ON PATTERN-AVOIDING FISHBURN PERMUTATIONS

JUAN B. GIL AND MICHAEL D. WEINER

ABSTRACT. The class of permutations that avoid the bivincular pattern $(231, \{1\}, \{1\})$ is known to be enumerated by the Fishburn numbers. In this paper, we call them *Fishburn permutations* and study their pattern avoidance. For classical patterns of length 3, we give a complete enumerative picture for regular and indecomposable Fishburn permutations. For patterns of length 4, we focus on a Wilf equivalence class of Fishburn permutations that are enumerated by the Catalan numbers. In addition, we also discuss a class enumerated by the binomial transform of the Catalan numbers and give conjectures for other equivalence classes of pattern-avoiding Fishburn permutations.

1. INTRODUCTION

Motivated by a recent paper by G. Andrews and J. Sellers [1], we became interested in the Fishburn numbers $\xi(n)$, defined by the formal power series

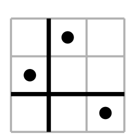
$$\sum_{n=0}^{\infty} \xi(n) q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1-q)^j).$$

They are listed as sequence A022493 in [5] and have several combinatorial interpretations. For example, $\xi(n)$ gives the:

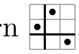
- ▷ number of linearized chord diagrams of degree n,
- ▷ number of unlabeled $(2+2)$-free posets on n elements,
- ▷ number of ascent sequences of length n,
- ▷ number of permutations in S_n that avoid the bivincular pattern $(231, \{1\}, \{1\})$.

For more on these interpretations, we refer the reader to [2] and the references there in.

In this note, we are primarily concerned with the aforementioned class of permutations. The fact that they are enumerated by the Fishburn numbers was proved in [2] by Bousquet-Mélou, Claesson, Dukes, and Kitaev, where the authors introduced bivincular patterns (permutations with restrictions on the adjacency of positions and values) and gave a simple bijection to ascent sequences. More specifically, a permutation $\pi \in S_n$ is said to contain the bivincular pattern $(231, \{1\}, \{1\})$ if there are positions $i < k$ with $\pi(i) > 1$, $\pi(k) = \pi(i) - 1$, such that the subsequence $\pi(i)\pi(i+1)\pi(k)$ forms a 231 pattern. Such a bivincular pattern may be visualized by the plot



where bold lines indicate adjacent entries and gray lines indicate an elastic distance between the entries.

We let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid the pattern , and since $|\mathscr{F}_n| = \xi(n)$ (see [2]), we call the elements of $\mathscr{F} = \bigcup_n \mathscr{F}_n$ *Fishburn permutations*. Further, we let $\mathscr{F}_n(\sigma)$ denote the class of Fishburn permutations that avoid the pattern σ.

Our goal is to study $F_n(\sigma) = |\mathscr{F}_n(\sigma)|$ for classical patterns of length 3 or 4. In Section 2, we give a complete picture for regular and indecomposable Fishburn permutations that avoid a classical pattern of length 3. In Section 3, we discuss patterns of length 4, focusing on a Wilf equivalence class of Fishburn permutations that are enumerated by the Catalan numbers C_n. We also prove the formula $F_n(1342) = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}$, and conjecture two other equivalence classes. Finally, in Section 4, we briefly discuss indecomposable Fishburn permutations that avoid a pattern of length 4. In Table 5, we make some conjectures based on our limited preliminary computations.

2. Avoiding patterns of length 3

Clearly, $\mathsf{Av}_n(231) \subset \mathscr{F}_n$. Now, since every Fishburn permutation that avoids the classical pattern 231 is contained in the set of regular 231-avoiding permutations, we get

$$\mathscr{F}_n(231) = \mathsf{Av}_n(231), \text{ and so } F_n(231) = C_n, \tag{2.1}$$

where C_n denotes the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Enumeration of the Fishburn permutations that avoid the other five classical patterns of length 3 is less obvious.

Theorem 2.1. *For $\sigma \in \{123, 132, 213, 312\}$, we have $F_n(\sigma) = 2^{n-1}$.*

Proof. First of all, note that for every σ of length 3, we have $F_1(\sigma) = 1$ and $F_2(\sigma) = 2$.

Case $\sigma = 132$: If $\pi \in \mathscr{F}_{n-1}(132)$, the permutations $1 \ominus \pi$ and $\pi \oplus 1$ are both in $\mathscr{F}_n(132)$. On the other hand, if τ is a permutation in $\mathsf{Av}_n(132)$ with $\tau(i) = n$ for some $1 < i < n$, then we must have $\tau(j) > \tau(k)$ for every $j \in \{1, \dots, i-1\}$ and $k \in \{i+1, \dots, n\}$. Thus $n - i = \tau(k')$ for some $k' > i$ and $n - i + 1 = \tau(j')$ for some $j' < i$. But this violates the Fishburn condition since $n - i + 1$ is the smallest value left from n and must therefore be part of an ascent in $\tau(1) \cdots \tau(i-1)\, n$. In other words, $\mathscr{F}_n(132)$ is the disjoint union of the sets $\{1 \ominus \pi : \pi \in \mathscr{F}_{n-1}(132)\}$ and $\{\pi \oplus 1 : \pi \in \mathscr{F}_{n-1}(132)\}$. Thus

$$F_n(132) = 2F_{n-1}(132) \text{ for } n > 1,$$

which implies $F_n(132) = 2^{n-1}$.

Case $\sigma = 123$: For $n > 2$, the permutation $(n-1)(n-2)\cdots 21n$ is the only permutation in $\mathscr{F}_n(123)$ that ends with n, and if $\pi \in \mathscr{F}_{n-1}(123)$, then $1 \ominus \pi \in \mathscr{F}_n(123)$.

Assume $\tau \in \mathscr{F}_n(123)$ is such that $\tau(i) = n$ for some $1 < i < n$. Since τ avoids the pattern 123, we must have $\tau(1) > \tau(2) > \cdots > \tau(i-1)$. Moreover, the Fishburn condition forces $\tau(i-1) = 1$, which implies $\tau(i+1) > \tau(i+2) > \cdots > \tau(n)$. In other words, τ may be any permutation with $\tau(i-1) = 1$, $\tau(i) = n$, and such that the entries left from 1 and right from n form two decreasing sequences. There are $\binom{n-2}{i-2}$ such permutations.

In conclusion, we have the recurrence

$$F_n(123) = F_{n-1}(123) + 1 + \sum_{i=2}^{n-1} \binom{n-2}{i-2} = F_{n-1}(123) + 2^{n-2},$$

which implies $F_n(123) = 2^{n-1}$.

Case $\sigma = 213$: For $n > 2$, the permutation $12\cdots n$ is the only permutation in $\mathscr{F}_n(213)$ that ends with n, and if $\pi \in \mathscr{F}_{n-1}(213)$, then $1 \ominus \pi \in \mathscr{F}_n(213)$.

Assume $\tau \in \mathscr{F}_n(213)$ is such that $\tau(i) = n$ for some $1 < i < n$. Since τ avoids the pattern 213, we must have $\tau(1) < \tau(2) < \cdots < \tau(i-1)$ and the Fishburn condition forces $\tau(j) = j$ for every $j \in \{1, \ldots, i-1\}$. Thus τ must be of the form $\tau = 1\cdots(i-1)n|\pi_R$, where π_R may be any element of $\mathscr{F}_{n-i}(213)$. This implies

$$F_n(213) = 1 + \sum_{i=1}^{n-1} F_{n-i}(213),$$

and we conclude $F_n(213) = 2^{n-1}$.

Case $\sigma = 312$: If $\pi \in \mathscr{F}_{n-1}(312)$, the permutation $1 \oplus \pi$ is in $\mathscr{F}_n(312)$. On the other hand, if τ is a permutation in $\mathsf{Av}_n(312)$ with $\tau(i) = 1$ for some $1 < i \le n$, then we must have $\tau(j) < \tau(k)$ for every $j \in \{1, \ldots, i-1\}$ and $k \in \{i+1, \ldots, n\}$. Moreover, the Fishburn condition forces $\tau(j) = i + 1 - j$ for every $j \in \{1, \ldots, i-1\}$. Thus τ must be of the form $\tau = i\cdots 21|\pi_R$, where $\pi_R = \emptyset$ if $i = n$, or $\pi_R \in \mathscr{F}_{n-i}(312)$ if $i < n$. This implies

$$F_n(312) = 1 + \sum_{i=1}^{n-1} F_{n-i}(312),$$

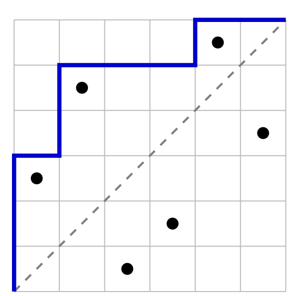
hence $F_n(312) = 2^{n-1}$. □

Theorem 2.2. *The set $\mathscr{F}_n(321)$ is in bijection with the set of Dyck paths of semilength n that avoid the subpath* UUDU. *Therefore, by* [4, Prop. 5] *we have*

$$F_n(321) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^j}{n-j}\binom{n-j}{j}\binom{2n-3j}{n-j+1}.$$

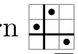
This is sequence [5, A105633].

Proof. This is a consequence of Krattenthaler's bijection between permutations in $\mathsf{Av}_n(321)$ and Dyck paths of semilength n, via the left-to-right maxima:



Example for $\pi = 351264 \in \mathsf{Av}_6(321)$, $\pi \notin \mathscr{F}_6$

Here we denote a vertical step by U and a horizontal step by D. Under this bijection, it is clear that an ascent $\pi_i < \pi_{i+1}$ in $\pi \in \mathsf{Av}_n(321)$ with $k = \pi_{i+1} - \pi_i$ generates the subpath $\mathsf{UDU^k}$ in the corresponding Dyck path D_π, and if $\pi_i - 1 = \pi_j$ for some $j > i+1$, then D_π

must necessarily contain the subpath $\mathsf{UUDU^k}$. Thus we have that π avoids the pattern  if and only if D_π avoids UUDU. □

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
123, 132, 213, 312	1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...	A000079
231	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
321	1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, ...	A105633

TABLE 1. σ-avoiding Fishburn permutations.

Indecomposable permutations. Let $\mathscr{F}_n^{\mathsf{ind}}(\sigma)$ be the set of indecomposable Fishburn permutations that avoid the pattern σ, and let ${}^iF_n(\sigma)$ denote the number of elements in $\mathscr{F}_n^{\mathsf{ind}}(\sigma)$. Observe that for every σ of length ≥ 3, we have ${}^iF_1(\sigma) = 1$ and ${}^iF_2(\sigma) = 1$.

We start with a fundamental known lemma, see e.g. [3, Lem. 3.1].

Lemma 2.3. *If a pattern σ is indecomposable, then the sequence $\lvert\mathsf{Av}_n(\sigma)\rvert$ is the* INVERT *transform of the sequence $\lvert\mathsf{Av}_n^{ind}(\sigma)\rvert$. That is, if $A^\sigma(x)$ and ${}^iA^\sigma(x)$ are the corresponding generating functions, then*

$$1 + A^\sigma(x) = \frac{1}{1 - {}^iA^\sigma(x)}, \text{ and so } {}^iA^\sigma(x) = \frac{A^\sigma(x)}{1 + A^\sigma(x)}.$$

In particular, since 231 is indecomposable, these identities are also valid for Fishburn permutations. The sequence $({}^iF_n)_{n\in\mathbb{N}}$ that enumerates indecomposable Fishburn permutations of length n starts with

$$1, 1, 2, 6, 23, 104, 534, 3051, 19155, 130997, \dots.$$

Theorem 2.4. *For $n > 1$, we have ${}^iF_n(123) = 2^{n-1} - (n-1)$.*

Proof. As discussed in the proof of Theorem 2.1, for $n > 2$ the set $\mathscr{F}_n(123)$ consists of elements of the form $1\ominus\pi$ with $\pi \in \mathscr{F}_{n-1}(123)$, and elements of the form $\tau = (i-1)\cdots 1n|\pi_R$ for some $1 < i \leq n$ and $\pi_R \in \mathscr{F}_{n-i}(123)$. This forces $\pi_R = \emptyset$ if $i = n$, and $\pi_R = (n-1)\cdots i$ if $i < n$. Thus the only decomposable elements of $\mathscr{F}_n(123)$ are the $n-1$ permutations

$$\begin{gathered} 1n(n-1)\cdots 32, \\ 21n(n-1)\cdots 3, \\ \vdots \\ (n-1)\cdots 321n. \end{gathered}$$

In conclusion, ${}^iF_n(123) = F_n(123) - (n-1) = 2^{n-1} - (n-1)$. □

Theorem 2.5. *For $n > 1$ and $\sigma \in \{132, 213\}$, we have ${}^iF_n(\sigma) = 2^{n-2}$.*

Proof. From the proof of Theorem 2.1, we know that for $n > 1$ every element of $\mathscr{F}_n(132)$ must be of the form $1 \ominus \pi$ or $\pi \oplus 1$ with $\pi \in \mathscr{F}_{n-1}(132)$. Since $\pi \oplus 1$ is decomposable and $1 \ominus \pi$ is indecomposable, we have

$$\mathscr{F}_n^{\mathsf{ind}}(132) = \{1 \ominus \pi : \pi \in \mathscr{F}_{n-1}(132)\}.$$

We also know that $\mathscr{F}_n(213)$ consists of elements of the form $1 \ominus \pi$ with $\pi \in \mathscr{F}_{n-1}(213)$, and elements of the form $\tau = (1 \cdots (i-1)) \oplus (1 \ominus \pi_R)$ for some $1 < i \leq n$ (with $\pi_R = \emptyset$ when $i = n$). Thus

$$\mathscr{F}_n^{\mathsf{ind}}(213) = \{1 \ominus \pi : \pi \in \mathscr{F}_{n-1}(213)\}.$$

In conclusion, if $\sigma \in \{132, 213\}$, we have ${}^{i}F_n(\sigma) = F_{n-1}(\sigma) = 2^{n-2}$. □

Theorem 2.6. *For $\sigma \in \{231, 312, 321\}$, we have*

$${}^{i}F^{\sigma}(x) = \frac{F^{\sigma}(x)}{1 + F^{\sigma}(x)}.$$

In particular, ${}^{i}F_n(231) = C_{n-1}$ and ${}^{i}F_n(312) = 1$.

Proof. This is a direct consequence of (2.1), Thm. 2.1, and Lem. 2.3. □

Theorem 2.7. *The enumerating sequence $a_n = {}^{i}F_n(321)$ satisfies the recurrence relation*

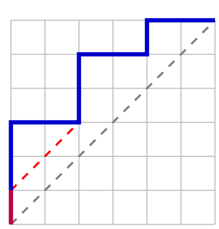
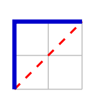
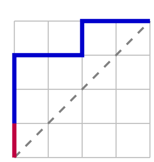
$$a_n = a_{n-1} + \sum_{j=2}^{n-2} a_j a_{n-j} \text{ for } n \geq 4,$$

with $a_1 = a_2 = a_3 = 1$. This is sequence [5, A082582].

Proof. We use the same Dyck path approach as in the proof of Theorem 2.2. Under this bijection, indecomposable permutations correspond to Dyck paths that do not touch the line $y = x$ except at the end points. Let $\mathcal{A}_n$ be the set of Dyck paths corresponding to $\mathscr{F}_n^{\mathsf{ind}}(321)$. We will prove that $a_n = |\mathcal{A}_n|$ satisfies the claimed recurrence relation. Clearly, for $n = 1, 2, 3$, the only indecomposable Fishburn permutations are 1, 21, and 312, which correspond to the Dyck paths UD, $\mathsf{U}^2\mathsf{D}^2$, and $\mathsf{U}^3\mathsf{D}^3$, respectively. Thus $a_1 = a_2 = a_3 = 1$.

Note that indecomposable permutations may not start with 1 or end with n. Moreover, every element of $\pi \in \mathscr{F}_n^{\mathsf{ind}}(321)$ must be of the form $m1\pi(3)\cdots\pi(n)$ with $m \geq 3$. Therefore, the elements of $\mathcal{A}_n$ have no peaks at the points $(0,1)$, $(0,2)$, or $(n-1,n)$, and for $n > 3$ their first return to the line $y = x+1$ happens at a lattice point $(x, x+1)$ with $x \in [2, n-1]$.

Dyck paths in $\mathcal{A}_n$ having $(n-1,n)$ as their first return to $y = x+1$, are in one-to-one correspondence with the elements of $\mathcal{A}_{n-1}$ (just remove the first U and the last D of the longer path). Now, for $j \in \{2, \ldots, n-2\}$, the set of paths $D \in \mathcal{A}_n$ having first return to $y = x+1$ at the point $(j, j+1)$ corresponds uniquely to the set of all pairs (D_j, D_{n-j}) with $D_j \in \mathcal{A}_j$ and $D_j \in \mathcal{A}_{n-j}$. For example,

 $\longleftrightarrow$  ,  .

This implies that there are $a_j a_{n-j}$ paths in $\mathcal{A}_n$ having the point $(j, j+1)$ as their first return to the line $y = x + 1$. Finally, summing over j gives the claimed identity. □

Our enumeration results for patterns of length 3 are summarized in Table 1 and Table 2.

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
123	1, 1, 2, 5, 12, 27, 58, 121, 248, 503, ...	A000325
132, 213	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, ...	A000079
231	1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
312	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...	A000012
321	1, 1, 1, 2, 5, 13, 35, 97, 275, 794, ...	A082582

TABLE 2. σ-avoiding indecomposable Fishburn permutations.

3. Avoiding patterns of length 4

In this section, we discuss the enumeration of Fishburn permutations that avoid a pattern of length 4. There are at least 13 Wilf equivalence classes that we break down into three categories: 10 classes with a single pattern, 2 classes with (conjecturally) three patterns each, and a larger class with eight patterns enumerated by the Catalan numbers.

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1342	1, 2, 5, 15, 51, 188, 731, 2950, ...	A007317
1432	1, 2, 5, 14, 43, 142, 495, 1796, ...	
2314	1, 2, 5, 15, 52, 200, 827, 3601, ...	
2341	1, 2, 5, 15, 52, 202, 858, 3910, ...	
3412	1, 2, 5, 15, 52, 201, 843, 3764, ...	A202062(?)
3421	1, 2, 5, 15, 52, 203, 874, 4076, ...	
4123	1, 2, 5, 14, 42, 133, 442, 1535, ...	A202061(?)
4231	1, 2, 5, 15, 52, 201, 843, 3765, ...	
4312	1, 2, 5, 14, 43, 143, 508, 1905, ...	
4321	1, 2, 5, 14, 45, 162, 639, 2713, ...	

TABLE 3. Equivalence classes with a single pattern

We will provide a proof for the enumeration of the class $\mathscr{F}_n(1342)$, but our main focus in this paper will be on the enumeration of the equivalence class given in Table 4.

For the remaining patterns we have the following conjectures.

Conjecture. $\mathscr{F}_n(2413) \sim \mathscr{F}_n(2431) \sim \mathscr{F}_n(3241)$.

Conjecture. $\mathscr{F}_n(3214) \sim \mathscr{F}_n(4132) \sim \mathscr{F}_n(4213)$.

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1234, 1243, 1324, 1423, 2134, 2143, 3124, 3142	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108

TABLE 4. Catalan equivalence class.

Our first result of this section involves the binomial transform of the Catalan numbers, namely the sequence [5, A007317].

Theorem 3.1.

$$F_n(1342) = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}.$$

Proof. Let $\mathcal{A}_{n,k}$ be the set of all permutations $\pi \in S_n$ such that

- $\pi(k) = 1$ and $\pi(1) > \pi(2) > \cdots > \pi(k-1)$,
- $\pi(k+1)\cdots\pi(n) \in \mathsf{Av}_{n-k}(231)$,

and let $\mathscr{A}_n = \bigcup_{k=1}^{n} \mathcal{A}_{n,k}$. Clearly,

$$|\mathscr{A}_n| = \sum_{k=1}^{n} |\mathcal{A}_{n,k}| = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}.$$

We will prove the theorem by showing that $\mathscr{A}_n = \mathscr{F}_n(1342)$.

First of all, since $\mathcal{A}_{n,k} \subset \mathsf{Av}_n(1342)$ and $\mathsf{Av}_{n-k}(231) = \mathscr{F}_{n-k}(231)$ for every n and k, we have $\mathscr{A}_n \subset \mathscr{F}_n(1342)$.

Going in the other direction, let $\pi \in \mathscr{F}_n(1342)$ and let k be such that $\pi(k) = 1$. Thus $\pi = \pi(1)\cdots\pi(k-1)\,1\,\pi(k+1)\cdots\pi(n)$, which implies $\pi(k+1)\cdots\pi(n) \in \mathsf{Av}_{n-k}(231)$. Now, if there is a $j \in \{1,\dots,k-2\}$ such that

$$\pi(1) > \cdots > \pi(j) < \pi(j+1),$$

then $\pi(j) - 1 = \pi(\ell)$ for some $\ell > j+1$, and the pattern $\pi(j)\pi(j+1)\pi(\ell)$ would violate the Fishburn condition. In other words, the entries left from $\pi(k) = 1$ must form a decreasing sequence, which implies $\pi \in \mathcal{A}_{n,k} \subset \mathscr{A}_n$.

Thus $\mathscr{F}_n(1342) \subset \mathscr{A}_n$, and we conclude that $\mathscr{A}_n = \mathscr{F}_n(1342)$. $\square$

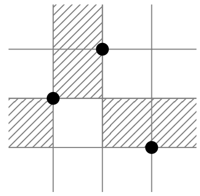
Theorem 3.2. *We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, hence $F_n(3142) = C_n$.*

Proof. Since 3142 contains the pattern 231, we have $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.

In order to prove the reverse inclusion, suppose there exists $\pi \in \mathscr{F}_n(3142)$ such that π contains the pattern 231. Let $i < j < k$ be the positions of the most-left closest 231 pattern contained in π. By that we mean:

- $\pi(k) < \pi(i) < \pi(j)$,
- $\pi(i)$ is the most-left entry of π involved in a 231 pattern,
- $\pi(j)$ is the first entry with $j > i$ such that $\pi(i) < \pi(j)$,
- $\pi(k)$ is the largest entry with $k > j$ such that $\pi(k) < \pi(i)$.

In other words, assume the plot of π is of the form



where no elements of π may occur in the shaded regions. It follows that, if ℓ is the position of $\pi(k)+1$, then $i \le \ell < j$. But this is not possible since, $\pi(\ell) < \pi(\ell+1)$ violates the Fishburn condition, and $\pi(\ell) > \pi(\ell+1)$ implies $\pi(k) > \pi(\ell+1)$ which forces the existence of a 3142 pattern. In conclusion, no permutation $\pi \in \mathscr{F}_n(3142)$ is allowed to contain a 231 pattern. Therefore, $\mathscr{F}_n(3142) \subseteq \mathscr{F}_n(231)$ and we obtain the claimed equality. $\square$

Theorem 3.3. $\mathscr{F}_n(1234) \sim \mathscr{F}_n(1243) \sim \mathscr{F}_n(2143) \sim \mathscr{F}_n(2134)$.

Proof. In order to prove the first and third Wilf equivalence relations, we use a bijection

$$\phi : \mathsf{Av}_n(\tau \oplus 12) \to \mathsf{Av}_n(\tau \oplus 21)$$

given by West in [6], which we proceed to describe.

For $\pi \in S_n$ and $\sigma \in S_k$, $k < n$, let $B_\pi(\sigma)$ be the set of maximal values of all instances of the pattern σ in the permutation π. For example, $B_\pi(\sigma) = \emptyset$ if π avoids σ, and for $\pi = 531968274$ we have $B_\pi(123) = \{4, 7, 8\}$.

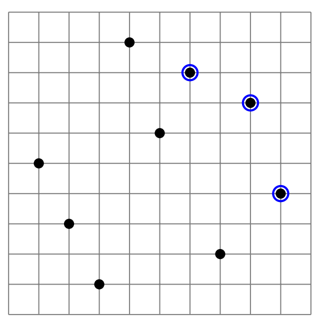


FIGURE 1. $\pi = 531968274$ and $B_\pi(123) = \{4, 7, 8\}$

For $\pi \in \mathsf{Av}_n(\tau \oplus 12)$, let ℓ be the number of elements in $B_\pi(\tau \oplus 1)$. If $\ell = 0$, we define $\phi(\pi) = \pi$. If $\ell > 0$, we let $i_1, \dots, i_\ell$ be the positions in π of the elements of $B_\pi(\tau \oplus 1)$ and define the permutation $\tilde\pi = \phi(\pi)$ by

$$\begin{cases} \tilde\pi(j) = \pi(j) & \text{if } j \notin \{i_1, \dots, i_\ell\} \\ \tilde\pi(i_k) = \pi(i_{\ell+1-k}) & \text{for } k \in \{1, \dots, \ell\} \end{cases}$$

Note that $\pi \in \mathsf{Av}_n(\tau \oplus 12)$ implies $\pi(i_1) > \cdots > \pi(i_\ell)$, and therefore $\tilde\pi(i_1) < \cdots < \tilde\pi(i_\ell)$. For example, for $\pi = 531968274$ and $\tau = 12$, we have $\ell = 3$ and $\tilde\pi = 531964278$.

It is easy to check that the map ϕ induces a bijection

$$\phi : \mathscr{F}_n(\tau \oplus 12) \to \mathscr{F}_n(\tau \oplus 21).$$

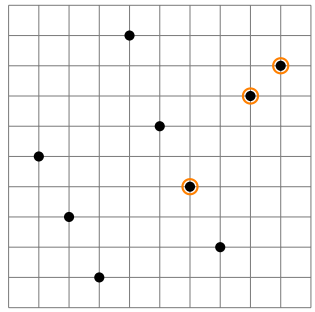
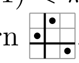
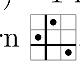


FIGURE 2. $\tilde{\pi} = \phi(531968274) = 531964278$

Indeed, if $\pi(i_k) \in B_\pi(\tau \oplus 1)$ is such that $\pi(i_k - 1) < \pi(i_k)$, then $\tilde{\pi}(i_k - 1) = \pi(i_k - 1)$ and the pair $\tilde{\pi}(i_k - 1), \tilde{\pi}(i_k)$ does not create a pattern .

On the other hand, if $\pi(i_1) > \cdots > \pi(i_k)$ is a maximal descent of elements from $B_\pi(\tau \oplus 1)$, and if $\pi(i_j) - 1 > 0$ (for $j \in \{1, \ldots, k\}$) is not part of that descent, then $\pi(i_j) - 1$ must be left from $\pi(i_1)$ and so the ascent $\tilde{\pi}(i_1) < \cdots < \tilde{\pi}(i_k)$ cannot create the pattern .

Thus, if $\pi \in \mathsf{Av}_n(\tau \oplus 12)$ is Fishburn, so is $\tilde{\pi} = \phi(\pi) \in \mathsf{Av}_n(\tau \oplus 21)$.

Finally, with a similar bijective map

$$\psi : \mathsf{Av}_n(12 \oplus \tau) \to \mathsf{Av}_n(21 \oplus \tau),$$

it can be shown that $\mathscr{F}_n(1234) \sim \mathscr{F}_n(2134)$ and $\mathscr{F}_n(1243) \sim \mathscr{F}_n(2143)$. □

Theorem 3.4. *$\mathscr{F}_n(1423) \sim \mathscr{F}_n(1243)$.*

Proof. Let $\alpha : \mathscr{F}_n(1423) \to \mathscr{F}_n(1243)$ be the map defined through the following algorithm.

Algorithm α: Let $\pi \in \mathscr{F}_n(1423)$ and set $\tilde{\pi} = \pi$.

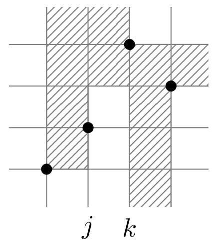
- Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(1243)$, let $i < j < k < \ell$ be the positions of the most-left 1243 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(k)$ to position j, shifting the entries at positions j through $k-1$ one step to the right.
- Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(1243)$, then return $\alpha(\pi) = \tilde{\pi}$; otherwise go to Step 1.

For example, for $\pi = 2135476 \in \mathscr{F}(1423)$, the above algorithm yields

$$\begin{array}{l} \mathbf{2}1\mathbf{354}76 \longrightarrow 21\mathbf{5}3476 \notin \mathsf{Av}(1243) \\ \qquad\quad \swarrow \\ \mathbf{2}1\mathbf{5}34\mathbf{76} \longrightarrow 21\mathbf{7}5346 \in \mathsf{Av}(1243) \end{array}$$

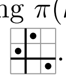
and so $\alpha(\pi) = 2175346 \in \mathscr{F}(1243)$.

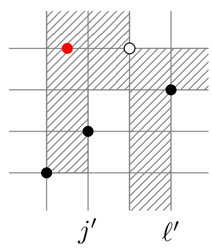
Observe that the map α changes every 1243 pattern into a 1423 pattern. To see that it preserves the Fishburn condition, let $\pi \in \mathscr{F}_n(1423)$ be such that $\pi(i)$, $\pi(j)$, $\pi(k)$, $\pi(\ell)$ form a most-left 1243 pattern. Thus, at first, π must be of the form

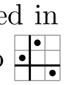


where no elements of π may occur in the shaded regions. In particular, we must have

$$\pi(j-1) < \pi(j) \text{ and } \pi(k-1) < \pi(k). \tag{3.1}$$

Hence the step of moving $\pi(k)$ to position j does not create a new ascent, and therefore it cannot create a pattern . After one iteration, $\tilde{\pi}$ takes the form



and if the most-left 1243 pattern $\tilde{\pi}(i)$, $\tilde{\pi}(j)$, $\tilde{\pi}(k)$, $\tilde{\pi}(\ell)$ contained in $\tilde{\pi}$ has its second entry at a position different from j', then $\tilde{\pi}$ must satisfy (3.1) and no  will be created.

Otherwise, if $j = j'$, then either $k = \ell'$ or $\ell = \ell'$. In the first case, we have $\tilde{\pi}(k-1) < \tilde{\pi}(k)$ and $\tilde{\pi}(k) < \tilde{\pi}(j-1)$, so moving $\tilde{\pi}(k)$ to position j does not create a new ascent. On the other hand, if $\ell = \ell'$, then $\tilde{\pi}(k) > \tilde{\pi}(j-1)$ but $\tilde{\pi}(j-1)-1$ must be left from $\tilde{\pi}(i)$. Therefore, also in this case, applying an iteration of α will preserve the Fishburn condition.

We conclude that, if π is Fishburn, so is $\alpha(\pi)$.

The reverse map $\beta : \mathscr{F}_n(1243) \to \mathscr{F}_n(1423)$ is given by the following algorithm.

Algorithm β: Let $\tau \in \mathscr{F}_n(1243)$ and set $\tilde{\tau} = \tau$.

Step 1: If $\tilde{\tau} \notin \mathsf{Av}_n(1423)$, let $i < j < k < \ell$ be the positions of the most-right 1423 pattern contained in $\tilde{\tau}$. Redefine $\tilde{\tau}$ by moving $\tilde{\tau}(j)$ to position k, shifting the entries at positions $j+1$ through k one step to the left.

Step 2: If $\tilde{\tau} \in \mathsf{Av}_n(1423)$, then return $\beta(\tau) = \tilde{\tau}$; otherwise go to Step 1.

In conclusion, the map α gives a bijection $\mathscr{F}_n(1423) \to \mathscr{F}_n(1243)$. $\square$

Theorem 3.5. *$\mathscr{F}_n(3142) \sim \mathscr{F}_n(3124) \sim \mathscr{F}_n(1324)$.*

Proof. We will define two maps

$$\mathscr{F}_n(3142) \xrightarrow{\alpha_1} \mathscr{F}_n(3124) \text{ and } \mathscr{F}_n(3124) \xrightarrow{\alpha_2} \mathscr{F}_n(1324)$$

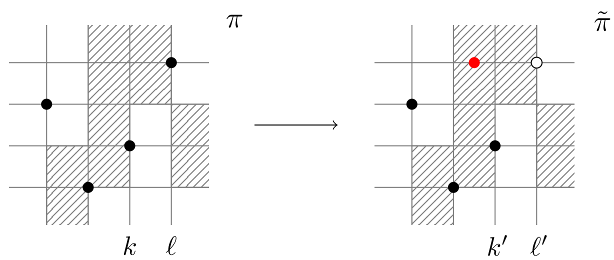
through algorithms similar to the one introduced in the proof of Theorem 3.4.

Algorithm α_1: Let $\pi \in \mathscr{F}_n(3142)$ and set $\tilde{\pi} = \pi$.

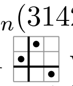
Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(3124)$, let $i < j < k < \ell$ be the positions of the most-left 3124 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(\ell)$ to position k, shifting the entries at positions k through $\ell - 1$ one step to the right.

Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(3124)$, then return $\alpha_1(\pi) = \tilde{\pi}$; otherwise go to Step 1.

As α in Theorem 3.4, the map α_1 is reversible and preserves the Fishburn condition. For an illustration of the latter claim, here is a sketch of a permutation $\pi \in \mathscr{F}_n(3142)$ having a most-left 3124 pattern, together with the sketch of $\tilde{\pi}$ after one iteration of α_1:



where no elements of the permutation π may occur in the shaded regions.

Since $\pi(k-1) < \pi(k)$ and $\pi(\ell - 1) < \pi(\ell)$, the Fishburn condition of π is preserved after the first iteration of α_1. Further, if $\tilde{\pi}$ has a most-left 3124 pattern with third entry at position k', then we must have $\ell > \ell'$ and $\tilde{\pi}(\ell) > \tilde{\pi}(i)$. If $\tilde{\pi}(\ell) < \tilde{\pi}(k'-1)$, no new ascent can be created when moving $\tilde{\pi}(\ell)$ to position k'. Otherwise, if $\tilde{\pi}(\ell) > \tilde{\pi}(k'-1)$, then either both entries were part of an ascent in the original π or every entry between $\tilde{\pi}(\ell')$ and $\tilde{\pi}(\ell)$ must be smaller than $\tilde{\pi}(j)$. Since $\pi \in \mathsf{Av}_n(3142)$, the latter would imply that $\tilde{\pi}(k'-1) - 1$ is left from $\tilde{\pi}(j)$. In any case, no pattern  will be created in the next iteration of α_1.

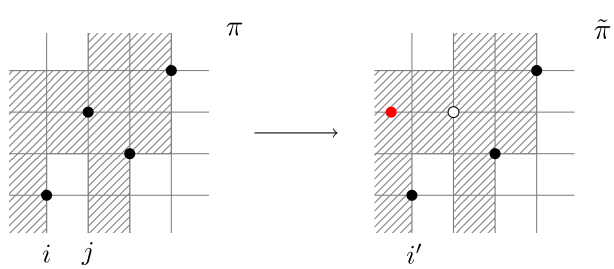
Since any later iteration of α_1 may essentially be reduced to one of the above cases, we conclude that α_1 preserves the Fishburn condition.

Algorithm α_2: Let $\pi \in \mathscr{F}_n(3124)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(1324)$, let $i < j < k < \ell$ be the positions of the most-left 1324 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(j)$ to position i, shifting the entries at positions i through $j - 1$ one step to the right.

Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(1324)$, then return $\alpha_2(\pi) = \tilde{\pi}$; otherwise go to Step 1.

This map is reversible and preserves the Fishburn condition. As before, we will illustrate the Fishburn property by sketching the plot of a permutation $\pi \in \mathscr{F}_n(3124)$ that contains a most-left 1324 pattern $\pi(i)$, $\pi(j)$, $\pi(k)$, $\pi(\ell)$, together with the sketch of the permutation $\tilde{\pi}$ obtained after one iteration of α_2:



Since no elements of the permutation π may occur in the shaded regions, we must have either $i = 1$ or $\pi(i-1) > \pi(j)$. Consequently, moving $\pi(j)$ to position i will not create a new ascent and the Fishburn condition will be preserved.

Similarly, if $\tilde{\pi}$ has a most-left 1324 pattern with first entry at a position different from i', or if $\tilde{\pi}(i) = \tilde{\pi}(i')$ and $\tilde{\pi}(j) < \tilde{\pi}(i'-1)$, then no new ascent will be created and the next $\tilde{\pi}$ will be Fishburn. It is not possible to have $\tilde{\pi}(i) = \tilde{\pi}(i')$ and $\tilde{\pi}(j) > \tilde{\pi}(i'-1)$.

In summary, α_1 and α_2 are both bijective maps. □

The following theorem completes the enumeration of the Catalan class (see Table 4).

Theorem 3.6. $\mathscr{F}_n(3142) \sim \mathscr{F}_n(2143)$.

Proof. Let $\gamma : \mathscr{F}_n(3142) \to \mathscr{F}_n(2143)$ be the map defined through the following algorithm.

Algorithm γ: Let $\pi \in \mathscr{F}_n(3142)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(2143)$, let $i < j < k$ be the positions of the most-left 213 pattern contained in $\tilde{\pi}$ such that $\tilde{\pi}(i)$, $\tilde{\pi}(j)$, $\tilde{\pi}(k)$, $\tilde{\pi}(\ell)$ form a 2143 pattern for some $\ell > k$. Let ℓ_m be the position of the smallest such $\tilde{\pi}(\ell)$, and let

$$Q = \{q \in [n] : \tilde{\pi}(i) \le \tilde{\pi}(q) < \tilde{\pi}(\ell_m)\}.$$

Redefine $\tilde{\pi}$ by replacing $\tilde{\pi}(\ell_m)$ with $\tilde{\pi}(i)$, adding 1 to $\tilde{\pi}(q)$ for every $q \in Q$.

Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(2143)$, then return $\gamma(\pi) = \tilde{\pi}$; otherwise go to Step 1.

For example, if $\pi = 4312576$, then $\gamma(\pi) = 5412673$ (after 2 iterations, see Figure 3).

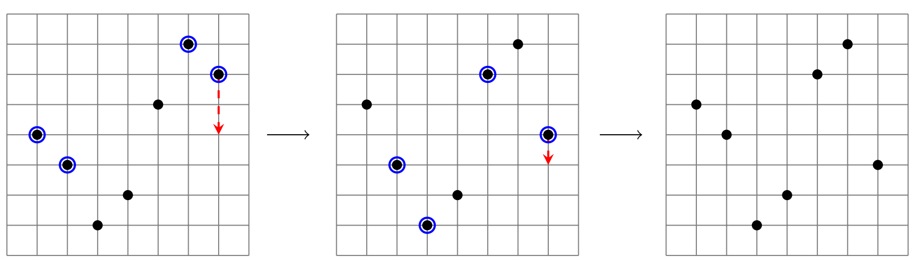
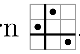


FIGURE 3. Algorithm γ: $4312576 \to 5312674 \to 5412673$.

The map γ is reversible. Moreover, observe that:

(a) since $\tilde{\pi}(\ell_m)$ is the smallest entry such that $\tilde{\pi}(i) < \tilde{\pi}(\ell_m) < \tilde{\pi}(k)$, replacing $\tilde{\pi}(\ell_m)$ with $\tilde{\pi}(i)$ (which is equivalent to moving the plot of $\tilde{\pi}(\ell_m)$ down to height $\tilde{\pi}(i)$) will not create any new ascent at position ℓ_m;

(b) since $\tilde{\pi}(i)$ is chosen to be the first entry of a most-left 2143 pattern, $\tilde{\pi}(i) - 1$ must be right from $\tilde{\pi}(i)$. Hence, replacing $\tilde{\pi}(i)$ by $\tilde{\pi}(i) + 1$ cannot create a new pattern .

In conclusion, γ preserves the Fishburn condition and gives the claimed bijection. □

4. Further remarks

In this paper, we have discussed the enumeration of Fishburn permutations that avoid a pattern of length 3 or a pattern of length 4. In Section 2, we offer the complete picture for patterns of length 3, including the enumeration of indecomposable permutations.

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
1234	1, 1, 2, 6, 22, 85, 324, 1204,...	
1243, 2134	1, 1, 2, 6, 21, 75, 266, 938,...	A289597(?)
1324	1, 1, 2, 6, 22, 84, 317, 1174, ...	
1342	1, 1, 2, 6, 22, 88, 367, 1568, ...	A165538(?)
1423, 3124	1, 1, 2, 6, 20, 68, 233, 805, ...	A279557(?)
1432	1, 1, 2, 6, 20, 71, 263, 1002, ...	
2143	1, 1, 2, 6, 19, 62, 207, 704, ...	A026012(?)
2314	1, 1, 2, 6, 23, 99, 450, 2109, ...	
2341	1, 1, 2, 6, 22, 91, 409, 1955, ...	
2413, 2431, 3241	1, 1, 2, 6, 22, 90, 395, 1823, ...	A165546(?)
3142	1, 1, 2, 5, 14, 42, 132, 429, ...	A000108
3214	1, 1, 2, 6, 20, 72, 275, 1096, ...	
3412	1, 1, 2, 6, 22, 90, 396, 1840, ...	
3421	1, 1, 2, 6, 22, 92, 423, 2088, ...	
4123	1, 1, 2, 5, 14, 43, 143, 507, ...	
4132, 4213	1, 1, 2, 5, 15, 51, 188, 732, ...	
4231	1, 1, 2, 6, 22, 90, 396, 1841, ...	
4312	1, 1, 2, 5, 15, 51, 188, 733, ...	
4321	1, 1, 2, 5, 17, 66, 279, 1256, ...	

TABLE 5.

Regarding patterns of length 4, we have proved the Wilf equivalence of eight permutation families counted by the Catalan numbers. We have also shown that $\mathscr{F}_n(1342)$ is enumerated by the binomial transform of the Catalan numbers. In general, there seems to be 13 Wilf equivalence classes of permutations that avoid a pattern of length 4, some of which appear to be in bijection with certain pattern avoiding ascent sequences ([5, A202061, A202062]). At this point in time, we don't know how the pattern avoidance of a Fishburn permutation is related to the pattern avoidance of an ascent sequence. It would be interesting to pursue this line of investigation.

Concerning indecomposable permutations, we leave the field open for future research. Note that as a direct consequence of Thm. 3.2 and Lem. 2.3, we get:

$$\lvert\mathscr{F}_n^{\mathsf{ind}}(3142)\rvert = C_{n-1}.$$

The study of other patterns is unexplored territory, and our preliminary data suggests the existence of 19 Wilf equivalence classes listed in Table 5.

We are particularly curious about the class $\mathscr{F}_n^{\mathsf{ind}}(1342)$ as it appears (based on limited data) to be equinumerous with the set $\mathsf{Av}_{n-1}(2413, 3421)$, cf. [5, A165538].

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Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601
E-mail address: `jgil@psu.edu`

E-mail address: `mdw8@psu.edu`