SURVIVAL AND COEXISTENCE FOR A SPATIAL POPULATION MODEL WITH FOREST FIRE EPIDEMICS

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Abstract. We investigate the effect on survival and coexistence of introducing forest fire epidemics to a certain two-species spatial competition model. The model is an extension of the one introduced by Durrett and Remenik [DR09], who studied a discrete time particle system running on a random 3-regular graph where occupied sites grow until they become sufficiently dense so that an epidemic wipes out large clusters. In our extension we let two species affected by independent epidemics compete for space, and we allow the epidemic to attack not only giant clusters, but also clusters of smaller order. Our main results show that, for the two-type model, there are explicit parameter regions where either one species dominates or there is coexistence; this contrasts with the behavior of the model without epidemics, where the fitter species always dominates. We also characterize the survival and extinction regimes for the particle system with a single species. In both cases we prove convergence to explicit dynamical systems; simulations suggest that their orbits present chaotic behavior.

1. Introduction

In the mathematical biology literature, resource competition between n species is widely modeled through Lotka-Volterra type ODEs of the form

$$\frac{dx_i(t)}{dt} \;=\; x_i(t)\Big(a_i - \sum_{j=1}^{n} b_{ij}x_j(t)\Big), \qquad i = 1, \dots, n$$

if time is taken to be continuous, and the analogous difference equations

$$x_i(m+1) - x_i(m) \;=\; x_i(m)\Big(a_i - \sum_{j=1}^{n} b_{ij}x_j(m)\Big), \qquad i = 1, \dots, n$$

if time is taken to be discrete, where $x_i \in [0,1]$ represents the density of the i-th species and the a_i's and b_{ij}'s are the parameters of the model. The term inside the parentheses determines the effect of inter-specific and intra-specific competition, and has the advantage of being simple enough for an easy interpretation of its coefficients while, at the same time, allowing the system to exhibit a rich asymptotic behavior, including fixed points, limit cycles and attractors. However, despite its ubiquitousness, the classical model seems inadequate to explain diverse and complex ecosystems, as conditions for stability become more restrictive for larger values of n; the same seems to be true regarding conditions for coexistence (see e.g. [HJ87; B$^+$16]), implying that, unless the parameters have been finely tuned, most species will be driven to extinction as a result of competition.

Even though it has been argued that natural selection alone may be able to tune the relevant parameters to yield a coexistence regime [AG04], a considerable amount of effort has been directed towards extending models such as Lotka-Volterra in ways that favor coexistence. Extensions of this sort include, for example, the addition of predators [MKo86; HS89; Sch97], of random fluctuations in the environment [ZY09; MMR02] and of diseases [HP85; SH06]; these extensions succeed in promoting biodiversity, but result in much more complicated models. An alternative way of extending the model is based on questioning the linear form of the inter-specific and intra-specific competition terms; indeed, for large population densities the intra-specific competition of a species has an increasingly important non-linear component, known

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as the *crowding effect*, which is overlooked in the original equations. The crowding effect is capable of effectively outbalancing the inter-specific competition effect for a significantly larger set of parameters, permiting coexistence even when n is large [HS02; Sev96; G$^+$18].

One important source for the crowding effect is the fact that at high population densities the connectedness between individuals tends to be high, making it easier for an infectious disease to spread through the population and giving rise to epidemic outbreaks. To the best of our knowledge, the effect that this phenomenon may have on coexistence has not been explored in the setting of competing spatial population models. This provides the main motivation for our paper.

The model which we will study is based on a particle system introduced by Durrett and Remenik [DR09], which we will refer to as the *moth model (MM)*. It is inspired by the gypsy moth, whose populations grow until they become sufficiently dense for the nuclear polyhedrosis virus (*Borralinivirus reprimens*, which strikes at larval stage and spreads between nearby hosts) to reduce them to a low level. The MM is a discrete time particle system which alternates between a growth stage akin to a discrete time contact process and a forest fire stage where an epidemic randomly destroys entire clusters of occupied sites. (Forest fire models, which were first introduced in [DS92], have received much interest as a prime example of a system showing self-organized criticality, see e.g. [RT09], but this is not the focus of our paper). [DR09] was devoted mostly to the study of the evolution of the density of occupied sites in the limit as the size of the system goes to infinity. Its main reult showed that the system converges to a discrete-time dynamical system which, for large enough rates of population growth, and as a result of the forest fire epidemic mechanism, is chaotic.

In this work we study an extension of the moth model to a case where there are multiple species competing for space, each one affected by a different disease. As expected, when birth rates are sufficiently large the evolution of the system still presents chaotic behavior. The main goal of our paper is to show that, in the case of two species, the introduction of forest fire epidemics can promote coexistence. The intuition behind this phenomenon is simple. Suppose that we have two species competing for space, in a situation where we would expect the fitter species to drive the other one to extinction. If we introduce forest fire epidemics into the system then the fitter species, which achieves higher densities, will be more susceptible to the destruction of very large occupied clusters. This will have the effect of periodically clearing space for the growth of the weaker species, which may then have a chance to survive.

Our main result shows that, for the particle system, and depending on the parameters of the model, the weaker species may die out quickly or it may in fact coexist with the fitter species for a relatively long time. We perform an analogous analysis of survival for the one-species particle system, complementing the [DR09] result for the limiting dynamical system.

Since the MM provides the basic setting for all of our results, we will begin by introducing it and the main results of [DR09] in some detail, and defer an overview of our extension and results until Sections 1.2 and 1.3. The detailed discussion of our results will be postponed until Section 2.

1.1. **The moth model.** The MM is a discrete time Markov process $\left(\eta_k^N\right)_{k\geq 0}$ taking values in $\{0,1\}^{G_N}$, where G_N is a finite, possibly random graph of size N, in which each vertex x is either occupied by a particle ($\eta_k^N(x)=1$) or empty ($\eta_k^N(x)=0$). The dynamics of the process at each time step is divided into two consecutive stages, *growth* and *epidemic*:

Growth: Each particle gives birth to a mean $\beta > 0$ number of individuals and then dies. Individuals born at site x are sent to a randomly chosen site in its *growth neighborhood* $\mathcal{N}_N(x) \subseteq G_N$.

Epidemic: Each site is attacked by an infection with probability α_N, independently across different sites. When an occupied site x is attacked, the infection wipes out the entire connected

component of occupied sites containing x. The occupied sites which survive the epidemic are the ones making up the population at the start of the next time step.

The main goal of [DR09] was to show that, for suitable choices of graphs G_N, and under some growth conditions on $\mathcal{N}_N$ and α_N, the trajectories described by the densities

$$\rho_k^N = \frac{1}{N} \sum_{x \in G_N} \eta_k^N(x) \tag{1.1}$$

converge to the orbit of a deterministic dynamical system which, for certain parameter values, is chaotic. The dynamical system obtained in [DR09] is defined by a map $h\colon [0,1] \to [0,1]$ of the form $h = g_0 \circ f_\beta$, where

$$f_\beta(p) = 1 - e^{-\beta p}$$

is the expected population density after the growth stage starting with density p and $g_0(q)$ is the expected density of sites that survive the epidemic stage when it attacks a population with density q which is uniformly spread (i.e. distributed according to a product measure with this density). The particular form of f_β can be guessed by approximating the spatially dependent model by its mean field version. The function g_0, on the other hand, depends heavily on the choice of the sequence of graphs G_N and the epidemic parameters α_N, which in [DR09] are assumed to be in the *weak epidemic regime* $\alpha_N \longrightarrow 0$, which implies that in the $N \to \infty$ limit the epidemic only attacks infinite connected components.

In the first part of [DR09], the authors take $\{G_N\}_{N\in\mathbb{N}}$ to be a sequence of random connected 3-regular graphs and work in the case of mean-field growth, where $\mathcal{N}_N(x) = G_N$ for all N. The mean-field assumption implies that after the growth stage the process looks like percolation on G_N, and since this graph looks locally like a 3-regular tree then one can hope to obtain explicit formulas: indeed, the probability that the root (and by consequence any vertex) is in an infinite component can be computed in terms of the survival probability of a binary branching process, and is given by

$$g_0(p) = \begin{cases} p & \text{if } p \le \frac{1}{2}, \\ \frac{(1-p)^3}{p^2} & \text{if } \frac{1}{2} < p \le 1. \end{cases} \tag{1.2}$$

Together with the above expression for f_β, this gives

$$h(p) = g_0 \circ f_\beta(p) = \begin{cases} 1 - e^{-\beta p} & \text{if } 0 \le p \le a_0, \\ \frac{e^{-3\beta p}}{(1-e^{-\beta p})^2} & \text{if } a_0 < p \le 1. \end{cases} \tag{1.3}$$

To keep the notation simple, in everything that follows we omit the dependence of h on the parameters of the model.

Throughout the paper we will use the notation $\mathsf{DS}(h)$ to denote the dynamical system $\big(h^n(p)\big)_{n\ge 0}$ defined from the iterates h^n of a given map h.

The following theorem states the precise convergence result for the evolution of the density ρ_k^N of occupied sites as $N \to \infty$:

Theorem 1.1 ([DR09], Thm. 2)**.** *Suppose that $(G_N)_{N\in\mathbb{N}}$ is a sequence of random connected 3-regular graphs and that $\mathcal{N}_N(x) = G_N$ for all x and N. Assume that the infection probability of the epidemic satisfies $\alpha_N \longrightarrow 0$ and $\alpha_N \log_2(N) \longrightarrow \infty$ as $N \to \infty$, and also that $\rho_0^N \longrightarrow p \in [0,1]$ in distribution as $N \to \infty$. Then the process $(\rho_k^N)_{k\ge 0}$ converges in distribution as $N \to \infty$ (on compact time intervals) to the (deterministic) orbit of $\mathsf{DS}(h)$ started at p.*

The behavior of $\mathsf{DS}(h)$ can be described as follows (see [DR09] for more details):

- If $\beta \le 1$ then for every $p \in [0,1]$ the sequence $h^k(p)$ decreases to 0 as $k \to \infty$.
- If $\beta \in (1, 2\log 2]$ then the orbit of $h^k(p)$ eventually gets trapped inside the interval $[0, \frac{1}{2}]$, where $h \equiv f_\beta$, which means that there are no epidemic outbreaks. Inside this interval, $h^k(p)$ converges to the only positive fixed point of f_β.

- If $\beta > 2\log 2$ then the orbit of $h^k(p)$ is still trapped inside the interval $[h(\frac{1}{2}), \frac{1}{2}]$ but there is no longer convergence to a fixed point. Indeed, since $\beta > 2\log 2$, the fixed point of f_β is larger than $\frac{1}{2}$, so the successive growth stages drive the density above this value, at which time the epidemic kicks in and forces a relatively large jump back to $[h(\frac{1}{2}), \frac{1}{2}]$.

Thus the case $\beta \leq 1$ corresponds to the *extinction* regime (at least for the limiting dynamical system), while for all $\beta > 1$ we have $\liminf_{k\to\infty} h^k(p) > 0$ (for all $p \geq 0$), which corresponds to *survival*.

The next result establishes the chaotic behavior of the orbits of h when $\beta > 2\log 2$ (the third of the cases above):

Theorem 1.2 ([DR09], Thm. 1)**.** *The dynamical system* $\mathsf{DS}(h)$ *restricted to the interval* $[h(\frac{1}{2}), \frac{1}{2}]$ *is chaotic for every* $\beta > 2\log 2$*. Furthermore, if* $\beta \in (2\log 2, 2.48]$*, then the system has an invariant measure,* $\mu = \mu \circ h^{-1}$*, which is absolutely continuous with respect to the Lebesgue measure.*

The notion of chaos in the first assertion of the theorem is the one given by Li and York [LY75] in their famous *period three implies chaos* theorem (see [DR09, Prop. 1.1] for more details). The authors also proved versions of Theorem 1.1 and of the second assertion of Theorem 1.2 (which is actually expected to hold for all $\beta > 2\log 2$) for the process running on the discrete torus with local growth, where newly born particles are sent to a local neighborhood with a diameter which grows suitably with N. However, in this case there is no explicit formula for g, nor numeric values for the critical parameters. It is precisely because of the availability of explicit formulas that, in everything that follows, we choose to work in the setting of random 3-regular graphs.

1.2. **The Multi-type Moth Model.** Our main interest in this paper is the study of the *multi-type moth model (MMM)*, a natural extension of the moth model which considers multiple species competing for space subject to the same sort of epidemics. We describe it formally next. Fix a graph G_N as before and let $m \in \mathbb{N}$, which will be the number of species. The MMM is a discrete time Markov chain $\left(\eta_k^N\right)_{k\geq 0}$ taking values in $\{0,\dots,m\}^{G_N}$; each site $x \in G_N$ can be occupied by an individual of type $i \in \{1,...,m\}$ ($\eta_k^N(x) = i$) or vacant ($\eta_k^N(x) = 0$). The process depends on two sets of parameters, $\vec{\beta} = (\beta(1),\dots,\beta(m)) \in \mathbb{R}_+^m$ and $\vec{\alpha}_N = (\alpha_N(1),\dots,\alpha_N(m)) \in [0,1]^m$, and as in the MM the dynamics of the process at each time step is divided into two consecutive stages:

Growth: An individual of type i at site $x \in G_N$ sends a Poisson$[\beta(i)]$ number of descendants to sites chosen uniformly at random in $\mathcal{N}_N(x) \subseteq G_N$. If a site receives individuals of more than one type, then the type of the site is chosen uniformly among the individuals it receives.

Epidemic: Each site x occupied by an individual of type i after the growth stage is attacked by an epidemic with probability $\alpha_N(i)$, independently across sites. The individual at x then dies along with its entire connected component of sites occupied by individuals of type i. This happens independently for $i = 1,\dots,m$.

Note that we have assumed that the offspring of each individual is Poisson distributed. Although it would be possible to work with more general offspring distributions, as in the MM, we opt to make this assumption in order to simplify the presentation and proofs.

If one suppresses the epidemic stage then our process turns into a multi-type contact process, for which it is relatively easy to prove that the fitter species (i.e. the one with the largest growth parameter $\beta(i)$) will outcompete and drive to extinction all the other ones. In our main result, Theorem 2.11, we show that the introduction of forest fire dynamics changes this picture, allowing two species to coexist even when they have different fitnesses. We remark, however, that in our model we are assuming that epidemics affect each species independently; this is natural when considering epidemics lacking cross-species transmission due to genetic distance, but is not a very realistic assumption if one thinks about the competition of different

species of trees and takes the forest fire metaphor literally. It seems, nevertheless, that this assumption is important for coexistence to arise in our setting, as we will discuss further in Section 2.2, where we present an example with non-specific epidemics in which the stronger species drives all the rest to extinction. It should be noted that this qualitative difference between epidemics with and without cross-species transmission is somewhat similar to the one found in the literature for predators, where the addition of a "specialist" predator to Lotka-Volterra systems can be more effective in promoting coexistence than the addition of a "generalist" predator (see [Sch97]).

Remark 1.3. A related model was studied by Chan and Durrett [CD06], who proved coexistence for the two-type, continuous time contact processes in $\mathbb{Z}^2$ with the addition of a different type of forest fires, which act by killing all individuals (regardless of their type, and regardless of whether they are connected) within blocks of a certain size. They showed that if the weaker competitor has a larger dispersal range then it is possible for the two species to coexist in the model with forest fires; this contrasts with Neuhauser's result [Neu92] for the model without forest fires for which such coexistence is impossible. Our context is different, since we work on a random graph with forest fires which travel only along neighbors of the same type and which have an unbounded range, and since all species use the same dispersal neigborhoods. The techniques we use are also different, and the results we obtain are of a slightly different nature. But the motivation is similar, and our results complement nicely with theirs.

As we already mentioned, all of our results will be proved in the case where G_N is a random connected 3-regular graph. The first step in our analysis of the MMM is an adaptation of Theorem 1.1 to the multi-type case, Theorem 2.2. More precisely, let $\{\rho_k^N\}_{k\geq 0}$ denote the sequence of density vectors obtained from $\{\eta_k^N\}_{k\geq 0}$ as

$$\rho_k^N = (\rho_k^{N,(1)}, \ldots, \rho_k^{N,(m)}) \qquad \text{with} \quad \rho_k^{N,(i)} = \frac{1}{N} \sum_{x \in G_N} \mathbb{1}_{\{\eta_k^N(x)=i\}}. \tag{1.4}$$

Then under suitable conditions on $\vec{\beta}$, $\vec{\alpha_N}$ and the growth neighborhood $\mathcal{N}_N(x)$, and assuming further that $\vec{\alpha_N}$ converges to some limiting $\vec{\alpha} \in [0,1]^m$, we show that ρ_k^N converges in distribution to a certain dynamical system $\mathsf{DS}(h)$ which is an m-dimensional analogue of the dynamical system we obtained for the MM.

We remark that in this work we are going beyond the weak epidemic regime of [DR09] by allowing the infection rates $\alpha_N(i)$ to converge to arbitrary values $\alpha(i) \in [0,1]$. This is natural from the biological point of view, as it incorporates into the model the effect of diseases with a fixed incidence rate. This generalization, which has the effect of modifying the $m \geq 1$ analog of g_0 (see (1.2)), has a major impact on the dynamical system, allowing the epidemics to kill not just infinite connected components but finite ones as well. In particular, for $\alpha(i) > 0$ the density of the type-i population no longer needs to be above the percolation parameter of the network for the epidemic to kick in, so we observe its effects at all times.

1.3. **Overview of the main results.**

1.3.1. *Phase diagram of the dynamical system.* Since our main interest is to understand whether the introduction of forest fire epidemics can promote coexistence, for simplicity we restrict our study of the phase diagram of $\mathsf{DS}(h)$ to the case of two species ($m = 2$). Our results (Theorems 2.7 and 2.9) show that, as expected, there exist parameter regions where domination occurs (that is, where the fittest species drives the other one to extinction) as well as other parameter regions where both species coexist (that is, where both coordinates of h^k remain bounded below as $k \to \infty$). The regions obtained in our theorems are defined through two explicit inequalities, (2.9) and (2.10), which are naturally expressed in terms of the parameters

$$\phi_i := (1 - \alpha(i))\beta(i),$$

which we will refer to as the *fitness* of each species (and corresponds to the effective birth rate of individuals after considering the probability that a newly born particle does not survive the epidemic stage due to an infection arising in its location). In particular, we find the following (see also Figure 5); here we assume that type 2 corresponds to the fitter species:

- Extinction is certain for any species with fitness value $\phi_i \leq 1$. This is analogous to the extinction for the case $\beta \leq 1$ described in the analysis of $\mathsf{DS}(h)$ for the MM.
- For every given fitness value $\phi_2 > 1$ of the stronger species we can choose ϕ_1 sufficiently close to, but larger than, 1, so that type 2 dominates.
- For any $\varepsilon > 0$ small we can choose ϕ_1 and ϕ_2 large but with relative fitness $\frac{\phi_1}{\phi_2} = \varepsilon$ such that both species coexist.

Note that, in view of the second and third points above, given any small $\varepsilon > 0$ we can choose two different sets of parameters with the same relative fitness ε so that in one case type 1 is driven to extinction while in the other case there is coexistence. Hence relative fitness does not provide enough information about the behavior of the system, which indicates that the effect of the forest fire epidemics is what is driving the qualitative difference in behavior.

As we have mentioned, even in the case of $m = 1$ our model provides an extension of the model studied in [DR09], as it drops the weak epidemics assumption by allowing for $\alpha_N \to \alpha > 0$. This extension is far from trivial at the level of the limiting dynamical system $\mathsf{DS}(h)$: as we will notice in Section 2.2.1, from numerical simulations it is clear that for each fixed $\alpha \in (0, 1)$ the bifurcation diagram of $\mathsf{DS}(h)$ develops *bifurcation cascades* (also known as *period-doubling bifurcations*) in β, such as those seen for example for the quadratic maps $x \longmapsto rx(1 - x)$, see Figure 1. See Section 2.2.1 for more details.

1.3.2. *Coexistence and survival for the particle systems.* Our main results concern the behavior of the particle system for finite N and for one and two species. The main idea is to show that the behavior of the limiting dynamical systems $\mathsf{DS}(h)$ provides a good guide for the behavior of the original process. Note, however, that the MMM is a finite state Markov chain for which the extinction time of all types is almost surely finite, so we need to change our notions of coexistence and survival when working at the level of the particle systems. To this end we follow the usual approach (see e.g. [Cox89; DL88]) where one characterizes the different phases of the system in terms of the behavior of the (random) extinction times as a function of the network size N. Our main result in the case with $m = 2$ is Theorem 2.11, which shows that there are parameter choices so that the weaker species dies out quickly while the fitter one survives for a relatively long time, and other parameter choices for which both species survive for a relatively long time.

The main challenge in proving results for our particle systems comes from the slow convergence of the empirical densities to the limiting dynamical system. This is intrinsic in the very nature of our model: it is hard to obtain a fine control on the distance between the finite system and its limit when the limiting system itself presents chaotic behavior, which makes it essentially impossible to predict its evolution. As a consequence, in our proof of coexistence we are not able to show that the extinction times of both species grow exponentially in N, as should be expected. For the case $m = 1$ (Theorems 2.5 and 2.6), on the other hand, we prove survival (when $\phi > 1$) and extinction (when $\phi \leq 1$) arguing directly on the particle system η_k^N (and not relying on the convergence to the dynamical system), and as a result we are able to prove that the expected extinction time does indeed grow exponentially at least for $\phi > \phi^*$ for some $\phi^* > 1$.

Outline. The rest of the article is organized as follows. In Section 2.1 we state our convergence results (discussed in Section 1.3.1). In Section 2.2 we state the results related to the MM (discussed in Section 1.3.1), while in Section 2.3 we state the results related to the MMM (the multi-type case discussed in Section 1.3.2). Last two sections also contain brief discussions

about the main aspects involved in the proofs of our results. The proof themselves are deferred to Sections 4, 5 and A, devoted to the MM, the MMM and some technical results respectively.

2. Results

2.1. **Convergence.** As discussed in Section 1.2, the starting point of our work is a convergence theorem for the MMM, analogous to the convergence proved in [DR09, Thm. 2] for the MM with weak epidemics. Analogously to (1.3), the limiting dynamical system will be given as $\mathsf{DS}(h)$ with h of the form $g_{\vec{\alpha}} \circ f_{\vec{\beta}}$, where $f_{\vec{\beta}}$ and $g_{\vec{\alpha}}$ describe the limiting densities after the growth and epidemic stage, respectively. In order to derive a good candidate for $f_{\vec{\beta}}$ we will focus for simplicity on the mean-field model ($\mathcal{N}_N(x) = G_N$), even though our result will be slightly more general, allowing for $\mathcal{N}_N(x) = B(x, r_N)$ (the ball around x of radius r_N in the natural graph distance for G_N) for r_N converging to infinity sufficiently fast. Recalling the Poisson assumption on the offspring distribution, the expected number of occupied sites after the growth stage with initial densities given by $p \in [0,1]^m$ is $1 - e^{-\sum_{i=1}^m \beta(i) p_i}$, and since in the process we let each site choose its type uniformly at random from the particles it receives, the expected density of sites occupied by type i after the growth stage is given by

$$f_{\vec{\beta}}^{(i)}(p) = \left(1 - e^{-\sum_{i=1}^m \beta(i) p_i}\right) \frac{\beta(i) p_i}{\sum_{i=1}^m \beta(i) p_i}. \tag{2.1}$$

The function $g_{\vec{\alpha}}$, on the other hand depends heavily on the particular choice G_N which, we recall, we always take to be a random 3-regular connected graph. In this case, and as explained in Section 1.1, the graph looks locally like a 3-regular tree, so in order to guess a candidate for $g_{\vec{\alpha}}$ we can pretend that the epidemic stage acts on the infinite 3-tree $\mathcal{T}$. Let us also assume for a moment that $m = 1$. Then we need to analyze the effect of the epidemic when attacking a configuration of particles distributed as site percolation on $\mathcal{T}$ with a given density q (whose distribution, i.e. a product measure on $\{0,1\}^{\mathcal{T}}$ where each vertex is occupied with probability q, we denote as $\mathbf{P}_q$). Note that if $\mathcal{C}_r$ denotes the connected component of occupied sites containing r then the probability that r survives is given by $(1-\alpha_N)^{|\mathcal{C}_r|} \mathbb{1}_{\{|\mathcal{C}_r|>0\}}$. As a consequence, we should expect that the limiting probability that a given site is occupied after the epidemic stage (when it attacks a system with a fraction q of occupied sites) be given by

$$g_\alpha(q) = \mathbf{P}_q(r \text{ is occupied, } r \text{ survives the epidemic}) = \mathbf{E}_q((1-\alpha)^{|\mathcal{C}_r|} \mathbb{1}_{\{|\mathcal{C}_r|>0\}})$$

(here r is any vertex of $\mathcal{T}$).

The right hand side can be computed explicitly:

Proposition 2.1. *For any $q \in [0,1]$,*

$$g_\alpha(q) = \begin{cases} 0 & \text{if } \alpha = 1, \\ \frac{\left(1 - \sqrt{1 - 4(1-\alpha)q(1-q)}\right)^3}{8(1-\alpha)^2 q^2} & \text{if } \alpha \in (0,1), \\ q & \text{if } \alpha = 0. \end{cases}$$

The explicit formula in the case $\alpha \in (0,1)$ (whose simple proof is included in Section 3) is related to the generating function of the Catalan numbers. Now in the general case, when $m \geq 1$, since the epidemics attack each species independently, we deduce that the density of sites occupied by type i after the epidemic stage acts on a population with initial densities $\vec{q} \in [0,1]^m$ should be given by

$$g_{\vec{\alpha}}^{(i)}(\vec{q}) = g_{\alpha(i)}(q_i). \tag{2.2}$$

We are ready to state our main convergence result. Given $p \in [0,1]^m$ define $h(p) = \big(h_1(p), \dots, h_m(p)\big)$ through

$$h_i(p) = g_{\alpha(i)} \circ f_{\vec{\beta}}(p).$$

Note that in the case $m = 1$, h_1 coincides with the function h defined above for the MM, which justifies our use of the same notation in both cases.

Theorem 2.2. *Consider the MMM with m types and with $\mathcal{N}_N(x) = B(x, r_N)$. Suppose that the sequences $\vec{\alpha_N}$ and r_N satisfy*

$$\alpha_N(i) \xrightarrow[N\to\infty]{} \alpha(i) \in [0,1], \qquad \alpha_N(i) r_N \xrightarrow[N\to\infty]{} \infty, \qquad \text{and} \qquad r_N \le \tfrac{1}{25}\log_2(N)^2 \alpha_N(i) \quad \forall\, N \in \mathbb{N}. \tag{2.3}$$

Suppose also that η_0^N is a product measure where each site is independently chosen to have type i with probability p_i. Then as $N \to \infty$, the density process $\left(\rho_k^N\right)_{k\ge 0}$ associated to the MMM converges in distribution (on compact time intervals) to the deterministic orbit, starting at $p = (p_1, \dots, p_m)$, of the dynamical system $\mathsf{DS}(h)$.

Note that the last two assumptions in (2.3) are trivially satisfied in the mean-field case $\mathcal{N}_N(x) = G_N$ (i.e. $r_N = \infty$) if $\alpha(i) > 0$. The proof of Theorem 2.2 is based on a relatively simple adaptation of the arguments of [DR09], needed to control the effect of the epidemic on finite components in order to go beyond the weak epidemics regime. It is worth noting (and will be clear from the proof) that in the mean-field case one could drop the product measure assumption on the initial condition (simply because the growth step returns a product measure anyway).

As discussed in the introduction, the behavior of the limiting dynamical system $\mathsf{DS}(h)$ provides us with informed guesses regarding the behavior of our particle systems. However, the above convergence result is not sufficient in order to prove that the behavior of the dynamical systems is in fact mirrored at the level of the finite MMM particle system; this requires quantitative estimates on the speed of convergence with an explicit control on the depedence on N. The approximation result that follows provides the necessary estimates in the case of mean-field growth. We believe that the result holds in the local growth setting of Theorem 2.2; however, the algebraic expressions involved become even more complicated, so for simplicity we choose, here and in basically all the other upcoming results, to restrict the discussion to the simpler mean-field setting.

Let

$$\theta_\alpha(N) = \begin{cases} e^{-\sqrt{\log(N)}} & \text{if } \alpha = 0 \\ N^{-\alpha/5} & \text{if } \alpha > 0. \end{cases}$$

Theorem 2.3. *Consider the mean-field MMM (i.e. $\mathcal{N}_N(x) = G_N$ for all $x \in G_N$) with m types. Suppose that the sequence $\vec{\alpha}_N$ converges to some $\vec{\alpha} \in [0,1]^m$ and satisfies*

$$-\alpha_N(i) \log N / \log \alpha_N(i) \longrightarrow \infty \tag{2.4}$$

for each i. Then for all $\delta > 0$ and $k \in \mathbb{N}$ there is a constant $C > 0$ depending only on δ and k such that for all $N \in \mathbb{N}$ and any initial condition η_0^N we have

$$\mathbb{P}\Big(\big\| \rho_k^N - h^k(\rho_0^N) \big\| > \delta \Big) \;\le\; C\theta_{\underline{\alpha}}(N), \tag{2.5}$$

where $\underline{\alpha} = \min\{\alpha(1), \dots, \alpha(M)\}$.

The main ingredient in the proof is Lemma 3.1, which uses a comparison with a branching process to estimate the difference between h and the expected density after one step.

2.2. **Results for the one-type model.**

2.2.1. *Phase diagram and bifurcation cascades.* We begin our study of the MM by briefly exploring the behavior of the limiting dynamical system. Recall our definition of the fitness parameter

$$\phi = \phi(\alpha, \beta) = \beta(1 - \alpha).$$

The following simple result establishes the desired phase transition between extinction and survival in the orbits of $\mathsf{DS}(h)$.

Proposition 2.4. *Let $\alpha \in [0,1]$ and $\beta > 0$.*

(i) *(Extinction) If* $\phi(\alpha,\beta) \leq 1$*, then*

$$h^k(p) \xrightarrow{k\to\infty} 0 \qquad \forall p \in [0,1].$$

(ii) *(Survival) if* $\phi(\alpha,\beta) > 1$*, then*

$$\liminf_{k\to\infty} h^k(p) > 0 \qquad \forall p \in (0,1).$$

Let us briefly comment on an interesting behavior which becomes apparent from numerical simulations of the orbits of $\mathsf{DS}(h)$: the *bifurcation cascades* which we mentioned in Section 2.2.1. These are sequences of period doubling bifurcations that occur as the parameter β is increased (for fixed $\alpha > 0$), and which accumulate at a certain finite value of β. Figure 1 (left) shows bifurcation diagrams for $\mathsf{DS}(h)$ which clearly suggest the occurence of this phenomenon in our system. This behavior contrasts with case $\alpha = 0$ where, as pointed out in [DR09] (see the discussion preceding Prop. 1.1 there), the system proceeds directly from a stable fixed point to a chaotic phase, without passing through period-doubling bifurcation; the parameter α has thus the effect of modulating the appearance of these bifurcation cascades.

The prototypical example of a dynamical system presenting this behavior is the one defined by the quadratic map $x \longmapsto rx(1-x)$, which has a first period doubling bifurcation occurring at $r = 3$ and then subsequent ones which continue up to $r \approx 3.56$, where a chaotic regime arises; this pattern is then repeated for larger values of r. This intricate behavior has been intensely studied since at least the 1970's, and presents an intriguing form of universality, which roughly states that the ratio of the gaps between subsequent period doubling bifurcations converges to a universal constant for a wide class of dynamical systems showing this type of cascades (see e.g. [Fei78; TC78], where several universality conjectures were settled). This area of dynamical systems continues to be developed to this day (see e.g. [SY11; JSY10]); we refer the reader to [TCF14] for a nice account. Our simulations suggest that cascades appear for all $\alpha \in (0,1)$ when β is increased above 1, but proving this appears to be difficult due to the algebraic structure of h (in particular, the bifurcation points do not have a simple analytic expression). Figure 1 (right) shows a simulation of the evolution of the MM for finite N and different values of β; note how some of the period doubling bifurcation behavior of the limiting system is still apparent in these simulations.

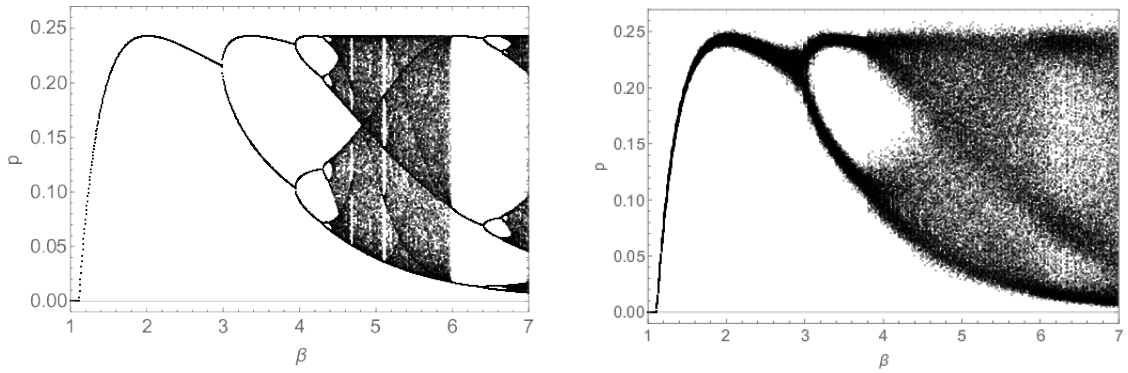


FIGURE 1. Left: Bifurcation diagram in β for $\mathsf{DS}(h)$ with $\alpha = 0.1$, showing the orbits of the system between iterations 900 and 1000 in the vertical direction for different values of β.

Right: Simulation of the evolution of the mean-field MM for $\alpha = 0.1$ and different values of β, from iteration 900 to 1000. Here $N \in \{20000, 40000, 100000\}$ (depending on β).

Figure 2 presents a schematic summary, partly based on simulations, of the behavior of the orbits of $\mathsf{DS}(h)$ as a function α and β.

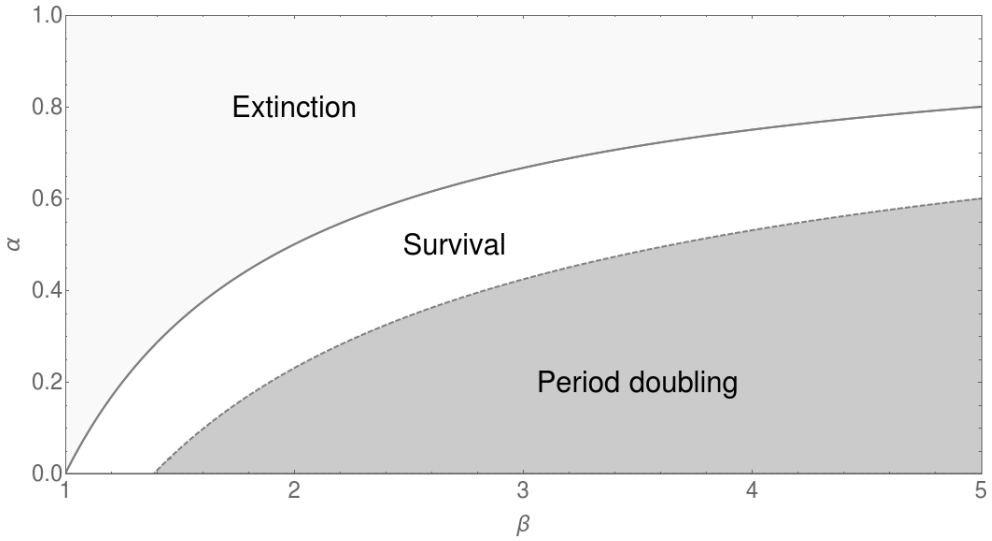


FIGURE 2. Approximate phase diagram of $\mathsf{DS}(h)$. The transition between extinction and survival is justified by Proposition 2.4, while the one governing the appearance of bifurcation cascades (dashed line) is based on simulations.

2.2.2. *Extinction and survival for the particle system.* We turn now to the dichotomy between extinction and survival at the level of the MM particle system for finite N. As discussed in the introduction, we will exhibit contrasting behaviors for the *absorption time*

$$\tau_N := \inf\{k \geq 1 \colon \eta_k^N(x) = 0 \ \forall\, x\} = \inf\{k \geq 1 \colon \rho_k^N = 0\}.$$

The following result is satisfied for any fixed N (the size of the graph G_N) and any fixed choice of $\alpha(N)$, but for the sake of concreteness one may think of the case $\alpha(N) \longrightarrow \alpha$ (or even $\alpha(N) = \alpha$).

Theorem 2.5. *For the mean-field* MM *and any $N \in \mathbb{N}$ we have:*

(i) (Extinction) *If $\phi(\alpha_N, \beta) \leq 1$, then for all $n \in \mathbb{N}$ and any initial density ρ_0^N*

$$\mathbb{P}(\tau_N \geq n) \leq \begin{cases} 1 - (1 - \phi(\alpha_N, \beta)^n)^N & \text{if } \phi(\alpha_N, \beta) < 1, \\ 1 - \left(1 - \frac{2}{n(1-\alpha_N)(\sigma^2 + \alpha_N \beta^2)}\right)^N & \text{if } \phi(\alpha_N, \beta) = 1, \end{cases}$$

where σ^2 is the variance of the offspring distribution of each particle in the growth stage. In particular, it follows that when $\phi(\alpha_N, \beta) < 1$ there is a $C > 0$ independent of N such that

$$\mathbb{E}(\tau_N) \leq C \log(N). \tag{2.6}$$

(ii) (Survival) *If $\phi(\alpha_N, \beta) > 1$ and $\rho_0^N \geq \bar{\rho}_0$ for some $\bar{\rho}_0 > 0$, then there exists $c > 0$ (depending only on $\bar{\rho}_0$ and α_N) such that*

$$\mathbb{P}(\tau_N \geq n) \geq \left(1 - \frac{c}{N}\right)^{3n}.$$

In particular, if we assume that $\alpha_N \log_2(N) \to \infty$ then

$$\mathbb{E}(\tau_N) \geq \frac{N}{4c}. \tag{2.7}$$

The proof of extinction is simple, and is based on a comparison with a branching process process where one essentially ignores the effect of epidemics. For survival the main idea of the proof is to keep track of isolated occupied sites, which are not affected by epidemic events coming from other sites.

We believe that in the extinction regime the process actually has exponential expected absorption times. In the next result we show that this is indeed the case, at least for large enough ϕ, under an additional (but reasonable) condition on our random graphs.

Recall that a k-independent set of a graph G is a subset I of its vertices such that, for any $x, y \in I$, $d_G(x, y) \geq k$. Given $0 \leq b < 1$ we define the events

$$\mathcal{R}_N(b) := \big\{G_N \text{ has a 3-independent set } I \text{ with } |I| \geq bN\big\}.$$

From [BDZ05, Thm. 1.1] there exists $b \approx 0.09$ such that

$$\mathbb{P}\big(\mathcal{R}_N(b)\big) \xrightarrow[N\to\infty]{} 1. \tag{2.8}$$

In words, our random 3-regular graphs contain a 3-independent set made out of fraction of at least b of its vertices with probability close to 1 as N becomes large. This justifies conditioning on $\mathcal{R}_N(b_1)$ in the coming theorem.

Theorem 2.6. *Fix b as in (2.8) and assume that $\phi(\alpha_N, \beta) > 1/b$ and that $\rho_0^N \geq \bar{\rho}_0$ for some $\bar{\rho}_0 > 0$. Then (in the case of mean-field growth) there is a $c > 0$, depending only on α_N and $\bar{\rho}_0$ such that*

$$\mathbb{P}\big(\tau_N \geq n \big| G_N \in \mathcal{R}_N(b)\big) \geq (1 - \exp(-cN))^{3n}.$$

In particular, if $\alpha_N \longrightarrow \alpha \in [0, 1)$ and $\alpha_N \log_2(N) \longrightarrow \infty$ as $N \to \infty$, then

$$\mathbb{E}\big(\tau_N | G_N \in \mathcal{R}_N(b)\big) \geq \begin{cases} 3\exp(cN) & \text{if } \alpha \in (0, 1), \\ 3\exp\Big(\frac{cN}{\log(N)^2}\Big) & \text{if } \alpha = 0. \end{cases}$$

2.3. **Results for the multi-type model.** In everything that follows we only consider the two-type case, $m = 2$.

2.3.1. *Phase diagram.* As for the MM, we begin by studying the behavior of the orbits of $\mathsf{DS}(h)$. The analysis is much more involved than the one for the one-type model, but it will provide us with a glimpse on the role that the forest fire dynamics can have in aiding coexistence. In fact, our results in this part, together with the above approximation result (Theorem 2.3), will constitute the basic ingredients for our later analysis of the particle system.

We are interested in identifying two different regimes for $\mathsf{DS}(h)$: we say that there is *domination* if one species goes extinct while the other one survives, i.e. if $\liminf_{k\to\infty} h^k(\vec{p})$ has one and only one vanishing coordinate, while we say that there is *coexistence* if both types survive, i.e. if the same liminf is strictly positive in both coordinates. Notice that once one species dies out, the behavior of the other one, say the one with type i, evolves according to the dynamical system given by $h_i = g_{\alpha(i)} \circ f_{\beta(i)}$ as in the one-type case.

Since we are interested in coexistence, we will restrict the discussion to the case when

$$\phi_i := \phi(\alpha(i), \beta(i)) > 1$$

for both $i = 1$ and $i = 2$; by Proposition 2.4 we know that if this fails then at least one of the species would die out even when facing no competition, whence it easily follows that coexistence would be impossible. For concreteness we will always assume type 2 is fitter than type 1, i.e. $\phi_1 < \phi_2$.

In order to ease notation, from now on we denote, for a given initial condition $p \in [0, 1]^2$ and any $i \in \{1, 2\}$

$$p_i^k = h_i^k(p).$$

Theorem 2.7 (Coexistence)**.** *There is a continuous, increasing function $z : [0, 1] \longrightarrow \mathbb{R}^+$ (defined in (5.20)) satisfying $z(0) = 2\log(2)$ and $z(1) < 4\log(2)$ such that the following holds. Suppose that $\phi_2 > z(\alpha(2))$ and*

$$\phi_1 \sqrt{\frac{2(1 - e^{-\kappa_2})(1 - e^{-\frac{\phi_2}{2}})}{\kappa_2 \phi_2}} \quad > \quad 1, \tag{2.9}$$

where κ_2 is the solution of $\kappa_2 = \beta(2)\, g_{\alpha(2)}(1-e^{-\kappa_2})$. Then for any initial condition $p \in (0,1)^2$ we have

$$\liminf_{k\to\infty} p_1^k > 0, \qquad \textit{and} \qquad \liminf_{k\to\infty} p_2^k > 0.$$

The intuition behind this result is the following. Consider the case where p_1 is very small. In that scenario the effect of the type 1 species on p_2^k is negligible, meaning that type 2 evolves as if it were alone. On the other hand, for p_1 small, the total growth of type 1 after one iteration will be roughly ϕ_1 times a factor smaller than 1 corresponding to the competition effect coming from p_2. What condition (2.9) states is that, on average, this competition effect coming from type 2 (represented by the square root factor) is not strong enough to compensate the growth produced by ϕ_1, allowing thus p_1 to move away from low density values.

Remark 2.8. In (2.9), the parameter ϕ_1 needs to grow roughly as $\sqrt{\phi_2 \log(\phi_2)}$ as a function of ϕ_2 in order for the left hand side to stay above 1. To see this, use the definition of κ_2 to write $\phi_2 = \frac{\kappa_2}{g_{\alpha(2)}(1-e^{-\kappa_2})} = \frac{\kappa_2}{(1-e^{-\kappa_2})G_{\alpha(2)}(1-e^{-\kappa_2})^3} = \frac{\kappa_2(1+\sqrt{1-4(1-\alpha(2))e^{-\kappa_2}(1-e^{-\kappa_2})})^3}{(1-e^{-\kappa_2})8e^{-3\kappa_2}}$, which says that κ_2 grows roughly as $\log(\phi_2)$, and then substitute this approximation in (2.9).

The next result states the domination counterpart to Theorem 2.7.

Theorem 2.9 (Domination)**.** *Let $a_1(x)$ be the solution of $a_1(x) = x(1-e^{-a_1(x)})$ and assume that ϕ_1 and ϕ_2 satisfy*

$$a_1(\phi_1) \;<\; \frac{\phi_2}{1-\alpha(2)} \min\Big\{ g_{\alpha(2)}(1-e^{-\frac{\phi_2}{2}}), g_{\alpha(2)}(1-e^{-a_1(\phi_1)}) \Big\}. \tag{2.10}$$

Then for any initial condition p with $p_2 \in (0,1)$ we have

$$p_1^k \xrightarrow[k\to\infty]{} 0 \qquad \textit{and} \qquad \liminf_{k\to\infty} p_2^k > 0.$$

Even though the condition given in (2.10) is again relatively obscure (see Figure 5 for an approximation of the associated region), the basic idea behind this result is simple. Starting from any initial condition we show that the orbit of the dynamical system eventually reaches a set B where p_1 decays exponentially. We then employ (2.10) to show that neither low nor high values of p_2 can take the dynamical system out of B, making it a "trapping" set where type 1 species dies out.

Remark 2.10. It is easy to see that $a_1(\phi_1)$ is increasing with respect to ϕ_1, with $a_1(1) = 0$, so for a given ϕ_2, any value of ϕ_1 sufficiently close to 1 satisfies (2.10).

Simulations suggest that if ϕ_2 is smaller than but sufficiently close to $2\log 2$ and $\alpha(1), \alpha(2) \in (0,1)$, then there exists ϕ_1 smaller than ϕ_2 such that coexistence holds. See Figure 4 (right) for a simulation which exhibits this behavior (note that both species have positive density to the left of the leftmost vertical line).

2.3.2. *Coexistence and domination in the MMM.* We arrive finally at the main results of the paper, which explore the possibility of domination and coexistence for the MMM. This is done by using the approximation theorem to transfer the properties of $\mathsf{DS}(h)$ derived in the last section to the associated particle systems for suitable families of parameters.

Let us stress again that, if we consider the MMM without epidemics, then the resulting process is nothing more than a multi-type contact process, for which it is known that the species with larger offspring parameter will always outcompete the other one (this has been proved for other choices of G_N, e.g. the result of [Neu92] mentioned in Remark 1.3, but in the current setting of mean-field growth it would be simple to prove). The upcoming results will show that, as advertised, there are choices of parameters for which there is coexistence even when one species has a larger offspring parameter, and hence that the introduction of forest fire dynamics can indeed lead to coexistence in a system which would otherwise show domination.

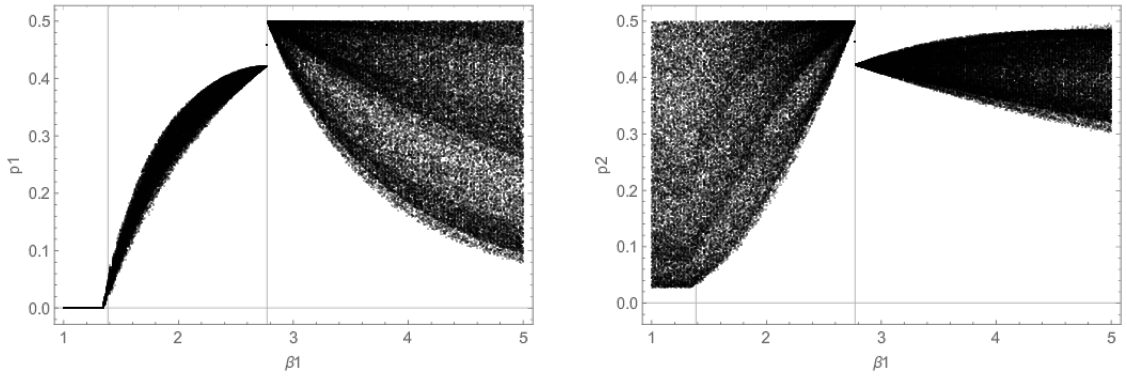


FIGURE 3. Bifurcation diagrams in $\beta(1)$ for type 1 on the left and type 2 on the right, with $\beta(2) = 4\log(2)$ and $\alpha(1) = \alpha(2) = 0$. From left to right, in each figure, the first vertical line is at $\phi_1 = 2\log 2$ and the second one at $\phi_1 = \phi_2$. These diagrams reflect theorems 2.9(b) and theorem 2.7. These diagrams depict regions corresponding to Theorems 2.9 (dominance of type 2 over type 1) and 2.7 (coexistence); coexistence corresponds to the region between the two vertical lines in both figures.

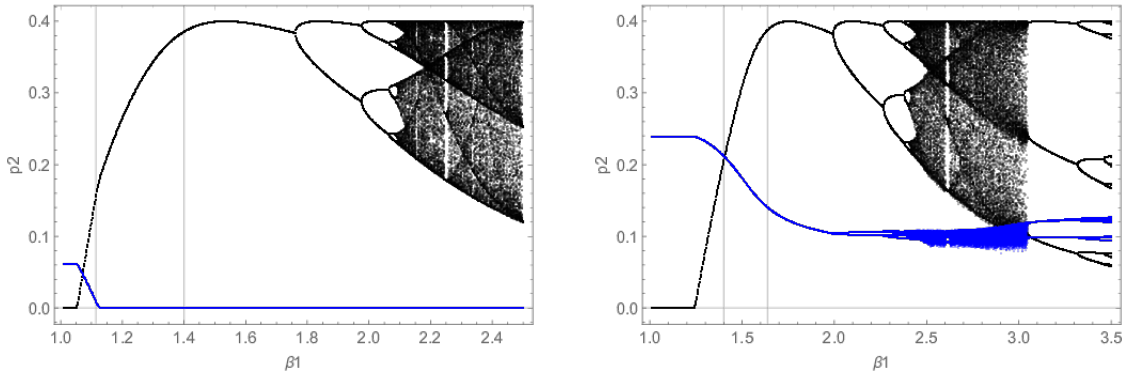


FIGURE 4. Bifurcation diagram for type 1 (black) and type 2 (blue). Left: $\beta(2) = 1.99\log(2)$, $\alpha(1) = 0.01$ and $\alpha(2) = 0.2$. From left to right, the first vertical line is at $\phi_1 = \phi_2$ while the second one is at $\phi_1 = 2\log 2$. Right: $\beta(2) = 2.6\log(2)$, $\alpha(1) = 0.01$ and $\alpha(2) = 0.1$. From left to right, the first vertical line is at $\phi_1 = 2\log 2$ while the second one is at $\phi_1 = \phi_2$.

Let

$$\tau_N^i = \inf\{k \geq 1 \colon \eta_k^N(x) \neq i \ \forall x \in G_N\} = \inf\{k \geq 1 : \rho_k^{N,(i)} = 0\}$$

denote the extinction time of the type i.

Theorem 2.11. *Consider the two-species mean-field MMM running on a random 3-regular graph G_N. Suppose that for each N the initial density of the process ρ_0^N is in $(0,1)^2$, and that the sequence $\vec{\alpha}_N$ satisfies the conditions in Theorem 2.3. Then there are constants $C = C(\rho_0^N) > 0$ and $\gamma \in (0,1)$ such that for $\underline{\alpha} = \min\{\alpha(1), \alpha(2)\}$ we have:*

(i) (Coexistence) *If $\vec{\alpha}_N$ and $\vec{\beta}$ satisfy the conditions of Theorem 2.7, then*

$$\mathbb{P}(\tau_N^1, \tau_N^2 \geq n) \ \geq \ \big(1 - C\,\theta_{\underline{\alpha}}(N)\big)^n. \tag{2.11}$$

(ii) (Domination of type 2 over type 1) *If $\vec{\alpha}_N$ and $\vec{\beta}$ satisfy the conditions of Theorem 2.9, then*

$$\mathbb{P}(\tau_N^2 \geq n) \ \geq \ \big(1 - C\,\theta_{\underline{\alpha}}(N)\big)^{n+1} \tag{2.12}$$

and

$$\mathbb{P}(\tau_N^1 \geq n) \ \leq \ 2 - (1-\gamma^n)^N - \big(1 - C\,\theta_{\underline{\alpha}}(N)\big)^n + C\,\theta_{\underline{\alpha}}(N). \tag{2.13}$$

In particular, if we assume that $\rho_0^N \longrightarrow p \in (0,1)^2$ as $N \to \infty$, then:

(i') *For $\vec{\alpha}$ and $\vec{\beta}$ satisfying the conditions of Theorem 2.7 and all $\varepsilon > 0$,*

$$\mathbb{P}\big(\tau_N^1, \tau_N^2 \geq 1/\theta_{\underline{\alpha}}(N)^{1-\varepsilon}\big) \xrightarrow{N\to\infty} 1. \tag{2.14}$$

(ii') *For $\vec{\alpha}$ and $\vec{\beta}$ satisfying the conditions of Theorem 2.9, and for all $\varepsilon > 0$, there is a $C' > 0$ depending only on p such that*

$$\mathbb{P}(\tau_N^1 \leq C' \log N) \xrightarrow{N\to\infty} 1 \quad \textit{and} \quad \mathbb{P}\big(\tau_N^2 \geq 1/\theta_{\underline{\alpha}}(N)^{1-\varepsilon}\big) \xrightarrow{N\to\infty} 1. \tag{2.15}$$

Recalling that $\theta_{\underline{\alpha}}(N)^{1-\varepsilon}$ is of larger order than $\log(N)$, this gives domination.

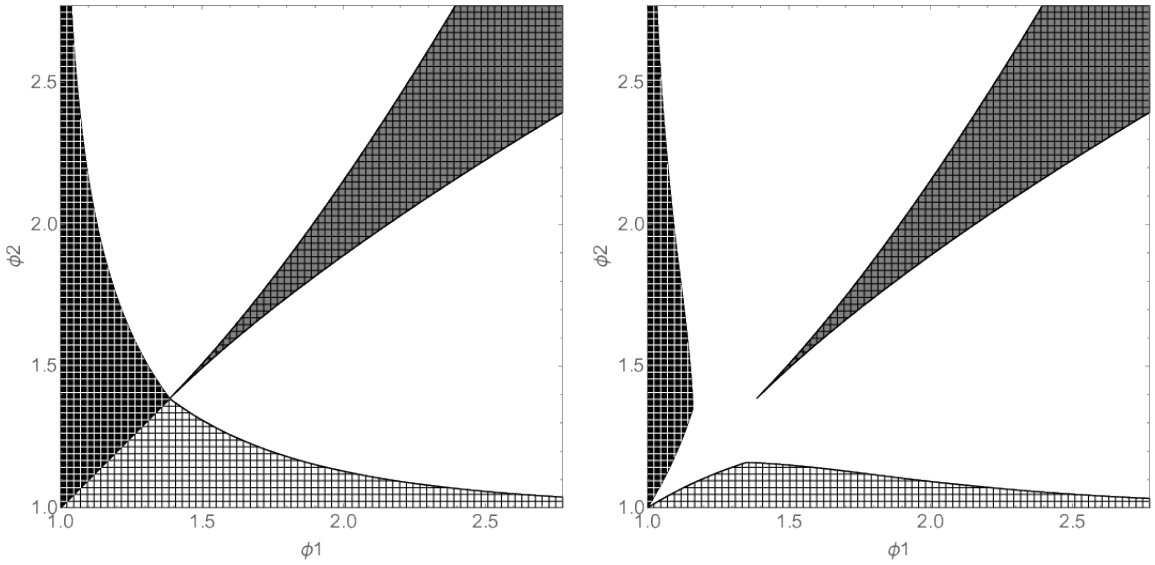


FIGURE 5. Summary of the domination and coexistence regimes for the MMM, for $\alpha(1) = \alpha(2) = 0$ on the left and $\alpha(1) = \alpha(2) = 0.1$ on the right. The white (resp. black) regions represent the domination regime of type 1 over type 2 (resp. type 2 over type 1); these regions are justified by Theorems 2.9 and 2.11. The gray regions roughly correspond the coexistence regime, and are justified by Theorems 2.7 and 2.11 (the coexistence regions are only approximate in the sense that they where plotted based on their asymptotic behavior: as $\phi_2 \to \infty$, ϕ_1 grows as $\sqrt{\phi_2 \log(\phi_2)}$, see Remark 2.8).

As a consequence if condition (2.4) is satisfied, then (in the mean-field case) we have that:

(a) Under the conditions of Theorem 2.7 there is coexistence, in the sense that with high probability both species are present in the system for an amount of time of order at least $\theta_{\underline{\alpha}}(N)^{-1}$.
(b) Under the conditions of Theorem 2.9 there is domination, in the sense that, with high probability, the extinction time of type 1 is at most of order $\log(N)$ while type 2 survives for at least an amount of time of order at least $\theta_{\underline{\alpha}}(N)^{-1}$.
(c) The possibilities for survival and extinction listed in Section 1.3.1 hold for the MMM.
(d) In particular, there exist $\phi_2 > 2\log(2)$ and $\phi_1 < \phi_1' < \phi_2$, such that in the MMM associated to (ϕ_1, ϕ_2) type 2 dominates over type 1 while the MMM associated to (ϕ_1', ϕ_2) is in the coexistence regime. This can be achieved, moreover, when $\alpha(1) = \alpha(2) = 0$.

Figure 5 contains a sketch of the regions of the phase diagram of the process which have been probed in Theorem 2.11, which in particular makes the existence of the parameter triplets $(\phi_1, \phi_1', \phi_2)$ referred to in (d) above apparent. In fact, as $\phi_2 \to \infty$ we have that ϕ_1' is of order $\sqrt{\phi_2 \log(\phi_2)}$ (see 2.8 for an explanation), and hence we can find $\phi_1' < \phi_2$ satisfying the corollary for $\beta(2)$ sufficiently large.

3. Proofs of the convergence and approximation results

We begin with the simple proof of the formula for g_α.

Proof of Proposition 2.1. Recall that $\mathcal{T}$ denotes an infinite 3-tree, $\mathbf{P}_p$ denotes the site percolation measure on $\mathcal{T}$ with density p, and $\mathcal{C}_r$ denotes the percolation cluster containing a given vertex r. The cases $\alpha = 0$ and $\alpha = 1$ are straightforward, so we turn to the case $\alpha \in (0,1)$, where we have

$$\mathbf{E}_p((1-\alpha)^{|\mathcal{C}_r|}\mathbb{1}_{|\mathcal{C}_r|>0}) = \sum_{n=1}^{\infty}(1-\alpha)^n \mathbf{P}_p(|\mathcal{C}_r| = n).$$

Let A_n be the number of possible connected components of size n in a 3-tree rooted at r, so that $\mathbf{P}_p(|\mathcal{C}_r| = n) = A_n p^n (1-p)^{n+2}$ (notice that $n+2$ is the number of vacant sites surrounding a cluster $\mathcal{C}_r$ of size n). Noting that a 3-tree is a root connected to three binary trees and recalling that the analog of A_n for a binary tree is given by the Catalan numbers C_n, we get

$$A_0 = 1 \quad \text{and} \quad A_{n+1} = \textstyle\sum_{i=0}^{n}\sum_{j=0}^{n-i} C_i C_j C_{n-i-j}. \tag{3.1}$$

Defining the generating functions $A(x) = \sum_{n=0}^{\infty} A_n x^n$ and $C(x) = \sum_{n=0}^{\infty} C_n x^n$, the above equation gives

$$A(x) = xC(x)^3 + 1 = x\Big(\tfrac{1-\sqrt{1-4x}}{2x}\Big)^3 + 1,$$

where we have used the explicit formula for $C(x)$ (see [Slo10]). Using this above yields

$$\begin{aligned}\mathbf{E}_p((1-\alpha)^{|\mathcal{C}_r|}\mathbb{1}_{|\mathcal{C}_r|>0}) &= \sum_{n=1}^{\infty}(1-\alpha)^n p^n (1-p)^{n+2} A_n = (1-p)^2\big(A((1-\alpha)p(1-p)) - 1\big) \\ &= \tfrac{\left(1-\sqrt{1-4(1-\alpha)p(1-p)}\right)^3}{8(1-\alpha)^2 p^2}.\end{aligned}$$

□

The proof of the convergence result, Theorem 2.2, is a relatively simple adaptation of the proof of [DR09, Thm. 4] for the one-species model running on the torus, so we defer it to the appendix. We turn next to the approximation result:

Proof of Theorem 2.3. Define the event $H_N = \{x \in G_N : G_N \cap B(x, L_N) \text{ is a finite 3-tree}\}$ with $L_N = \log_2(N)/5$. Observe first that, since $\delta > 0$ is arbitrary, and from the uniform continuity of h, we only need to prove the statement of the theorem for $k = 1$. Even further, it is enough to show that for any fixed $j \in \{1, \dots, m\}$ and $\delta > 0$ we can find C such that

$$\mathbb{P}\Big(\big|\rho_1^{N,(j)} - h_j(\rho_0^N)\big| > \delta \Big) \;\le\; C\theta_{\alpha(j)}(N). \tag{3.2}$$

Fix then any such j and, as in [DR09], define $\tilde{\eta}_1^N$ as $\tilde{\eta}_1^N(x) = j$ if the vertex x belongs to H_N and at time $\frac{1}{2}$ it is occupied by an individual of type j that survives the epidemic when one ignores infections arising outside $B(x, L_N)$. Defining $\tilde{\rho}_1^N$ analogously to ρ_1^N as the density of $\tilde{\eta}_1^N$, the event inside the probability (3.2) is contained in

$$\begin{aligned}\mathsf{Q}_N := \Big\{ &\big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big| + \big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\big| \\ &+ |\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})| + |\mathbb{E}(\tilde{\rho}_1^{N,(j)}) - h_j(\rho_0^N)| > \delta\Big\},\end{aligned}$$

and using Markov's inequality we obtain

$$\begin{aligned}\mathbb{P}(\mathsf{Q}_N) \le\; &\tfrac{4}{\delta}\mathbb{E}\Big(\big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big|\Big) + \tfrac{4}{\delta}\mathbb{E}\Big(\big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\big|\Big) \\ &+ \tfrac{4}{\delta}\mathbb{E}\Big(|\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})|\Big) + \mathbb{P}\Big(|\mathbb{E}(\tilde{\rho}_1^{N,(j)}) - h_j(\rho_0^N)| > \tfrac{\delta}{4}\Big),\end{aligned} \tag{3.3}$$

so the result will follow by showing that each term in on the right hand side is bounded by $C\theta_{\alpha(j)}(N)$ for some C independent of ρ_0^N. For the first term we use the bound

$$\mathbb{E}\Big(\big|\rho_1^{N,(j)} - \tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N|\big|\Big) \leq \frac{\mathbb{E}(G_N \setminus H_N)}{N} \leq 4N^{-3/5},$$

the second inequality by [DR09, Lem. 3.2], while for the third term we use

$$\begin{aligned}\mathbb{E}\Big(|\tilde{\rho}_1^{N,(j)} - \mathbb{E}(\tilde{\rho}_1^{N,(j)})|\Big)^2 &\leq \mathrm{Var}(\tilde{\rho}_1^{N,(j)}) \leq N^{-2}\,|\{(x,y) \in H_N \times H_N,\ d(x,y) \leq 2L_N\}| \\ &\leq N^{-2}\sum\nolimits_{x\in G_N} |B(x,L_N)| \ = \ N^{-2}(2N \cdot N^{2/5}) \ \leq \ 2N^{-3/5},\end{aligned}$$

where we used independence between any pair of events of the form $x \in \tilde{\eta}_1^{N,(j)}$ and $y \in \tilde{\eta}_1^{N,(j)}$ for $x, y \in H_N$ with $d(x,y) > 2L_N$. Hence for both terms we obtain a bound $\frac{16}{\delta}N^{-3/10}$, which (since $\alpha < 1$) is smaller than $C\theta_{\alpha(j)}(N)$.

In order to control the second and fourth terms in (3.3) observe that by translation invariance we can fix any vertex $r \in G_N$ and use the definition of $\tilde{\eta}_1^N$ to express $\mathbb{E}(\tilde{\rho}_1^{N,(j)})$ as

$$\mathbb{E}(\tilde{\rho}_1^{N,(j)}) \ = \ \mathbb{P}(\tilde{\eta}_1^N(r) = j) \ = \ \mathbb{E}\Big(\mathbb{1}_{\{r\in\eta_{1/2}^{N,(j)}\cap H_N\}}(1-\alpha_N(j))^{|\mathcal{C}_r^j \cap B(r,L_N)|}\Big),$$

where $\mathcal{C}_r^j$ is the type j connected component containing r. Now, the event $r \in H_N$ implies that $B(r, L_N)$ is a 3-tree, and by the mean-field assumption for the growth stage, at time $1/2$ each vertex is occupied by a type j individual independently with probability $q = f_{\vec{\beta}}^{(j)}(\rho_0^N)$. As a result, $|\mathcal{C}_r^j \cap B(r, L_N)|$ will be the size of the cluster containing r in the percolated 3-tree, which we represent as the total amount of individuals of a Galton-Watson process $Z_0, Z_1, \ldots, Z_{L_N-1}$. More precisely since a 3-tree can be seen as a vertex connected to the root of three binary trees, we set the offspring distribution of the first generation of the Galton-Watson process to be a Binomial$[3, q]$ and of all subsequent generations to be a Binomial$[2, q]$, with $Z_0 = \mathbb{1}_{\{r\in\eta_{1/2}^{N,(j)}\}}$, giving the expression

$$\begin{aligned}\mathbb{E}(\tilde{\rho}_1^{N,(j)}) &= \ \mathbb{E}\big(\mathbf{1}_{\{r\in H_N\}} Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big) \\ &= \ \mathbb{P}(r \in H_N)\,\mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big),\end{aligned} \tag{3.4}$$

where the second equality comes from the fact that given the event $r \in H_N$, the variables $Z_0, Z_1, \ldots, Z_{L_N-1}$ do not depend on the particular realization of G_N.

The next result, whose proof we postpone to the appendix, allows us to control $\mathbb{E}(\tilde{\rho}_1^{N,(j)})$:

Lemma 3.1. *Take a sequence $(\alpha_N)_{N\geq 0} \subseteq [0,1]$ converging to some α, and a Galton-Watson process $Z_0, Z_1, \ldots$ as above. Assuming that $-\alpha_N \log N / \log \alpha_N \longrightarrow \infty$, there is a $C > 0$ independent of q such that for all N,*

$$\big|\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big) - g_{\alpha_N}(q)\big| \leq C\theta_\alpha(N). \tag{3.5}$$

The same bound holds for $\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}=0\}}\big)$.

Using Lemma 3.1, (3.4), the uniform convergence of $g_{\alpha_N(j)}$ to $g_{\alpha(j)}$, and that $\mathbb{P}(0 \in H_N) \longrightarrow 1$, we deduce that there is are C, N_0 independent of ρ_0^N such that $|\mathbb{E}(\rho_1^{N,(j)}) - h_j(\rho_0^N)| < C\theta_\alpha(N)$ for all $N \geq N_0$. In particular, we deduce

$$\mathbb{P}\big(|\mathbb{E}(\tilde{\rho}_1^{N,(j)}) - h^j(\rho_0^N)| > \tfrac{\delta}{4}\big) \ \leq \ \tfrac{4}{\delta}C\theta_{\alpha(j)}(N),$$

so it only remains to control the second term in (3.3). Notice that $\tilde{\rho}_1^{N,(j)} - \frac{1}{N}|\eta_1^{N,(j)} \cap H_N|$ corresponds by definition to the fraction of vertices x in H_N which at time $\frac{1}{2}$ are occupied by an individual of type j that survives the restricted epidemic but not the unrestricted one. In

particular, for any such vertex there must be an open path to the boundary of $B(x, L_N)$ used by the unrestricted infection to kill x, so we deduce

$$\begin{aligned}\mathbb{E}\Big(\Big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\Big|\Big) &\le \mathbb{E}\big(\mathbb{1}_{\{r\in\eta_{1/2}^{N,(j)}\cap H_N\}}(1-\alpha_N(j))^{|\mathcal{C}_r^j\cap B(r,L_N)|}\mathbb{1}_{\{\mathcal{C}_r^j\not\subseteq B(r,L_N)\}}\big)\\ &\le \mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}>0\}}\big),\end{aligned}$$

where the variables $Z_0, \dots, Z_{L_N-1}$ are defined as before. This last bound is equal to

$$\mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\big) - \mathbb{E}\big(Z_0(1-\alpha_N(j))^{Z_0+Z_1+\cdots+Z_{L_N-1}}\mathbb{1}_{\{Z_{L_N-1}=0\}}\big),$$

but from Lemma 3.1 both therms are at distance at most $C\theta_\alpha(N)$ from $g_{\alpha_N(j)}(q)$, so

$$\mathbb{E}\Big(\Big|\tfrac{1}{N}|\eta_1^{N,(j)} \cap H_N| - \tilde{\rho}_1^{N,(j)}\Big|\Big) \le 2C\theta_\alpha(N),$$

giving the result. □

4. Proofs for the one-type model

Proof of Theorem 2.5(i). We start by sampling the graph G_N, which will remain fixed for the rest of the argument, and labeling its vertices as $\{1, \dots, N\}$. The proof is based on a simple coupling of the MM on G_N with a branching process $(Z_n)_{n\in\mathbb{N}}$. To this end we consider stacks $\{(\mathsf{O}_i^j)_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of i.i.d. random variables distributed according to the offspring distribution of the MM model and stacks $\{(\mathsf{E}_i^j)_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of i.i.d. random variables with $\mathbb{P}(\mathsf{E}_i^j = 1) = 1-\mathbb{P}(\mathsf{E}_i^j = 0) = \alpha$. We use the subset of these stacks of random variables with subindices $i \le N$ to define the MM process $(\eta_n)_{n\in\mathbb{N}}$ on G_N with a given initial configuration η_0 in the obvious way, using O_i^j to determine the offspring of the individual at site i and time j (if occupied) and E_i^j similarly for the occurence of an epidemic event. Using these variables we define the branching process as follows: we set $Z_0 = |\eta_0|$ and then use the random variable $\mathsf{O}_i^j(1-\mathsf{E}_i^j)$ to determine the offspring of i-th individual of Z_{j-1}. Z corresponds to a version of the MM where there is no restriction on the number of individuals per site, and where the epidemics attack each individual independently with probability α but they are not transmitted; it is clear then that $|\eta_n| \le Z_n$ for all n.

Let now $\tau_k^{\mathrm{GW}} = \inf\{n \ge 1 : Z_n = 0\}$ be the extinction time of the Galton-Watson process started with k particles. From branching process theory we know that, since the mean offspring is ϕ_N, then starting with 1 particle we have that

$$\mathbb{P}(\tau_1^{\mathrm{GW}} \ge n) \le \begin{cases}(\phi_N)^n & \text{if } \phi_N < 1,\\ \frac{2}{\mathsf{Var}(\mathsf{X}_1^1)n} & \text{if } \phi_N = 1.\end{cases}$$

Focusing on the case $\phi_N < 1$, it follows that, since $|\eta_0| \le N$, there exists a $c > 0$ such that

$$\begin{aligned}\mathbb{P}(\tau_N \ge n) &\le \mathbb{P}(\tau_{|\eta_0|}^{\mathrm{GW}} \ge n) = \mathbb{P}\left(\max_{i\in\{1,2,\dots,|\eta_0|\}}\tau_1^{\mathrm{GW}}(i) \ge n\right)\\ &\le 1-(1-(\phi_N)^n)^{|\eta_0|} \le 1-(1-(\phi_N)^n)^N \le 1-\exp\left(-c(\phi_N)^n N\right).\end{aligned}$$

The next-to-last bound is what we wanted. The last bound yields the estimate on the expectation: in fact, for $K_0 = \log_{\phi_N^{-1}}(N)$ there exists $C > 0$, such that

$$\textstyle\sum_{n\in\mathbb{N}}(1-\exp(-c(\phi_N)^n N)) \le K_0 + \sum_{n\ge K_0}(1-\exp(-c(\phi_N)^n N)) \le K_0 + \frac{c}{1-\phi_N} \le C\log(N).$$

The same arguments yield the result in the case $\phi_N = 1$. □

The proof of Theorem 2.5(ii) will be adapted from that of Theorem 2.6, so we turn to that proof next.

Proof of Theorem 2.6. The basic idea of the proof is to keep track of the isolated particles in each stage (growth and epidemic). Notice that since we are in the mean-field case we can suppose without loss of generality that we start from a product measure (the growth step returns a product measure anyway). We will need an upper bound on the number of empty

sites. We will say that a site is infected by the epidemic if the epidemic attacks the site, irrespective of whether the site is occupied or not. Let EP_i denote the number of sites infected by the epidemic at time i. Since the mean number of sites attacked by the epidemic is $\alpha_N N$, a Chernoff bound yields $\mathbb{P}(|\mathsf{EP}_i - \alpha_N N| \le \alpha_N N/2) \ge 1 - 2\exp(-\alpha_N^2 N/2)$. In particular (and uniformly on the initial condition we have)

$$\mathbb{P}(\mathsf{D}_i) \ge 1 - 2\exp(-\alpha_N^2 N/2) \qquad \text{with} \quad \mathsf{D}_i = \{\rho_i \le 1 - \alpha_N/2\}. \tag{4.1}$$

Note that the process after the growth stage (started with density p) is just site percolation on G_N with parameter $f_\beta(p)$. We say that a vertex u of G_N is *isolated* at time $i+1/2$ if u is open in $\eta_{i+1/2}$ but all its neighbors are closed, and we let $\mathsf{X}_u^i = \mathbb{1}_{\{u \text{ is isolated at time i+1/2}\}}$. The family $\{\mathsf{X}_u^i\}_{u\in G_N}$ is of course not independent; in order to produce an independent family we will divide the graph into a set of disjoint claws, by which we mean one vertex joined with its three neighbors.

By hypothesis the graph has a 3-independent set I_N of size at least bN, which implies that it has at least bN disjoint claws (each given by a vertex in I_N together with its three neighbors).

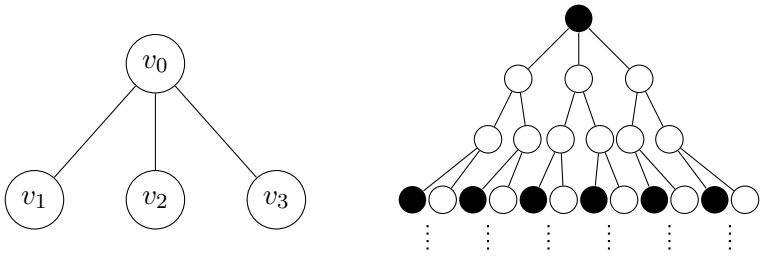


FIGURE 6. The left figure depicts a claw, also called a cherry or $K_{1,3}$ in the literature. The right figure depicts the local behavior of a 3-independent set on the 3-regular tree (black vertices belong to I_N).

Consider the family $(\mathsf{X}_u^i)_{u\in I_N}$, and notice that it is made of independent random variables. Let $\mathsf{p}_{i+1/2}$ be the density of isolated particles after the growing stage in the i-th iteration of the system. Let also p_{i+1} be the density of isolated sites that survive the epidemic:

$$\mathsf{p}_{i+1} = \frac{1}{N}\sum\nolimits_{v\in G_N} \eta_{i+1}(v)\mathsf{X}_v^i.$$

Let $\rho_{I_N} = |I_N|/N$ and define $\mathsf{Iso}_N(\rho) = (1-e^{-\beta\rho})e^{-3\beta\rho}\rho_{I_N}$; this quantity will be important in the developments that follow as it represents the expected number of isolated particles after the growth stage in I_N. More formally, starting with density ρ, the probability that a given site is empty after the growth stage is by translation invariance equal to the expectation of the density after the growth stage starting with density ρ, and this expectation is equal to $(1-\beta/N)^{\rho N} \approx e^{-\beta\rho}$. For simplicity we will use the function Iso_N not just as an approximation for the expectation, but instead of the actual expectation; this may be justified from the fact that this approximation has a rate of convergence which is much faster than the approximations which we do in what follows, but we leave these details to the reader. Notice that we can suppose that we sample the graph, we choose I_N, and then we run the process, so that the function Iso at this point is deterministic; we omit the dependence on the graph and on N. Define $\mathsf{Iso}(\rho) := (1-e^{-\beta\rho})e^{-3\beta\rho}b$ and notice that $\mathsf{Iso}_N(x) \ge \mathsf{Iso}(x)$ for all $x \in [0,1]$, since we are conditioning on $G_N \in \mathcal{R}_N(b)$. Let

$$m = m(\delta) = \min_{x\in[\delta,1-\delta]} \mathsf{Iso}(x).$$

This minimum is attained at one of the boundary values, as can be checked from simple properties of $\mathsf{Iso}(x)$. In particular, $\mathsf{Iso}'(x)$ vanishes at $x = \bar{x}$ defined as the unique point

satisfying $\exp(\beta\bar{x}) = 4/3$, which is a maximum. The values of the function are strictly positive inside $[0,1]$, hence $m > 0$. Also 0 is not attractive since the derivative there is $b\beta > 1$.

Lemma 4.1. *For small enough $\varepsilon > 0$ there exists $\bar{\delta} \in (0, 1/2)$ such that for every $\delta \in (0, \bar{\delta})$ satisfying $m(\delta)(1-\alpha_N)(1-2\varepsilon) \le x \le 1 - \alpha_N/2$, we have $m(\delta) \le \mathsf{Iso}(x)$.*

Proof. Notice that $\lim_{x\to 0} \frac{\mathsf{Iso}(x)}{x} = b\beta$, so for $\gamma \ge 0$ small, there exists θ_γ such that if $0 \le x \le \theta_\gamma$, then $\frac{\mathsf{Iso}(x(1-\alpha_N)(1-2\varepsilon))}{x(1-\alpha_N)(1-2\varepsilon)} \ge \frac{b\beta}{1+\gamma}$. But since $1 < (1-\alpha_N)\beta b$, the conclusion follows if we choose γ and ε such that $(1-\alpha_N)\beta b\frac{(1-2\varepsilon)}{(1+\gamma)} \ge 1$, since it is enough then to consider $\bar{\delta}$ such that $m \le \theta_\gamma$. □

Fix some small $\varepsilon > 0$ and choose δ such that $m(\delta)(1-\alpha_N)(1-2\varepsilon) \le \rho_0^N$; this can be achieved since $m(\delta)$ is decreasing in δ. The event $\{\rho_0^N \le 1 - \alpha_N/2\}$, on the other hand, happens with high probability thanks to (4.1). Hence with our parameter choices we may apply Lemma 4.1. Now we have all the elements to prove the theorem. Define the events

$$\begin{aligned}
\mathsf{A}_i &= \{m(1-\alpha_N)(1-2\varepsilon) \le \rho_i \le 1 - \alpha_N/2\},\\
\mathcal{A}_i &= \cap_{j=0}^i \mathsf{A}_j,\\
\mathsf{B}_{i+1/2} &= \{|\mathsf{p}_{i+1/2} - \mathsf{Iso}_N(\rho_i)| \le \varepsilon \mathsf{Iso}_N(\rho_i)\},\\
\mathsf{C}_{i+1} &= \{|\mathsf{p}_{i+1} - \mathsf{p}_{i+1/2}(1-\alpha_N)| \le \varepsilon(1-\varepsilon)(1-\alpha_N)\mathsf{p}_{i+1/2}\},\\
\mathsf{E}_i &= \{\rho_i \ge m(1-\alpha_N)(1-2\varepsilon)\}.
\end{aligned}$$

Observe that, from the previous comment and (4.1), $\mathbb{P}(\mathsf{A}_0) > 1 - \exp(-\alpha_N^2 N/2)$ by choosing $\delta > 0$ sufficiently small. Also observe that $\mathbb{P}(\tau_N \ge n+1) \ge \mathbb{P}(\mathcal{A}_{n+1}) = \mathbb{P}(\mathsf{A}_0)\prod_{i=0}^n \mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i)$ and that we have the decomposition

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i) &= \mathbb{P}(\mathsf{E}_{i+1}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i) \ge \mathbb{P}(\mathsf{C}_{i+1}, \mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i)\\
&= \mathbb{P}(\mathsf{C}_{i+1}|\mathcal{A}_i, \mathsf{B}_{i+1/2}, \mathsf{D}_i)\mathbb{P}(\mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1})\mathbb{P}(\mathsf{D}_{i+1}|\mathcal{A}_i);
\end{aligned} \tag{4.2}$$

the inequality follows from $\rho_i \ge \mathsf{p}_i$ and noting that on the events $\mathsf{B}_{i+1/2}$ and C_{i+1} we have

$$\begin{aligned}
|\mathsf{p}_{i+1} - \mathsf{Iso}_N(\rho_i)(1-\alpha_N)| &\le \varepsilon \mathsf{p}_{i+1/2}(1-\alpha_N) + (1-\alpha_N)\varepsilon \mathsf{Iso}_N(\rho_i)\\
&\le \varepsilon(1-\alpha_N)(1-\varepsilon)\mathsf{Iso}_N(\rho_i) + (1-\alpha_N)\varepsilon\mathsf{Iso}_N(\rho_i) \le 2\varepsilon(1-\varepsilon)(1-\alpha_N)\mathsf{Iso}_N(\rho_i),
\end{aligned}$$

which together with Lemma 4.1 gives $\rho_{i+1} \ge \mathsf{p}_{i+1} \ge \mathsf{Iso}_N(\rho_i)(1-\alpha_N)(1-2\varepsilon) \ge m(1-\alpha_N)(1-2\varepsilon)$ as needed. We need to bound the product in the second line of (4.2). The bound for the third factor is obtained from (4.1) The middle factor can be bounded using a Chernoff bound similarly to (4.1) (notice that independence is crucial here again),

$$\begin{aligned}
\mathbb{P}(\mathsf{B}_{i+1/2}|\mathcal{A}_i, \mathsf{D}_{i+1}) &= 1 - \mathbb{P}(\mathsf{B}_{i+1/2}^c|\mathcal{A}_i, \mathsf{D}_{i+1}) \ge 1 - \mathbb{E}(\mathbb{E}(2e^{-2N\mathsf{Iso}_N(\rho_i)^2\varepsilon^2}|\rho_i, \mathcal{A}_i, \mathsf{D}_{i+1})|\mathcal{A}_i, \mathsf{D}_{i+1})\\
&\ge 1 - 2\exp(-2Nm^2\varepsilon^2).
\end{aligned}$$

For the first factor we use that, conditional on $\mathsf{B}_{i+1/2}$ and $\mathcal{A}_i$, $\mathsf{p}_{i+1/2} \ge (1-\varepsilon)\mathsf{Iso}_N(\rho_i) \ge m(1-\varepsilon)$, and that each isolated particle lives independently from the others with probability $(1-\alpha_N)$; a similar estimate then gives

$$\mathbb{P}(\mathsf{C}_{i+1}|\mathsf{B}_{i+1/2}, \mathcal{A}_i, \mathsf{D}_{i+1}) \ge 1 - 2\exp(-2Nm^2(1-\alpha_N)^2\varepsilon^2(1-\varepsilon)^4).$$

The conclusion is that $\mathbb{P}(\mathcal{A}_{i+1}|\mathcal{A}_i) \ge (1-e^{-cN})^3$ for some c which depends on α_N and in ρ_0, and hence

$$\mathbb{P}(\tau_N \ge n+1) \ge (1-e^{-cN})^{3(n+1)} \tag{4.3}$$

as desired.

To obtain an estimate on the expected value appearing in the theorem we need to sum the right hand side of (4.3) in n. Note that in the above bounds, ε, δ and m are fixed, so we only need to understand how the constant c in (4.3) depends on α_N. Notice first that (since $\alpha \in [0,1)$) the dependence on α_N comes only from our bound on $\mathbb{P}(\mathsf{A}_0)$, namely

$\mathbb{P}(\mathsf{A}_0) > 1 - \exp(-\alpha_N^2 N/2)$. If $\alpha_N \to \alpha \in (0,1)$ then there is nothing to prove. Otherwise, if $\alpha_N \to 0$, the condition $\alpha_N \log_2(N) \longrightarrow \infty$ gives a similar bound, since fixing $M \in \mathbb{N}$ one gets $\alpha_N \geq \frac{M}{\log_2(N)}$ for large enough N, and then there exists $c' > 0$ such that $\mathbb{P}(\mathsf{A}_0) \geq 1 - \exp\left(-c' N (\log_2(N))^{-2}\right)$, which again gives us the bound we want. □

Proof of Theorem 2.5(ii). We will just we explain how to adapt the proof of Theorem 2.6 to obtain this result. We use the whole graph instead of a 3-independent set, and since the variables are now dependent, we change the Chernoff bounds to Chebyshev bounds. Consequently we use $\mathsf{Isop}(x) = (1 - e^{-\beta\rho})e^{-3\beta\rho}$ instead of Iso, and we bound the middle factor on the second line of (4.2) by

$$\begin{aligned}\mathbb{P}(\mathsf{B}_{i+1/2} | \mathcal{A}_i, \mathsf{D}_{i+1}) &= 1 - \mathbb{P}(\mathsf{B}^c_{i+1/2} | \mathcal{A}_i, \mathsf{D}_{i+1}) = 1 - \mathbb{E}(\mathbb{1}_{\mathsf{B}^c_{i+1/2}} | \mathcal{A}_i, \mathsf{D}_{i+1}) \\ &\geq 1 - \mathbb{E}\Big(\tfrac{\mathsf{Var}(\mathsf{p}_{i+1/2})}{\varepsilon^2 \mathsf{Isop}(\rho_i)^2} | \mathcal{A}_i, \mathsf{D}_{i+1}\Big) \geq 1 - \tfrac{10}{\varepsilon^2 N \underline{m}^2},\end{aligned}$$

where $\underline{m} := \min_{x \in [m(1-\alpha)(1-2\varepsilon), 1-\alpha/2]} \mathsf{Isop}(x)$, and where the inequality is obtained from site-percolation with parameter $f_\beta(\rho_i)$, at time i, and the following computation, which uses the independence between X^i_v and X^i_u for $u \in B(v,2)$.

$$\begin{aligned}\mathsf{Var}\Big(\textstyle\sum_{v \in G_N} \mathsf{X}^i_v\Big) &= \mathbb{E}\left(\left(\textstyle\sum_{v \in G_N} \mathsf{X}^i_v\right)^2\right) - \mathbb{E}\left(\textstyle\sum_{v \in G_N} \mathsf{X}^i_v\right)^2 \\ &= \textstyle\sum_{v \in G_N, u \in B(v,2)} \mathbb{E}\left(\mathsf{X}^i_v \mathsf{X}^i_u\right) + \sum_{v \in G_N, u \notin B(v,2)} \mathbb{E}\left(\mathsf{X}^i_v \mathsf{X}^i_u\right) - N^2 \mathsf{Iso}(\rho_i) \\ &\leq 10N + (N^2 - 10N)\mathsf{Iso}(\rho_i) - N^2 \mathsf{Iso}(\rho_i) \\ &\leq 10N(1 - \mathsf{Iso}(\rho_i)^2) \leq 10N.\end{aligned}$$ □

Now we turn to the results concerning the dynamical system $\mathsf{DS}(h)$. The following proposition will help us prove Proposition 2.4.

Proposition 4.2. $\mathsf{DS}(h)$ *has* 0 *as a unique attractive fixed point if* $\alpha \in [0,1]$, $\beta \in (0,\infty]$ *satisfy*

$$\phi(\alpha, \beta) \leq 1.$$

If this condition does not hold, i.e. $\phi(\alpha,\beta) > 1$, *then* 0 *is a repulsive fixed point.*

Proof. The origin is clearly a fixed point, and it is attractive if $\lim_{p \to 0} h'(p) \leq 1$ and repulsive otherwise. A simple computation gives $\lim_{p \to 0} h'(p) = \phi(\alpha,\beta)$, so all that remains is to prove the uniqueness in the case $\phi(\alpha,\beta) \leq 1$. This will follow from showing that in this case the identity function is lies above h. In order to achieve this, write $h(p) = g_\alpha(1 - e^{-\beta p}) = (1-\alpha)(1-e^{-\beta p})G_\alpha(1-e^{-\beta p})^3$ (see (5.6)) so that, since the function G_α is decreasing in α (by Lemma 5.4) it is enough to study the case $\alpha = 1 - \frac{1}{\beta}$, meaning that our assertion is equivalent to proving that $p - (1-e^{-\beta p})G_{1-1/\beta}(1-e^{-\beta p})^3$ is positive for all $p \in (0,1]$ and for all $\beta > 1$. Again, since $G_{1-1/\beta}$ is decreasing in β, and since $\frac{1-e^{-\beta p}}{\beta}$ is also decreasing in β, it is enough to prove that $p - \frac{\left(1-\sqrt{1-4e^{-p}(1-e^{-p})}\right)^3}{8(1-e^{-p})^2} > 0$ for all $p \in (0,1]$. But the infimum of the right hand side is attained (strictly) as $p \to 0$, where its value is 0, which yields the result. □

Proof of Proposition 2.4. The case $\alpha = 0$ corresponds to the setting of [DR09] while the case $\alpha = 1$ is trivial, so we restrict to $\alpha \in (0,1)$. In this case, from Proposition 5.5 the function g_α has a unique critical point x_0, which is a maximum. Without loss of generality we can suppose that $p \in [0, f_\beta^{-1}(p^*_\alpha)]$, otherwise apply h once to make this happen. Because it is the composition of two increasing functions, h is increasing inside $[0, f_\beta^{-1}(p^*_\alpha)]$. Using a restricted version of $h : [0, f_\beta^{-1}(p^*_\alpha)] \longrightarrow [0, f_\beta^{-1}(p^*_\alpha)]$, we get that 0 is the unique fixed point according to Proposition 4.2. Finally, from Proposition 2.3.5 in [HK03] the property holds in the first case. For the second case it is enough to notice that h is positive in $(0,1)$, hence the repulsive behavior of 0 in this regime yields the conclusion. □

5. Proofs of the multi-type results

5.1. **Interior-recurrent sets.** As discussed in Section 2, our approach to prove Theorem 2.11 consists in using Theorem 2.3 to show that the particle system follows the behavior observed for the dynamical system in Theorems 2.7 and 2.9. However, in order to apply our approximation theorem, we need more information about $\mathsf{DS}(h)$ than just the definitions of *coexistence* and *domination*. These definitions only explicit the behavior of $\mathsf{DS}(h)$ in the long term, giving no control of the initial part of the orbits, where the randomness of the particle system might have a large impact. With this in mind we introduce a concept which will draw most of our attention in this section:

Definition 5.1. We say that a set $A \subseteq [0,1]^2$ is *interior-recurrent* for $\mathsf{DS}(h)$ if there are $0 < \delta' < \delta$ and $\bar{k} \in \mathbb{N}$ such that

(i) $\forall p \in A,\ d(p, A^c) > \delta \implies d(h(p), A^c) \geq \delta'$,

(ii) $\forall p \in A,\ d(p, A^c) \leq \delta \implies d(h^k(p), A^c) \geq \delta'$ for some $k \leq \bar{k}$.

In words, a set A is interior-recurrent if the dynamical system cannot exit its interior using jumps larger than a certain size δ and if every time it gets to distance smaller than δ to the boundary, it takes a bounded number of steps for it to go back to the interior of A, where by *interior* here we mean a certain subset of A bounded away from the boundary.

Interior-recurrent sets will play a crucial role in showing that the particle system tracks closely the behavior of $\mathsf{DS}(h)$ (at least for a fixed, finite number of steps). The next proposition shows that, thanks to the approximation result Theorem 2.3, the control on $\mathsf{DS}(h)$ furnished by interior-recurrent sets can be transferred to the particle system.

Proposition 5.2. *Let $(\eta_k^N)_{k\in\mathbb{N}}$ be the mean-field MMM with parameters satisfing the conditions in Theorem 2.3, and assume that its initial condition ρ_0^N lies within an interior-recurrent set A with parameters δ, δ' and $\bar{k}$. Then there is a $C > 0$ depending only on A such that*

$$\mathbb{P}\big(\rho_k^N \notin A,\ \forall k \in \{1, 2, \dots, \bar{k}\}\big) \ \leq\ C\theta_{\underline{\alpha}}(N). \tag{5.1}$$

Proof. From Theorem 2.3 there is a $C > 0$ independent of ρ_0^N and N such that for any $k \leq \bar{k}$, we have

$$\mathbb{P}\Big(\big\|\rho_k^N - h^k(\rho_0^N)\big\| > \delta'\Big) \ \leq\ C\theta_{\underline{\alpha}}(N).$$

Now we use the interior-recurrent property of A. If $d(\rho_0^N, A^c) \geq \delta$ then $d(h(\rho_0^N), A^c) > \delta'$, so the left hand side of (5.1) is bounded by

$$\mathbb{P}\big(\rho_1^N \notin A\big) \ \leq\ \mathbb{P}\big(\|\rho_1^N - h(\rho_0^N)\| > \delta'\big) \ \leq\ C\theta_{\underline{\alpha}}(N).$$

Otherwise, if $d(\rho_0^N, A^c) < \delta$, then there is a $k \leq \bar{k}$ such that $d(h^k(\rho_0^N), A^c) > \delta'$, and the same argument shows that the left hand side of (5.1) is bounded by $C\theta_{\underline{\alpha}}(N)$. □

In the next result we use the concept of interior-recurrence to provide a stronger version of coexistence and domination for $\mathsf{DS}(h)$.

Theorem 5.3. *Consider the dynamical system $\mathsf{DS}(h)$ with initial condition $p^0 \in (0,1)^2$:*

(i) (Coexistence) Assume that the parameters $\vec{\alpha}$ and $\vec{\beta}$ satisfy the conditions in Theorem 2.7. Then there is a compact interior-recurrent set $A \subseteq (0,1)^2$ which contains p^0 as an interior point.

(ii) (Domination) Assume that the parameters $\vec{\alpha}$ and $\vec{\beta}$ satisfy the conditions in Theorem 2.9. Then, there are $\gamma_1, \gamma_2 \in (0,1)$ and an interior-recurrent set B (independent of p^0) with parameter $\bar{k} = 1$ such that for all $p \in B$

$$(1 - \alpha(1)) f_{\vec{\beta}}^{(1)}(p) \leq \gamma_1 p_1 \quad \text{and} \quad \gamma_2 < p_2. \tag{5.2}$$

Furthermore, there is a $k \in \mathbb{N}$ such that $h^k(p^0)$ is an interior point of B.

Before turning to the proof of Theorem 5.3, we show how it yields all the results in Section 2.3.

Proof of Theorem 2.9. By Theorem 5.3 we know that the orbit of $\mathsf{DS}(h)$ eventually reaches an interior-recurrent set B with parameter $\bar{k} = 1$, which satisfies (5.2) for some $\gamma_1, \gamma_2 \in (0,1)$. Now, since $\bar{k} = 1$, from Definition 5.1 we deduce that the set B is actually trapping for the dynamical system, meaning that $h(p) \in B$ for all $p \in B$. Since $\gamma_2 < p_2$ for $p \in B$, we deduce that $\liminf_{k\to\infty} h_2^k(p) \geq \gamma_2 > 0$; similarly, since $h_1(p) \leq (1-\alpha(1))f^{(1)}_{\vec{\beta}}(p_1) \leq \gamma_1 p_1$ for $p \in B$ (the first inequality follows from comparing with a system where we let the epidemic attack but not spread), we deduce that $\lim_{k\to\infty} h_1^k(p) = 0$. This shows domination of type 2. □

Proof of Theorem 2.7. By Theorem 5.3 we know that there is a compact interior-recurrent set $A \subseteq (0,1)^2$ containing p^0, and thus we know that every time the dynamical system leaves A, it spends at most $\bar{k}$ units of time in A^c. Thus the orbit of $\mathsf{DS}(h)$ is contained in $A_{\bar{k}} := \cup_{l=0}^{\bar{k}} h^l(A)$, which is also compact. Since $A_{\bar{k}} \subseteq (0,1)^2$ (otherwise it would contain an orbit that never returns to A), compactness implies that it must be bounded away from the axes, so in particular we deduce that $\liminf_{k\to\infty} h_i^k(p) > 0$ for $i = 1, 2$, giving coexistence. □

Proof of Theorem 2.11. Consider the mean-field MMM and assume first that the parameters of the model satisfy the conditions of Theorem 2.7. As in the proof of the previous theorem, Theorem 5.3 gives a set $A \subseteq (0,1)^2$ containing ρ_0^N which is interior-recurrent for the dynamical system $\mathsf{DS}(h)$. Let σ_n denote the n-th return time of the dynamical system to A. By Proposition 5.2 we have $\mathbb{P}(\sigma_1 > \bar{k}) \leq C\theta_{\underline{\alpha}}(N)$ for some $C > 0$ which depends only on A, and since the bound is uniform on the initial condition, by the strong Markov property we get

$$\mathbb{P}\big(\sigma_1 \leq \bar{k},\ \sigma_2 - \sigma_1 \leq \bar{k}, \ldots,\ \sigma_n - \sigma_{n-1} \leq \bar{k}\big) \ \geq\ (1 - C\theta_{\underline{\alpha}}(N))^n.$$

The event on the left hand side implies in particular that $\sigma_n < \infty$ a.s., but since $\sigma_n \geq n$ it follows that $\rho_k^N \in A$ for some $k \geq n$. Since both species have to be alive to lie within A, on this event both τ_N^1 and τ_N^2 must be larger than n, so we conclude (2.11).

To deduce (2.14) we start by repeating the above argument for an interior-recurrent set A containing p (using that A contains ρ_0^N for large enough N), yielding a constant $C' > 0$ such that $\mathbb{P}(\tau_N^1, \tau_N^2 \geq n) \ \geq\ \big(1 - C'\theta_{\underline{\alpha}}(N)\big)^n$ for all $N \in \mathbb{N}$. Taking $n = \theta_{\underline{\alpha}}(N)^{-(1-\varepsilon)}$ gives the result. (2.14).

We turn now our attention to the MMM with parameters satisfying the conditions of Theorem 2.9. Under this assumption Theorem 5.3 gives an interior-recurrent set B with parameter $\bar{k} = 1$ which satisfies (5.2). Assume first that $\rho_0^N \in B$; we will explain later how to treat the case $\rho_0^N \notin B$.

Since $\bar{k} = 1$, Definition 5.1 implies that regardless of the value of $d(\rho_0^N, B^c)$ we have $d(h(\rho_0^N), B^c) > \delta'$, so Theorem 2.3 gives some $C > 0$ depending only on B such that $\mathbb{P}(\rho_1^N \notin B) \leq \mathbb{P}\big(\delta' < \big\|\rho_1^N - h(\rho_0^N)\big\|\big) \leq C\theta_{\underline{\alpha}}(N)$. Since the bound is uniform over $\rho_0^N \in B$, an application of the strong Markov property gives that, for any $n \in \mathbb{N}$,

$$\mathbb{P}(\rho_k^N \in B\ \forall k \leq n) \ \geq\ (1 - C\theta_{\underline{\alpha}}(N))^n. \tag{5.3}$$

Noticing that $\gamma_2 < p_2$ for all $p \in B$ we deduce that the event on the left hand side implies $\tau_N^2 \geq n$, yielding (2.12).

To deduce (2.13) observe first that (as in the proof of Theorem 2.9) the number of type 1 individuals at time 1 is dominated by a Poisson random variable with parameter $(1 - \alpha_N(1))f^{(1)}_{\vec{\beta}}(\rho_0^N)$, which is less than $\gamma_1\rho_0^N$ when $\rho_0^N \in B$. From this it is easy to see that, on the event $\mathcal{E}_n = \{\rho_k^N \in B\ \forall k \leq n\}$, the process $(\rho_k^N)_{k\leq n}$ is stochastically dominated by a subcritical Galton-Watson process starting with $\rho_0^N N$ individuals and with offspring distribution Poisson$[\gamma_1]$. By (5.3) and branching processes theory we get

$$\mathbb{P}\left(\tau_N^1 \geq n\right) \leq \mathbb{P}\left(\mathcal{E}_n \cap \{\tau_N^1 > n\}\right) + \mathbb{P}(\mathcal{E}_n^c) \leq 1 - (1-\gamma^n)^N + 1 - (1 - C\theta_{\underline{\alpha}}(N))^n \tag{5.4}$$

with $\gamma = \gamma_1$, as desired.

Suppose now that $\rho_0^N \notin B$ and observe that from Theorem 5.3 there is some $k \in \mathbb{N}$ depending only on ρ_0^N such that $h^k(\rho_0^N)$ is an interior point of B. Let $\varepsilon > 0$ be small so that B contains the ball of center ρ_0^N with radius ε. Using Theorem 2.3 there is a $\bar{C}$ depending on k and ε such that

$$\mathbb{P}(\rho_k^N \notin B) \;\le\; \mathbb{P}\big(\big\|\rho_k^N - h^k(\rho_0^N)\big\| > \varepsilon\big) \;\le\; \bar{C}\theta_{\underline{\alpha}}(N), \tag{5.5}$$

so the general proof of (2.12) and (2.13) follows from restricting to the event on the left hand side above and restarting the process at time k.

Finally, to conclude (2.15) observe that these inequalities would follow directly from (2.12), (2.13), and our definition of $\theta_{\underline{\alpha}}(N)$ if the parameter C was independent from ρ_0^N. However, neither the bound in (5.3) nor the one in (5.4) depend on ρ_0^N, so the only parameter dependent on ρ_0^N is $\bar{C}$ in (5.5). Now under the additional assumption $\rho_0^N \longrightarrow p$ we may take $k \ge 0$ such that $h^k(p)$ is an interior point of B (such k exists because of Theorem 5.3), and then choose $\varepsilon > 0$ such that B contains the ball centered at p with radius ε. Since $\rho_0^N \to p$, for large enough N we have $\|h^k(\rho_0^N) - h^k(p)\| < \frac{\varepsilon}{2}$, so from Theorem 2.3 we obtain

$$\mathbb{P}(\rho_k^N \notin B) \;\le\; \mathbb{P}\big(\big\|\rho_k^N - h^k(p)\big\| > \varepsilon\big) \;\le\; \mathbb{P}\big(\big\|\rho_k^N - h^k(\rho_0^N)\big\| > \tfrac{1}{2}\varepsilon\big) \;\le\; \bar{C}\theta_{\underline{\alpha}}(N),$$

for some $\bar{C}$ depending only on p. □

The rest of this section is devoted to the proof of Theorem 5.3, which is rather lengthy and technical, so we divide it into three parts. In Section 5.2 we present some preliminary notation and functions which will be used to facilitate the analysis of the trajectories of $\mathsf{DS}(h)$, as well as some technical results about them. Using these results we prove the coexistence part of the theorem in Section 5.3, and the extinction part in Section 5.4.

5.2. **Preliminaries.** We begin this section by decomposing the function g_α as

$$g_\alpha(x) = (1-\alpha)xG_\alpha(x)^3 \qquad \text{with} \quad G_\alpha(x) = \frac{1-\sqrt{1-4(1-\alpha)x(1-x)}}{2(1-\alpha)x}. \tag{5.6}$$

Lemma 5.4. *The function $G_\alpha : [0,1] \to [0,1]$ satisfies the following:*

(1) When $\alpha = 0$, it is given by

$$G_0(x) \;=\; \begin{cases} 1 & \text{if } x \le 1/2 \\ \frac{1-x}{x} & \text{if } x > 1/2 \end{cases}.$$

(2) It is decreasing as a function of both α and x, with $G_\alpha(0) = 1$ and $G_\alpha(1) = 0$ for all $\alpha \in [0,1)$.

(3) As $\alpha \to 1$, it converges monotonically to $G_1(x) := 1 - x$.

We omit the simple proof of this result. Let now $\bar{p} \in [0,1]^2$ be the maximum possible density achieved after the epidemic stage, that is

$$\bar{p}_i = \sup_{x\in[0,1]} g_{\alpha(i)}(x).$$

Since $g_\alpha \le (1-\alpha)g_0$, it is easily seen that $\bar{p}_i \le \frac{1-\alpha(i)}{2}$. By definition, except maybe for the initial value p^0, the orbit of $\mathsf{DS}(h)$ lies within $[0,\bar{p}]$, where the next result provides control on the behavior of g_α:

Proposition 5.5. *There is a single value $x_0 \in [0,1/2]$ where g_α attains its global maximum. This value is characterized as the solution of $G_\alpha(x_0) = x_0 + \frac{1}{2}$ and satisfies:*

(1) If $\alpha > 0$, this is the only critical point of g_α in $[0,1]$.

(2) If $\phi_i < 2\log 2$, then for any p with $p_i \le \bar{p}_i$ we have $f_{\vec{\beta}}^{(i)}(p) < x_0$. In particular, $g'_{\alpha(i)} \circ f_{\vec{\beta}}^{(i)}(p) \ge 0$ for all $p \in [0,\bar{p}]$.

Even if g_α is not monotone, the last result still yields sufficient information about the growth of h:

Proposition 5.6. *For each $i = 1, 2$ define $l_i : [0,1]^2 \to \mathbb{R}^+$ as $l_i(p) = h_i(p)/p_i$. Then:*

(1) The function $f^{(1)}_{\vec{\beta}}(p)$ is increasing in p_1 and decreasing in p_2.
(2) The function $l_1(p)$ is decreasing in p_1.
(3) If $\phi_1 < 2\log 2$, then $h_1(p)$ is increasing in p_1 and decreasing in p_2. In particular, in this case l_1 is also decreasing in p_2.

The reason why we are interested in l_i is that, from the relation $h_i(p) = l_i(p)p_i$, it is enough to bound l_i in order to show exponential growth or decay of the density of a species. This is precisely what we do in the next result.

Proposition 5.7. *For any small $\varepsilon > 0$ define κ_ε as the unique solution of*

$$g_{\alpha(1)}(1 - e^{-\beta(1)\kappa_\varepsilon}) \quad = \quad (1-\varepsilon)\kappa_\varepsilon.$$

Under the assumption $\phi_2 > \phi_1$ there are $\bar{c}, \varepsilon, \varepsilon' > 0$ small such that for all $c \le \bar{c}$:

(i) For all $0 < p_1 < \kappa_\varepsilon$ it holds that

$$p_2 \in (0, c) \implies l_2(p) > 1 + \varepsilon', \tag{5.7}$$

$$p_2 \in (c, \bar{p}_2) \implies h_2(p) > (1 + \varepsilon')c. \tag{5.8}$$

(ii) Under the additional assumption $\phi_2 > 2\log 2$, the property above holds for all $p_1 > 0$.
(iii) If $\phi_1 < 2\log 2$, then:

$$p_1 \in (0, \kappa_\varepsilon) \implies h_1(p) \le (1 - \varepsilon')\kappa_\varepsilon. \tag{5.9}$$

$$p_1 \in (\kappa_\varepsilon, \bar{p}_1) \implies l_1(p) \le 1 - \varepsilon'. \tag{5.10}$$

Properties (i) and (ii) state that when the stronger species starts at a low density, it starts growing exponentially until it reaches a certain threshold value c, which becomes a lower bound for its density from that time onwards. Property (iii), on the other hand, states that if the fitness of the weaker species is below $2\log 2$, then its density decays exponentially until it reaches a trapping set $[0, \kappa_\varepsilon]$.

The proofs of the last three propositions are mostly calculus, so we defer them to the appendix.

5.3. **Proof of Theorem 5.3(i).** As we just discussed, Proposition 5.7 already provides a good control on the behavior of the stronger species, so our main focus will be on the weaker one. Assuming that the conditions of Theorem 2.7 are satisfied, our approach consists in analyzing the dynamical system when the density of the weaker species is at low values. In that situation we will approximate h by a simpler function $\bar{h}$, and show that for this particular dynamical system the density p_1 tends to grow on average.

The approximating map $\bar{h} : [0,1]^2 \to [0,1]^2$ which we will use is the linear approximation of h in the first component,

$$\bar{h}(p) \;=\; \begin{pmatrix} h_1(0,p_2) + p_1 \frac{\partial h_1}{\partial p_1}(0,p_2) \\ h_2(0,p_2) \end{pmatrix} \;=\; \begin{pmatrix} \phi_1 p_1 \frac{1-e^{-\beta(2)p_2}}{\beta(2)p_2} \\ h_2(0,p_2) \end{pmatrix}.$$

The next result states that this approximation is good uniformly on p_2:

Proposition 5.8. *For all $k \in \mathbb{N}$ we have $\lim_{p_1 \to 0} \frac{\bar{h}_1^k(p)}{h_1^k(p)} = 1$ uniformly on $p_2 \in [0,1]$.*

Proof. Let $\Sigma_k(p) = \beta(1)h_1^k(p) + \beta(2)h_2^k(p)$ and $\overline{\Sigma}_k(p) = \beta(2)\bar{h}_2^k(p)$. Using these values and the definition of h and $\bar{h}$ it is fairly simple to see that

$$\frac{\bar{h}_1^k(p)}{h_1^k(p)} \;=\; \frac{\bar{h}_1^{k-1}(p)}{h_1^{k-1}(p)} \frac{\psi(\overline{\Sigma}_{k-1}(p))}{\psi(\Sigma_{k-1}(p))} (G_{\alpha(1)})^{-3} \circ f^{(1)}_{\vec{\beta}} \circ h_1^{k-1}(p) \tag{5.11}$$

where ψ is defined as $\psi(x) = \frac{1-e^{-x}}{x}$, which is uniformly continuous and bounded away from 0 for $x \in [0,1]$. Noticing that $\Sigma_k(p)$ and $\overline{\Sigma}_k(p)$ converge to the same value as $p_1 \to 0$, the last two factors on the right hand side of (5.11) converge to 1 uniformly, so $\bar{h}_1^k(p)/h_1^k(p)$ converges to 1 uniformly if $\bar{h}_1^{k-1}(p)/h_1^{k-1}(p)$ does. Since $\bar{h}_1^0(p) = h_1^0(p) = p_1$, the result follows by repeating the argument k times. □

Thanks to this proposition we can approximate h by $\bar{h}$ whenever p_1 is small enough, independently of the value of p_2. The resulting dynamical system $(q^k)_{k\in\mathbb{N}}$ can be realized by first running the one-dimensional MM for type 2 by itself, and then using its trajectory to compute the values of q_1^n as

$$q_1^n \;=\; q_1^0 \prod_{k=0}^{n-1} \frac{\phi_1(1-e^{-\beta(2)q_2^j})}{\beta(2)q_2^j} \;=\; q_1^0 \left(\frac{\phi_1}{\beta(2)} e^{\bar{\varphi}^n(q_2^0)}\right)^n \tag{5.12}$$

with

$$\bar{\varphi}^n(x) \;=\; \frac{1}{n}\sum_{k=0}^{n-1} \varphi(h_2^k(0,x)),$$

where $\varphi(x) = \log(\frac{1-e^{-\beta(2)x}}{x})$. This suggests that it will be useful to study the following observable of the orbit $(q_2^k)_{k\in\mathbb{N}}$:

$$\bar{\varphi}(x) \;=\; \lim_{n\to\infty} \bar{\varphi}^n(x).$$

The limit exists for all $x \le \bar{p}_2$: in fact, since $\bar{p}_2 \le \frac{1}{2}$, $h_2^k(0,x) \ge 2 - 2e^{-\beta(2)/2} > 1$ thanks to the assumption $\phi_2 > z(\alpha(2)) > 2\log 2$, so by monotonicity of φ the summands in the definition of $\bar{\varphi}(x)$ are positive and bounded. In view of (5.12), $\frac{\phi_1}{\beta(2)}e^{\bar{\varphi}}$ can be interpreted as the average growth of type 1 when taking into account the effect of type 2. In order to control this growth we define η to be the smallest possible value of $\bar{\varphi}$, that is

$$\eta = \inf_{x\in[0,\bar{p}_2]} \bar{\varphi}(x).$$

The following result shows that controlling η properly allows us to make q_1^k grow to be as large as we want:

Lemma 5.9. *Suppose that the conditions of Theorem 2.7 hold. If $\frac{\phi_1}{\beta(2)}e^{\eta} > 1$, then for all $M > 0$ there exists $\bar{k} \in \mathbb{N}$ satisfying the following property: for all $q_2^0 \in [0,\bar{p}_2]$, there is a $0 \le k \le \bar{k}$ such that*

$$\prod_{j=0}^{k-1} \frac{\phi_1(1-e^{-\beta(2)q_2^j})}{\beta(2)q_2^j} \;>\; M. \tag{5.13}$$

Proof. From the hypothesis we know that there exists $\delta > 0$ such that $\phi_1 = \beta(2)e^{-\eta}(1+2\delta)$. Taking $\varepsilon > 0$ small enough such that $(1-\varepsilon)(1+2\delta) > 1+\delta$, for each q_2^0 we can find $\underline{k} \in \mathbb{N}$ such that for all $k \ge \underline{k}$

$$\frac{\phi_1}{\beta(2)} \exp\left(\bar{\varphi}^k(q_2^0)\right) \;>\; (1-\varepsilon)\frac{\phi_1}{\beta(2)} \exp\left(\bar{\varphi}(q_2^0)\right) \;\ge\; (1-\varepsilon)\frac{\phi_1}{\beta(2)}e^{\eta} \;>\; 1+\delta,$$

where the first inequality follows from convergence of $\bar{\varphi}^k$ to $\bar{\varphi}$. Using the definition of $\bar{\varphi}^k$ we obtain $\frac{\phi_1}{\beta(2)}\left(\prod_{j=1}^{k-1}\frac{1-e^{-\beta(2)q_2^j}}{q_2^j}\right)^{1/k} > 1+\delta$ for all $k \ge \underline{k}$. In particular we find that for each q_2^0 there is some $k \ge \underline{k}$ such that

$$\left(\frac{\phi_1}{\beta(2)}\right)^k \prod_{j=0}^{k-1} \frac{1-e^{-\beta(2)q_2^j}}{q_2^j} \;>\; M.$$

For k fixed call O_k the set of all q_2^0 satisfying the inequality above for that given value of k. From the continuity of $\bar{h}$ each O_k is open, and from the previous argument, each q_2^0 belongs

to some k, so $(O_k)_{k\in\mathbb{N}}$ is an open cover of $[0,\bar{p}_2]$, which necessarily contains a finite subcover. Taking $\bar{k}$ to be the largest index of the subcover gives the result. □

The next result shows that, in our setting, (ii) in the definition of interior-recurrence follows directly from Proposition 5.8 and Lemma 5.9. The idea is simple: as long as the trajectory of p_1^k stays small then the system is well approximated by $\mathsf{DS}(\bar{h})$, but by the last proposition the first component of this system gets large, which hints at a contradiction.

Proposition 5.10. *Suppose that the conditions of Lemma 5.9 are satisfied. If $p_2^0 \in (0,1)$, then there is a $\bar{c} \in (0, p_1^0)$ satisfying the following: for all $c \le \bar{c}$ we can find $\bar{k} \in \mathbb{N}$ such that for all $n \in \mathbb{N}$*

$$p_1^n \ge c \quad \Longrightarrow \quad \exists k \le \bar{k} \text{ such that } p_1^{n+k} > \tfrac{3}{2r}c \tag{5.14}$$

with $r = \inf_{p\le\bar{p}} l_1(p)$.

In other words, after the trajectory gets above a certain threshold parameter c, it cannot stay below c for more than $\bar{k}$ consecutive steps.

Proof of Proposition 5.10: Let $M = \frac{2}{r^2}$, choose $\bar{k}$ as in Lemma 5.9 and use the uniform convergence proved in Proposition 5.8 to choose $\delta_0 > 0$ such that

$$p_1 < \delta_0 \quad \Longrightarrow \quad \bar{h}_1^k(p)/h_1^k(p) < 4/3 \quad \forall p_2 \in [0,1], \ \ \forall k = 1,\dots,\bar{k}. \tag{5.15}$$

Define now $c = \min\{\frac{2}{3}\delta_0, \frac{1}{2}p_1^0\}$. We prove (5.14) by contradiction as follows. Suppose that for some $n \in \mathbb{N}$ we have $p_1^n \ge c > p_1^{n+1}$ and that there is no $k \le \bar{k}$ such that $p_1^{n+k} > \frac{3}{2r}c$. From our choice of c we know each p_1^{n+k} is smaller than δ_0, so from (5.15), for each $k \le \bar{k}$ we have

$$p_1^{n+k} = h_1^k(p^n) \ge \tfrac{3}{4}\bar{h}_1^k(p^n) = \tfrac{3}{4}p_1^{n+1}\Big(\tfrac{\phi_1}{\beta(2)}\Big)^k \prod_{j=0}^{k-1} \tfrac{1-e^{-\beta(2)q_2^j}}{q_2^j}. \tag{5.16}$$

However, for the specific value of k given in Lemma 5.9 with initial condition p_1^{n+1}, we can bound the right hand side in (5.16) by $\frac{3}{2r^2}p_1^{n+1}$. This is a contradiction with our assumption $p_1^{n+k} < \frac{3}{2r}c$ because

$$p_1^{n+k} > \tfrac{3}{2r^2}p_1^{n+1} = \tfrac{3}{2r^2}l_1(p_1^n)p_1^n \ge \tfrac{3}{2r^2}rc = \tfrac{3}{2r}c, \tag{5.17}$$

where the last inequality follows from the definition of r and the assumption $p_1^n \ge c$. □

Using the results obtained so far, as well as Proposition 5.7, we are finally ready to prove Theorem 5.3(i). To do so observe that after one iteration, the orbit of $\mathsf{DS}(h)$ lies within $[0,\bar{p}_1]\times[0,\bar{p}_2]$, so we may assume that p^0 lies in this set. Let us assume that $\phi_1 e^{\eta} > \beta(2)$; we will show that this is the case below. Then given $r = l_1(\bar{p})$ we can take c_1 so that (5.14) holds for all $c \le c_1$. Next, observe that from the hypotheses of Theorem 2.7 we have that $\phi_2 > z(\alpha(2)) > 2\log 2$ so we can take c_2 so that the statement of Proposition 5.7(ii) holds for all $c \le c_2$. We claim that the set

$$A \ := \ [c_1,\bar{p}_1]\times[c_2,\bar{p}_2]$$

satisfies the properties required in Theorem 5.3(i). Indeed, it is clear that the set is compact and contains p^0 as an interior point. To see that the set is interior-recurrent, notice that from (5.8) in Proposition 5.7, for any $p_2 \in (c_2,\bar{p}_2)$ we have $h_2(p_2) > (1+\varepsilon')c_2$ independently of p_1, so that both requirements for interior-recurrent are satisfied with $\bar{k} = 1$ in the second component. To deduce the same for the first component notice that from the definition of r, we have that $p_1 > \frac{c_1}{r}$ implies that $p_1^1 > c_1$, and from Proposition 5.10 there is $\bar{k}$ such that $\frac{c_1}{r} > p_1 \ge c_1$ implies that there is a $k \le \bar{k}$ such that $p_1^k > \frac{3}{2r}c_1$, so both requirements for interior-recurrence are satisfied in this component as well.

To finish the proof we need to show that the hypotheses assumed in Theorem 2.7 imply that $\phi_1 e^{\eta} > \beta(2)$. To see this notice that φ is decreasing and that for any value of p_2^0 we have

$p_2^k \le \bar{p}_2 \le \frac{1-\alpha(2)}{2}$, so in particular

$$\varphi(p_2^k) \ge \varphi\big(\tfrac{1-\alpha(2)}{2}\big) = \log\big(\tfrac{2(1-e^{-\phi_2/2})}{1-\alpha(2)}\big), \tag{5.18}$$

so the expression on the right hand side provides a lower bound for η. We can improve this bound by reasoning as follows. Take κ_2 as defined in Theorem 2.7 so that in particular $P_2 := \frac{\kappa_2}{\beta(2)}$ is a fixed point of $h_2(0,\cdot)$. Now, since the function $x \to 1 - e^{-\beta(2)x}$ is increasing, from Proposition 5.5 it follows that the function $h_2(0,\cdot)$ has a unique critical point P_1, where it attains its maximum. Assume for a moment that $P_1 \le P_2$. Then $h_2(0,\cdot)$ is decreasing on $[P_2, \frac{1}{2}]$, so that

$$P_2 \le x \quad \Longrightarrow \quad h_2(0,x) \le h_2(0,P_2) = P_2.$$

In words, every time q_2^k in the orbit of q_2 which is larger than P_2 is followed by an element q_2^{k+1} which is smaller than P_2, so at least half of the points in the orbit lie in $[0, P_2]$. Bounding by $\varphi(P_2)$ the value of φ in this interval, and by $\varphi(\frac{1-\alpha(2)}{2})$ the value outside of it, we obtain

$$\bar{\varphi}(p_2^0) \;\ge\; \tfrac{1}{2}\Big[\log\Big(\tfrac{1-e^{-\beta(2)P_2}}{P_2}\Big) + \log\Big(\tfrac{2-2e^{-\phi_2/2}}{1-\alpha(2)}\Big)\Big] \quad \forall p_2^0 \in [0,1/2].$$

This is then also a lower bound for η, and it is easy to see that, assuming this bound, $\phi_1 > e^{-\eta}\beta(2)$ is equivalent to condition (2.9).

It only remains to show that indeed $P_1 \le P_2$. From the facts that P_1 is the unique maximizer of $h_2(0,\cdot)$ and that P_2 is a fixed point of this function one checks that the inequality is equivalent to

$$P_1 \le g_{\alpha(2)}(1 - e^{-\beta(2)P_1}). \tag{5.19}$$

From the above definition of P_1, $x_0 = 1 - e^{-\beta(2)P_1}$ is a critical point of $g_{\alpha(2)}$, and thus by Proposition 5.5 it satisfies $G_{\alpha(2)}(x_0) = x_0 + \frac{1}{2}$. Replacing these equalities into (5.19) we obtain that x_0 must satisfy

$$\phi_2 x_0 (x_0 + \tfrac{1}{2})^3 + \log(1 - x_0) \ge 0.$$

Now, from Lemma 5.4, the equality $G_{\alpha(2)}(x_0) = x_0 + \frac{1}{2}$ defines an implicit function $x_0(\alpha(2))$ which is strictly decreasing in $\alpha(2)$ and satisfies $x_0(0) = 1/2$ and $x_0(1) = 1/4$. Solving for ϕ_2, the inequality above becomes

$$\phi_2 > z(\alpha(2)) := \frac{-\log(1-x_0)}{x_0(x_0+1/2)^3}, \tag{5.20}$$

which is satisfied by our hypothesis on ϕ_2 so we conclude that $P_1 \le P_2$ as needed.

5.4. **Proof of Theorem 5.3(ii).** We want to prove that there is an interior-recurrent set B where the stronger species survives while the density of the weaker one decays exponentially. The cornerstone of this section is the following result:

Lemma 5.11. *Assume that conditions of Theorem 2.9 hold. Take $\bar{c}$, ε and ε' as in Proposition 5.7 and for sufficiently small $c < \bar{c}$ let*

$$B_1 \;=\; \Big\{p \in [0,\kappa_\varepsilon] \times [c, \bar{p}_2],\; l_1(p) < 1\Big\},$$

where κ_ε is defined in Proposition 5.7 as the solution of $g_{\alpha(1)}(1 - e^{-\beta(1)\kappa_\varepsilon}) = (1-\varepsilon)\kappa_\varepsilon$. Then

$$\sup_{p\in B_1} l_1 \circ h(p) < 1 \quad \textit{and} \quad \inf_{l_1(p)\ge 1} l_2(p) > 1. \tag{5.21}$$

Proof. We begin by observing that $\phi_1 < 2\log 2$. To see this, observe first that from the assumption $\phi_1 < \phi_2$ the bound holds trivially if $\phi_2 \le 2\log 2$, so we only need to worry about the case $\phi_2 > 2\log 2$, where condition (2.10) gives

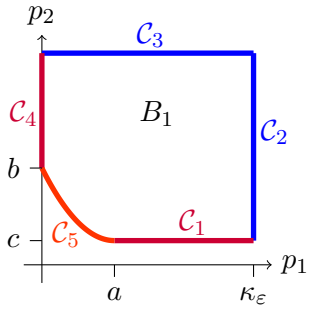
$$a_1(\phi_1) \;<\; \tfrac{\phi_2}{1-\alpha(2)} g_{\alpha(2)}(1 - e^{-\frac{\phi_2}{2}}) \;\le\; 8\phi_2(1 - e^{-\frac{\phi_2}{2}})e^{-\frac{3\phi_2}{2}},$$

where we have used that $G_{\alpha(2)}(x) \le 2(1-x)$. The function on the right hand side is decreasing in $(2\log 2, +\infty)$, so $a_1(\phi_1) \le 16\log 2(1 - e^{-\frac{2\log 2}{2}})e^{-3\log 2} = \log 2$, and thus $\phi_1 < 2\log 2$,

using the definition and monotonicity of $a_1(x)$. Thanks to this bound on ϕ_1, h_1 satisfies the monotonicity stated in (3) of Proposition 5.6.

From Proposition 5.6 we also know that l_1 is strictly decreasing on both p_1 and p_2, so the level set $\{l_1(p) = 1\}$ defines a strictly decreasing function $p_2 = s(p_1)$, for which there are values a and b such that $l_1(a, c) = l_1(0, b) = 1$. Using these values we can easily characterize B_1 as a set bounded by the curves

$$\begin{aligned}
\mathcal{C}_1 &:= \{(p_1, c),\ a \le p_1 \le \kappa_\varepsilon\}\\
\mathcal{C}_2 &:= \{(\kappa_\varepsilon, p_2),\ c \le p_2 \le \bar{p}_2\}\\
\mathcal{C}_3 &:= \{(p_1, \bar{p}_2),\ 0 \le p_1 \le \kappa_\varepsilon\}\\
\mathcal{C}_4 &:= \{(0, p_2),\ b \le p_2 \le \bar{p}_2\}\\
\mathcal{C}_5 &:= \{(p_1, s(p_1)),\ 0 \le p_1 \le a\}
\end{aligned}$$



We will make use of the following lemma; its proof is postponed.

Lemma 5.12.

$$\sup_{p\in B_1} l_1 \circ h(p) \quad = \quad \max_{p\in \mathcal{C}_1\cup\mathcal{C}_4\cup\mathcal{C}_5} l_1 \circ h(p). \tag{5.22}$$

Thus in order to obtain the first statement in (5.21) we need to find the maximum of $l_1 \circ h$ on each set $\mathcal{C}_1$, $\mathcal{C}_4$ and $\mathcal{C}_5$ separately.

Consider first $\mathcal{C}_1$. From Proposition 5.6 we know that $l_1(\cdot, 0)$ is strictly decreasing and since $\phi_1 < 2\log 2$, the same proposition states that $h_1(\cdot, 0)$ is strictly increasing on $(0, \kappa_\varepsilon]$. As a result, the function $p_1 \mapsto l_1(h_1(p_1, 0), 0)$ is strictly decreasing with no critical points on any interval $[u, \kappa_\varepsilon]$, so its derivative is negative and bounded away from zero. Since all the functions involved in the argument are smooth, if c is sufficiently small we also obtain that $\frac{\partial}{\partial p_1} l_1 \circ h$ is negative and bounded away from zero on $\mathcal{C}_1$. We conclude that $l_1 \circ h$ is maximized at the point (a, c), so we need to show that its value at that point is less than 1. Indeed, using the definition of a, we obtain $h_1(a, c) = a$, and since $a < \kappa_\varepsilon$ we can use Proposition 5.7 to deduce that $h_2(a, c) > c$, where the inequality follows from Proposition 5.7; from the monotonicity of l_1 we deduce now that $l_1 \circ h(a, c) < l_1(a, c) = 1$.

Next consider $\mathcal{C}_4$. Here we have $p_1 = 0$, which greatly simplifies the analysis since

$$h_1(0, p_2) = 0, \qquad h_2(0, p_2) = g_{\alpha(2)}(1 - e^{-\beta(2)p_2}), \qquad l_1 \circ h = \phi_1 \tfrac{1-e^{-\beta(2)h_2}}{\beta(2)h_2}.$$

Indeed, from the particular form of $l_1 \circ h$ on this set, the condition $l_1 \circ h < 1$ is equivalent to $\frac{1-e^{-\beta(2)h_2}}{\beta(2)h_2} < \frac{1-e^{-a_1(\phi_1)}}{a_1(\phi_1)}$ from the definition of $a_1(\phi_1)$. Now, since the function $\frac{1-e^{-x}}{x}$ is decreasing we obtain

$$l_1 \circ h(0, p_2) < 1 \quad \Longleftrightarrow \quad a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-\beta(2)p_2}). \tag{5.23}$$

Observe now that l_1 is decreasing, so it is maximized at the points where h_2 attains its minimum. From the special form of h_2 given above, we deduce from Proposition 5.5 that h_2 is minimized either where p_2 is maximal or minimal. From this we conclude that the maximum value of l_1 on $\mathcal{C}_4$ is either $l_1 \circ h(0, \bar{p}_2)$ or $l_1 \circ h(0, b)$. Now from (5.23) we see that $l_1 \circ h(0, \bar{p}_2) < 1$ if and only if $a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-\beta(2)\bar{p}_2})$, which follows from $\bar{p}_2 < \frac{1-\alpha(2)}{2}$ and (2.10). To deal with $l_1(0, b)$ we observe that that $a_1(\phi_1) = \beta(2)b$, so (5.23) shows that $l_1 \circ h(0, b) < 1$ if and only if $a_1(\phi_1) < \beta(2) g_{\alpha(2)}(1 - e^{-a_1(\phi_1)})$, which follows directly from (2.10).

Finally for $\mathcal{C}_5$ it will be enough to show that

$$\inf_{p\in\mathcal{C}_5} \left[\phi_2 G^3_{\alpha(2)} \circ f^{(2)}_{\vec{\beta}} - \phi_1 G^3_{\alpha(1)} \circ f^{(1)}_{\vec{\beta}}\right](p) > 0. \tag{5.24}$$

Indeed, if (5.24) is satisfied then multiplying the inequality by $\frac{1-e^{-\Sigma_p}}{\Sigma_p}$, with $\Sigma_p = \beta(1)p_1 + \beta(2)p_2$, gives $l_2(p) > l_1(p) = 1$, and this implies $p_2 < h_2(p)$, which implies $l_1(h) = l_1(p_1, h_2) <$

$l_1(p) = 1$. To prove (5.24) recall that $s(p_1)$ is a decreasing function, which means that $f^{(1)}_{\vec{\beta}}(p_1, s(p_1))$ is increasing and $f^{(2)}_{\vec{\beta}}(p_1, s(p_1))$ is decreasing. It follows that on $\mathcal{C}_5$ the function in (5.24) is increasing on p_1, so the infimum is positive if the inequality holds at $(0, b)$, which in this case follows from assumption (2.10).

To complete the proof we need to show that $\inf_{p:\, l_1(p)\geq 1} l_2(p) > 1$, but l_2 is decreasing in p_2 and the maximal values of p_2 within the region given by $l_1 \leq 1$ are found at $l_1 = 1$. This way, it is enough to show that $\inf_{l_1(p)=1} l_2(p) > 1$, and this is analogous to the proof of (5.24). □

It remains to prove Lemma 5.12, which follows from similar monotonicity arguments.

Proof of Lemma 5.12. Observe that, since $f^{(2)}_{\vec{\beta}}$ is increasing in p_2 and decreasing in p_1, the level sets $\{f^{(2)}_{\vec{\beta}}(p) = \gamma\}$ define strictly increasing functions $p_2 = r_\gamma(p_1)$. On these level sets h_2 is clearly constant and h_1 is increasing in p_1; this last statement follows from the monotonicity of $g_{\alpha(1)}$ (proved in Proposition 5.5) and from $f^{(1)}_{\vec{\beta}}(p_1, r_\gamma(p_1)) + \gamma = (f^{(1)}_{\vec{\beta}} + f^{(2)}_{\vec{\beta}})(p_1, r_\gamma(p_1)) = 1 - \exp(-\beta(1)p_1 - \beta(2)r_{\gamma(p_1)})$, which implies that $f^{(1)}_{\vec{\beta}}$ increases in p_1. Since l_1 is decreasing in both arguments, at each level set $l_1(h)$ attains its maximum at points of minimal values of p_1. Our claim then is a result of the fact that each point $p \in A$ belongs to a level set $f^{(2)}_{\vec{\beta}} \equiv \gamma$ which attains a minimal value of p_1 at $\mathcal{C}_1 \cup \mathcal{C}_4 \cup \mathcal{C}_5$. □

The rest of the proof of Theorem 5.3.2 consists of modifying B_1 until obtaining the interior-recurrent set B required in the lemma. As a first step, observe that from Lemma 5.11 there is some $\gamma \in (0, 1)$ such that $\sup_{p\in B_1} l_1 \circ h(p) = \gamma$. We will build an interior-recurrent set B_2 simply by modifying slightly the definition of B_1. Define

$$B_20 = \left\{p \in [0, \kappa_\varepsilon] \times [c, \bar{p}_2],\ l_1(p) < \bar{\gamma}\right\}$$

for some $\bar{\gamma} \in (\gamma, 1)$. We claim that this set is interior-recurrent with parameter $\bar{k} = 1$. Indeed, take any $p \in B_2$, then, from our choice of parameters:

- From Proposition 5.7.(iii) we have $h_1(p) \leq (1 - \varepsilon')\kappa_\varepsilon$.
- Since $p_1 \leq \kappa_\varepsilon$, from Proposition 5.7.(i) we have $h_2(p) \leq (1 + \varepsilon')c$.
- From Lemma 5.11 we have $\sup_{p\in B_2} l_1 \circ h(p) \leq \sup_{p\in B_1} l_1 \circ h(p) = \gamma$.

This way, there is some $\delta > 0$ such that $d(h(p), B_2^c) > \delta$ uniformly on $p \in B_2$, which proves the claim. To show that the dynamical system reaches B_2, it suffices to show that it reaches B_1 in finite time. Fix an initial condition p^0. If $p^0_1 > \kappa_\varepsilon$, then by Proposition 5.7(iii) we have $p^1_1 \leq (1 - \varepsilon')p^0_1$, and we may repeat the argument until the trajectory reaches $[0, \kappa_\varepsilon] \times [0, \bar{p}_2]$, where it remains forever. From this point on we assume that $p^0_2 > c$, since if this is not satisfied we use Proposition 5.7.(i) to obtain $p^1_2 > p^0_2(1+\varepsilon')$, and then repeat the argument to show that the sequence eventually reaches $[0, \kappa_\varepsilon] \times [c, \bar{p}_2]$, where it remains forever. Hence to finish the proof it is enough to consider an initial condition p^0 inside this set and show that there is some finite k such that $l_1(p^k) < 1$. Suppose this is not the case. Then for all $n \in \mathbb{N}$ we have $l_1(p^n) \geq 1$, but from Lemma 5.11 this implies that there is some $\varepsilon > 0$ such that $l_2(p_n) > 1 + \varepsilon$ for all n. In particular, $p^{n+1}_2 > (1 + \varepsilon)p^n_2$ for all n and hence $p^n_2 \to \infty$, which is impossible since $p_2 \in [0, 1]$. We conclude that the dynamical system reaches B_1.

It remains to show that there are γ_1 and γ_2 such that

$$(1 - \alpha(1))f^{(1)}_{\vec{\beta}}(p) \leq \gamma_1 p_1 \quad \text{and} \quad \gamma_2 < p_2.$$

Taking $\gamma_2 = c$ the second inequality is trivially satisfied. The main problem is that in B_2 the decay we get is of the form $h_1(p) \leq \bar{\gamma}p_1$, which is not as strong as the one we need. However, once inside B_2 we have $p^k_1 \longrightarrow 0$, so in particular it is easy to see that for each δ, the set $B_\delta \subseteq B_2$ given by

$$B_\delta := \left\{p \in [0, \delta] \times [c, \bar{p}_2],\ l_1(p) < \bar{\gamma}\right\}$$

is also interior-recurrent and satisfies the same properties as B_2. Indeed, once the dynamical system reaches B_2, p_1^n decreases exponentially so it reaches B_δ. For any $\varepsilon > 0$ we can take δ sufficiently small, so that for any $p_1 < \delta$ we have $G^3_{\alpha(1)} \circ f^{(1)}_{\vec{\beta}}(p) \geq 1-\varepsilon$. Choosing ε sufficiently small, we use the inequality above to conclude that $(1-\alpha(1))f^{(1)}_{\vec{\beta}}(p) \leq \frac{\bar{\gamma}}{1-\varepsilon}p_1$, and the result then follows taking $\gamma_1 = \frac{\bar{\gamma}}{1-\varepsilon}$.

Appendix A. Technical proofs

A.1. **Proofs from Section 3.** As we mentioned, the proof of the convergence result, Theorem 2.2, is an adaptation of the proof of [DR09, Thm. 4] for the one-species model running on the torus, so we will only explain what needs to be changed. The extension to $m > 1$ is relatively straightforward, so we will focus first on the adaptations needed to drop the weak epidemics assumption $\alpha_N \longrightarrow 0$. The main step in their proof consists in considering *bad* sites, which are sites x such that the density of occupied sites in the ball of radius r_N around it is far from the global density of occupied sites, and then proving (see their Prop. 5.1) that if the system starts with a small enough density of bad points then the density of bad points remains small after one time step.

We introduce the following definitions:

$$B(x,r) = \{y \in R_N : d(x,y) \leq r\}, \qquad V(r) = |B(x,m)| = 3 \cdot 2^r - 2,$$
$$d_k^{N,(i)}(x) = \frac{1}{V(r_N)} \sum_{y \in B(x,r_N)} \eta_k^{N,(i)}(y), \quad G_k^N(\varepsilon) = \{x \in G_N : \textstyle\sum_{i=1}^m |d_k^{N,(i)}(x) - h_i^k(p)| < \varepsilon\}.$$

In the next lemma we will use the same random variables defined in the proof of [DR09, Thm. 4], only changing their C_0 by $\mathcal{C}_r$.

Lemma A.1. *Assume $m = 1$. Given $\varepsilon > 0$ there exists N sufficiently large such that*

$$\mathbb{E}\big(|\tilde{\rho}_{k+1}^N - \hat{\rho}_{k+1}^N|\big) \leq \varepsilon.$$

Proof. We will use $\delta_1 > 0$, $\delta_2 > 0$ as small as needed. Changing the proof of convergence given for $|d_k^N(x) - d_k^N(0)|$ in [DR09, Lem. 5.4] by

$$\begin{aligned}
&\mathbb{P}\left(|d_k^N(i) - d_k^N(0)| > \delta_1 \text{ for some } x \in B(0,l_N)\right)\\
&\quad \leq V(l_N)\sup_{x\in B(0,l_N)} \mathbb{P}\big(|d_k^N(x) - d_k^N(0)| > \delta_1\big)\\
&\quad = V(l_N)\sup_{x\in B(0,l_N)} \mathbb{P}\Big(\Big|\tfrac{1}{V(r_N)}\Big(\textstyle\sum_{y\in B(x,r_N)\setminus B(0,r_N)} \eta_k^N(y) - \sum_{y\in B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big)\Big| > \delta_1\Big)\\
&\quad \leq \sup_{x\in B(0,l_N)} \frac{V(l_N)\mathsf{Var}\Big(\Big|\sum_{y\in B(x,r_N)\setminus B(0,r_N)} \eta_k^N(y) - \sum_{y\in B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big|\Big)}{V(r_N)^2\delta_1^2}\\
&\quad \leq \sup_{x\in B(0,l_N)} \frac{V(l_N)\mathsf{Var}\Big(\sum_{y\in B(x,r_N)\setminus B(0,r_N)\cup B(0,r_N)\setminus B(x,r_N)} \eta_k^N(y)\Big)}{V(r_N)^2\delta_1^2}\\
&\quad \leq \frac{2V(l_N)V(r_N)\mathsf{Var}\big(\eta_k^N(0)\big)}{V(r_N)^2\delta_1^2}
\end{aligned}$$

gives

$$\mathbb{P}\left(|d_k^N(i) - d_k^N(0)| > \delta_1 \text{ for some } i \in B(0,l_N)\right) \leq \tfrac{2V(l_N)}{V(r_N)\delta_1^2} \leq \tfrac{1}{\delta_1^2} 2^{l_N - r_N + 2} \longrightarrow 0.$$

Now define $Y(\delta) = \#\left(\xi_{1/2}^{h_k(p)+2\delta_1} \setminus \xi_{1/2}^{h_k(p)}\right)$. For the inequality in (5.4) in [DR09] in our case we consider the following bound

$$\begin{aligned}
&\mathbb{P}\left(\xi_1^{h_k(p)+2\delta_1,N}(0)=0, \xi_1^{h_k(p),N}(0)=1, \#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\right)\\
&\qquad\leq \mathbb{E}\left(1-\left(\tfrac{(1-\alpha)}{2}\right)^{Y(\delta_1)}\mathbb{1}_{\{\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\}}\right)\\
&\qquad\leq \textstyle\sum_{i=0}^{\infty}\left(1-\left(\tfrac{(1-\alpha)}{2}\right)^{i}\right)\mathbb{P}\Big(Y(\delta_1)=i\Big|\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\Big).
\end{aligned}$$

The last term converges to 0 when $\delta_1 \to 0$, because (here A_j comes from (3.1))

$$\begin{aligned}
&\mathbb{P}\left(\xi_{1/2}^{h_k(p)+2\delta_1}=\xi_{1/2}^{h_k(p)}\Big|\#\xi_{1/2}^{h_k(p)+2\delta_1}<\infty\right)\\
&\qquad= \frac{e^{-\beta\left(h_k(p)+2\delta_1\right)}+\sum_{j=1}^{\infty}A_j\left(1-e^{-\beta h_k(p)}\right)^j e^{-(j+2)\beta\left(h_k(p)+2\delta_1\right)}}{1-\mathbb{P}_q(|\mathcal{C}_r|=\infty)} \xrightarrow[\delta_1\to 0]{} 1,
\end{aligned}$$

where $q = 1-e^{-\beta\left(h_k(p)+2\delta_1\right)}$. The last limit is obtained using the proof of Proposition 2.1 and the Dominated Convergence Theorem. The conclusion follows as in [DR09]. □

Proof of Theorem 2.2. The case $m = 1$ follows by changing Lemma 5.4 in [DR09, Thm. 4] by Lemma A.1 above. The proof for the case $m \geq 2$ is just an adaptation of the case $m = 1$ multiple species, here we show the key points. In these adaptations one should always use $\|\cdot\|_1$ instead of $|\cdot|$. Recall that the evolution in the growth step of a given site x depends on the local density $d_k^{N,(i)}$. Given that each occupied site x of type i sends a Poisson$[\beta(i)/V(r_N)]$ number of births to each of its $V(r_N)$ neighbors in $B(x, r_N)$ it follows that each site receives a Poisson$[\beta(i)d_k^{N,(i)}]$ number of births of type i and a total Poisson$[\sum_{i=1}^m \beta(i)d_k^{N,(i)}]$ number of births. Then, given $\eta_k^N(x)$, the site x has a particle of type i after the growing stage with probability

$$\mathbb{P}\Big(\eta_{k+1/2}^N(x)=i\Big)=\left(1-\exp\left(\textstyle\sum_{j=1}^m\beta(j)d_k^{N,(j)}\right)\right)\frac{\beta(i)d_k^{N,(i)}}{\sum_{j=1}^m\beta(j)d_k^{N,(j)}}.$$

The random variables $\tilde{\eta}_k^N$, η_k^N and $\hat{\eta}_k^N$ have to be extended for multi-species and the coupling between these has to be reformulated accordingly. These are simple adaptations so they are left to the reader, together with the remainder of the proof. □

Proof of Lemma 3.1. For each $k \geq 2$ define

$$W_k = \sqrt{\mathbb{E}_1\big((1-\alpha_N)^{Z_2+\cdots+Z_k}\big)}$$

where $\mathbb{E}_1$ stands for the law of the Galton-Watson process with $Z_1 = 1$. Since Z_0 is a Bernoulli random variable with parameter q, we clearly have (with the obvious notation)

$$\begin{aligned}
\mathbb{E}\big(Z_0(1-\alpha_N)^{Z_0+\cdots+Z_{L_N-1}}\big) &= q(1-\alpha_N)\mathbb{E}\big((1-\alpha_N)^{Z_1+\cdots+Z_{L_N-1}}\big)\\
&= q(1-\alpha_N)\mathbb{E}\Big((1-\alpha_N)^{Z_1}\big(\mathbb{E}_1\big(1-\alpha_N\big)^{Z_2+\cdots+Z_{L_N-1}}\big)^{Z_1}\Big)\\
&= q(1-\alpha_N)r((1-\alpha_N)W_{L_N-1}^2)
\end{aligned}$$

where $r(x) = (qx+1-q)^3$ is the probability generating function of a Binomial$[3, q]$ random variable. To obtain an expression for W_{L_N-1} we study the sequence $(W_k)_{k\geq 2}$ which, using the same reasoning as above, satisfies the quadratic recurrence equation

$$W_{k+1} \;=\; q(1-\alpha_N)W_k^2 + 1 - q \tag{A.1}$$

with initial condition $W_2 = (1-\alpha_N)q + 1 - q$. This recurrence equation has two fixed points, $\frac{1\pm\sqrt{1-4q(1-q)(1-\alpha_N)}}{2q(1-\alpha_N)}$; the one with a plus is repulsive while the one with a minus is attractive, so all orbits starting in $[0, 1]$ converge to the latter which we call $\overline{W}$. From its definition we have

$r((1-\alpha_N)\overline{W}^2) = \Big[q(1-\alpha_N)\overline{W}^2 + 1 - q\Big]^3 = \overline{W}^3$, and observing that $g_{\alpha_N}(q) = q(1-\alpha_N)\overline{W}^3$, we deduce that (3.5) is equivalent to

$$q(1-\alpha_N)\Big|r((1-\alpha_N)W_{L_N-1}^2) - r((1-\alpha_N)\overline{W}^2)\Big| \le C\theta_\alpha(N). \tag{A.2}$$

Since $q(1-\alpha_N) \le 1$ and $|r(a) - r(b)| \le 3|a-b|$ for all $a, b \in [0,1]$, it will be enough to show that $|W_{L_N-1} - \overline{W}| \le C\theta_\alpha(N)$. To this end we notice that, from the definition of $\overline{W}$,

$$\begin{aligned} \big|W_{k+1} - \overline{W}\big| &= \Big|\big[q(1-\alpha_N)W_k^2 + 1 - q\big] - \big[q(1-\alpha_N)\overline{W}^2 + 1 - q\big]\Big| \\ &= q(1-\alpha_N)\big|W_k - \overline{W}\big|\big(W_k + \overline{W}\big) \le q(1-\alpha_N)\big|W_k - \overline{W}\big|\big(1 + \overline{W}\big), \end{aligned} \tag{A.3}$$

but it can be easily deduced that $q(1+\overline{W}) \le 1$, thus

$$\big|W_{k+1} - \overline{W}\big| \le (1-\alpha_N)\big|W_k - \overline{W}\big| \tag{A.4}$$

for all $k \ge 2$. In particular, we obtain

$$|W_{L_N-1} - \overline{W}| \le 2(1-\alpha_N)^{L_N-2} \le Ce^{-\alpha_N L_N} = CN^{-\frac{\alpha_N}{5\log(2)}},$$

where the last equality follows from the definition of L_N. If $\alpha \neq 0$, then for N large the exponent is smaller than $-\frac{\alpha}{5}$, giving the result. When $\alpha = 0$, we need to improve this bound. To do so, we use (A.4) to bound the distance between the $\frac{L_N}{2}$-th term of the sequence and $\overline{W}$, obtaining the similar expression;

$$|W_{L_N/2} - \overline{W}| \le 2e^{-\frac{\alpha_N(L_N-2)}{2}} \le Ce^{2\log\alpha_N} = C(\alpha_N)^2,$$

where in the second inequality we used condition (2.4) to bound the exponent (this is valid for N large, hence the C factor). Noticing that W_k converges monotonically to $\overline{W}$, the above bound is valid for all W_k with $k \ge \frac{L_N}{2}$, so we can restart the sequence at the $\frac{L_N}{2}$-th term to improve the bound in (A.3) to

$$\big|W_{k+1} - \overline{W}\big| = q(1-\alpha_N)\big|W_k - \overline{W}\big|\big(W_k + \overline{W}\big) \le q(1-\alpha_N)\big|W_k - \overline{W}\big|\big(C(\alpha_N)^2 + 2\overline{W}\big).$$

But $2q(1-\alpha_N)\overline{W} = 1 - \sqrt{1 - 4q(1-q)(1-\alpha_N)} \le 1 - \sqrt{\alpha_N}$, giving $\big|W_{k+1} - \overline{W}\big| \le \big|W_k - \overline{W}\big|\big[1 - \sqrt{\alpha_N} + C(\alpha_N)^2\big]$ for all $k \ge \frac{L_N}{2}$. In particular,

$$|W_{L_N-1} - \overline{W}| \le 2\big[1 - \sqrt{\alpha_N} + C(\alpha_N)^2\big]^{L_N/2} \le Ce^{-\frac{\sqrt{\alpha_N}\log N}{20}} \le Ce^{-\sqrt{\log N}},$$

where we used that $\alpha_N \log N \to \infty$ as $N \to \infty$. □

A.2. **Proofs from Section 5.2.**

Proof of Proposition 5.5. We prove only the case $\alpha > 0$; the case $\alpha = 0$ is much easier to handle. Observe first that $G_\alpha(x)$ satisfies

$$G_\alpha(x)\sqrt{1 - 4(1-\alpha)x(1-x)} = -G_\alpha(x) + 2 - 2x, \tag{A.5}$$

$$G'_\alpha(x) = \frac{G_\alpha(x)-1}{x\sqrt{1-4(1-\alpha)x(1-x)}} = \frac{G_\alpha(x)-1}{x[1-2(1-\alpha)xG_\alpha(x)]}. \tag{A.6}$$

To find the maximum of g_α we impose the first order condition $0 = g'_\alpha(x) = xG_\alpha^3(x)\Big[\frac{1}{x} + \frac{3G'_\alpha(x)}{G_\alpha(x)}\Big]$. The factor $xG_\alpha^3(x)$ equals 0 only at 0 and 1, so $g'_\alpha(x) = 0$ only if the factor in brackets vanishes. It is left to the reader that together with (A.5) and (A.6), the condition above gives $G_\alpha(x) = x + 1/2$. This way, since $G_\alpha \le 1$, every critical point of the function must lie in $[0, 1/2]$.

The first part of the proposition will follow if we show that at every critical point x_0 we have $g''_\alpha(x_0) < 0$ (so every critical point is a maximum, and hence there can be only one). Now, since $g'_\alpha(x_0) = 0$, $g''_\alpha(x_0) = g_\alpha(x_0)\Big[\frac{3G''_\alpha(x_0)}{G_\alpha(x_0)} - \frac{4}{3x_0^2}\Big]$, and it is enough to show that $G''_\alpha(x_0) < 0$. Using (A.5) and (A.6) we find $G''_\alpha(x) = \frac{[G_\alpha(x)-1]2(1-\alpha)x[2G_\alpha(x)+xG'_\alpha(x)]}{[x(1-2(1-\alpha)xG_\alpha)]^2}$, which is negative as

soon as $2G_\alpha(x_0) + xG'_\alpha(x_0) > 0$ since $G_\alpha \le 1$. By (A.5) and (A.6) this is equivalent to $3 - 4x > G_\alpha(x_0)$, which is satisfied because $0 \le x \le 1/2$.

To prove the second part of the proposition write $\Sigma_p = \beta(1)p_1 + \beta(2)p_2$ so that

$$f^{(i)}_{\vec\beta}(p) \;=\; \tfrac{1-e^{-\Sigma_p}}{\Sigma_p}\beta(i)p_i.$$

Since $x \mapsto \frac{1-e^{-x}}{x}$ is decreasing, it follows that $f^{(i)}_{\vec\beta}(p) \le 1 - e^{-\beta(i)p_i} \le 1 - e^{-\beta(i)g_{\alpha(i)}(x_0)}$ so it will be enough to prove that $1 - e^{-\beta(i)g_{\alpha(i)}(x_0)} \le x_0$. Since x_0 is characterized by $G_{\alpha(i)}(x_0) = x_0 + 1/2$, it is enough to show that $V(x_0) := \phi_i x_0\big(\frac12 + x_0\big)^3 + \log(1 - x_0) \le 0$. But, in fact, V is non-positive on the entire interval $[0, 1/2]$. Indeed, $V(0) = 0$ and $V(1/2) = \frac{\phi_i}{2} - \log 2$, which is negative from our assumption $\phi_i < 2\log 2$, so it is enough to prove that the inequality holds at the critical points of V; this follows from $V'(x) = \phi_i(\frac12 + x)^2(\frac12 + 4x) - \frac{1}{1-x}$, $V''(x) = \phi_i(\frac12 + x)(3 + 12x) - \frac{1}{(1-x)^2}$, so whenever $V'(x_1) = 0$ we have $(1 - x_1)V''(x_1) = \phi_i(x_1 + 1/2)[-16x_1^2 + 13x_1/2 + 11/4]$, which is positive in $[0, 1/2]$, giving that x_1 is a minimum. □

Proof of Proposition 5.6. We keep the notation Σ_p used in the previous proof. For the dependence of $f^{(1)}_{\vec\beta}$ on p_1 we write the function as $(1 - e^{-\Sigma_p})\frac{\beta(1)p_1}{\Sigma_p}$ which, for fixed p_2, is the product of two increasing functions. For the dependence of $f^{(1)}_{\vec\beta}$ on p_2, on the other hand, we write $f^{(1)}_{\vec\beta}$ as $\frac{1-e^{-\Sigma_p}}{\Sigma_p}\beta(1)p_1$; the factor on the left is decreasing on p_2 while the one on the right is constant. This gives (1). Next observe that $l_1(p) = \phi_1\frac{1-e^{-\Sigma_p}}{\Sigma_p}G^3_{\alpha(1)} \circ f^{(1)}_{\vec\beta}(p)$ and the same analysis shows that $f^{(1)}_{\vec\beta}$ is increasing and G_α is decreasing, giving (2).

If $\phi_1 < 2\log 2$, then from Proposition 5.5 we know that $g'_{\alpha(i)} \circ f^{(i)}_{\vec\beta}(p) \ge 0$, so h^1 satisfies the same monotonicity as $f^{(1)}_{\vec\beta}$ on each argument. Since $l_1(p) = \frac{h^1(p)}{p_1}$, it must behave as h^1 with respect to p_2. This gives (3). □

Proof of Proposition 5.7. Again we keep definition of Σ_p used in the proof of Proposition 5.5. To prove (5.7) we take c small (to be fixed later) and suppose that $p_2 < c$. Observing that $f^{(2)}_{\vec\beta}(p) = \frac{1-e^{-\Sigma_p}}{\Sigma_p}\beta(2)p_2$ we deduce that $\frac{1-e^{-\beta(1)p_1}}{\beta(1)p_1}\beta(2)p_2 < f^{(2)}_{\vec\beta}(p) < \beta(2)p_2$, so from the assumption $p_2 < c$ and the monotonicity of G_α, we deduce

$$l_2(p) = (1 - \alpha(2))\frac{f^{(2)}_{\vec\beta}(p)}{p_2}G^3_{\alpha(2)} \circ f^{(2)}_{\vec\beta}(p) \ge \phi_2\tfrac{1-e^{-\beta(1)p_1}}{\beta(1)p_1}G^3_{\alpha(2)}(\beta(2)c). \tag{A.7}$$

Since the fraction is decreasing on p_1 we obtain a lower bound by taking $p_1 = \kappa_\varepsilon$ and using its definition to obtain $l_2 \ge (1 - \varepsilon)\frac{\phi_2}{\phi_1}\frac{G^3_{\alpha(2)}(\beta(2)c)}{G^3_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon})}$. But $\frac{\phi_2}{\phi_1} > 1$, and as $c \to 0$ we have $G_{\alpha(2)}(\beta(2)c) \to 1$, so taking first ε small and then c sufficiently small, the right hand side is larger than $1 + \varepsilon'$ for some ε'.

For (5.8), Proposition 5.5 gives that $g_{\alpha(2)}$ has a single critical point which is a maximum, so $h_2 = g_{\alpha(2)} \circ f^{(2)}_{\vec\beta}$ is minimized either when $f^{(2)}_{\vec\beta}$ is minimized or maximized. Remembering that $f^{(2)}_{\vec\beta}$ decreases with p_1 and increases with p_2, we conclude that the minimum of h_2 over the set $[0, \kappa_\varepsilon] \times [c, \bar p_2]$ is obtained either at $(0, \frac{1-\alpha(2)}{2})$ or at (κ_ε, c). We already saw that at $p = (\kappa_\varepsilon, c)$ we have $h_2(p) = l_2(p)p_2 > (1 + \varepsilon')c$, meaning that we need only to control h_2 at $(0, \frac{1-\alpha(2)}{2})$, which is equal to $g_{\alpha(2)}(1 - e^{-\phi_2/2})$ and the result follows by taking c small enough so that $g_{\alpha(2)}(1 - e^{-\phi_2/2}) > (1 + \varepsilon')c$.

To get (ii) in the proposition we need to extend the above properties for a general value of p_1. We proceed analogously, but when computing (A.7) we use the additional information

$\phi_2 > 2\log 2$ to improve the lower bound without imposing any restriction on p_1. Indeed, since $\phi_1 < \phi_2$ we deduce that $\beta(1)p_1 \leq \frac{\phi_2}{2}$ so, from monotonicity of $\frac{1-e^{-x}}{x}$,

$$l_2(p) \geq \phi_2 \tfrac{1-e^{-\beta(1)p_1}}{\beta(1)p_1} G^3_{\alpha(2)}(\beta(2)c) \geq 2(1-e^{-\phi_2/2}) G^3_{\alpha(2)}(\beta(2)c),$$

but $2(1-e^{-\phi_2/2}) > 1$ from the assumption on ϕ_2, so taking c sufficiently small we conclude again that $l_2(p) > 1+\varepsilon'$ for some ε' small. The proof of the second property is exactly the same as in 5.8.

We turn finally to (5.9) and (5.10). Notice that, since $\phi_1 < 2\log 2$, from Proposition 5.6 we know that h_1 is increasing in p_1 and decreasing in p_2, so using the definition of κ_ε we deduce

$$p_1 < \kappa_\varepsilon \implies h_1(p) \leq h_1(\kappa_\varepsilon, 0) = g_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon}) = (1-\varepsilon)\kappa_\varepsilon,$$

which proves (5.9). To prove (5.10) we use a similar argument with l_1, which we know is decreasing in both arguments, so that

$$\kappa_\varepsilon < p_1 \implies l_1(p) \leq l_1(\kappa_\varepsilon, 0) = \tfrac{g_{\alpha(1)}(1-e^{-\beta(1)\kappa_\varepsilon})}{\kappa_\varepsilon} = (1-\varepsilon),$$

and the result follows. □

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