MACDONALD TREES AND DETERMINANTS OF REPRESENTATIONS FOR FINITE COXETER GROUPS

ARVIND AYYER, AMRITANSHU PRASAD, AND STEVEN SPALLONE

ABSTRACT. Every irreducible odd dimensional representation of the *n*'th symmetric or hyperoctahedral group, when restricted to the (n-1)'th, has a unique irreducible odd-dimensional constituent. Furthermore, the subgraph induced by odd-dimensional representations in the Bratteli diagram of symmetric and hyperoctahedral groups is a binary tree with a simple recursive description. We survey the description of this tree, known as the Macdonald tree, for symmetric groups, from our earlier work. We describe analogous results for hyperoctahedral groups.

A partition λ of n is said to be *chiral* if the corresponding irreducible representation V_{λ} of S_n has non-trivial determinant. We review our previous results on the structure and enumeration of chiral partitions, and subsequent extension to all Coxeter groups by Ghosh and Spallone. Finally we show that the numbers of odd and chiral partitions track each other closely.

1. Summary of results

This article is mainly a survey of the main results of three papers [1] and [2] and [7] concerning the application of the dyadic arithmetic of partitions to the representation theory of finite Coxeter groups.

Recall that Young's lattice is the set Λ of integer partitions, partially ordered by containment of Young diagrams. It has a unique minimal element \emptyset , the trivial partition of 0. Its Hasse diagram is known as Young's graph. For each $\lambda \in \Lambda$, let f_{λ} denote the number of saturated chains in Λ from \emptyset to λ . This number is also the number of standard tableaux of shape λ , and equals the dimension of the irreducible representation of the symmetric group associated with λ . The numbers f_{λ} can be computed by the hook-length formula of Frame, Robinson and Thrall [6, Theorem 1].

We say that a partition λ is odd if f_{λ} is odd. In Ref. [1] it is shown that the subgraph induced by the subset of odd partitions in Young's graph forms an incomplete binary tree rooted at \emptyset . See Figure 1 for an illustration of the first few rows of this tree. Moreover, this binary tree has a simple recursive structure that gives a structural explanation for a formula independently due to McKay [15] and Macdonald [13] (stated as Theorem 1 here) for the number of odd partitions of n. Macdonald generalized this formula to other Coxeter groups in Ref. [14]. In Section 2, we begin by explaining these results for the symmetric group. We then present our new results for the hyperoctahedral groups, which explain Macdonald's formulas.

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Our results for the symmetric group have been used by Bessenrodt, Giannelli, Kleshchev, Miller, Navarro, Olsson and Tiep to propose bijective McKay correspondences for symmetric groups and study their representation-theoretic properties [3, 8, 9, 10, 11].

The irreducible complex representations of S_n are parametrized by partitions of n. Let λ be a partition and $(\rho_{\lambda}, V_{\lambda})$ be the associated irreducible representation of S_n . Let det : $GL_{\mathbf{C}}(V_{\lambda}) \to \mathbf{C}^*$ denote the determinant function. The composition det $\circ \rho_{\lambda} : S_n \to \mathbf{C}^*$, being a multiplicative character of S_n , is either the trivial character or the sign character. We say that λ is a *chiral partition* if det $\circ \rho_{\lambda}$ is the sign character. For each $\lambda \in \Lambda$, let g_{λ} denote the number of saturated chains in Λ

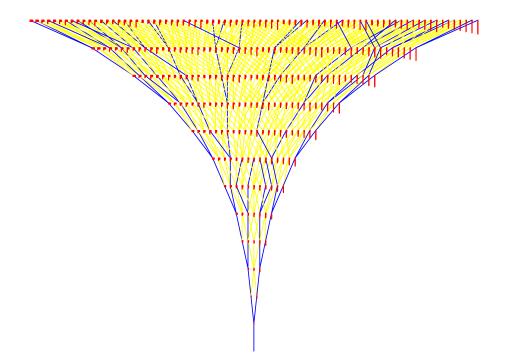


FIGURE 1. The first thirteen rows of the Macdonald tree (blue edges) in Young's graph (yellow edges).

from (1,1) to λ , which is also the number of standard tableaux T of shape λ such that 2 occurs in the first column of T. Note that $g_{(n)} = 0$ for each n. It turns out (see Section 3.1) that a partition λ is chiral if and only if g_{λ} is odd. The most striking result of Ref. [2] is a closed formula for the number of chiral partitions of n. This answers an open question posed in [18, Exercise 7.55(b)]. In Section 3, we review this result as well as analogues of chiral partitions for all Coxeter groups. Finally, in Section 4, we show that the enumeration problem of odd and chiral partitions for the symmetric groups are closely related.

We appeal to standard textbooks in representation theory (see, for example, [12]) for basic definitions involving partitions such as arm-length, leg-length and content.

2. Macdonald trees

Despite having nice combinatorial properties, the Bratteli diagrams of systems of Coxeter groups are objects of great complexity when viewed as directed graphs. A powerful manifestation of this complexity is the difficulty in providing a closed formula for the number of partitions of n. In this section, we shall examine the subgraphs induced in Bratteli diagrams of Coxeter groups by odd-dimensional representations. For symmetric and hyperoctahedral groups, these subgraphs turn out to be rooted binary trees (where every node has up to two branches) whose structure can be captured in a few succinct axioms (Theorems 3 and 8). We call these trees *Macdonald trees* after Ian G. Macdonald, whose techniques involving cores and quotients we used. The structure of these trees explains the enumerative results of Macdonald [13] and McKay [15]. At the time we found our results, we only knew about the work of Macdonald. It was later brought to our attention that McKay had published the same results a year before.

We begin with an illustrative toy case in Section 2.1. We then recall our results for the symmetric group in Section 2.2. Our new results for the hyperoctahedral group are explained in Section 2.3. We show that the subgraph of odd-dimensional representations for the demihyperoctahedral group is not a tree in Section 2.4.

2.1. Toy case: Pascal's triangle. To better appreciate the recursive structure of Macdonald trees it is instructive to begin with the much simpler example of Pascal's triangle. Consider the graph P whose nodes are pairs (a, b), where a and b are non-negative integers. Pairs (a, b) and (c, d) are connected by an edge if either a = c and |b - d| = 1 or |a - c| = 1 and b = d. This graph is the Hasse diagram of the poset $\mathbf{N} \times \mathbf{N}$, where \mathbf{N} denotes the non-negative integers with the usual linear order.

It is easy to see that the number of number of saturated chains in $\mathbf{N} \times \mathbf{N}$ from (0,0) to (a,b) is the binomial coefficient $C(a,b) = \frac{(a+b)!}{a!b!}$. Call a pair $(a,b) \in P$ an odd pair if C(a,b) is odd. Call it a one-dimensional pair if C(a,b) = 1 (i.e., a = 0 or b = 0). It is a now well-known discovery of Lucas (see [20, Chapter 17]) that the subset of P consisting of odd pairs is a combinatorial version of the Sierpiński gasket. We revisit this connection from the point of view of graphs. Let L denote the subgraph induced in P by odd pairs. Let L_k denote the subgraph of L consisting of its first 2^k rows (namely odd pairs (a, b) with $0 \le a + b < 2^k$). Then L_1 is the

binary tree \bigvee . All the nodes in L_1 are one-dimensional. Moreover,

- For each $k \ge 1$, L_{k+1} can be obtained from L_k as follows: to each onedimensional node in the top row of L_k , add an edge and attach one copy of L_k .
- One of the one-dimensional nodes in each copy of L_k so attached becomes a one-dimensional node in the top row of L_{k+1} .

Let $\ell(n)$ be the number of odd pairs summing to n and $\nu(n)$ be the number of 1's in the binary expansion of n. The two axioms suffice to construct L by recursion (see Fig. 2), and imply that, if $2^k \leq n < 2^{k+1}$, then $\ell(n) = 2\ell(n-2^k)$. It then follows that $\ell(n) = 2^{\nu(n)}$. Our results on Macdonald trees of symmetric and hyperoctahedral groups are more subtle analogs of this phenomenon.

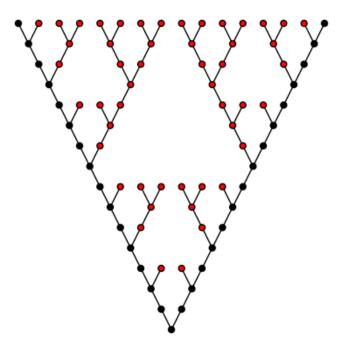


FIGURE 2. The binary tree L_4 . One-dimensional nodes are black.

2.2. Odd partitions of symmetric groups. We will need some definitions. A partition $\lambda \vdash n$ is said to be a *hook* if $\lambda = (k, 1^{n-k})$ for some $k, 1 \leq k \leq n$. In the special case when k equals 1 or n, we say that the partition is *one-dimensional*.

Let a(n) be the number of odd partitions of n. Let n be a positive integer with binary expansion given by

(1)
$$n = \epsilon + 2^{k_1} + 2^{k_2} + \dots + 2^{k_r}, \epsilon \in \{0, 1\}, \ 0 < k_1 < k_2 < \dots < k_r.$$

Then we have the following result:

Theorem 1 ([13, 15]). If n is an integer having binary expansion given in (1), then

$$a(n) = 2^{k_1 + \dots + k_r}.$$

In other words, if we express n in binary as a sum of powers of 2, then a(n) is the *product* of those powers of 2. A recursive version of Theorem 1 is:

Theorem 2. We have a(0) = 1, and moreover, for any positive integer n, let k be such that $2^k \leq n < 2^{k+1}$. Then

$$a(n) = 2^k a(n - 2^k).$$

Our first result is an interpretation of Theorem 2 in terms of Young's lattice. Let M denote the subgraph induced in Young's graph by the set of odd partitions. The first thirteen rows of M inside Young's graph are shown in Figure 1. Let M_k denote the first 2^k rows of M (consisting of partitions of 0 through $2^k - 1$).

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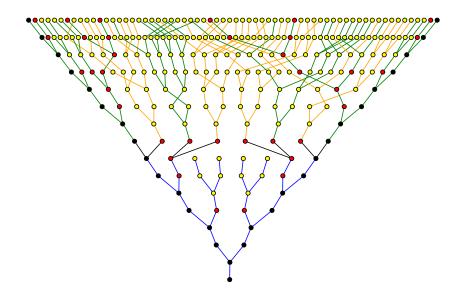


FIGURE 3. The recursive construction of M_4 from M_3 . M_3 is shown in blue. Two copies of M_3 (shown in green and orange) are attached to each of the four odd-dimensional hooks of rank seven. The black nodes represent one-dimensional partitions, the red nodes represent higher-dimensional hooks, and the remaining nodes are yellow.

Theorem 3 (Recursive description of M). M_2 is the rooted tree



For each $k \geq 2$, we have:

- (3.1) The graph M_{k+1} is obtained from M_k by attaching two copies of M_k to each hook in the top row of M_k via an edge.
- (3.2) The hooks in the top row of M_{k+1} are the hooks in M_k that ascend from the one-dimensional nodes in the top row of M_k .
- (3.3) The one-dimensional nodes in the top row of M_{k+1} are one-dimensional nodes in one of the two copies of M_k that ascend from the one-dimensional nodes in the top row of M_k .

In particular, M_{k+1} is obtained from M_k by attaching copies of M_k to certain leaves. The entire Macdonald tree M can be constructed by starting with M_2 and applying the recursive rules above. The fact that M is a tree means that an odd-dimensional irreducible representation of S_n , when restricted to S_{n-1} , contains exactly one odd-dimensional representation of S_{n-1} .

Remark 4. The rules in Theorem 3 are consistent with the following observations:

(4.1) The graph M_k is a rooted binary tree with $2^{\binom{k}{2}}$ nodes in its top row.

- (4.2) Of these nodes, 2^{k-1} are hooks.
- (4.3) Two of these hooks are one-dimensional.
- (4.4) The subgraph of hooks in M is isomorphic to the subgraph L of odd pairs.

Figure 3 shows how M_4 is constructed from M_3 . Theorem 3 may be regarded as a structural explanation of Theorem 2, similar to the explanation for the number of odd numbers in the *n*'th row of Pascal's triangle in Section 2.1. The main ingredient in the proof of Theorem 3 is the following recursive description of odd partitions and their cover relations (see [1, Lemma 2]).

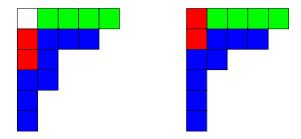


FIGURE 4. The odd partition $\lambda = (5, 4, 2, 2, 1, 1)$ on the left and the unique odd partition that it covers, $\mu = (5, 4, 2, 1, 1, 1)$, on the right. Both can be reduced to their 2-cores, (1) and \emptyset respectively, by successively removing unique rim-hooks of size 8 (blue), 4 (green) and 2 (red) with the same leg-lengths.

Theorem 5. Let n be a positive integer, and let k be the non-negative integer such that $2^k \leq n < 2^{k+1}$. A partition $\lambda \vdash n$ is odd if and only if

- (5.1) λ has a 2^k -hook, and
- (5.2) the partition of $n 2^k$ obtained from λ by removing this hook (the 2^k -core of λ) is odd.

Moreover, the odd partition λ is uniquely determined by its 2^k -core and the leglength of its 2^k -hook. Finally, if $2^k < n < 2^{k+1}$, and μ is an odd partition of n-1, then μ is covered by λ if and only if the 2^k -hooks of λ and μ have the same leg-length and the 2^k -core of μ is covered by the 2^k -core of λ .

Theorem 5 provides fast algorithms to sequentially enumerate all odd partitions of n and to generate a uniformly random odd partition of n. See Section 4 for an example.

Repeated applications of Theorem 5 show that any odd partition of n of the form (1) can be reduced to a partition of size ϵ by successively removing rims of hooks of size 2^{k_r} , $2^{k_{r-1}}$, and so on. See Figure 4 for an example.

2.3. Odd-dimensional representations of hyperoctahedral groups. The *n*'th hyperoctahedral group is the wreath product $\mathbb{B}_n = S_2 \wr S_n = S_2^n \rtimes S_n$. The normal subgroup $S_2^n \triangleleft \mathbb{B}_n$ has two S_n -invariant characters, the trivial character, and the character ε whose restriction to each factor S_2 is its non-trivial character.

For $\lambda \vdash n$, consider the irreducible representations ρ_{λ}^{0} and ρ_{λ}^{1} of \mathbb{B}_{n} on the representation space V_{λ} of S_{n} defined by:

 $\rho_{\lambda}^{0}(xw) = \rho_{\lambda}(w)$ and $\rho_{\lambda}^{1}(xw) = \epsilon(x)\rho_{\lambda}(w)$, for $x \in S_{2}^{n}$, and $w \in S_{n}$.

Now suppose that n = a + b for two non-negative integers a and b, and that α is a partition of a, and β a partition of b. The pair (α, β) is called a *bipartition* of nand we write $(\alpha, \beta) \models n$ in this case. The number of bipartitions of n is denoted $p_2(n)$ and it is well-known that [17, Sequence A000712]

$$\sum_{n \ge 0} p_2(n) x^n = \prod_{j \ge 1} \frac{1}{(1 - x^j)^2}.$$

Consider the subgroup $\mathbb{B}_a \times \mathbb{B}_b \cong S_2^n \rtimes (S_a \times S_b) < \mathbb{B}_n$. Define

(2)
$$\rho_{\alpha\beta} = \operatorname{ind}_{\mathbb{B}_a \times \mathbb{B}_b}^{\mathbb{B}_n} \rho_{\alpha}^0 \boxtimes \rho_{\beta}^1,$$

where \boxtimes is the external tensor product. Then

$$\{\rho_{\alpha\beta} \mid \alpha \vdash a, \beta \vdash b, a+b=n\}$$

is a complete set of representatives for the set of isomorphism classes of irreducible representations of \mathbb{B}_n . Moreover, the restriction of $V_{\alpha\beta}$ to \mathbb{B}_{n-1} contains $V_{\gamma\delta}$ if and only if either $\gamma \in \alpha^-$ and $\delta = \beta$, or $\gamma = \alpha$ and $\delta \in \beta^-$. Here λ^- is the set of all partitions that are obtained from the Young diagram of λ by removing a corner box. Thus the Bratteli diagram of the family \mathbb{B}_n of hyperoctahedral groups turns out to be the Hasse diagram of the poset of bipartitions, partially ordered under containment. This poset is the *Cartesian product* [19, Section 3.2] of Young's lattice with itself. We denote this Bratteli diagram by Y^2 .

We say that a bipartition (α, β) is *odd* if the dimension of $\rho_{\alpha\beta}$ is odd. Equation (2) implies that the dimension of $\rho_{\alpha\beta}$ is given by:

$$f_{\alpha\beta} = \frac{n!}{a!b!} f_{\alpha} f_{\beta}.$$

As a consequence, (α, β) is an odd bipartition if and only if α and β are both odd partitions, and the binomial coefficient C(a, b) is odd. Let bin(n) denote the set of place values where 1 occurs in the binary expansion of n. By Lucas's theorem (see [5], for example), C(a, b) is odd if and only if there exists a subset $S \subset bin(n)$ such that bin(a) = S and $bin(b) = bin(n) \setminus S$. These considerations, combined with Macdonald's formula for counting the number of odd partitions, lead to his formula for the number $a_2(n)$ of odd bipartitions of n from [14]:

Theorem 6 (Macdonald [14]). If n is an integer having binary expansion given in (1), then

$$a_2(n) = a(2n).$$

A recursive version of Theorem 6 is:

Theorem 7. We have $a_2(0) = 1$, and for any positive integer n, let k be such that $2^k \leq n < 2^{k+1}$. Then

$$a_2(n) = 2^{k+1}a_2(n-2^k).$$

As in the case of symmetric groups, Theorem 7 has an explanation in terms of the Bratteli diagram of hyperoctahedral groups.

Let M^2 denote the subgraph induced in Y^2 by odd bipartitions. The above description of odd bipartitions easily implies that M^2 is also an incomplete binary tree. Indeed, given an odd bipartition (α, β) , with $|\alpha| = a$, and $|\beta| = b$, if n = a+b is odd, then exactly one of a and b is odd. If a is odd, then the unique odd bipartition covered by (α, β) in Y^2 is (γ, β) , where γ is the unique odd partition covered by α . Also, the bijection $(\alpha, \beta) \mapsto (\beta, \alpha)$ is an automorphism of Y^2 . We call M^2 the hyperoctahedral Macdonald tree. Let M_k^2 denote the first 2^k rows of M^2 , consisting of bipartitions with weights 0 through $2^k - 1$. Call a bipartition (α, β) a hook if either $\alpha = \emptyset$ or $\beta = \emptyset$, and the non-empty component is a hook. A bipartition (α, β) is said to be one-dimensional $f_{\alpha\beta} = 1$.

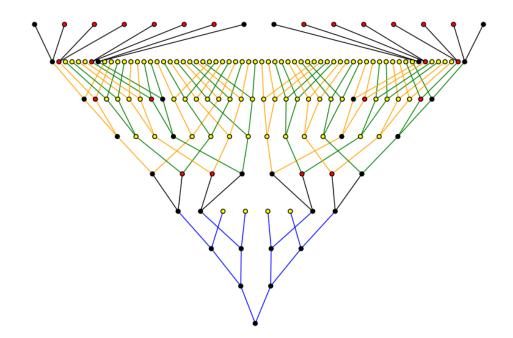


FIGURE 5. The recursive construction of M_3^2 from M_4^2 . Two copies of M_2^2 (shown in orange and green) are attached to each of the four hook-nodes in M_2^2 of rank 3. The black nodes represent one-dimensional bipartitions, while the red nodes represent higher-dimensional hook bipartitions.

Theorem 8 (Recursive description of M^2). M_2^2 is the rooted tree with a distinguished left branch and a right branch.



Nodes colored black in the top row of this tree are one-dimensional (and therefore also hooks). The remaining nodes in the top row are not hooks. For $k \ge 2$ we have:

- (8.1) M_{k+1}^2 is obtained from M_k^2 by attaching two copies of M_k^2 to each hook in the top row of M_k^2 via an edge.
- in the top row of M_k² via an edge.
 (8.2) The hooks in the top row of the left (respectively, right) branch of M_{k+1}² are the hooks in the left (respectively, right) branch of one copy of M_k² that ascends from each one-dimensional node in the top row of M_k².

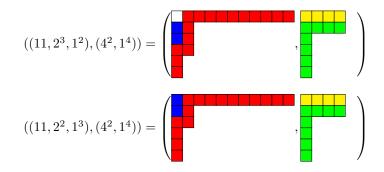


FIGURE 6. The odd bipartition in the second row is the unique odd bipartition covered by the odd bipartition in the first row.

(8.3) The one-dimensional nodes in the left (respectively, right) branch of the top row of M_{k+1}^2 are one of the two one-dimensional nodes in in the left (respectively, right) branch of one copy of M_k^2 that ascends from a one-dimensional node of the left (respectively, right) branch of M_k^2 .

All of M^2 can be reconstructed from M_2^2 by recursively applying these rules. The fact that M^2 is a tree means that an odd-dimensional irreducible representation of \mathbb{B}_n , when restricted to \mathbb{B}_{n-1} , contains exactly one odd-dimensional representation of \mathbb{B}_{n-1} .

Remark 9. The recursive rules in Theorem 8 are consistent with the following observations:

- (9.1) M_k^2 is a rooted binary tree with $2^{\binom{n+1}{2}}$ nodes in its topmost row.
- (9.2) Of these, 2^k nodes are hooks. 2^{k-1} of these hooks are on the left branch, and 2^{k-1} of them are on the right branch.
- (9.3) Of these hooks, 4 are one-dimensional; 2 on the left branch, and 2 on the right branch.

Theorem 8 can be proved using a characterization of vertices and edge relations in M^2 (the analog of Theorem 5).

Theorem 10. Let n be a positive integer and let k be the unique non-negative integer such that $2^k \leq n < 2^{k+1}$. A bipartition (α, β) of n is odd if and only if either α or β has a hook of length 2^k , and the bipartition $(\tilde{\alpha}, \tilde{\beta})$ obtained after removing the rim of this hook is an odd bipartition of $n - 2^k$. Moreover the odd bipartition (α, β) is uniquely determined by $(\tilde{\alpha}, \tilde{\beta})$, the leg-length of the hook which was removed, and the knowledge of whether it was removed from α or from β .

2.4. **Demihyperoctahedral groups.** We denote by \mathbb{D}_n the kernel of $\varepsilon : \mathbb{B}_n \to \{\pm 1\}$, where ε is the character defined at the beginning of Section 2.3. It is the Weyl group of type D_n , and called the *demihyperoctahedral group*. The irreducible representations of \mathbb{D}_n are of two kinds:

• Let (α, β) be a bipartition of n such that $\alpha \neq \beta$. Then the irreducible representation $\rho_{\alpha\beta}$ of \mathbb{B}_n remains irreducible when restricted to \mathbb{D}_n , and is isomorphic to the restriction of $\rho_{\beta\alpha}$ to \mathbb{D}_n .

• Let n > 0 be even, and let α be any partition of n/2. Then the irreducible representation $\rho_{\alpha\alpha}$ of \mathbb{B}_n , when restricted to \mathbb{D}_n , is a sum of two nonisomorphic irreducible representations of \mathbb{D}_n , which are denoted by $\rho_{\alpha\alpha}^+$ and $\rho_{\alpha\alpha}^-$.

The representation $\rho_{\alpha\alpha}^-$ is the twist of $\rho_{\alpha\alpha}^+$ by any element $e \in \mathbb{B}_n \setminus \mathbb{D}_n$. For n > 0, we have

$$\operatorname{Irr}(\mathbb{D}_n) = \{\operatorname{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha\beta} \mid (\alpha,\beta) \models n, \alpha \neq \beta\} \prod \left\{ \rho_{\alpha\alpha}^+, \rho_{\alpha\alpha}^- \mid \alpha \vdash \frac{n}{2} \right\},\$$

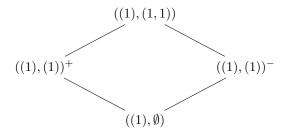
with the second set nonempty only when n is even. The total number of representations of \mathbb{D}_n is therefore equal to $\frac{1}{2}p_2(n)$ when n is odd, and equal to

$$\frac{1}{2}p_2(n) + \frac{3}{2}p\left(\frac{n}{2}\right)$$

when n is even. The branching rules are as follows: for every partition α that is covered by a partition β in Young's lattice,

- ρ[±]_{αα} occurs in the restriction of ρ_{αβ}.
 ρ_{αβ} occurs in the restrictions of ρ[±]_{ββ}.
 ρ_{αγ} occurs in the restriction of ρ_{βγ} for every partition γ.

Let Y_D^2 denote the resulting Bratteli diagram. It turns out that the subgraph induced in Y_D^2 by odd-dimensional representations is no longer a tree. For example,



is a cycle.

3. Determinants of representations of Coxeter groups

In this section, we collect all the results about irreducible representations of Coxeter groups for which the determinant homomorphism is nontrivial. The results for the symmetric group are presented in Section 3.1 and summarize the work in Ref. [2]. The Solomon principle is explained in Section 3.2. This is used to explain the results for all other Coxeter groups, which are taken from Ref. [7], and appear in later subsections.

3.1. Symmetric groups. Recall that a partition λ of n is said to be *chiral* if the corresponding irreducible representation V_{λ} of S_n has non-trivial determinant. Let b(n) denote the number of chiral partitions of n. The first few entries of the sequence b(n) [17, Sequence A272090] are

 $0, 1, 2, 3, 5, 4, 8, 12, 20, 8, 16, 24, 40, 32, 64, 88, 152, 16, 32, 48, \ldots$

The numbers b(n) also have a simple combinatorial interpretation. In Young's seminormal form, V_{λ} has a basis indexed by standard tableaux of shape λ . The basis vector corresponding to a tableau T is an eigenvector for $\rho_{\lambda}(s_1)$; the eigenvalue is -1 if 2 occurs in the first column of T and +1 otherwise. Thus λ is chiral if and

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only if g_{λ} , the number of standard tableaux where 2 occurs in the first column, is odd. Let

$$C(\lambda) = \sum_{(i,j)\in\lambda} (j-i),$$

be the sum of contents of cells in the Young diagram of λ .

Theorem 11 (Stanley [18, Exercise 7.55(a)]). Given a partition λ of n, we have

(3)
$$g_{\lambda} = \frac{f_{\lambda}\left(\binom{n}{2} - C(\lambda)\right)}{2\binom{n}{2}}$$

The partition λ of n is chiral if and only if g_{λ} is odd.

We now state the main result of Ref. [2].

Theorem 12. If n is an integer having binary expansion given in (1), then the number of chiral partitions of n is given by

$$b(n) = a(n) \left(\frac{1}{2} + \sum_{v=1}^{k_1 - 1} 2^{vk_1 - \frac{v^2 + 3v + 4}{2}} + \epsilon \cdot 2^{\binom{k_1}{2} - k_1} \right).$$

For any integer m, let $v_2(m)$ denote the 2-*adic valuation* of m, that is, the largest integer v such that 2^v divides m. Theorem 12 is a consequence of the following more refined enumerative result.

Theorem 13. If n is an integer having binary expansion (1), then the number $b_v(n)$ of chiral partitions λ of n for which $v_2(f_{\lambda}) = v$ is given by

$$b_{v}(n) = 2^{k_{2} + \dots + k_{r}} \times \begin{cases} 2^{k_{1} - 1} & \text{if } v = 0, \\ 2^{(v+1)(k_{1} - 2) - \binom{v}{2}} & \text{if } 0 < v < k_{1}, \\ \epsilon 2^{\binom{k_{1}}{2}} & \text{if } v = k, \\ 0 & \text{if } v > k_{1}. \end{cases}$$

Theorem 13 follows from the characterization of chiral partitions in Theorem 15. The relevant combinatorial data for this characterization is the notion of 2-core towers of partitions, which is roughly the following. To each partition $\lambda \vdash n$, we associate a binary tree, each node of which is a partition. Each of these partitions is a 2-core. The number of non-trivial rows in the 2-core tower of λ is at most the number of digits in the binary expansion of n. We refer to Refs [2, 16] for the details. Let $w_i(\lambda)$ denote the sum of the sizes of the partitions in the *i*'th row of the 2-core tower of λ . We then have

$$n = \sum_{i=0}^{\infty} w_i(\lambda) 2^i.$$

Let $\nu(n)$ denote the number of times 1 occurs in the binary expansion of n (for n as in (1), $\nu(n) = r + \epsilon$). Also let $w(\lambda) = \sum_{i \ge 0} w_i(\lambda)$. Define the 2-deviation of λ as

$$e_2(\lambda) = w(\lambda) - \nu(|\lambda|).$$

The following result gives a formula for the 2-adic valuation $v_2(f_{\lambda})$.

Theorem 14 ([16, Proposition 6.4]). For any partition λ , $v_2(f_{\lambda}) = e_2(\lambda)$.

In particular, a partition is odd if and only if its 2-core tower has at most one cell in each row. Theorem 14 characterizes partitions parameterizing representations with dimensions of specified 2-adic valuation in terms of their 2-core towers.

Theorem 15 ([2, Theorem 6]). Let λ be a partition of n = a + b written as in (1) with 2-quotient (α, β) with $|\alpha| = a$ and $|\beta| = b$. Then λ is chiral if and only if one of the following holds:

- (15.1) λ is odd, and
 - (a) if n is even, then $k_1 \in bin(a)$.
 - (b) if n is odd, then $k_1 \in bin(b)$.
- (15.2) $\operatorname{core}_2 \lambda = \emptyset \text{ or } (1), \text{ and }$
 - (a) α and β are odd,
 - (b) $bin(a) \cap bin(b) = \{j\}$, with $j = v_2(a) = v_2(b)$.
- (15.3) $\operatorname{core}_2 \lambda = (2, 1)$ and $\phi(\alpha, \beta)$ is odd.

In particular, one can infer whether a partition is chiral or not by looking at its 2-core tower. From (15.1), it follows that half of the odd partitions are chiral and half are achiral.

Not only does Theorem 15 prove the enumerative results in Theorems 12 and 13, it also provides fast algorithms to:

- sequentially enumerate all chiral partitions of n with given $v_2(f_{\lambda})$, and
- generate a uniformly random chiral partition of n with given $v_2(f_{\lambda})$.

See Section 4 for an example.

The proof of Theorem 15 takes as its starting point the *Solomon principle*, which we now discuss in the context of a general Coxeter group.

3.2. Solomon principle. Let ρ be a representation of a Coxeter group W. In this section we show how to infer det ρ from its character. So let (W, S) be a Coxeter group (S is a certain set of generators of order 2; see [4]). There is a unique multiplicative character ε_W so that $\varepsilon_W(s) = -1$ for each $s \in S$, namely

$$\varepsilon_W(w) = (-1)^{\ell(w)},$$

where $\ell(w)$ is the length of w with respect to S.

For the Coxeter groups of type A_n , D_n , E_6 , E_7 , E_8 , H_3 , H_4 , and $I_2(p)$ with p odd, the trivial character and ε_W are the only multiplicative characters. This is equivalent to the abelianization W_{ab} having order 2.

Proposition 16. Suppose $|W_{ab}| = 2$, and let $s \in S$. If ρ is a representation of W, then

$$\det \rho = (\varepsilon_W)^b,$$

where

$$b = \frac{\dim \rho - \chi_{\rho}(s)}{2}.$$

Proof. Let a be the multiplicity of 1 as an eigenvalue of $\rho(s)$, and b be the multiplicity of -1. Then dim $\rho = a + b$ and $\chi_{\rho}(s) = a - b$.

We attribute this approach to counting chiral partitions to L. Solomon; see [18, Exercise 7.55].

The abelianization of the other Coxeter groups $(B_n, F_4, \text{ and } I_2(p) \text{ for } p \text{ even})$ are Klein 4 groups. For these, fix two non-conjugate simple reflections $s_1, s_2 \in S$, and multiplicative characters ω_1, ω_2 so that $\omega_1(s_1) = -1, \omega_1(s_2) = 1, \omega_2(s_1) = 1$, $\omega_2(s_2) = -1$. Then $\varepsilon_W = \omega_1 \cdot \omega_2$ and the multiplicative characters of W are $\{1, \omega_1, \omega_2, \omega_1 \cdot \omega_2\}$. A similar proof to that of Proposition 16 gives:

Proposition 17. Suppose $|W_{ab}| = 4$, and let $s_1, s_2, \omega_1, \omega_2$ be as above. If ρ is a representation of W, then

$$\operatorname{let} \rho = (\omega_1)^{x_1} (\omega_2)^{x_2},$$

(

where

and

$$x_1 = \frac{\dim \rho - \chi_{\rho}(s_1)}{2},$$
$$x_2 = \frac{\dim \rho - \chi_{\rho}(s_2)}{2}.$$

3.3. Hyperoctahedral groups. Recall that ε denotes the character of $\mathbb{B}_n = S_2^n \rtimes S_n$ whose restriction to each factor S_2 is its non-trivial character. Write sgn⁰ for the composition of the projection $\mathbb{B}_n \to S_n$ with the sign character of S_n . Finally write sgn¹ = $\varepsilon \cdot$ sgn⁰. The multiplicative characters of \mathbb{B}_n are then $1, \varepsilon,$ sgn⁰, and sgn¹. For ω a multiplicative character of \mathbb{B}_n , put

$$N_{\omega}(n) = \#\{(\alpha, \beta) \models n \mid \det \rho_{\alpha, \beta} = \omega\}.$$

In this section we will give closed formulas for $N_{\omega}(n)$ when $\omega \neq 1$. (If we had a closed formula for $N_1(n)$, then we would have a closed formula for $p_2(n)$, which seems out of reach.)

The determinants of the representations of \mathbb{B}_n can be computed either through the Solomon formulas, combined with the Frobenius character formula, or through a well-known formula for the determinant of an induced representation. (See [7], for example.) Let

$$x_{\alpha\beta} = f_{\alpha}f_{\beta}\binom{n-1}{a,b-1},$$

and

$$y_{\alpha\beta} = f_{\alpha}f_{\beta}\binom{n-2}{a-1,b-1} + f_{\beta}g_{\alpha}\binom{n-2}{a-2,b} + f_{\alpha}g_{\beta}\binom{n-2}{a,b-2}.$$

Theorem 18. For a bipartition (α, β) , we have

$$\det \rho_{\alpha\beta} = \varepsilon^{x_{\alpha\beta}} \cdot (\operatorname{sgn}^0)^{y_{\alpha\beta}}.$$

Since the parities of f_{λ} and g_{λ} can be read from the 2-core tower of λ by Theorems 14 and 15, the parities of $x_{\alpha\beta}$ and $y_{\alpha\beta}$ are also determined by the same tower. In particular, this allows for closed formulas for

$$N_{\omega}(a,b) = \#\{\alpha \vdash a, \beta \vdash b \mid \det \rho_{\alpha,\beta} = \omega\},\$$

at least when ω is nontrivial. From this, the $N_{\omega}(n)$ are computed by the identity

$$N_{\omega}(n) = \sum_{a+b=n} N_{\omega}(a,b).$$

Here is the result: For $n \ge 2$, and $k = v_2(n)$, we have

(4)
$$N_{\varepsilon}(n) = \begin{cases} \frac{1}{4}a_2(n) & \text{if } n \text{ is odd,} \\ \frac{1}{8}a_2(n)\left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}}\right) & \text{if } n \text{ is even.} \end{cases}$$

For n as in (1), let $n' = n - \epsilon$. The following equations compute $N_{\text{sgn}^0}(n)$ in all cases [7]. Let $k = v_2(n')$.

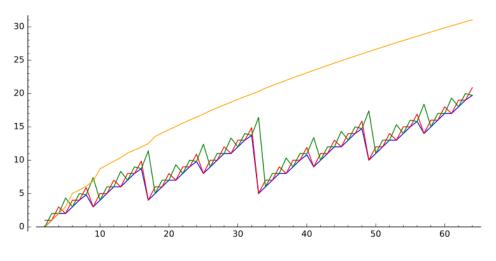


FIGURE 7. $N_1(n)$ (orange), $N_{\mathrm{sgn}^0}(n)$ (green), $N_{\mathrm{sgn}^1}(n)$ (red) and $N_{\varepsilon}(n)$ (blue).

(18.1) If $n \equiv 1 \mod 4$, then

$$N_{\mathrm{sgn}^{0}}(n) = \frac{1}{4}a_{2}(n)\left(1 + 3 \cdot 2^{\binom{k}{2}-1} + 2^{\binom{k}{2}-k+1} + \sum_{j=2}^{k-1} \left(2^{\binom{k}{2}-\binom{j}{2}} + 2^{\binom{k}{2}-j}\right)\right).$$

(18.2) If $n \equiv 3 \mod 4$, then $N_{\text{sgn}^0}(n) = \frac{1}{2}a_2(n)$. (18.3) If *n* is even, then

$$N_{\rm sgn^0}(n) = \frac{1}{8}a_2(n)\left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}}\right).$$

For $n \geq 2$, and $k = v_2(n)$, we have

$$N_{\text{sgn}^{1}}(n) = \begin{cases} \frac{1}{4}a_{2}(n) & \text{if } n \text{ is odd} \\ \frac{1}{8}a_{2}(n)\left(2+2^{k}+\sum_{j=1}^{k-1}2^{\binom{k}{2}}-\binom{j}{2}(2^{j}-1)\right) & \text{if } n \text{ is even.} \end{cases}$$

Since $N_1(n) + N_{\varepsilon}(n) + N_{\mathrm{sgn}^0}(n) + N_{\mathrm{sgn}^1}(n) = p_2(n)$, one implicitly has a formula for $N_1(n)$ as well. Figure 7 shows a logplot of each $N_{\omega}(n)$, with $N_1(n)$ in orange, $N_{\mathrm{sgn}^0}(n)$ in green, $N_{\mathrm{sgn}^1}(n)$ in red, and $N_{\varepsilon}(n)$ in blue. Figure 7 suggests the following inequalities, which have been shown to hold for $n \ge 10$:

- $\begin{array}{ll} (18.1) & N_{\varepsilon}(n) = N_{{\rm sgn}^1}(n) < N_{{\rm sgn}^0}(n) < N_1(n), \, {\rm for} \, \, n \, \, {\rm odd} \\ (18.2) & N_{\varepsilon}(n) = N_{{\rm sgn}^0}(n) < N_{{\rm sgn}^1}(n) < N_1(n), \, {\rm for} \, \, n \, \, {\rm even}. \end{array}$

3.4. Demihyperoctahedral groups. The group \mathbb{D}_n , for $n \geq 2$, has two multiplicative characters: 1 and sgn, where sgn is the restriction of sgn^0 (or of sgn^1) to \mathbb{D}_n . To avoid confusion with earlier notation, let us write

$$N'_{\omega}(n) = \#\{\rho \in \operatorname{Irr}(\mathbb{D}_n) \mid \det \rho = \omega\},\$$

for $\omega = 1$, sgn. Since the sum of $N'_1(n)$ and $N'_{\text{sgn}}(n)$ is equal to this number, so we need only count $N'_{sgn}(n)$.

The restrictions of $\rho_{\alpha\beta}$ with $\alpha \neq \beta$ to \mathbb{D}_n are irreducible, and their determinants are given by restriction. On the other hand, the restrictions of $\rho_{\alpha\alpha}$ split into two irreducible pieces $\rho_{\alpha\alpha}^{\pm}$ with identical determinants. To compute these determinants, we apply the Solomon formula, using that $\chi_{\rho_{\alpha\alpha}^{\pm}}(s_1) = \frac{1}{2}\chi_{\rho_{\alpha\alpha}}(s_1)$. Unless $n = 2^k$ or $n = 2^k + 2$ for some k, one gets the trivial character. If n is a power of 2, then $\rho_{\alpha\alpha}^{\pm}$ has determinant sgn if and only if α is odd and achiral. If $n = 2^k + 2$ for some k, then $\rho_{\alpha\alpha}^{\pm}$ has determinant sgn if and only if α is odd. This leads to the following count:

Theorem 19. Let $n \ge 4$. We have

$$N_{\rm sgn}'(n) = \begin{cases} \frac{1}{2}(N_{\rm sgn^0}(n) + N_{\rm sgn^1}(n)) + \frac{1}{2}n & \text{if } n = 2^k \text{ for some } k \ge 1, \\ \frac{1}{2}(N_{\rm sgn^0}(n) + N_{\rm sgn^1}(n)) + n - 2 & \text{if } n = 2^k + 2 \text{ for some } k \ge 1, \\ \frac{1}{2}(N_{\rm sgn^0}(n) + N_{\rm sgn^1}(n)) & \text{otherwise.} \end{cases}$$

Formulas for $N_{\omega}(n)$ were given in Section 3.3.

3.5. Other Coxeter groups. The remaining Coxeter groups are either dihedral or exceptional. The dihedral case is facile, and the exceptional cases are finite in number and the determinants may be computed by available character tables. See [7] for these remaining cases.

4. Comparison of odd and chiral partitions

It turns out that the functions a(n) and b(n) track each other closely (see Figure 8).

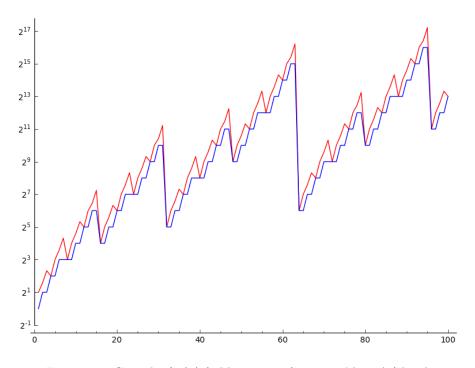


FIGURE 8. Growth of a(n) (odd partitions) against b(n+2) (chiral partitions) on a log-linear scale.

Theorem 20. For every positive integer n,

$$\frac{2}{5} \le \frac{a(n)}{b(n+2)} \le 1.$$

Moreover, a(n)/b(n+2) = 1 if and only if n is divisible by 4.

See [2] for the proof. Theorem 20 says that a(n) is a good proxy for estimating the growth of b(n). The order of the sequence a(n) fluctuates widely; when $n = 2^k$, $a(n) = 2^k$ and when $n = 2^k - 1$, $a(n) = 2^{k(k-1)/2}$. In any case, a(n) (and thus also b(n)) is dwarfed by the growth of the partition function:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$$
 as $n \to \infty$.

For example, Theorem 12 predicts a relatively large value for b(4097) (compared to neighboring integers), but even so, the probability of a partition of 4097 being chiral is

 $b(4097)/p(4097) \approx 2.4236148775415833 \times 10^{-47},$

which is astronomically small. Therefore, algorithms for sampling uniformly random partitions will be very ineffective for sampling uniformly random odd and chiral partitions. But using Theorem 5 and Theorem 15, one may easily generate these of size 4097 very fast. A sample run of our code, which is available at http://www.imsc.res.in/~amri/chiral.sage, gives random odd and chiral partitions of 4095 and 4097 respectively instantaneously on a conventional office desktop:

```
sage: random_odd_dim_partition(4095).frobenius_coordinates()
([677, 491, 148, 65, 24, 6, 2, 1, 0],
[1556, 446, 346, 206, 107, 7, 3, 1, 0])
sage: random_chiral_partition(4097).frobenius_coordinates()
([1879, 272, 152, 27, 20, 19, 8, 2, 0],
[1015, 239, 168, 103, 100, 43, 32, 7, 2])
```

These programs also allow us to enumerate odd and chiral partitions efficiently. Our code for chiral partitions based on Theorem 15 also provides algorithms for generating random chiral partitions with dimension having fixed 2-adic valuation, and for enumerating all chiral partitions of n with dimension having fixed 2-adic valuation.

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References

- Arvind Ayyer, Amritanshu Prasad, and Steven Spallone. Odd partitions in Young's lattice. Séminaire Lotharingien de Combinatoire, 75:B75g, 2016.
- [2] Arvind Ayyer, Amritanshu Prasad, and Steven Spallone. Representations of symmetric groups with non-trivial determinant. J. Combin. Theory Ser. A, 150:208–232, 2017.

- [3] Christine Bessenrodt, Eugenio Giannelli, and Jørn B. Olsson. Restriction of odd degree characters of S_n. SIGMA Symmetry Integrability Geom. Methods Appl., 13:Paper No. 070, 10, 2017.
- [4] Nicolas Bourbaki. Lie groups and Lie algebras. Chapters 4-6. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.
- [5] N. J. Fine. Binomial coefficients modulo a prime. American Mathematical Monthly, 54:589– 592, 1947.
- [6] James Sutherland Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric group. *Canad. J. Math*, 6:316–324, 1954.
- [7] Debarun Ghosh and Steven Spallone. Determinants of representations of Coxeter groups. Journal of Algebraic Combinatorics, to appear.
- [8] Eugenio Giannelli. Characters of odd degree of symmetric groups. J. Lond. Math. Soc. (2), 96(1):1–14, 2017.
- [9] Eugenio Giannelli, Alexander Kleshchev, Gabriel Navarro, and Pham Huu Tiep. Restriction of odd degree characters and natural correspondences. *Int. Math. Res. Not. IMRN*, (20):6089– 6118, 2017.
- [10] Eugenio Giannelli and Alexander R. Miller. Character restrictions and reflection groups. preprint, available at https://www.mat.univie.ac.at/~miller/GM-Final.pdf, 2018.
- [11] I. M. Isaacs, Gabriel Navarro, Jørn B. Olsson, and Pham Huu Tiep. Character restrictions and multiplicities in symmetric groups. J. Algebra, 478:271–282, 2017.
- [12] Gordon Douglas James and Adalbert Kerber. The Representation Theory of the Symmetric Group, volume 16 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.
- [13] Ian G. Macdonald. On the degrees of the irreducible representations of symmetric groups. Bulletin of the London Mathematical Society, 3(2):189–192, 1971.
- [14] Ian G. Macdonald. On the degrees of the irreducible representations of finite Coxeter groups. Journal of the London Mathematical Society, 2(6):298–300, 1973.
- [15] John McKay. Irreducible representations of odd degree. J. Algebra, 20:416–418, 1972.
- [16] Jørn B. Olsson. Combinatorics and representations of finite groups, volume 20 of Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen. Universität Essen, 1993.
- [17] Neil J. A. Sloane. Online encyclopedia of integer sequences. http://oeis.org/.
- [18] Richard P. Stanley. Enumerative Combinatorics, volume 2. Cambridge University Press, 2001.
- [19] Richard P. Stanley. Enumerative Combinatorics, volume 1. Cambridge University Press, second edition, 2012.
- [20] Ian Stewart. How to Cut a Cake: And other mathematical conundrums. Oxford University Press, 2006.
- [21] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. http://combinat.sagemath.org.
- [22] The Sage Developers. Sage Mathematics Software (Version 6.10), 2015. http://www.sagemath.org.

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