}$ 2·A000272	$2, 6, 32, 250, 2592$
non-splitting	labeled rooted trees on $n+1$ vertices with improperly 2-colored non-root vertices	—	$0, 4, 78, 1568, 36670$

TABLE 2. Number of RB-operators of nonzero weight on a sum of n fields (up to conjugation with an automorphism)

Class of RB-operators	Description	OEIS [18]	first 5 values
all	rooted trees on $n+1$ vertices with 2-colored non-root vertices	A000151	$2, 7, 26, 107, 458$
splitting	rooted trees on $n+1$ vertices with properly 2-colored non-root vertices	A000106 ?!	$2, 5, 12, 30, 74$
inner-splitting	rooted trees on $n+1$ vertices (twice)	2·A000081	$2, 4, 8, 18, 40$
non-splitting	rooted trees on $n+1$ vertices with improperly 2-colored non-root vertices	—	$0, 2, 14, 77, 384$

Let us write down all non-splitting pairwise nonisomorphic RB-operators for $n = 2, 3$.

Statement 5. Up to ϕ, we have the following non-splitting pairwise nonisomorphic RB-operators

a) for $n = 2$: $R(e_1) = e_2$, $R(e_2) = 0$;

b) for $n = 3$:

(RB1) $R(e_1) = e_2 + e_3$, $R(e_2) = e_3$, $R(e_3) = 0$,

(RB2) $R(e_1) = e_2 + e_3$, $R(e_2) = e_3$, $R(e_3) = -e_3$,

(RB3) $R(e_1) = e_2 + e_3$, $R(e_2) = -e_2 - e_3$, $R(e_3) = -e_3$,

(RB4) $R(e_1) = e_2 + e_3$, $R(e_2) = R(e_3) = 0$,

(RB5) $R(e_1) = e_2 + e_3$, $R(e_2) = -e_2$, $R(e_3) = 0$,

(RB6) $R(e_1) = e_2$, $R(e_2) = R(e_3) = 0$,

(RB7) $R(e_1) = e_2$, $R(e_2) = 0$, $R(e_3) = -e_3$.

PROOF. a) Non-splitting case appears only when the graph T' is non-empty and improperly 2-colored. Up to ϕ, we may assume that two vertices are colored in white.

b) Cases (RB1)–(RB3) correspond to improperly 2-colorings of the graph T' with $V(T') = \{1,2,3\}$ and $E(T') = \{(1,2),(2,3)\}$. Cases (RB4), (RB5) correspond to improperly 2-colorings of the graph T' with $E(T') = \{(1,2),(1,3)\}$. Finally, cases (RB6), (RB7) correspond to improperly 2-colorings of the graph T' with $E(T') = \{(1,2)\}$.

Statement 6. Up to ϕ, we have the following splitting but not inner-splitting pairwise nonisomorphic RB-operators:

a) for $n = 2$: $R(e_1) = -e_1$, $R(e_2) = 0$;

b) for $n = 3$:

(RB1′) $R(e_1) = e_2$, $R(e_2) = 0$, $R(e_3) = -e_3$,

(RB2′) $R(e_1) = -e_1$, $R(e_2) = R(e_3) = 0$.

4 RB-induced algebra structures on a sum of fields

Let C be an associative algebra and R be an RB-operator on C of weight λ. Then the space C under the product

$$x \circ_R y = R(x)y + xR(y) + \lambda xy \tag{5}$$

is an associative algebra [14, 13]. Let us denote the obtained algebra as C^R. It is easy to see that $C^{\phi(R)} \cong C^R$.

Let us denote by Ab_n the n-dimensional algebra with zero (trivial) product.

Theorem 3. Given an algebra $A = Fe_1 \oplus \ldots \oplus Fe_n$ and an RB-operator R of weight λ on A, we have $A^R \cong \begin{cases} \mathrm{Ab}_n, & \lambda = 0, \\ A, & \lambda \neq 0. \end{cases}$

Proof. If $\lambda = 0$, then $R = 0$ [12] and $x \circ_R y = 0$. For $\lambda \neq 0$, we may assume that $\lambda = 1$, since rescalling of the product does not exchange the algebraic structure.

Let us prove the statement by induction on n. For $n = 1$, we have either $R = 0$ or $R = -\mathrm{id}$. Due to (5) we get either $x \circ y = xy$ or $x \circ y = -xy$, in both cases $A^R \cong A$.

Suppose that we have proved Theorem 3 for all numbers less n. Let a graph $T' = T'(R)$ with n vertices be disconnected, denote by $T_1, \ldots, T_k$ the connected components of T'. As earlier, we define $A = A_1 \oplus \ldots A_k$ for $A_s = \mathrm{Span}\{e_j \mid j \in V(T_s)\}$ and define R_s as the induced RB-operator $R|_{A_s}$. By the induction hypothesis, $A_s^R \cong A_s$ for every s and so $A = A_1 \oplus \ldots \oplus A_k \cong A_1^R \oplus \ldots \oplus A_k^R = A^R$.

Now consider the case when T' is connected. We may assume that e_1 corresponds to the vertex 1, the only source in G. Note the space $I_1 = \mathrm{Span}\{e_j \mid j \geq 2\}$ is an ideal in A^R which is isomorphic to $Fe_2 \oplus \ldots \oplus Fe_n$ by the induction hypothesis. Up to ϕ, we may assume that the vertex 1 in T' is colored in white and $2, \ldots, t$ is a list of all neighbours of 1 in T'. Let us consider the one-dimensional space I_2 in A^R generated by the vector $a = e_1 - c(2)e_2 - \ldots - c(t)e_t$, where

$$c(i) = \begin{cases} 1, & i \text{ is colored in white}, \\ -1, & i \text{ is colored in black}. \end{cases}$$

In terms of the matrix entries, $c(i) = 1 + 2r_{ii}$. We may assume that $c(2) = c(3) = \ldots = c(s) = 1$ and $c(s+1) = \ldots = c(t) = -1$ for some $s \in \{2, \ldots, t\}$.

By (5) we compute the product of a with e_k for $k > t$:

$$a \circ e_k = (e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) \circ e_k \\ = R(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t)e_k.$$

Since k is connected with only one vertex from $2, \ldots, t$ (due to (SF2)), say j, we have

$$a \circ e_k = R(e_1 - c(j)e_j)e_k = e_k - c(j)(1 + 2r_{jj})e_k = (1 - (c(j))^2)e_k = 0.$$

Analogously we can check that $a \circ e_k = 0$ for all $k > 1$. Thus, I_2 is an ideal in A^R.

Now, we calculate

$$a \circ a = e_1 \circ (e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) \\ = R(e_1)(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) + e_1 \\ = (e_2 + \ldots + e_s + e_{s+1} + \ldots + e_t)(e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t) + e_1 \\ = e_1 + e_2 + \ldots + e_s - e_{s+1} - \ldots - e_t = a$$

and so I_2 is isomorphic to F.

Summarising, we have $A^R = I_1 \oplus I_2 \cong (Fe_2 \oplus \ldots \oplus Fe_n) \oplus F \cong A$. Theorem is proved.

Acknowledgements

The main part of the paper was done while working in Sobolev Institute of Mathematics in 2017. The research is supported by RSF (project N 14-21-00065).

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