Conditional gambler's ruin problem with arbitrary winning and losing probabilities with applications

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Abstract

In this paper we provide formulas for a conditional game duration in a finite state-space one-dimensional gambler's ruin problem with arbitrary winning $p(n)$ and losing $q(n)$ probabilities (*i.e.,* they depend on the current fortune). The formulas are stated in terms of the parameters of the system. Beyer and Waterman (1977) showed that for the classical gambler's ruin problem the distribution of a conditional absorption time is symmetric in p and q. Our formulas imply that for non-constant winning/losing probabilities the expectation of conditional game duration is symmetric in these probabilities (*i.e.,* it is the same if we exchange $p(n)$ with $q(n)$) as long as a ratio $q(n)/p(n)$ is constant. Most of the formulas are applied to a non-symmetric random walk on a polygon.

Keywords: Gambler's ruin problem, conditional absorption time, random walk on a polygon, birth and death chain

1. Introduction

The classical gambler's ruin problem is following. Having initially i dollars, $1 \leq i \leq N-1$, in one step we either win one dollar (*i.e.,* we move to $i+1$) with probability $p \in (0,1)$, or we lose one dollar (*i.e.,* we move to $i-1$) with probability $q = 1-p$. The game ends when the player reaches N (wins the game) or 0 (goes broke). The typical questions one can ask are:

- What is the probability of winning (*i.e.,* reaching N before 0)?
- What is the (expected) game duration?
- What is the (expected) conditional game duration (*i.e.,* game duration given we win or given we lose)?
- Is the (expected) conditional game duration symmetric in p and q?

Similarly, one can consider random walk on $\mathbf{Z}_{m+1} = \{0, \ldots, m\}$: being at state i we either move clockwise with a probability $p \in (0,1)$ (*i.e.,* from i to $i+1$ mod $(m+1)$) or we move counterclockwise with a probability $1-p$ (*i.e.,* we move from i to $i-1$ mod $(m+1)$). We will refer to this as to *the classical random walk on a polygon* (cf. [Sar06]). Assuming we start at i, the typical questions one can ask are:

- What is the probability that all vertices have been visited before the particle returns to i?
- What is the probability that the last vertex visited is j ?
- What is the expected number of moves needed to visit all the vertices?
- What is the expected additional number of moves needed to return to i after visiting all the vertices?

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All above questions were answered in the classical settings. Several generalizations were studied. The probability of winning in a gambler's ruin problem with general winning and losing probabilities (*i.e.,* $p(i)$ being probability of moving from i to $i+1$ and $q(i)$ being the probability of moving from i to $i-1$, with $p(i)+q(i) \leq 1$, $i \in \{1,\ldots,N-1\}$) goes back to Parzen [Par62], revisited in [ES09]. Siegmund duality based proof is given in [Lor17] (where more general, multidimensional, game is considered). In [Len09] the questions related to the conditional game duration are answered for the classical gambler's ruin problem with ties allowed, *i.e.,* $p+q \leq 1$ (with probability $1-(p+q)$ we can stay at a given state). In [Lef08] author considers specific generalization, namely $p(i) = q(i) = \frac{1}{2(2ci+1)}, c \geq 0$ (thus the probability of staying is $1 - \frac{1}{2ci+1}$) and answers the question about the winning probability and the expected game duration (and also considers the corresponding diffusion process). In this paper we present formulas for the expected (conditional) absorption time in terms of parameters of the system (*i.e.,* winning/losing probabilities $p(i), q(i)$). Similar problem was considered in [ES00], the recursion for the expected conditional game duration is given therein (equations (3.4) and (3.5)), however it is not solved in its general form – later on author considers only constant winning/losing probabilities. In [GMZ12] (similar results with different proofs are presented in [MZ16]) the generating function of absorption time (including a conditional one) is given in terms of eigenvalues of a transition matrix and eigenvalues of a truncated transition matrix. The questions for the classical random walk on a polygon were answered in [Sar06]. Some generalizations (rather then allowing arbitrary winning/losing probabilities, symmetric random walks on tetrahedra, octahedra, and hexahedra, are considered) are studied in [SM17].

In 1977 in [BW77] it was shown that for a classical gambler's ruin problem with $p(n) = p = 1-q(n) = 1-q$, the distribution of a conditional game duration is symmetric in p and q, *i.e.,* it is the same as in a game with $p' = q$ and $q' = p$. In 2009 in [Len09] it was extended to a case $p+q < 1$ (*i.e.,* the classical case with ties allowed). In this paper we show that that the expected conditional game duration is symmetric also for non-constant winning/losing probabilities $p(n), q(n)$ as long as $q(n)/p(n)$ is constant (thus, including for example the spatially non-homogeneous case).

In Section 2 we introduce gambler's ruin problem with arbitrary winning and losing probabilities $p(i), q(i)$ together with main results. In Section 2.1 the main result is applied to constant $r(i) = r = q(i)/p(i)$, in Section 2.2 it is applied to non-homogeneous case, whereas the classical case is recalled in Section 2.3. The main example is given in Section 2.4. The results are applied to a random walk on polygon in Section 3. Last Section 4 contains proofs of main results.

2. Gambler's ruin problem

Fix an integer $N \geq 2$. Let

$$\mathbf{p} = (p(0), p(1), \ldots, p(N)), \quad \mathbf{q} = (q(0), q(1), \ldots, q(N)),$$

where $p(0) = q(0) = p(N) = q(N) = 0$ and $p(i), q(i) > 0, p(i)+q(i) \leq 1$ for $i \in \{1, 2\ldots, N-1\}$. Consider a Markov chain $\mathbf{X} = \{X_k\}_{k\geq 0}$ on $\mathbb{E} = \{0,1,\ldots,N\}$ with transition probabilities

$$\mathbf{P}_X(i,j) = \begin{cases} p(i) & \text{if} \quad j = i+1, \\ q(i) & \text{if} \quad j = i-1, \\ 1-(p(i)+q(i)) & \text{if} \quad j = i. \end{cases}$$

We will refer to $\mathbf{X}$ starting at i as to the (gambler's ruin) *game* $G(\mathbf{p}, \mathbf{q}, 0, i, N)$. Note that the chain will eventually end up in either in N (the *winning* state) or in 0 (the *losing* state). To simplify some notation, let $r(i) = \frac{q(i)}{p(i)}$ for $i \in \{1,\ldots,N-1\}$.

Define $\tau_j = \inf\{k : X_k = j\}$. We will study the following *smaller* games $G(\mathbf{p}, \mathbf{q}, j, i, k)$ with k as the *winning* state and j as the *losing* $(j \le i \le k)$. Let us define:

$$\begin{aligned}
\rho_{j:i:k} &= P(\tau_k < \tau_j | X_0 = i), \\
T_{j:i:k} &= \inf\{n \ge 0 : X_n = j \text{ or } X_n = k | X_0 = i\}, \\
W_{j:i:k} &= \inf\{n \ge 0 : X_n = j \text{ or } X_n = k | X_0 = i, X_n = k\}, \\
B_{j:i:k} &= \inf\{n \ge 0 : X_n = j \text{ or } X_n = k | X_0 = i, X_n = j\}.
\end{aligned}$$

In other words: $\rho_{j:i:k}$ is the probability that a gambler starting with i dollars wins in the smaller game; $T_{j:i:k}$ is the distribution of a game the duration (time till gambler either wins or goes broke); $W_{j:i:k}$ is the distribution of $T_{j:i:k}$ conditioned on $X_{T_{j:i:k}} = k$ (winning) and similarly $B_{j:i:k}$ is the distribution of $T_{j:i:k}$ conditioned on $X_{T_{j:i:k}} = j$ (losing).

Notation. For given rates $\mathbf{p}, \mathbf{q}$ by $\mathbf{p} \leftrightarrow \mathbf{q}$ we understand new rates $\mathbf{p}' = \mathbf{q}, \mathbf{q}' = \mathbf{p}$. For some random variable R (one of ρ, T, W, B) for a game with rates $\mathbf{p}, \mathbf{q}$, by $R(\mathbf{p} \leftrightarrow \mathbf{q})$ we understand the random variable defined for game with rates $\mathbf{p}' = \mathbf{q}, \mathbf{q}' = \mathbf{p}$ (and similarly, e.g., $ER(\mathbf{p} \leftrightarrow \mathbf{q})$ is an expectation of R defined for such a game). We say that R (ER) is *symmetric in* $\mathbf{p}$ and $\mathbf{q}$ if $R \overset{distr}{=} R(\mathbf{p} \leftrightarrow \mathbf{q})$ ($ER = ER(\mathbf{p} \leftrightarrow \mathbf{q})$).

By $f(n) = \Theta(g(n))$ we mean $\exists(c_1, c_2 > 0)\ \exists(n_0)\ \forall(n > n_0)\ c_1 g(n) \le f(n) \le c_2 g(n)$. In this section we use the convention: empty sum equals 0, empty product equals 1; however in Section 3 we use some nonstandard notation, see details on page 12.

The results for $\rho_{j:i:k}$ - as already mentioned - are known. We state them below for completeness (and because we will also need them later). We have

Theorem 2.1. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, 1, \ldots, N\}$ described above. We have*

$$\rho_{j:i:k} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} r(s)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} r(s)},$$

$$ET_{j:i:k} = \frac{\sum_{n=j+1}^{k-1}[d_n \sum_{s=j+1}^{n} \frac{1}{p(s)d_s}]}{\sum_{n=j}^{k-1} d_n} \sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1} \left[d_n \sum_{s=j+1}^{n} \frac{1}{p(s)d_s} \right], \tag{1}$$

where $d_s = \prod_{i=j+1}^{s} \frac{q(i)}{p(i)} = \prod_{i=j+1}^{s} r(i)$ (with convention $d_j = 1$).

The proof of Theorem 2.1 is postponed to Section 4.1.1.

Next theorem (our main contribution) gives the formulas for $EW_{0:i:k}$ and $EB_{0:i:k}$. First, let us introduce some necessary notation. With some abuse of notation let us extend

$$\rho_{j:i:k} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} r(s)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} r(s)}$$

for $k < i$ (but still $k > j$).

For given integers n, m, k such that $n \le m, k \in \{0, \lfloor (m-n+1)/2 \rfloor\}$ define

$$\mathbf{j}_k^{n,m} = \big\{\{j_1, j_2, \ldots, j_k\} : j_1 \ge n+1, j_k \le m, j_i \le j_{i+1} - 2 \text{ for } 1 \le i \le k-1\big\}. \tag{2}$$

For given $\mathbf{p}, \mathbf{q}$ and $\mathbf{j} \in \mathbf{j}_k^{n,m}$ define

$$\delta_{\mathbf{j}}^{n,m} = (-1)^k \prod_{s \in j} r(s) \prod_{s \in \{n,\ldots,m\} \setminus \mathbf{j} \cup \mathbf{j}-1} 1 + r(s), \tag{3}$$

where $\{n, \ldots, m\}$ is an empty set for $n > m$ and $\mathbf{j} - 1 = \{j_1 - 1, j_2 - 1, \ldots, j_k - 1\}$ for $\mathbf{j} = \{j_1, j_2, \ldots, j_k\}$. Finally, let

$$\xi_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m}. \tag{4}$$

Now we are ready to state our main theorem.

Theorem 2.2. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, 1, \ldots, N\}$ described above. We have*

$$EW_{0:i:N} = EW_{0:1:N} - EW_{0:1:i}, \quad \textit{where} \tag{5}$$

$$EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} \tag{6}$$

Moreover, we have

$$EB_{0:i:N} = EW'_{0:N-i:N}, \tag{7}$$

where $W'_{0:N-i:N}$ is defined for a gambler's ruin problem with rates $p'(i) = q(N-i)$ and $q'(i) = p(N-i)$ for $i \in \mathbb{E}$.

The proof of Theorem 2.2 is postponed to Section 4.1.2.

2.1. Constant $r(n) = r = \frac{q(n)}{p(n)}$

In this section we will apply Theorems 2.1 and 2.2 to a gambler's ruin problem with constant $r = \frac{q(i)}{p(i)}$. The winning probabilities $\rho_{0:i:N}$ are known (they are the same as in the classical formulation of the problem), we will focus on a game duration. We have

Corollary 2.3. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, \ldots, N\}$ with constant $r = \frac{q(i)}{p(i)}$. We have*

$$r = 1: \quad ET_{j:i:k} = \frac{i-j}{k-j} \sum_{n=j+1}^{k-1} \sum_{s=j+1}^{n} \frac{1}{p(s)} - \sum_{n=j+1}^{i-1} \sum_{s=j+1}^{n} \frac{1}{p(s)},$$

$$ET_{0:i:N} = \frac{i}{N} \sum_{n=1}^{N-1} \sum_{s=1}^{n} \frac{1}{p(s)} - \sum_{n=1}^{i-1} \sum_{s=1}^{n} \frac{1}{p(s)},$$

$$r \ne 1: \quad ET_{j:i:k} = \frac{r^j - r^i}{r^j - r^k} \sum_{n=j+1}^{k-1} \left[r^n \sum_{s=j+1}^{n} \frac{r^{-s}}{p(s)} \right] - \sum_{n=j+1}^{i-1} \left[r^n \sum_{s=j+1}^{n} \frac{r^{-s}}{p(s)} \right],$$

$$ET_{0:i:N} = \frac{1-r^i}{1-r^N} \sum_{n=1}^{N-1} \left[r^n \sum_{s=1}^{n} \frac{r^{-s}}{p(s)} \right] - \sum_{n=1}^{i-1} \left[r^n \sum_{s=1}^{n} \frac{r^{-s}}{p(s)} \right]$$

Proof. We have $d_k = \prod_{j=1}^{k} r = r^k$. Simple recalculations of (1) yield the result. $\square$

For constant r we have that $\delta_{\mathbf{j}}^{n,m}$ (given in (3)) for all $i \in \{1, \ldots, N-1\}$ depends on $\mathbf{j}$ only through k, thus

$$\xi_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m} = C_k^{n,m} (-r)^k (1+r)^{m+1-n-2k}, \tag{8}$$

where $C_k^{n,m} = |\mathbf{j}_k^{n,m}|$. Moreover, we have $|\mathbf{j}_k^{n,m}| = T(m+1-n, k)$, where $T(n,k) = \binom{n-k}{k}$ is the number of subsets of 1,2,...,n-1 of size k and containing no consecutive integers [1].

The proof of the next corollary requires the following lemma.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $r \geq 0$. We have*

$$\sum_{k=0}^{n} \binom{n-k}{k} \left(-\frac{r}{(1+r)^2}\right)^k = \begin{cases} \dfrac{1-r^{n+1}}{(1+r)^n(1-r)} & \textit{if } r \neq 1, \\[2ex] \dfrac{n+1}{2^n} & \textit{if } r = 1. \end{cases} \tag{9}$$

The proof of Lemma 2.4 is given in Section 4.1.2.

Remark 2.5. Note that the assertion of Lemma 2.4 can be stated in the following form (simply substituting $c = \frac{r}{(1+r)^2}$): for $n \in \mathbb{N}$ and $c \in (0, 1/4]$ we have

$$\sum_{k=0}^{n} \binom{n-k}{k} (-c)^k = \begin{cases} \dfrac{1-\gamma^{n+1}}{(1+\gamma)^n(1-\gamma)}, \quad \text{where } \gamma = \dfrac{1-2c+\sqrt{1-4c}}{2c}, & \text{if } c \in (0, 1/4), \\[2ex] \dfrac{n+1}{2^n} \quad \text{if } c = 1/4. \end{cases}$$

These sums for $c \in \{-1, 1\}$ were known ($F(n)$ is the n-th Fibonacci number):

$$\sum_{k=0}^{n} \binom{n-k}{k} = F(n+1),$$

$$\sum_{k=0}^{n} \binom{n-k}{k} (-1)^k = \begin{cases} 1 & \text{if } n \bmod 6 \in \{0,1\}, \\ 0 & \text{if } n \bmod 6 \in \{2,5\}, \\ -1 & \text{if } n \bmod 6 \in \{3,4\}. \end{cases}$$

We will give formulas for $EW_{0:1:i}$ for several cases ($EW_{0:i:N}$ can be calculated via (5)).

Corollary 2.6. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, \ldots, N\}$ with constant $r = \frac{q(i)}{p(i)}$. We have:*

$$r = 1: \quad EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} = \sum_{n=1}^{i-1} \frac{n/i}{p(n)} (i-n).$$

$$r \neq 1: \quad EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} = \sum_{n=1}^{i-1} \frac{\frac{1-r^n}{1-r^i}(1-r^{i-n})}{p(n)(1-r)}. \tag{10}$$

[1] http://oeis.org/A011973

Additionally, if $p(n)=p$ is constant (so is $q(n)$ then, since $r(n)$ is constant) we have

$$r=1: \quad EW_{0:1:i} \;=\; \frac{1}{p}\sum_{n=1}^{i-1}\frac{n}{i}(i-n)=\frac{(i-1)(i+1)}{6p}, \tag{11}$$

$$\begin{aligned} r\neq 1: \quad EW_{0:1:i} \;&=\; \frac{1}{p}\sum_{n=1}^{i-1}\frac{\frac{1-r^n}{1-r^i}(1-r^{i-n})}{1-r}=\frac{1}{p(1-r^i)(1-r)}\sum_{n=1}^{i-1}(1-r^n)(1-r^{i-n}) \\ &=\; \frac{i(1+r^i)-(1+r)\frac{1-r^i}{1-r}}{p(1-r^i)(1-r)}=\frac{1}{p(1-r)}\left(i\frac{1+r^i}{1-r^i}-\frac{1+r}{1-r}\right). \end{aligned}$$

Proof. We will only show case $r=1$, general $p(n)$ (the proof for $r\neq 1$ is very similar). Let us calculate $\xi_s^{n+1,i-1}$ first. From (8) and form of $C_k^{n,m}$ for $r=1$ we have

$$\xi_s^{n+1,i-1}=C_s^{n+1,i-1}(-1)^s 2^{i-n-1-2s}=2^{i-n-1}\binom{i-n-1-s}{s}\left(-\frac{1}{4}\right)^s.$$

From Theorem 2.2 (eq. (6)) and the fact that $\rho_{0:n:i}=n/i$ (since $r=1$) we have

$$\begin{aligned} EW_{0:1:i} \;&=\; \sum_{n=1}^{i-1}\frac{n/i}{p(n)}\sum_{s=0}^{\lfloor (i-1-n)/2\rfloor}\xi_s^{n+1,i-1} \\ &=\; \sum_{n=1}^{i-1}\frac{n/i}{p(n)}2^{i-n-1}\sum_{s=0}^{\lfloor (i-1-n)/2\rfloor}\binom{i-n-1-s}{s}\left(-\frac{1}{4}\right)^s \\ &\overset{\text{Lemma 2.4}}{=}\; \sum_{n=1}^{i-1}\frac{n/i}{p(n)}2^{i-n-1}\frac{i-n-1+1}{2^{i-n-1}}=\sum_{n=1}^{i-1}\frac{n/i}{p(n)}(i-n), \end{aligned}$$

what finishes the proof. □

In 1977 Beyer and Waterman [BW77] showed that for a classical case *i.e.,* for constant birth $p(n)=p$ and death $q(n)=q$ rates such that $p+q=1$, the distribution of $W_{0:i:N}$ is symmetric in p and q (*i.e.,* it has the same distribution for birth rate $p'=q$ and death rate $q'=p$). In 2009 Lengyel [Len09] showed that this holds also for the classical case with ties allowed, *i.e.,* $p+q<1$. In the following theorem we show that $EW_{0:i:N}$ is symmetric in $\mathbf{p}$ and $\mathbf{q}$ (*i.e.,* it is the same for case with birth deaths $p'(n)=q(n)$ and death rates $q'(n)=p(n)$) as long as $r(n)=\frac{q(n)}{p(n)}$ is constant.

Theorem 2.7. *Consider the gambler's ruin problem on $\mathbb{E}=\{0,\ldots,N\}$ with constant $r=\frac{q(i)}{p(i)}$. We have*

$$EW_{0:i:N}=EW_{0:i:N}(\mathbf{p}\leftrightarrow\mathbf{q}),$$

(i.e., $W_{0:i:N}$ is symmetric in $\mathbf{p}$ and $\mathbf{q}$).

Proof. By (5) it is enough to show that $EW_{0:1:i}=EW_{0:1:i}(\mathbf{p}\leftrightarrow\mathbf{q})$.

Let $W_{0:1:i}$ be defined for rates $\mathbf{p}$ and $\mathbf{q}$, whereas $W'_{0:1:i}$ be defined for rates $\mathbf{p'}=\mathbf{q}$ and $\mathbf{q'}=\mathbf{p}$, thus $r'=1/r$. Since $r=\frac{q(n)}{p(n)}$, we have $p'(n)=q(n)=rp(n)$.

$$\begin{aligned} EW'_{0:1:i} \;&=\; \sum_{n=1}^{i-1}\frac{1}{p'(n)}\frac{(1-\frac{1}{r^n})}{(1-\frac{1}{r^i})}\frac{(1-\frac{1}{r^{i-n}})}{(1-\frac{1}{r})}=\sum_{n=1}^{i-1}\frac{1}{rp(n)}\frac{r^i(1-r^n)}{r^n(1-r^i)}\frac{r(1-r^{i-n})}{r^{i-n}(1-r)} \\ &=\; \sum_{n=1}^{i-1}\frac{1}{p(n)}\frac{(1-r^n)}{(1-r^i)}\frac{(1-r^{i-n})}{(1-r)}, \end{aligned}$$

what is equal to (10). □

It is natural to state the following conjecture.

Conjecture 2.8. *Consider the gambler's ruin problem on* $\mathbb{E} = \{0, \ldots, N\}$ *with constant* $r = \frac{q(i)}{p(i)}$*. Then, the distribution of* $W_{0:i:N}$ *is symmetric in* $\mathbf{p}$ *and* $\mathbf{q}$.

2.2. The spatially non-homogeneous case

In this Section we consider gambler's ruin problem with birth rates $p(n) = \frac{p}{2cn+1}$ and death rates $q(n) = \frac{q}{2cn+1}$, where c is a non-negative constant. This is often called spatially non-homogeneous gambler's ruin problem. We will thus still consider case with constant $r(n)$, but with specific rates. As far as we are aware, all results in this section, except the one for $p(n) = q(n) = 1/2$, are new.

Corollary 2.9. *Consider spatially non-homogeneous gambler's ruin problem. We have*

$$\begin{aligned} r = 1 : ET_{0:i:N} &= \frac{1}{2p}\left(iN\left(1+\frac{2c}{3}N\right) - i^2\left(1+\frac{2c}{3}i\right)\right), \\ r \neq 1 : ET_{0:i:N} &= \frac{1}{p(r-1)}\left(\frac{1-r^i}{1-r^N}\left(-cN^2 - N\frac{(cr+c)}{r-1} - N\right) + ci^2 + i\frac{(cr+c)}{r-1} + i\right). \end{aligned}$$

Proof. Applying Corollary 2.3 we have:

- Case $r = 1$

$$\begin{aligned} ET_{0:i:N} &= \frac{i}{N}\sum_{n=1}^{N-1}\sum_{s=1}^{n}\frac{1}{p(s)} - \sum_{n=1}^{i-1}\sum_{s=1}^{n}\frac{1}{p(s)} = \frac{i}{N}\sum_{n=1}^{N-1}\sum_{s=1}^{n}\frac{2cn+1}{p} - \sum_{n=1}^{i-1}\sum_{s=1}^{n}\frac{2cn+1}{p} \\ &= \frac{1}{p}\left(\frac{i}{N}\sum_{n=1}^{N-1}n(cn+c+1) - \sum_{n=1}^{i-1}n(cn+c+1)\right) \\ &= \frac{1}{p}\left(\frac{i}{N}\frac{1}{6}(N-1)(N(2c(N+1)+3) - \frac{1}{6}(i-1)(i(2c(i+1)+3)\right) \\ &= \frac{1}{2p}\left(iN\left(1+\frac{2c}{3}N\right) - i^2\left(1+\frac{2c}{3}i\right)\right). \end{aligned}$$

- Case $r \neq 1$

$$\begin{aligned} ET_{0:i:N} &= \frac{1-r^i}{1-r^N}\sum_{n=1}^{N-1}\left[r^n\sum_{s=1}^{n}\frac{r^{-s}}{p(s)}\right] - \sum_{n=1}^{i-1}\left[r^n\sum_{s=1}^{n}\frac{r^{-s}}{p(s)}\right] \\ &= \frac{1}{p}\left(\frac{1-r^i}{1-r^N}\sum_{n=1}^{N-1}\left[r^n\sum_{s=1}^{n}r^{-s}(2cs+1)\right] - \sum_{n=1}^{i-1}\left[r^n\sum_{s=1}^{n}r^{-s}(2cs+1)\right]\right). \end{aligned}$$

We have

$$\sum_{s=1}^{n} r^{-s}(2cs+1) = \frac{r^{-n}}{(r-1)^2}\left(2cr^{n+1} - 2cnr + 2cn - 2cr + r^{n+1} - r^n - r + 1\right)$$

and

$$\begin{aligned}
&\sum_{n=1}^{k-1}\left[r^n\frac{r^{-n}}{(r-1)^2}\left(2cr^{n+1}-2cnr+2cn-2cr+r^{n+1}-r^n-r+1\right)\right]\\
&= \frac{1}{(r-1)^2}\quad\Big(-\tfrac{2cr(r-r^k)}{r-1}-c(k-1)kr+c(k-1)k-2cr(k-1)\\
&\qquad -\frac{r(r-r^k)}{r-1}+\frac{r-r^k}{r-1}+r-kr+k-1\Big)\\
&= \frac{1}{(r-1)^2}\left(-ck^2(r-1)+\frac{(2cr+r-1)(r^k-1)}{r-1}-k(cr+c+r-1)\right)\ .
\end{aligned}$$

Thus,

$$\begin{aligned}
ET_{0:i:N} &= \frac{1}{p(r-1)^2}\Big\{\ \frac{1-r^i}{1-r^N}\Big(\ -cN^2(r-1)+\tfrac{(2cr+r-1)(r^N-1)}{r-1}-N(cr+c+r-1)\Big)\\
&\qquad -\Big(-ci^2(r-1)+\tfrac{(2cr+r-1)(r^i-1)}{r-1}-i(cr+c+r-1)\Big)\Big\}\\
&= \frac{1}{p(r-1)^2}\Big\{\ \frac{1-r^i}{1-r^N}\Big(\ -cN^2(r-1)-N(cr+c+r-1)\Big)\\
&\qquad +ci^2(r-1)+i(cr+c+r-1)\Big\}\\
&= \frac{1}{p(r-1)}\left(\frac{1-r^i}{1-r^N}\left(-cN^2-N\frac{(cr+c)}{r-1}-N\right)+ci^2+i\frac{(cr+c)}{r-1}+i\right),
\end{aligned}$$

what was to be shown.

□

Remark 2.10. Note that for $p=q=1/2$ we have $ET_{0:i:N}=iN\left(1+\frac{2c}{3}N\right)-i^2\left(1+\frac{2c}{3}i\right)$, *i.e.,* we obtained Proposition 2.1 from [Lef08].

Concerning the conditional game duration (because of (7) it is enough to provide formula only for $EW_{0:i:N}$) we have

Corollary 2.11. *Consider spatially non-homogeneous gambler's ruin problem. We have*

$$\begin{aligned}
r=1:\quad EW_{0:i:N} &= \frac{(N^2-1)(cN+1)}{6p}-\frac{(i^2-1)(ci+1)}{6p},\\
r\neq 1:\quad EW_{0:i:N} &= \frac{cN+1}{p(1-r)}\left(\frac{r+1}{r-1}-N\frac{r^N+1}{r^N-1}\right)-\frac{ci+1}{p(1-r)}\left(\frac{r+1}{r-1}-i\frac{r^i+1}{r^i-1}\right).
\end{aligned}$$

Proof. Applying Corollary 2.6 we have:

- $r=1$

$$EW_{0:1:i}=\sum_{n=1}^{i-1}\frac{n/i}{p(n)}(i-n)=\frac{1}{p}\sum_{n=1}^{i-1}\frac{n}{i}(i-n)(2cn+1)=\frac{(i-1)(i+1)(ci+1)}{6p}.$$

- $r \neq 1$

$$\begin{aligned} EW_{0:1:i} &= \sum_{n=1}^{i-1} \frac{(1-r^n)(1-r^{i-n})}{p(n)(1-r^i)(1-r)} = \frac{1}{p}\sum_{n=1}^{i-1} \frac{(1-r^n)(1-r^{i-n})}{(1-r^i)(1-r)}(2cn+1) \\ &= \frac{(ci+1)((r+1)(r^i-1) - i(r-1)(r^i+1))}{p(1-r^i)(1-r)^2} \\ &= \frac{ci+1}{p(1-r)}\left(\frac{r+1}{r-1} - i\frac{r^i+1}{r^i-1}\right). \end{aligned}$$

Applying (5), *i.e.*, $EW_{0:i:N} = EW_{0:1:N} - EW_{0:1:i}$, completes the proof. □

2.3. The classical case.

For constant winning/losing probabilities we recover known results (all given in Sarkar [Sar06]). We state them here for completeness and will indicate how they can be derived from our more general results.

Corollary 2.12. *Consider the gambler's ruin problem on* $\mathbb{E} = \{0, 1, \ldots, N\}$ *with constant winning/losing probabilities* $p(i) = p, q(i) = q, i = 1, \ldots, N-1, p+q = 1$. *We have*

$$\rho_{0:i:N} = \begin{cases} \frac{1-r^i}{1-r^N} & \text{if r} = 1, \\ \frac{i}{N} & \text{if r} \neq 1, \end{cases}$$

$$ET_{0:i:N} = \begin{cases} i(N-i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left(i - N\frac{r^i-1}{r^N-1}\right) & \text{if r} \neq 1, \end{cases}$$

$$EW_{0:i:N} = \begin{cases} \frac{1}{3}(N-i)(N+i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left[N\frac{r^N+1}{r^N-1} - i\frac{r^i+1}{r^i-1}\right] & \text{if r} \neq 1, \end{cases}$$

$$EB_{0:i:N} = \begin{cases} \frac{1}{3}i(2N-i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left[N\frac{r^N+1}{r^N-1} - (N-i)\frac{r^{N-i}+1}{r^{N-i}-1}\right] & \text{if r} \neq 1, \end{cases}$$

Results for $ET_{0:i:N}$ follows from Corollary 2.3 (case $r = 1$); $EW_{0:i:N}$ from Corollary 2.6 eq. (11) followed by (5); $EB_{0:i:N}$ follows from results on $EW_{0:i:N}$ and Theorem 2.2 (eq. (7)).

2.4. Example

Fix an integer N and some $p, q > 0$. Consider a gambler's ruin problem with rates

$$p(i) = \frac{p(1+\alpha_1 i)}{2ci+1}, \quad q(i) = \frac{q(1+\alpha_2 i)}{2ci+1},$$

with fixed $\alpha_1, \alpha_2, c \geq 0$ such that $p(i), q(i) > 0, p(i)+q(i) \leq 1, i \in \{1, ..., N\}$. We want to calculate $EW_{0:1:N}$.

2.4.1. $N = 3$

We have

$$\mathbf{p} = \left(0, \frac{p(1+\alpha_1)}{2c+1}, \frac{p(1+2\alpha_1)}{2c+1}, 0\right), \quad \mathbf{q} = \left(0, \frac{q(1+\alpha_2)}{2c+1}, \frac{q(1+2\alpha_2)}{2c+1}, 0\right).$$

Note that in general (for $\alpha_1 \neq \alpha_2$) $r(n) = \frac{q(n)}{p(n)} = \frac{q}{p}\frac{(1+\alpha_2 n)}{(1+\alpha_1 n)}$ is non-constant, thus we will apply Theorem 2.2. Eq. (6) takes form

$$EW_{0:1:3} = \sum_{n=1}^{2} \frac{\rho_{0:n:3}}{p(n)} \sum_{s=0}^{\lfloor (2-n)/2 \rfloor} \xi_s^{n+1,2} = \frac{\rho_{0:1:3}}{p(1)}\xi_0^{2,2} + \frac{\rho_{0:2:3}}{p(2)}\xi_0^{3,2}.$$

We need winning probabilities $\rho_{0:1:3}$ and $\rho_{0:2:3}$, which can be calculated from Theorem 2.1:

$$\rho_{0:i:3} = \frac{\sum_{n=1}^{i}\prod_{s=1}^{n-1} r(s)}{\sum_{n=1}^{3}\prod_{s=1}^{n-1} r(s)} = \frac{1+(i-1)r(1)}{1+r(1)+r(1)r(2)} = \frac{1+(i-1)\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}}{1+\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}+\frac{q^2}{p^2}\frac{(1+\alpha_2)(1+2\alpha_2)}{(1+\alpha_1)(1+2\alpha_1)}} =: \frac{1+(i-1)\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}}{\gamma(p,q,\alpha_1,\alpha_2)}.$$

We also need $\xi_0^{2,2}$ and $\xi_0^{3,2}$. We have $\mathbf{j}_0^{2,2} = \mathbf{j}_0^{3,2} = \{\emptyset\}$, thus

$$\xi_0^{2,2} = \delta_{\mathbf{j}}^{2,2} = 1 + r(2) = 1 + \frac{q}{p}\frac{1+2\alpha_2}{1+2\alpha_1}, \quad \xi_0^{3,2} = \delta_{\mathbf{j}}^{3,2} = 1$$

(in the latter the second product was 1, since $\{3,\ldots,2\} \equiv \emptyset$).

Finally,

$$EW_{0:1:3} = \frac{1}{p\gamma(p,q,\alpha_1,\alpha_2)}\left[\frac{2c+1}{1+\alpha_1}\left(1+\frac{q}{p}\frac{(1+2\alpha_2)}{(1+2\alpha_1)}\right) + \left(1+\frac{q}{p}\frac{(1+\alpha_2)}{(1+\alpha_1)}\right)\frac{4c+1}{1+2\alpha_1}\right]. \tag{12}$$

Special cases:

- $\alpha_1 = \alpha_2 = \alpha$. Then (12) reduces to

$$EW_{0:1:3} = \frac{1+\frac{q}{p}}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}\left(\frac{2c+1}{1+\alpha} + \frac{4c+1}{1+2\alpha}\right). \tag{13}$$

  Note that in this case $r(n) = \frac{q}{p}$ is constant, thus (13) could be derived easier using Corollary 2.6:

$$r = 1: \quad EW_{0:1:3} = \sum_{n=1}^{2} \frac{n}{3}\frac{2cn+1}{p(1+\alpha_1 n)}(3-n) = \frac{2}{3p}\left(\frac{2c+1}{1+\alpha} + \frac{4c+1}{1+2\alpha}\right),$$

$$r \neq 1: \quad EW_{0:1:3} = \sum_{n=1}^{2} \frac{\frac{1-r^n}{1-r^3}(1-r^{3-n})}{(1-r)}\frac{2cn+1}{p(1+\alpha_1 n)} = \frac{1-r^2}{p(1-r^3)}\left(\frac{2c+1}{1+\alpha} + \frac{4c+1}{1+2\alpha}\right),$$

  what is equivalent to (13) in both cases. *Note* also that this is not a spatially non-homogeneous case as long as $\alpha > 0$.

- $\alpha_1 = \alpha_2 = 0$. Then (12) (and thus (13)) reduces to

$$EW_{0:1:3} = \frac{2\left(1+\frac{q}{p}\right)}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}\left(\frac{3c+1}{1+\alpha}\right). \tag{14}$$

  Note that this is a spatially non-homogeneous case, thus (14) could be derived from Corollary 2.11 (we skip the calculations).

- $\alpha_1 = \alpha_2 = 0$ and $c = 0$, then (14) reduces to

$$EW_{0:1:3} = \frac{2\left(1+\frac{q}{p}\right)}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}.$$

  This situation corresponds to a gambler's ruin problem with constant birth and death rates. In particular, for $p = q = 1/2$ we have $EW_{0:1:3} = \frac{8}{3}$ what agrees with Example 1 in [Len09].

2.4.2. General $N \geq 3, p = q$ and $\alpha_2 = \alpha_1 = 1$

We thus have $p(i) = \frac{p(1+i)}{2ci+1}, q(i) = \frac{q(1+i)}{2ci+1}$. This is constant $r(n) = \frac{q(n)}{p(n)} = \frac{q}{p} = 1$ case, which is however not spatially non-homogeneous. We skip the lengthy calculations, but we can obtain $EW_{0:1:N}$ from Corollary 2.6 (H_N is the N-th harmonic number):

$$\begin{aligned} EW_{0:1:N} &= \sum_{n=1}^{N-1} \frac{n(N-n)(2cn+1)}{pN(1+n)} \\ &= \tfrac{1}{p}\left(\tfrac{c}{3}(N-5)(N+2)+\tfrac{1}{2}(3+N)\right)+\tfrac{1}{Np}(2c-1)(1+N)H_N = \tfrac{c}{3p}N^2+\Theta(N), \end{aligned}$$

which for $p(i) = p(1+i), q(i) = q(1+i)$ (*i.e.,* for $c = 0$) simplifies to

$$EW_{0:1:N} = \frac{N+3}{2p} - \frac{1}{Np}(N+1)H_N = \frac{N}{2p} + \Theta(log(N)).$$

2.4.3. General $N \geq 3, p = q$ and $\alpha_2 = \alpha_1 = \alpha$

$$\begin{aligned} EW_{0:1:i} &= \sum_{n=1}^{i-1} \frac{n(i-n)(2cn+1)}{pi(1+\alpha n)} \\ &= \frac{1}{6\alpha^4 pi}\left(\alpha(i-1)(\alpha^2 i(2c(i+1)+3)+\alpha(6-6ci)-12c)+\right. \\ &\quad \left. 6(\alpha-2c)(\alpha i+1)\left[\psi\left(1+\tfrac{1}{\alpha}\right)-\psi\left(i+\tfrac{1}{\alpha}\right)\right]\right), \end{aligned}$$

where ψ is a digamma function. It is known that $\psi(m) = H_{m-1} - \gamma$, where $\gamma = 0.5772156..$ is a known Euler–Mascheroni constant. Let us assume that $\alpha = \frac{1}{m}$ and m is an integer. Then $\psi\left(1+\frac{1}{\alpha}\right) - \psi\left(i+\frac{1}{\alpha}\right) = H_m - H_{i+m-1}$.

3. Random walk on a polygon

Fix an integer $m \geq 2$. Let

$$\mathbf{p} = (p(0), p(1), \ldots, p(m)), \quad \mathbf{q} = (q(0), q(1), \ldots, q(m)),$$

where $p(i), q(i) > 0, p(i)+q(i) \leq 1$ for $i \in \{0\ldots, m\}$. Consider the following random walk $X \equiv \{X_t\}_{t\in\mathbb{N}}$ on $\mathbb{E} = Z_{m+1}$. Being in state i we move to the state $i+1$ with probability $p(i)$, we move to the state $i-1$ with probability $q(i)$, and we do nothing with the remaining probability. We will refer to this walk as to a *random walk on a polygon*. Throughout the paper, in then context of a random walk on a polygon, all additions and substractions are performed modulo $m+1$. The notation intentionally resembles that of gambler's ruin problem. Throughout the Section we consider fixed $\mathbf{p}, \mathbf{q}$ and $m \geq 2$ (and omit subscripts $\mathbf{p}, \mathbf{q}$ in random variables below). We are interested in:

$$
\begin{aligned}
A_i &= \{X : X_0 = i, X_n = i, \forall_{0<t<n} X_t \neq i, \forall_{k\in\mathbb{E}} \exists_{0\leq t\leq n} X_t = k\} \\
L_{i,j} &= \{X : X_0 = i, X_n = j, \forall_{0<t<n} X_t \neq j, \forall_{k\in\mathbb{E}} \exists_{0\leq t\leq n} X_t = k\} \\
V_{i,j} &= \inf\{n \geq 1 : X_0 = i, X_n = j, \forall_{k\in\mathbb{E}} \exists_{0\leq t\leq n} X_t = k\} \\
V_i &= \inf\{n \geq 1 : X_0 = i, \forall_{k\in\mathbb{E}} \exists_{0\leq t\leq n} X_t = k\} \\
R_i &= \inf\{n_2 \geq 1 : X_0 = i, X_{n_1+n_2} = i, n_1 = \inf\{n \geq 1 : \forall_{k\in\mathbb{E}} \exists_{0\leq t\leq n} X_t = k\}\}
\end{aligned}
$$

In other words: A_i is the event that the process starting at i will return for the first time to i after all other vertices are visited; $L_{i,j}$ is the event that the process starting at i will reach for the first time state j after visiting all other vertices; $V_{i,j}$ is the number of steps of the process starting at i to reach for the first time state j after visiting all other vertices; V_i is the number of steps of the process starting at i needed to visit all vertices; R_i is the number of additional steps for the process starting at i needed to reach i after visiting all the vertices.

For $j \preceq i \preceq k$, where $\preceq$ is a cyclic order, *i.e.,* $j \leq i \leq k$ or $i \leq k \leq j$ or $k \leq j \leq i$, let $G(\mathbf{p}, \mathbf{q}, j, i, k)$ denote a gambler's ruin game with i being a starting state, j being a losing state and k being a winning state. Note that independently of j, i, k, winning and losing probabilities $\mathbf{p}, \mathbf{q}$ are fixed.

Notation. In contrast to a usual notation neither $\sum_{k=s}^{t} a_k = 0$ nor $\prod_{k=s}^{t} a_k = 1$ for $t < s-1$. Since we are considering operations in Z_{m+1}, we define

$$
\text{For } t < s \leq m, s-t > 1: \quad \sum_{k=s}^{t} a_k := a_s + a_{s+1} + \ldots + a_m + a_0 + \ldots + a_t,
$$

$$
\prod_{k=s}^{t} a_k := a_s \cdot a_{s+1} \cdot \ldots \cdot a_m \cdot a_0 \cdot \ldots \cdot a_t,
$$

$$
\text{For } s = t+1 \bmod m+1: \quad \sum_{k=s}^{t} a_k = 0 \qquad \prod_{k=s}^{t} a_k := 1.
$$

In all other cases we use usual sums and products. Using this notation, we are ready to state our results.

Theorem 3.1. *Consider the random walk on a polygon described above. We have*

$$P(A_i) = \frac{1}{1+r(i)} \left(\frac{1}{\sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{1}{\sum_{n=i+2}^{i} \prod_{s=n}^{i} \left(\frac{1}{r(s)}\right)} \right) \tag{15}$$

$$P(L_{i,j}) = \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=n}^{j-1} \frac{1}{r(s)}} \right) \tag{16}$$

$$\begin{aligned} EV_{i,j} &= \rho_{j+1:i:j-1} \left(EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j} \right) \\ &\quad + (1-\rho_{j+1:i:j-1}) \left(EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j} \right) \end{aligned} \tag{17}$$

$$EV_i = \sum_{j=i+1}^{i-1} P(L_{i,j}) EV_{i,j} \tag{18}$$

$$ER_i = \sum_{k=i+1}^{i-1} P(L_{i,k}) ET_{i:k:i} \tag{19}$$

The proof of Theorem 3.1 is postponed to Section 4.2.1.

Constant $r(n) = r = \frac{q(n)}{p(n)}$*..*

In this case the starting point does not matter, we consider $i = 0$. Note that $P(A_i)$ and $P(L_{i,j})$ depend on $p(n)$ and $q(n)$ only through $r(n)$, thus they must reduce to known results for constant birth $p(n) = p$ and death $q(n) = q$ rates (see (3.1) and (3.3) in [Sar06]). Indeed, substituting $r(n) = r$ to (15) and (16) yields

Corollary 3.2. *Consider the random walk on polygon with constant* $r(n) = \frac{q(n)}{p(n)}$*, then we have*

$$P(A_0) = \begin{cases} \frac{1}{m} & \text{if } r = 1, \\ \frac{r-1}{r+1} \frac{r^m+1}{r^m-1} & \text{if } r \neq 1, \end{cases}$$

$$P(L_{0,j}) = \begin{cases} \frac{1}{m} & \text{if } r = 1, \\ \frac{r^{m-j}(r-1)}{r^m-1} & \text{if } r \neq 1. \end{cases}$$

We skip the formulas for $EV_{0,j}$, EV_0 and ER_0 in this case, noting that they can be derived from Corollaries 2.3 and 2.6.

Constant $q(n) = q, p(n) = p$*..*

First, let us recall formulas for EV_0, ER_0 for the case $p + q = 1$.

Corollary 3.3. *[Sar06] Consider the random walk on a polygon with constant* $q(n) = q, p(n) = p, p+q = 1$*. We have*

$$EV_0 = \begin{cases} \frac{m(m+1)}{2} & \text{if } r = 1, \\ \frac{r+1}{r-1} \left[m - \frac{1}{r-1} - \frac{m^2}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1} \right] & \text{if } r \neq 1, \end{cases}$$

$$ER_0 = \begin{cases} \frac{1}{6}(m+1)(m+2) & \text{if } r = 1, \\ \frac{r+1}{r-1} \left[\frac{r}{r-1} - \frac{m(m+2)}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1} \right] & \text{if } r \neq 1, \end{cases}$$

In the case $p+q \le 1$ note that $EB_{j:i:k} = \frac{1}{p(1+r)} EB^1_{j:i:k}$, $EW_{j:i:k} = \frac{1}{p(1+r)} EW^1_{j:i:k}$, $ET_{j:i:k} = \frac{1}{p(1+r)} ET^1_{j:i:k}$, where superscript 1 denotes the case $p+q=1$. Thus Theorem 3.1 implies $EV_0 = EV_0^1, ER_0 = ER_0^1$, *i.e.*, we have

Corollary 3.4. *Consider the random walk on a polygon with constant $q(n)=q, p(n)=p$. We have*

$$EV_0 = \begin{cases} \frac{m(m+1)}{4p} & \text{if } r=1, \\ \frac{1}{p(r-1)}\left[m - \frac{1}{r-1} - \frac{m^2}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

$$ER_0 = \begin{cases} \frac{1}{12p}(m+1)(m+2) & \text{if } r=1, \\ \frac{1}{p(r-1)}\left[\frac{r}{r-1} - \frac{m(m+2)}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

4. Proofs

4.1. Gambler's ruin problem

4.1.1. Proof of Theorem 2.1

Proof of Theorem 2.1. Consider the birth and death chain with j and k ($j<k$) as recurrent absorbing states ($p(j)=q(j)=p(k)=q(k)=0$). First step analysis yields (for $j<i<k$)

$$ET_{j:i:k} = p(i)(1+ET_{j:i+1:k}) + q(i)(1+ET_{j:i-1:k}) + (1-q(i)-p(i))(1+ET_{j:i:k}),$$

thus

$$ET_{j:i+1:k} = ET_{j:i:k} + \frac{q(i)}{p(i)}\left(ET_{j:i:k} - ET_{j:i-1:k} - \frac{1}{q(i)}\right). \tag{20}$$

Since $ET_{j:j:k}=0$, we have:

$$ET_{j:j+2:k} = ET_{j:j+1:k}\left(1+\frac{q(j+1)}{p(j+1)}\right) - \frac{q(j+1)}{p(j+1)}\frac{1}{q(j+1)}.$$

Recall that $d_s = \prod_{i=j+1}^{s} \frac{q(i)}{p(i)}$ (where $d_j=1$), iterating the above equations yields:

$$ET_{j:i:k} = ET_{j:j+1:k}\sum_{s=j}^{i-1} d_s - \sum_{s=j+1}^{i-1}\left[d_s \sum_{m=j+1}^{s} \frac{1}{p(m)d_m}\right], \tag{21}$$

what can be checked by induction. Plugging (21) into (20) we have:

$$
\begin{aligned}
&ET_{j:i+1:k} = \\
&\qquad ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \\
&\qquad +\frac{q(i)}{p(i)}\quad\left(ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]\right. \\
&\qquad\qquad \left. -ET_{j:j+1:k}\sum_{n=j}^{i-2} d_n - \sum_{n=j+1}^{i-2}[d_n \sum_{m=j+1}^{n}\frac{1}{p(s)d_s}] - \frac{1}{q(i)}\right) \\
&= ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \\
&\qquad + \frac{q(i)}{p(i)}\left(ET_{j:j+1:k}d_{i-1} - d_{i-1}\sum_{s=j+1}^{i-1}\frac{1}{p(s)d_s} - d_i\frac{1}{d_i q(i)}\right) \\
&= ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] + ET_{j:j+1:k}d_i - d_i\sum_{s=j+1}^{i-1}\frac{1}{p(s)d_s} - d_i\frac{1}{d_i p(i)} \\
&= ET_{j:j+1:k}\sum_{n=j}^{i} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] - d_i\sum_{s=j+1}^{i}\frac{1}{p(s)d_s} \\
&= ET_{j:j+1:k}\sum_{n=j}^{i} d_n - \sum_{n=j+1}^{i}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \qquad .
\end{aligned}
$$

Since $ET_{j:k:k} = 0$, we have:

$$
0 = ET_{j:j+1:k}\sum_{n=j}^{k-1} d_n - \sum_{n=j+1}^{k-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \Rightarrow ET_{j:j+1:k} = \frac{\sum\limits_{n=j+1}^{k-1}\left[d_n \sum\limits_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]}{\sum\limits_{n=j}^{k-1} d_n},
$$

thus

$$
ET_{j:i:k} = \frac{\sum_{n=j+1}^{k-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]}{\sum_{n=j}^{k-1} d_n}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right],
$$

what was to be shown.

□

4.1.2. Proof of Lemma 2.4 and Theorem 2.2

Proof of Lemma 2.4. Denote by $f(n)$ lhs of (9) and by $h(n)$ its rhs. We will show that generating functions of f and h are equal. Let us start with $\mathfrak{g}_f(x)$, the generating function of f at x:

$$\mathfrak{g}_f(x) =$$

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)x^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^{n+k} = \sum_{k=0}^{\infty}\left(-\frac{r}{(1+r)^2}\right)^k x^k \sum_{n=0}^{\infty}\binom{n}{k}x^n
\end{aligned}$$

Applying $\sum_{n=0}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}$ we have

$$\begin{aligned}
\mathfrak{g}_f(x) &= \sum_{k=0}^{\infty}\left(-\frac{r}{(1+r)^2}\right)^k x^k \frac{x^k}{(1-x)^{k+1}} = \frac{1}{1-x}\sum_{k=0}^{\infty}\left(\frac{-rx^2}{(1+r)^2(1-x)}\right)^k \\
&= \frac{1}{(1-x)}\frac{(1+r)^2(1-x)}{(1+r)^2(1-x)+rx^2} = \frac{(1+r)^2}{(1+r)^2(1-x)+rx^2}.
\end{aligned}$$

On the other hand, the generating function of h is following:

$$\begin{aligned}
\mathfrak{g}_h(x) &= \sum_{n=0}^{\infty} h(n)x^n = \sum_{n=0}^{\infty}\frac{1-r^{n+1}}{(1+r)^n(1-r)}x^n = \frac{1}{(1-r)}\left(\sum_{n=0}^{\infty}\frac{1}{(1+r)^n}x^n - \sum_{n=0}^{\infty}\frac{r^n}{(1+r)^n}x^n\right) \\
&= \frac{1}{(1-r)}\left(\sum_{n=0}^{\infty}\frac{1}{(1+r)^n}x^n - \sum_{n=0}^{\infty}\frac{r^n}{(1+r)^n}x^n\right) = \frac{1}{(1-r)}\left(\frac{1+r}{1+r-x} - r\frac{1+r}{1+r-xr}\right) \\
&= \frac{1+r}{(1-r)}\frac{1+r-xr-r-r^2-xr}{(1+r-x)(1+r-xr)} = \frac{1+r}{(1-r)}\frac{(1+r)(1-r)}{(1+r)^2-(1+r)(x+xr)+x^2r} \\
&= \frac{(1+r)^2}{(1+r)^2(1-x)+rx^2},
\end{aligned}$$

thus $\mathfrak{g}_h(x) = \mathfrak{g}_f(x)$, what finishes the proof.

□

The following lemma will be needed in a proof of Theorem 2.2.

Lemma 4.1. *Consider the gambler's ruin problem with general rates* $\mathbf{p}, \mathbf{q}$. *Define*

$$\begin{aligned}
a_i &= -\frac{\rho_{0:i:i+1}}{p(i)} = -\frac{\rho_{0:i:i+1}}{p(i)}, \\
b_i &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)}, \\
c_i &= -\frac{q(i)}{p(i)}\rho_{0:i-1:i+1}.
\end{aligned}$$

Then, for all $N \geq 1$ *we have*

$$\prod_{j=2}^{N}\begin{pmatrix} b_j & c_j & a_j \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 & A_N \\ 1 & 0 & A_{N-1} \\ 0 & 0 & 1\end{pmatrix},$$

where

$$A_M = -\sum_{n=1}^{M} \frac{1}{p(n)} \rho_{0:n:M+1} \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} \xi_k^{n+1,M}$$

and $\xi_k^{n+1,M}$ *was defined in (4).*

Proof. Recall that $\mathbf{j}_k^{n,m}$ was defined in (2) as

$$\mathbf{j}_k^{n,m} = \big\{\{j_1, j_2, \ldots, j_k\} : j_1 \geq n+1, j_k \leq m, j_i \leq j_{i+1} - 2 \text{ for } i \in \{1, k-1\}\big\}.$$

For given $\mathbf{p}, \mathbf{q}$, b_n, c_n and $\mathbf{j} \in \mathbf{j}_k^{n,m}$ define

$$D_{\mathbf{j}}^{n,m} = b_n b_{n+1} \ldots b_{j_1-2} c_{j_1} b_{j_1+1} b_{j_1+2} \ldots b_{j_2-2} c_{j_2} \ldots b_{j_{k-1}+1} b_{j_{k-1}+2} \ldots b_{j_k-2} c_{j_k} b_{j_k+1} b_{j_k+2} \ldots b_m$$

and let

$$S_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} D_{\mathbf{j}}^{n,m}.$$

Let

$$\begin{aligned} \alpha_i &= -\frac{1}{p(i)}, \\ \beta_i &= \frac{(p(i)+q(i))}{p(i)} = 1 + r(i), \\ \gamma_i &= -\frac{q(i)}{p(i)} = -r(i). \end{aligned}$$

$D_{\mathbf{j}}^{n,m}$ can be rewritten as

$$\begin{aligned} D_{\mathbf{j}}^{n,m} &= \rho_{0:n:m+1} \beta_n \beta_{n+1} \cdots \beta_{j_1-2} \gamma_{j_1} \beta_{j_1+1} \beta_{j_1+2} \cdots \beta_{j_2-2} \gamma_{j_2} \cdots \\ &\quad \cdot \beta_{j_{k-1}+1} \beta_{j_{k-1}+2} \ldots \beta_{j_k-2} \gamma_{j_k} \beta_{j_k+1} \beta_{j_k+2} \cdots \beta_m \\ &= (-1)^k \prod_{s \in \mathbf{j}} r(s) \prod_{s \in \{n,\ldots,m\} \setminus \mathbf{j} \cup \mathbf{j}-1} 1 + r(s) = \rho_{0:n:m+1} \delta_{\mathbf{j}}^{n,m}. \end{aligned}$$

Thus $S_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} D_{\mathbf{j}}^{n,m} = \rho_{0:n:m+1} \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m} =: \rho_{0:n:m+1} \xi_k^{n,m}$ and A_M can be rewritten as

$$A_M = \sum_{n=1}^{M} a_n \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} S_k^{n+1,M}.$$

We will show this by induction.

- For $M = 1$ we have

$$A_1 = \sum_{n=1}^{1} a_n \sum_{k=0}^{\lfloor (1-n)/2 \rfloor} S_k^{n+1,1} = a_1 \sum_{k=0}^{\lfloor 0/2 \rfloor} S_k^{2,1} = a_1 S_0^{2,1} = a_1.$$

- For $N \geq M \geq 2$ assuming $A_M = \sum_{n=1}^{M} a_n \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} S_k^{n+1,M}$ we shall prove that $A_{N+1} = b_{N+1} A_N + c_{N+1} A_{N-1} + a_{N+1}$. We have

$$
\begin{aligned}
&b_{N+1}A_N + c_{N+1}A_{N-1} + a_{N+1} = \\
&= b_{N+1}\sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N-n)/2 \rfloor} S_k^{n+1,N} + c_{N+1}\sum_{n=1}^{N-1} a_n \sum_{k=0}^{\lfloor (N-n-1)/2 \rfloor} S_k^{n+1,N-1} + a_{N+1} \\
&= \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N-n)/2 \rfloor} b_{N+1} \sum_{\mathbf{j}_k^{n+1,N}} D^{n+1,N}_{\mathbf{j}_k^{n+1,N}} + \sum_{n=1}^{N-1} a_n \sum_{k=0}^{\lfloor (N-n-1)/2 \rfloor} c_{N+1} \sum_{\mathbf{j}_k^{n+1,N-1}} D^{n+1,N-1}_{\mathbf{j}_k^{n+1,N-1}} + a_{N+1} \\
&= \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N+1-n)/2 \rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}:j_k\neq N+1} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}} \\
&\quad + \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N+1-n)/2 \rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}:j_k = N+1} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}} + a_{N+1} \\
&= \sum_{n=1}^{N+1} a_n \sum_{k=0}^{\lfloor (N+1-n)/2 \rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}} = \sum_{n=1}^{N+1} a_n \sum_{k=0}^{\lfloor (N+1-n)/2 \rfloor} S_k^{n+1,N+1} = A_{N+1}
\end{aligned}
$$

what finishes the proof.

□

Proof of Theorem 2.2. First step analysis yields (for $N > i > 1$):

$$
\begin{aligned}
EW_{0:i:N} &= (1 + EW_{0:i-1:N})P(X_1 = i-1|X_0 = i, X_T = N) \\
&\quad +(1 + EW_{0:i:N})P(X_1 = i|X_0 = i, X_T = N) \\
&\quad +(1 + EW_{0:i+1:N})P(X_1 = i+1|X_0 = i, X_T = N).
\end{aligned}
$$

We have $EW_{0:N:N} = 0$ and for simplicity we also set $EW_{0:0:N} = 0$. We have

$$
\begin{aligned}
P(X_1 = i-1|X_0 = i, X_T = N) &= \tfrac{P(X_1=i-1|X_0=i)P(X_T=N|X_1=i-1)}{P(X_T=N|X_0=i)} = \frac{q(i)\rho_{0:i-1:N}}{\rho_{0:i:N}} = q(i)\rho_{0:i-1:i}, \\
P(X_1 = i|X_0 = i, X_T = N) &= \frac{(1-p(i)-q(i))\rho_{0:i:N}}{\rho_{0:i:N}} = 1 - p(i) - q(i), \\
P(X_1 = i+1|X_0 = i, X_T = N) &= \frac{p(i)\rho_{0:i+1:N}}{\rho_{0:i:N}} = p(i)\rho_{0:i+1:i}.
\end{aligned}
$$

For $i = 1$ we have

$$
EW_{0:1:N} = [1 + EW_{0:1:N}](1 - p(1) - q(1)) + [1 + EW_{0:2:N}]p(1)\rho_{0:2:1},
$$

thus

$$
EW_{0:2:N} = \frac{(p(1)+q(1)-1)\rho_{0:1:2}}{p(1)} - 1 + \frac{(p(1)+q(1))\rho_{0:1:2}}{p(1)} EW_{0:1:N}.
$$

For $1 \le i \le N$ we have

$$
EW_{0:i:N} = (1 + EW_{0:i-1:N})q(i)\rho_{0:i-1:i} + (1 + EW_{0:i:N})(1 - p(i) - q(i)) + (1 + EW_{0:i+1:N})p(i)\rho_{0:i+1:i} \quad (22)
$$

and

$$
\begin{aligned}
EW_{0:i+1:N} &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} - \frac{q(i)}{p(i)}\rho_{0:i-1:i+1} - 1 - \frac{\rho_{0:i:i+1}}{p(i)} \\
&\quad + \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} EW_{0:i:N} - \frac{q(i)}{p(i)}\rho_{0:i-1:i+1} EW_{0:i-1:N}, \\
&= b_i + c_i - 1 + a_i + b_i EW_{0:i:N} + c_i EW_{0:i-1:N} \\
&\overset{(*)}{=} a_i + b_i EW_{0:i:N} + c_i EW_{0:i-1:N}, \qquad (23)
\end{aligned}
$$

where a_i, b_i, c_i were defined in Lemma 4.1 and in $(*)$ we used the fact that

$$
\begin{aligned}
b_i + c_i &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} - \frac{q(i)}{p(i)}\rho_{0:i:i+1} \\
&= \frac{p(i)+q(i)}{p(i)} \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} - \frac{q(i)}{p(i)} \frac{\sum_{n=1}^{i-1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \frac{q(i)}{p(i)}\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) - \frac{q(i)}{p(i)}\sum_{n=1}^{i-1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \frac{q(i)}{p(i)}\sum_{n=i}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \prod_{k=1}^{i}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = \frac{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = 1.
\end{aligned}
$$

Equations (22) and (23) can be written in a matrix form:

$$
\begin{pmatrix} EW_{0:i+1:N} \\ EW_{0:i:N} \\ 1 \end{pmatrix} = \begin{pmatrix} b_i & c_i & a_i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:i:N} \\ EW_{0:i-1:N} \\ 1 \end{pmatrix}. \qquad (24)
$$

Note that $c_1 = -\frac{q_1}{p_1}W_0^2 = -\frac{q_1}{p_1}0 = 0$ and

$$
b_1 = \frac{(p(1)+q(1))\rho_{0:1:2}}{p(1)} = \frac{p(1)+q(1)}{p(1)} \frac{\sum_{n=1}^{1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{2}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = \frac{1+\frac{q(1)}{p(1)}}{1} \frac{1}{1+\frac{q(1)}{p(1)}} = 1,
$$

thus using (24) recursively we obtain

$$
\begin{aligned}
\begin{pmatrix} 0 \\ EW_{0:N-1:N} \\ 1 \end{pmatrix} = \begin{pmatrix} EW_{0:N:N} \\ EW_{0:N-1:N} \\ 1 \end{pmatrix} &= \prod_{j=2}^{N-1} \begin{pmatrix} b_j & c_j & a_j \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:1:N} \\ EW_{0:0:N} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & A_{N-1} \\ 1 & 0 & A_{N-2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:1:N} \\ 0 \\ 1 \end{pmatrix},
\end{aligned}
$$

where A_N is given in Lemma 4.1, what implies

$$EW_{0:1:N} = -A_{N-1}$$

and thus proves (6). Equation (5) follows from the fact that $W_{0:1:N} \stackrel{(distr)}{=} W_{0:1:i} + W_{0:i:N}$ (Markov property, moreover $W_{0:1:i}$ and $W_{0:1:i}$ are independent).

□

4.2. Random walk on a polygon

4.2.1. Proof of Theorem 3.1

Proof of eq. (15) . Let F_i denote the event that at the first time we leave state i (recall, ties are allowed) we move clockwise. Similarly, let F_i^c denotes the event that at the first time we leave state i we move counterclockwise. We have

$$\begin{aligned} P(F_i) &= \frac{p(i)}{p(i)+q(i)} = \frac{1}{1+r(i)}, \\ P(F_i^c) &= \frac{q(i)}{p(i)+q(i)} = \frac{r(i)}{1+r(i)} \end{aligned}$$

and

$$P(A_i) = P(F_i)P(A_i|F_i) + P(F_i^c)P(A_i|F_i^c) = \frac{1}{1+r(i)}P(A_i|F_i) + \frac{r(i)}{1+r(i)}P(A_i|F_i^c).$$

- For $P(A_i|F_i)$ we have: we start at $i+1$ and we have to reach $i-1$ before reaching i. This is the probability of winning in the game $G(\mathbf{p}, \mathbf{q}, i, i+1, i-1)$. We thus have

$$P(A_i|F_i) = \rho_{i:i+1:i-1} = \frac{1}{\sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)}.$$

- Similarly for $P(A_i|F_i^c)$ we have: we start at $i-1$, and we have to reach $i+1$ before reaching i which corresponds to losing in the game $G(\mathbf{p}, \mathbf{q}, i+1, i-1, i)$. We thus have

$$P(A_i|F_i^c) = 1 - \rho_{i+1:i-1:i} = 1 - \frac{\sum_{n=i+2}^{i-1} \prod_{s=i+2}^{n-1} r(s)}{\sum_{n=i+2}^{i} \prod_{s=i+2}^{n-1} r(s)} = \frac{\prod_{s=i+2}^{i-1} r(s)}{\sum_{n=i+2}^{i} \prod_{s=i+2}^{n-1} r(s)} = \frac{1}{\sum_{n=i+2}^{i} \prod_{s=n}^{i-1} \left(\frac{1}{r(s)}\right)}.$$

Finally

$$\begin{aligned} P(A_i) &= \frac{1}{(1+r(i)) \sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{r(i)}{(1+r(i)) \sum_{n=i+2}^{i} \prod_{s=n}^{i-1} \left(\frac{1}{r(s)}\right)} \\ &= \frac{1}{(1+r(i)) \sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{1}{(1+r(i)) \sum_{n=i+2}^{i} \prod_{s=n}^{i} \left(\frac{1}{r(s)}\right)}. \end{aligned}$$

□

Proof of eq. (16). Let us define $T_1 = \inf\{t : X_t = j-1 \vee X_t = j+1 | X_0 = i\}$ and consider separately two cases when at T_1 we are at $j-1$ or $j+1$. The first one corresponds to winning, whereas the second one corresponds to losing in the game $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$. The winning probability is

$$\rho_{j+1:i:j-1}.$$

In the first case (when we get to the $j-1$ before $j+1$) vertex j will be the last one if we reach $j+1$ earlier - this can be interpreted as losing in the game $G(\mathbf{p}, \mathbf{q}, j+1, j-1, j)$, what happens with probability:

$$1 - \rho_{j+1:j-1:j} = 1 - \frac{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} = \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)}.$$

In the second case (when we get to the $j+1$ before $j-1$) vertex j will be the last one if we reach $j-1$ earlier - this can be interpreted as winning in the game $G(\mathbf{p}, \mathbf{q}, j, j+1, j-1)$, what happens with probability:

$$\rho_{j:j+1:j-1} = \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)}.$$

Finally:

$$\begin{aligned}
P(L_{i,j}) &= (1 - \rho_{j+1:i:j-1})\rho_{j:j+1:j-1} + \rho_{j+1:i:j-1}(1 - \rho_{j+1:j-1:j}) \\
&= \left(1 - \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)}\right) \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \\
&= \frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \\
&= \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\left(\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)\right)\left(\prod_{s=j+2}^{j-1} r(s)\right)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \right) \\
&= \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=n}^{j-1} \frac{1}{r(s)}} \right).
\end{aligned}$$

□

Proof of eqs. (17), (18) and (19) . Let us start with the expectation of $V_{i,j}$ – number of steps to visit all vertices starting at i when j is the last visited vertex. As earlier, let $T_1 = \inf\{t : X_t = j-1 \vee X_t = j+1\}$. We have two cases:

- If $X_{T_1} = j-1$ (and j was the last visited vertex) then the expected game time consists of: expected time to win in $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$, expected time to lose in $G(\mathbf{p}, \mathbf{q}, j+1, j-1, j)$ and expected duration of the game $G(\mathbf{p}, \mathbf{q}, j, j+1, j)$. That is:

$$EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j}$$

- If $X_{T_1} = j+1$ (and j was last visited vertex) then the expected game time consists of: expected time to lose in $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$, expected time to win in $G(\mathbf{p}, \mathbf{q}, j, j+1, j-1)$ and expected duration of the game $G(\mathbf{p}, \mathbf{q}, j, j-1, j)$. That is:

$$EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j}$$

Now, conditioning on X_{T_1}, we obtain:

$$\begin{aligned} EV_{i,j} &= \rho_{j+1:i:j-1} \left(EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j} \right) \\ &+ (1 - \rho_{j+1:i:j-1}) \left(EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j} \right). \end{aligned}$$

Equations (18) and (19) are simply obtained by conditioning on the states.

□

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