Diophantine properties of fixed points of Minkowski question mark function.

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Abstract

We consider irrational fixed points of the Minkowski question mark function $?(x)$, that is irrational solutions of the equation $?(x) = x$. It is easy to see that there exist at least two such points. Although it is not known if there are other fixed points, we prove that the smallest and the greatest fixed points have irrationality measure exponent equal to 2. We give more precise results about the approximation properties of these fixed points. Moreover, in Appendix we introduce a condition from which it follows that there are only two irrational fixed points.

1 Introduction

For $x \in [0,1]$ we consider its continued fraction expansion

$$x = [a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}, \quad a_j \in \mathbb{Z}_+$$

which is unique and infinite when $x \notin \mathbb{Q}$ and finite for rational x. Each rational x has just two representations

$$x = [a_1, a_2, \ldots, a_{n-1}, a_n], \quad \text{and} \quad x = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1], \quad \text{where} \quad a_n \geq 2.$$

By

$$\frac{p_k}{q_k} := [a_1, \ldots, a_k]$$

we denote the kth convergent fraction to x. By B_n we denote the nth level of the Stern-Brocot tree, that is

$$B_n := \{x = [a_1, \ldots, a_k] : a_1 + \ldots + a_k = n + 1\}.$$

In [8] Minkowski introduced a special function $?(x)$ which may be defined as the limit distribution function of sets B_n. This function was rediscovered and studied by many authors (see [7],[6],[1],[4],[11]). For rational or irrational $x = [a_1, a_2, \ldots, a_n, \ldots]$ the formula

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k-1}} \tag{1}$$

introduced by Denjoy [2, 3] and Salem [13] may be considered as one of the equivalent definitions of the function $?(x)$. It is known that $?(x)$ is a continuous strictly increasing function, its derivative $?'(x)$ exists almost everywhere in $[0,1]$ in the sense of Lebesgue measure, and $?'(x) = 0$ for $x \in \mathbb{Q}$.

As $?(x)$ is continuous and

$$?(0) = 0, \quad ?\left(\frac{1}{2}\right) = \frac{1}{2}, \quad ?(1) = 1.$$

we see that there exist at least two points $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_2 \in \left(\frac{1}{2}, 1\right)$ with

$$?(x_j) = x_j, \quad j = 1, 2.$$

A folklore conjecture states that

Conjecture 1. *The Minkowski question mark function $?(x)$ has exactly five fixed points. There is only one irrational fixed points of $?(x)$ from $(0, \frac{1}{2})$ interval.*

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This conjecture has not yet been proved (for certain discussion see survey preprint by Moshchevitin [9]). We present an equivalent statement to this conjecture in the Appendix. However, if there are more then one fixed points on the interval $(0; \frac{1}{2})$, then their first 4000 partial quotients in continued fraction expansion coincide. Although we do not know if there are exactly two irrational fixed points of $?(x)$, we are able to say something about Diophantine properties of some of them. In the present paper we give explicit lower bounds for the irrationality measure of the smallest and the greatest fixed points of X, that is lower bounds of the form

$$\left|x - \frac{p}{q}\right| > \frac{1}{q^2 \cdot I(q)} = \frac{1}{q^{2+\delta(q)}}, \quad \delta(q) \geq 0$$

satisfied by all $p, q \in \mathbb{Z}, q \geq q_0$, where the dependence $I(q)$ on q is explicit and q_0 is given. Usually the infimum $\inf_{q \in \mathbb{Z}_+}(2 + \delta(q))$ is called irrationality measure (or exponent) of x.

2 Main results

Our first result establishes some properties of the continued fraction expansion of certain fixed points of $?(x)$.

Theorem 1. *Let $x = [a_1, \ldots, a_n, \ldots]$ be the smallest or the greatest fixed point of Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then $a_1 = 2$ and*

$$a_{n+1} \leq \sum_{i=1}^{n} a_i. \tag{2}$$

for all $n \in \mathbb{N}$

We give a proof of Theorem 1 in Section 4. The following theorem is a stronger version of Theorem 1. It uses some new geometrical considerations.

Theorem 2. *Denote $\kappa_1 = 2\log_2(\frac{\sqrt{5}+1}{2}) - 1 \approx 0.38848383\ldots$. Let x be fixed point from Theorem 1, then*

$$a_{n+1} < \kappa_1 \sum_{i=1}^{n} a_i + 2\log_2\left(\sum_{i=1}^{n} a_i\right). \tag{3}$$

for all $n \geq 1$.

Formula (3) gives an explicit irrationality measure lower estimate for the fixed points under considerations.

Theorem 3. *Let x be fixed point from Theorem 1, then*

$$\left|x - \frac{p}{q}\right| > \frac{1}{\left(\kappa_1\left(\frac{2}{\log 2}\log q + \log_2\frac{2}{9}\right) + \frac{2}{\log 2}\log\left(\frac{2}{\log 2}\log q + \log_2\frac{2}{9}\right) + 1\right)q^2},$$

for all $q > q_0 \in \mathbb{N}, p \in \mathbb{N}$

We give a proof of Theorem 3 in Section 6.

3 Preliminaries

By F_k we denote kth Fibonacci number, that is

$$F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}.$$

Lemma 3.1. *Let $?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k]$ and $\sum_{i=1}^{n-1} a_i = s + 1 > 2$, then $\sum_{i=1}^{k} b_i > s + 1$.*

Proof. Reducing (1) to a common denominator we get

$$?([a_1, a_2, \ldots, a_{n-1}]) = \frac{2^{a_2+\ldots+a_{n-1}} - 2^{a_3+\ldots+a_{n-1}} + \ldots + (-1)^{n-1} \cdot 2^{a_{n-1}} + (-1)^n}{2^{\sum\limits_{i=1}^{n-1} a_i - 1}}. \tag{4}$$

Here the denominator is equal to 2^s, and the numerator is an odd number. Let us consider level B_s of the Stern-Brocot tree, which contains the number $[a_1, a_2, \ldots, a_{n-1}]$. The greatest denominator on this level is equal to F_{s+2}. We know that for all $s > 2$ one has $F_{s+2} < 2^s$. This means that the image of the number $[a_1, a_2, \ldots, a_{n-1}]$, given by the formula (4), belongs to level B_{s+k} for some $k \in \mathbb{N}$, since the denominator of the image is greater than the greatest denominator on the level B_s. □

Corollary 3.1. *Minkowski question mark function has exactly 3 rational fixed points:* 0, $\frac{1}{2}$ *and* 1.

Proof. We see that $F_{s+2} = 2^s$ only for $s = 0, 1$, that is for numbers from the $0th$ and the $1st$ levels of the Stern-Brocot tree, and there are numbers $0, \frac{1}{2}$ and 1 only. For every other rational number, the sum of its partial quotients increases under the map $?(x)$. So the number is not mapped onto itself. □

The following lemma about the values of Minkowski function at rational points is related to a famous statement known as "Folding lemma" (see [12]).

Lemma 3.2. *Let s be an arbitrary nonnegative integer and*

$$?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k], \;\; b_k \neq 1.$$

Consider the number

$$\theta = [a_1, a_2, \ldots, a_{n-1}, a_n], \;\; \textit{where} \;\; a_n = \sum_{i=1}^{n-1} a_i + s.$$

Then

1. *If $n \equiv k (mod\ 2)$, then* $?(\theta) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]$.
2. *If $n \equiv k + 1 (mod\ 2)$ then* $?(\theta) = [b_1, b_2, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \ldots, b_1]$.

Proof. We know that $b_k \neq 1$. Let us choose one of the representations $\frac{p_l}{q_l} = [b_1, b_2, \ldots, b_k]$ or $\frac{p_l}{q_l} = [b_1, b_2, \ldots, b_k - 1, 1]$ so that the length l of the continued fraction expansion is of the same parity as $n + 1$, that is $l \equiv n + 1 \pmod 2$, and $l = k$ or $l = k + 1$. From (4) we know that $q_l = 2^{\sum\limits_{i=1}^{n-1} a_i - 1}$.
Without loss of generality suppose that $n \equiv k + 1 \pmod 2$, then

$$?(\theta) = ?([a_1, a_2, \ldots, a_{n-1}]) + \frac{(-1)^{n+1}}{2^{2 \cdot \sum\limits_{i=1}^{n-1} a_i - 1 + s}} = \frac{p_l}{q_l} + \frac{(-1)^{n+1}}{2^{s+1} \cdot q_l^2} = \frac{p_l}{q_l} + \frac{(-1)^l}{2^{s+1} \cdot q_l^2} = \frac{p_l q_l 2^{s+1} - (p_l q_{l-1} - q_l p_{l-1})}{2^{s+1} q_l^2} =$$

$$= \frac{p_l(2^{s+1} - \frac{q_{l-1}}{q_l}) + p_{l-1}}{q_l(2^{s+1} - \frac{q_{l-1}}{q_l}) + q_{l-1}} = [b_1, b_2, \ldots, b_k, 2^{s+1} - \frac{q_{l-1}}{q_l}] = [b_1, b_2, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \ldots, b_1].$$

In the last equality we use $-x = 0 + \cfrac{1}{-1 + \cfrac{1}{1 + \cfrac{1}{x - 1}}}$. □

Remark 1. H.Niederreiter [10] proved that if m is a power of 2, then there exists an odd integer a with $1 \leq a \leq m$ such that all partial quotients in continued fraction expansion of a/m are bounded by 3. In fact, he took iterations of Minkowski question mark function of the continued fractions of a special form, where each partial quotient is equal to the sum of all previous ones or to the sum of all previous ones plus 1.

Lemma 3.3. *Let $a_1, \dots, a_{n-1}$ be the partial quotients of a fixed point x, then, depending on the parity of n, the next partial quotient a_n satisfies one of the following systems*

1. *If n is even, then a_n satisfies* $\begin{cases} [a_1, \dots, a_{n-1}, a_n] <?([a_1, \dots, a_{n-1}, a_n + 1]), \\ [a_1, \dots, a_{n-1}, a_n + 1] >?([a_1, \dots, a_{n-1}, a_n]). \end{cases}$

2. *If n is odd, then a_n satisfies* $\begin{cases} [a_1, \dots, a_{n-1}, a_n] >?([a_1, \dots, a_{n-1}, a_n + 1]), \\ [a_1, \dots, a_{n-1}, a_n + 1] <?([a_1, \dots, a_{n-1}, a_n]). \end{cases}$

Proof. Let $[a_1, \dots, a_n, \dots]$ be the fixed point. Let us show 1).
Since n is even, from the continued fraction theory we know

$$[a_1, \dots, a_n] < [a_1, \dots, a_n, \dots] < [a_1, \dots, a_n + 1].$$

That is $[a_1, \dots, a_n]$ and $[a_1, \dots, a_n + 1]$ lie on opposite sides with respect to x_0, and hence their images lie on different sides too, and we have

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] <?([a_1, \dots, a_{n-1}, a_n + 1]), \\ [a_1, \dots, a_{n-1}, a_n + 1] >?([a_1, \dots, a_{n-1}, a_n]). \end{cases}$$

Case 2) can be treated similarly, since for odd n one has

$$[a_1, \dots, a_n + 1] < [a_1, \dots, a_n, \dots] < [a_1, \dots, a_n].$$

□

The following lemma localizes fixed points.

Lemma 3.4. *All fixed points of ?(x) inside the interval* $(0, \frac{1}{2})$ *belong to the interval* $(\frac{2}{5}, \frac{3}{7})$

Proof. First of all, we will show that there are no fixed points on the interval $(0, \frac{1}{3})$. Decompose the interval $(0, \frac{1}{3})$ into the union of subintervals $(0, \frac{1}{3}) \setminus \mathbb{Q} = \bigcup_{n=3}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \setminus \mathbb{Q}$. Assume that for some n_0 there exists $x_0 \in (\frac{1}{n_0+1}, \frac{1}{n_0})$ such that $?(x_0) = x_0$. Then

$$\frac{1}{n_0 + 1} < x_0 =?(x_0) <? \left(\frac{1}{n_0}\right) = \frac{1}{2^{n_0-1}}.$$

So $n_0 + 1 > 2^{n_0+1}$, and this is not true for $\forall n_0 \geq 3$.
This means that the first partial quotient is equal to 2 (1 is excluded, since we are on the interval $(0, \frac{1}{2})$).

Now we show that there are no fixed points on the interval $(\frac{4}{9}, \frac{1}{2})$. Consider the decomposition $(\frac{4}{9}, \frac{1}{2}) \setminus \mathbb{Q} = \bigcup_{n=4}^{\infty} (\frac{n}{2n+1}, \frac{n+1}{2n+3}) \setminus \mathbb{Q}$. Assume that for some n_0 there exists $x \in (\frac{n_0}{2n_0+1}, \frac{n_0+1}{2n_0+3})$ such that $?(x) = x$.

$$\frac{n_0 + 1}{2n_0 + 3} > x_0 =?(x_0) >? \left(\frac{n_0}{2n_0 + 1}\right) = \frac{1}{2} - \frac{1}{2^{1+n_0}}.$$

We get $2^{n_0} < 2n_0 + 3$, which holds only for $n_0 = 1, 2, 3$.

To show that there are no fixed points on the interval $(\frac{1}{3}, \frac{2}{5})$, we observe that $?(?(\frac{2}{5})) =?(\frac{3}{8}) = \frac{5}{16} < \frac{1}{3}$ and $?(?(\frac{3}{7})) =?(\frac{7}{16}) = \frac{29}{64} > \frac{4}{9}$. Hence the image of the whole interval $(\frac{1}{3}, \frac{2}{5})$ under the the second iteration of $?(x)$ belongs to $(0, \frac{1}{3})$ □

Lemma 3.4 means that the continued fraction expansion of every fixed point on the $(0, \frac{1}{2})$ is of the form $[2, 2, \dots]$. The next statement is an obvious property of continuous functions. We formulate it without a proof.

Lemma 3.5. *Let $f(x)$ be a continuous function. Consider an interval $[a, b]$ such that the endpoints of this segment are fixed points of $f(x)$, and there are no fixed points inside $[a, b]$. Then $f(x) - x$ does not change sign on (a, b).*

4 Proof of Theorem 1

Let us prove this theorem for the left fixed point of $?(x)$ on the interval $(0, \frac{1}{2})$. Denote it by $x = [a_1, \ldots, a_n, \ldots]$. We shall prove by contradiction. Assume that there exist $n \geq 3$ such that

$$a_n \geq \sum_{i=1}^{n-1} a_i. \tag{5}$$

Consider $[a_1, \ldots, a_{n-1}]$ and let $?([a_1, \ldots, a_{n-1}]) = [b_1, \ldots, b_k], b_k \neq 1$. Now we distinguish cases 1) - 4) with several subcases. In each of them we will deduce a contradiction. We present a very detailed exposition of the case 1) below. Cases 2) - 4) are quite similar. We establish them with less details.
1) n - odd, k - odd. Then by Lemma 3.2, $?([a_1, a_2, \ldots, a_n]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]$.
By Lemma 3.3 a_n should satisfy

$$\begin{cases} [a_1, \ldots, a_{n-1}, a_n] > ?([a_1, \ldots, a_{n-1}, a_n + 1]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \ldots, b_1], \\ [a_1, \ldots, a_{n-1}, a_n + 1] < ?([a_1, \ldots, a_{n-1}, a_n]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]. \end{cases} \tag{6}$$

Now we consider 3 subcases.
1.1) $k \leq n - 1$. Then by Lemma 3.1 we have $\sum_{i=1}^{k} b_i > \sum_{i=1}^{n-1} a_i$, hence there exists $i \in \{1, \ldots, k\}$ such that $a_i \neq b_i$. By considering a partial quotient with the smallest index $i \leq k$ for which $a_i \neq b_i$, we get that the system (6) is incompatible by the rules of comparison of continued fractions.
1.2) $k > n$. If there is $i \in \{1, \ldots, n-1\}$ such that $a_i \neq b_i$, then similarly to case 1.1) system (6) is incompatible. Hence we assume that for all $i \in \{1, \ldots, n-1\}$ $a_i = b_i$.
Let us consider 4 variants.
1.2.1) $b_n \leq a_n - 1$. The first inequality of (6) can be rewritten as

$$[a_1, \ldots, a_{n-1}, a_n] > [a_1, a_2, \ldots, a_{n-1}, b_n, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \ldots, b_1]. \tag{7}$$

But n is odd and $b_n \leq a_n - 1$. So (7) can not be true.
1.2.2) $b_n \geq a_n + 2$. The second inequality of (6) can be rewritten as

$$[a_1, \ldots, a_{n-1}, a_n + 1] < [a_1, a_2, \ldots, a_{n-1}, b_n, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]. \tag{8}$$

But n is odd and $b_n \geq a_n + 2$. So (8) can not be true.
1.2.3) $b_n = a_n + 1$. Let us rewrite system (6) under the assumptions of this case in the form

$$\begin{cases} [a_1, \ldots, a_n] > ?([a_1, \ldots, a_n + 1]) = [a_1, \ldots, a_n + 1, b_{n+1}, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \ldots, a_1], \\ [a_1, \ldots, a_n + 1] < ?([a_1, \ldots, a_n]) = [a_1, \ldots, a_n + 1, b_{n+1}, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, a_1]. \end{cases}$$

Second inequality fails since the value of continued fraction is always less than an odd convergent.
1.2.4) $b_n = a_n$. Now the equality $?([a_1, \ldots, a_{n-1}]) = [b_1, \ldots, b_k]$ gives

$$?([a_1, \ldots, a_{n-1}]) = [a_1, \ldots, a_{n-1}, a_n, \ldots, b_k] > [a_1, \ldots, a_{n-1}].$$

The last inequality is due to the fact that n is odd. But $[a_1, \ldots, a_{n-1}] < x$ is also a convergent for our fixed point x. So

$$[a_1, \ldots, a_{n-1}] < ?([a_1, \ldots, a_{n-1}]) < [a_1, \ldots, a_n, \ldots] = x.$$

We will show that this contradicts to Lemma 3.5. Indeed, by Lemma 3.3 there are no fixed points on the interval $(0, \frac{1}{3})$, and so $x > \frac{1}{3}$. Consider the segment $[0, x]$. It satisfies conditions of Lemma 3.5 and for the fraction $\frac{1}{4}$ holds $?(\frac{1}{4}) = \frac{1}{8} < 1/4$. Hence by Lemma 3.5 the inequality $?(y) < y$ holds for every $y \in (0, x)$, but we have constructed the number $y = [a_1, \ldots, a_{n-1}] \in (0, x)$ with $?(y) > y$.
1.3) $k = n$. Similarly to the case 1.2) we get $a_i = b_i$ for all $i = 1, \ldots, n-1$. Let us consider 4 subcases.

1.3.1) $b_n \le a_n$. In this case we deduce a contradiction similarly to the case 1.2.1)
1.3.2) $b_n \ge a_n + 3$. In this case we deduce a contradiction similarly to the case 1.2.2)
1.3.3) $b_n = a_n + 1$. Equality $?([a_1, \dots, a_{n-1}]) = [b_1, \dots, b_k]$ under the assumptions of this case gives

$$?([a_1, \dots, a_{n-1}]) = [a_1, \dots, a_{n-1}, a_n + 1] > [a_1, \dots, a_{n-1}].$$

But this contradicts to Lemma 3.5 as in the case 1.2.4).
1.3.4) $b_n = a_n + 2$. System (6) gives now

$$\begin{cases} [a_1, \dots, a_n] > [a_1, \dots, a_n + 1, 1, 2^{s+2} - 1, a_n, \dots, a_1], \\ [a_1, \dots, a_n + 1] < [a_1, \dots, a_n + 1, 1, 2^{s+1} - 1, a_n, \dots, a_1]. \end{cases}$$

The second inequality fails since odd convergent is always greater than the value of the continued fraction.

2) n - even, k - even. Then by Lemma 3.2 $?([a_1, a_2, \dots, a_n]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1]$.
By Lemma 3.3 a_n should satisfy

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] < ?([a_1, \dots, a_{n-1}, a_n + 1]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \dots, b_1], \\ [a_1, \dots, a_{n-1}, a_n + 1] > ?([a_1, \dots, a_{n-1}, a_n]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1]. \end{cases} \tag{9}$$

As before, we consider 3 cases
2.1) $k \le n - 1$. In this case we deduce a contradiction similarly to the case 1.1)
2.2) $k > n$. In the same way as in 1.2), we come to four options
2.2.1) $b_n \le a_n - 1$. In this case we deduce a contradiction similarly to the case 1.2.1)
2.2.2) $b_n \ge a_n + 2$. In this case we deduce a contradiction similarly to the case 1.2.2)
2.2.3) $b_n = a_n + 1$. Then the system (4) can be rewritten as follows

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] < ?([a_1, \dots, a_{n-1}, a_n + 1]) = [a_1, \dots, a_n + 1, b_{n+1}, \dots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \dots, a_1], \\ [a_1, \dots, a_{n-1}, a_n + 1] > ?([a_1, \dots, a_{n-1}, a_n]) = [a_1, \dots, a_n + 1, b_{n+1}, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, a_1]. \end{cases}$$

The second inequality fails since even convergent is always smaller than the value of the continued fraction.
2.2.4) $b_n = a_n$. Let us consider the even convergent $[a_1, \dots, a_n]$ of our fixed point, or rather its image. Taking into account the assumptions and using Lemma 3.2, we obtain
$?([a_1, \dots, a_n]) = [a_1, \dots, a_n, b_{n+1}, \dots, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, a_1] > [a_1, \dots, a_n]$. We get contradiction in the same way as in 1.2.4), since we found a number less than the *the smallest* fixed point, the image of which is greater than this number.
2.3) $k = n$. Similarly to the case 1.2) we come to the assumption that for all $i \in \{1, \dots, n - 1\}$ $a_i = b_i$ and so we consider 4 cases
2.3.1) $b_n \le a_n$. In this case we deduce a contradiction similarly to the case 1.2.1)
2.3.2) $b_n \ge a_n + 3$. In this case we deduce a contradiction similarly to the case 1.2.2)
2.3.3) $b_n = a_n + 1$. We have $?([a_1, \dots, a_n]) = [a_1, \dots, a_n, 1, 2^{s+1} - 1, a_n, \dots, a_1] > [a_1, \dots, a_n]$. Now we get a contradiction as in 1.2.4).
2.3.4) $b_n = a_n + 2$. Rewriting the system (4) under the assumptions, we get

$$\begin{cases} [a_1, \dots, a_n] < [a_1, \dots, a_n + 1, 1, 2^{s+2} - 1, a_n, \dots, a_1], \\ [a_1, \dots, a_n + 1] > [a_1, \dots, a_n + 1, 1, 2^{s+1} - 1, a_n, \dots, a_1]. \end{cases}$$

The second inequality fails since even convergent is always smaller than the value of the continued fraction.

Cases

$$3)\ n - \text{even},\ k - \text{ odd} \qquad \text{and} \qquad 4)\ n - \text{ odd},\ k - \text{even}$$

follow the similar argument. For example, in case 3) by Lemma 3.2,

$$?([a_1, a_2, \dots, a_n]) = [b_1, \dots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1].$$

By Lemma 3.3, a_n satisfies

$$\begin{cases}[a_1,\ldots,a_{n-1},a_n] < [b_1,\ldots,b_k,2^{s+2}-1,1,b_k-1,b_{k-1},\ldots,b_1],\\ [a_1,\ldots,a_{n-1},a_n+1] > [b_1,\ldots,b_k,2^{s+1}-1,1,b_k-1,b_{k-1},\ldots,b_1],\end{cases}$$

and we need to consider only two subcases (subcase $n=k$ is impossible since parity of n and k is different)

$$3.1)\, k \le n-1 \qquad \text{and} \qquad 3.2)\, k > n. \tag{10}$$

In the case 3.2), analogously to the case 1.2), we come to four options

$$3.2.1)\, b_n = a_n+1, \qquad 3.2.2)\, b_n = a_n, \qquad 3.2.3)\, b_n \le a_n - 1, \qquad 3.2.4)\, b_n \ge a_n+2. \tag{11}$$

In every subcase we get a contradiction.
In the case 4) we have by Lemma 3.2

$$?([a_1,a_2,\ldots,a_n]) = [b_1,\ldots,b_k,2^{s+1}-1,1,b_k-1,b_{k-1},\ldots,b_1],$$

and by Lemma 3.3, a_n satisfies

$$\begin{cases}[a_1,\ldots,a_{n-1},a_n] > [b_1,\ldots,b_k,2^{s+2}-1,1,b_k-1,b_{k-1},\ldots,b_1],\\ [a_1,\ldots,a_{n-1},a_n+1] < [b_1,\ldots,b_k,2^{s+1}-1,1,b_k-1,b_{k-1},\ldots,b_1].\end{cases}$$

Now we consider the same subcases as in (10), and the second subcase splits into the same subcases as in (11). We exhausted all possibilities, getting a contradiction in each of them, hence the assumption (5) was false. □

Remark 2. To prove Theorem 1 for the greatest fixed point y we should consider the segment $[y,\frac{1}{2}]$ and the number $\frac{3}{7}$ which belongs to it (by Lemma 3.4) to deduce a contradiction using Lemma 3.5.
Remark 3. One can see that a slight generalization of Theorem 1 can be proven not only for the smallest and the greatest fixed points, but for any fixed point x which is isolated and unstable at least at one side.
For example, x is isolated and unstable at the left side if and only if

$$\exists \varepsilon > 0 : \forall y \in (x-\varepsilon, x) \;\text{ holds }\; |y-x| < |?(y)-x|.$$

For the isolated and unstable fixed point instead of Theorem 1 one can show that the inequality (2) is valid for all *n large enough.*

5 Proof of Theorem 2

Consider an arbitrary continued fraction $[a_1,\ldots,a_n,\ldots]$. Denote $S_n = a_1+\ldots+a_n$. We need the following lemma from ([5], Theorem 4).

Lemma 5.1. *Denote* $\varphi = \frac{\sqrt{5}+1}{2}$. *For any* $n \in \mathbb{N}$ *one has*

$$q_n \le F_{S_n+1} \le \varphi^{S_n}. \tag{12}$$

Now we are ready to prove Theorem 2.

Proof. First of all, one can easily see that Theorem 2 holds for $n < 64$, as we know 64 first partial quotients of x. The corresponding sequence is OEIS A058914. Suppose that n is even. Then $?(\frac{p_n}{q_n}) < \frac{p_n}{q_n} < x = x$. Hence

$$\frac{1}{(a_{n+1}+1)q_n^2} < x - \frac{p_n}{q_n} < ?(x) - ?\left(\frac{p_n}{q_n}\right) < ?\left(\frac{p_{n+1}}{q_{n+1}}\right) - ?\left(\frac{p_n}{q_n}\right) = \frac{1}{2^{S_n+a_{n+1}-1}} \tag{13}$$

We obtain the inequality

$$\frac{(a_{n+1}+1)q_n^2}{2^{S_n+a_{n+1}-1}} > 1 \tag{14}$$

Suppose that $a_{n+1} \geq \kappa_1 S_n + 2\log_2 S_n$. We apply the upper estimate from Lemma 5.1 and use the fact that $\frac{x}{2^x}$ is strictly decreasing function for $x \geq 4$.

$$1 < \frac{(a_n+1)q_n^2}{2^{S_n+a_{n+1}-1}} < \frac{(\kappa_1 S_n + 2\log_2 S_n + 1)\varphi^{2S_n}}{2^{S_n(\kappa_1+1)+2\log_2 S_n - 1}} < \frac{2(\kappa_1 S_n + 2\log_2 S_n + 1)}{S_n^2} \tag{15}$$

One can easily see that

$$\frac{2(\kappa S_n + 2\log_2 S_n + 1)}{S_n^2} < 1$$

for $S_n \geq 4$. We obtain a contradiction.

The case when n is odd is slightly more complicated. Now we have $?(\frac{p_{n+1}}{q_{n+1}}) < \frac{p_{n+1}}{q_{n+1}} < x = x$. Using the same argument we obtain that

$$\frac{(a_{n+2}+1)q_{n+1}^2}{2^{S_n+a_{n+1}+a_{n+2}-1}} > 1.$$

As $q_{n+1} < (a_{n+1}+1)q_n$ and $\frac{a_{n+2}+1}{2^{a_{n+2}}} < 1$,

$$\frac{(a_{n+1}+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} > 1. \tag{16}$$

Suppose that $a_{n+1} \geq \kappa_1 S_n + 2\log_2 S_n$. Similarly to the previous case, we apply Lemma 5.1 and use the fact that $\frac{x^2}{2^x}$ is strictly decreasing function for $x \geq 7$. We have

$$1 < \frac{(a_n+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} < \frac{(\kappa S_n + 2\log_2 S_n + 1)^2\varphi^{2S_n}}{2^{S_n(\kappa+1)+2\log_2 S_n - 1}} < \frac{2(\kappa S_n + 2\log_2 S_n + 1)^2}{S_n^2}. \tag{17}$$

From (17) one can easily see that

$$\kappa_1 + \frac{2\log_2 S_n}{S_n} + \frac{1}{S_n} > \frac{1}{\sqrt{2}}. \tag{18}$$

But (18) is not true for $S_n \geq 64$ and we obtain a contradiction. □

Remark 4. One can see that we can prove an even stronger statement, namely

$$a_{n+1} + \ldots + a_{n+k} < \kappa_1 s_n + 2k\log_2 s_n$$

From the (16) we know that $\forall n$

$$\frac{(a_{n+1}+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} > 1.$$

Consider this inequality for $n = m+k-1$:

$$\frac{(a_{m+k}+1)^2 q_{m+k-1}^2}{2^{S_{m+k}-1}} > 1.$$

Let us estimate that fraction on the left as:

$$1 < \frac{(a_{m+k}+1)^2 q_{m+k-1}^2}{2^{S_{m+k}-1}} < \frac{2(a_{m+k}+1)^2 \cdot \ldots \cdot (a_{m+1}+1)^2 q_m^2}{2^{s_m+a_{m+1}+\ldots+a_{m+k}}}.$$

By assuming that $a_{m+1} + \ldots + a_{m+k} \geq \kappa_1 s_m + 2k\log_2 s_m$ and applying the similar argument as in the proof of Theorem 2, we will get

$$\frac{\kappa_1}{k} + 2\frac{\log_2 s_m}{s_m} + \frac{1}{s_m} > \frac{1}{2^{1/(2k)}},$$

what will obviously fail even for small $k, m \in \mathbb{N}$.
Remark 5. Theorem 2 provides a (non-optimal) upper estimate on partial quotients of x. However, their mean behavior is much simpler. Denote

$$\lambda_i = \frac{i + \sqrt{i^2 + 4}}{2}, \qquad \kappa_2 = \frac{5 \log \lambda_4 - 4 \log \lambda_5}{0.5 \log 2 + \log \lambda_4 - \log \lambda_5} \approx 4.40104874\ldots \tag{19}$$

In [4] Dushistova, Moshchevitin and Kan proved that

Lemma 5.2 ([4], Theorem 3)**.** *Let for an irrational number x there exists a constant C such that for all natural t one has*

$$S_x(t) \geq \kappa_2 t - C.$$

Then $?'(x)$ exists and equals 0.

The fact that $?(x - \delta) < x - \delta$ for any positive δ implies that in our case $S_x(t) < \kappa_2 t - C$ for some C. Easy calculations show that one can take $C = 0$. Now we have an obvious consequence of Theorem 2 and Lemma 5.2.

Corollary 5.1. *Let x be fixed point from Theorem 1, then*

$$a_{n+1} < \kappa_1 \kappa_2 n + 2 \log_2(\kappa_2 n). \tag{20}$$

for all $n \geq 2$.

6 Proof of Theorem 3

Proof. From (14) and (16) we deduce that for any n

$$\frac{(a_{n+1} + 1)^2 q_n^2}{2^{S_n + a_{n+1} - 1}} > 1. \tag{21}$$

As $\frac{(a_{n+1}+1)^2}{2^{a_{n+1}-1}} < \frac{9}{2}$, we have

$$\frac{9}{2} 2^{S_n} < q_n^2$$

or

$$S_n < \frac{2}{\log 2} \log q_n + \log_2 \frac{2}{9}$$

As $a_{n+1} < \kappa_1 S_n + 2 \log_2 S_n$, we have

$$a_{n+1} < \kappa_1 \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{2}{9} \right) + \frac{2}{\log 2} \log \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{2}{9} \right)$$

Consider an arbitrary convergent continued fraction to x. As

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{(a_{n+1} + 1) q_n^2},$$

we see that

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{(\kappa_1 \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{2}{9} \right) + \frac{2}{\log 2} \log \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{2}{9} \right) + 1) q_n^2}.$$

□

Appendix

The following statement reduces the problem of fixed points of $?(x)$ to the properties of values of $?(x)$ at rational points only.

Theorem 4. *Conjecture 1 follows from the inequality*

$$\left|?\left(\frac{p}{q}\right)-\frac{p}{q}\right|>\frac{1}{2q^2} \tag{22}$$

for all $p,q\in\mathbb{Z}, q>2$.

Proof. Suppose that there are at least two fixed points from $(0,\frac{1}{2})$ interval, namely x_1 and x_1'. Consider an arbitrary rational number $\frac{p}{q}$ between them. Note that the sequence of iterations $\underbrace{?(?(\ldots?}_{n \text{ iterations}}(\frac{p}{q})\ldots)$ is monotonic and converges to some fixed point in $[x_1,x_1']$ interval. Without loss of generality one can say that the sequence is decreasing. Denote this point by x_1'', it may coincide with x_1. Consider an arbitrary even convergent fraction to x_1'', which lies between x_1'' and $\frac{p}{q}$. Denote it by $\frac{p_{2n-1}}{q_{2n-1}}$. As $\frac{p_{2n-1}}{q_{2n-1}}>?(\frac{p_{2n-1}}{q_{2n-1}})>x_1''$, we have

$$0<\frac{p_{2n-1}}{q_{2n-1}}-?\left(\frac{p_{2n-1}}{q_{2n-1}}\right)<\frac{p_{2n}}{q_{2n}}-x_1''<\frac{1}{a_{2n}q_{2n-1}^2}.$$

Then, if $a_{2n}\geq 2$, we have a contradiction with (22). Now suppose that for all n big enough one has $a_{2n}=1$. We know that

$$\frac{1}{q_{2n-1}^2}>\frac{p_{2n-1}}{q_{2n-1}}-x_1''>?\left(\frac{p_{2n-1}}{q_{2n-1}}\right)-?(x_1'')$$

The right-hand side of the previous inequality may be estimated as follows:

$$?\left(\frac{p_{2n-1}}{q_{2n-1}}\right)-?(x_1'')>\frac{1}{2^{S_{2n-1}+a_{2n}}-1}-\frac{1}{2^{S_{2n-1}+a_{2n}+a_{2n+1}}-1}>\frac{1}{2^{S_{2n-1}+a_{2n}}}$$

Hence, as $a_{2n}=1$,

$$\frac{q_{2n-1}^2}{2^{S_{2n-1}+1}}<1 \tag{23}$$

Now suppose that $?\left(\frac{p_{2n}}{q_{2n}}\right)<\frac{p_{2n}}{q_{2n}}$. Then, by (23) we have

$$\frac{(a_{2n+1}+1)q_{2n}^2}{2^{S_{2n-1}+a_{2n+1}}}>1 \tag{24}$$

From (23) and (14) we obtain that

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2\frac{a_{2n+1}+1}{2^{a_{2n+1}-1}}>1 \tag{25}$$

Note that

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2=1+[a_{2n-1},a_{2n-2},a_{2n-2},\ldots]<1+[1,a_{2n-2},1,\ldots] \tag{26}$$

As we already mentioned, there exists K such that $a_{2k}=1$ for all $k>K$. Hence one can say that

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2<\varphi+\varepsilon_n<1.62^2=2.6244 \quad \text{for } n \text{ large enough.} \tag{27}$$

Here ε_n is some function, which exponentially tends to 0 as n tends to infinity. Now we can see that $a_{2n+1}\leq 4$, because if $a_{2n+1}\geq 5$, from (26) we have

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2\frac{a_{2n+1}+1}{2^{a_{2n+1}-1}}<2.6244\frac{6}{16}<1 \tag{28}$$

and we obtain a contradiction. Hence there exists N such that for any $n > N$ we have $a_n \le 4$. By ([4], Theorem 3), we have $?'(x_1'') = +\infty$. As

$$\frac{?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) - ?(x_1'')}{\frac{p_{2n-1}}{q_{2n-1}} - x_1''} < 1$$

for all n large enough, we obtain a contradiction. Then $?\left(\frac{p_{2n}}{q_{2n}}\right) > \frac{p_{2n}}{q_{2n}}$ and we have

$$\begin{cases} 0 < \frac{p_{2n-1}}{q_{2n-1}} - ?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) < \frac{p_{2n-1}}{q_{2n-1}} - x_1'' \\ 0 < ?\left(\frac{p_{2n}}{q_{2n}}\right) - \frac{p_{2n}}{q_{2n}} < x_1'' - \frac{p_{2n}}{q_{2n}} \end{cases}$$

A classical theorem states that for any n at least one of the inequalities $\frac{p_{2n-1}}{q_{2n-1}} - x_1'' < \frac{1}{2q_{2n-1}^2}$ and $x_1'' - \frac{p_{2n-1}}{q_{2n-1}} < \frac{1}{2q_{2n}^2}$ holds. That finishes the proof. □

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