DISSECTIONS OF STRANGE q-SERIES

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Dedicated to George E. Andrews on his 80th birthday

ABSTRACT. In a study of congruences for the Fishburn numbers, Andrews and Sellers observed empirically that certain polynomials appearing in the dissections of the partial sums of the Kontsevich-Zagier series are divisible by a certain q-factorial. This was proved by the first two authors. In this paper we extend this strong divisibility property to two generic families of q-hypergeometric series which, like the Kontsevich-Zagier series, agree asymptotically with partial theta functions.

1. Introduction

Recall the usual q-series notation

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.1}$$

and let $\mathcal{F}(q)$ denote the Kontsevich-Zagier "strange" function [13, 14],

$$\mathcal{F}(q) := \sum_{n\geq 0}(q;q)_n.$$

This series does not converge on any open subset of $\mathbb{C}$, but it is well-defined both at roots of unity and as a power series when q is replaced by $1-q$. The coefficients $\xi(n)$ of

$$\mathcal{F}(1-q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots$$

are called the Fishburn numbers, and they count a number of different combinatorial objects (see [11] for references).

Andrews and Sellers [4] discovered and proved a wealth of congruences for $\xi(n)$ modulo primes p. For example, we have

$$\begin{aligned} \xi(5n+4) \equiv \xi(5n+3) \equiv 0 \pmod{5}, \\ \xi(7n+6) \equiv 0 \pmod{7}. \end{aligned} \tag{1.2}$$

In subsequent work of the first two authors, Garvan, and Straub [1, 6, 12], similar congruences were obtained for prime powers and for generalized Fishburn numbers.

Taking a different approach, Guerzhoy, Kent, and Rolen [7] interpreted the coefficients in the asymptotic expansions of functions $P^{(1)}_{a,b,\chi}(e^{-t})$ defined in (1.8) below in terms of special values of L-functions, and proved congruences for these coefficients using divisibility

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properties of binomial coefficients. These congruences are inherited by any function whose expansion at $q = 1$ agrees with one of these expansions; these include the function $\mathcal{F}(q)$ and, more generally, the Kontsevich-Zagier functions described in Section 5 below. See [7] for details.

Although the congruences (1.2) bear a passing resemblance to Ramanujan's congruences for the partition function $p(n)$, it turns out that they arise from a divisibility property of the partial sums of $\mathcal{F}(q)$. For positive integers N and s consider the partial sums

$$\mathcal{F}(q;N) := \sum_{n=0}^{N} (q;q)_n$$

and the s-dissection

$$\mathcal{F}(q;N) = \sum_{i=0}^{s-1} q^i A_s(N,i,q^s).$$

Let $S(s) \subseteq \{0,1,\dots s-1\}$ denote the set of reductions modulo s of the set of pentagonal numbers $m(3m+1)/2$, where $m \in \mathbb{Z}$. The key step in the proof of Andrews and Sellers is to show that if p is prime and $i \notin S(p)$ then we have

$$(1-q)^n \mid A_p(pn-1,i,q). \tag{1.3}$$

This divisibility property is also important for the proof of the congruences in [6, 12]. Andrews and Sellers [4] observed empirically that $(1-q)^n$ can be strengthened to $(q;q)_n$ in (1.3). The first two authors showed that this divisibility property holds for any s. To be precise, define

$$\lambda(N,s) = \left\lfloor \frac{N+1}{s} \right\rfloor. \tag{1.4}$$

Then we have

Theorem 1.1 ([1])**.** *Suppose that s and N are positive integers and that $i \notin S(s)$. Then*

$$(q;q)_{\lambda(N,s)} \mid A_s(N,i,q). \tag{1.5}$$

The proof of (1.5) relies on the fact that the Kontsevich-Zagier function satisfies the "strange identity"

$$\mathcal{F}(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n \left(\frac{12}{n}\right) q^{(n^2-1)/24}.$$

Here the symbol " = " means that the two sides agree to all orders at every root of unity (this is explained fully in Sections 2 and 5 of [13]). In this paper we show that a analogue of Theorem 1.1 holds for a wide class of "strange" q-hypergeometric series—that is, q-series which agree asymptotically with partial theta functions.

To state our result, let F and G be functions of the form

$$F(q) = \sum_{n=0}^{\infty} (q;q)_n f_n(q), \tag{1.6}$$

$$G(q) = \sum_{n=0}^{\infty} (q;q^2)_n g_n(q), \tag{1.7}$$

where $f_n(q)$ and $g_n(q)$ are polynomials. (Functions of the form (1.6) are said to lie in the *Habiro ring* [8].) Note that $F(q)$ is not necessarily well-defined as a power series in q, but it has a power series expansion at every root of unity ζ. In other words $F(\zeta e^{-t})$ has a meaningful definition as a formal power series in t whose coefficients are expressed in the usual way as the "derivatives" of $F(\zeta e^{-t})$ at $t = 0$. This is explained in detail in the next section. Likewise, $G(q)$ has a power series expansion at every odd-order root of unity.

We will consider partial theta functions

$$P_{a,b,\chi}^{(\nu)}(q) := \sum_{n\geq 0} n^{\nu}\chi(n)q^{\frac{n^2-a}{b}}, \tag{1.8}$$

where $\nu \in \{0, 1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi : \mathbb{Z} \to \mathbb{C}$ is a function satisfying the following properties:

$$\chi(n) \neq 0 \quad \text{only if} \quad \frac{n^2 - a}{b} \in \mathbb{Z}, \tag{1.9}$$

and for each root of unity ζ,

$$\text{the function } n \mapsto \zeta^{\frac{n^2-a}{b}}\chi(n) \text{ is periodic and has mean value zero.} \tag{1.10}$$

These assumptions are enough to ensure that for each root of unity ζ, the function $P_{a,b,\chi}^{(\nu)}(\zeta e^{-t})$ has an asymptotic expansion as $t \to 0^+$ (see Section 3 below). We note that (1.10) is satisfied by any odd periodic function. To see this, suppose that χ is odd with period T, and let ζ be a kth root of unity. Set $M = \operatorname{lcm}(T, bk)$. Then we have

$$\zeta^{\frac{(M-n)^2-a}{b}}\chi(M - n) = -\zeta^{\frac{n^2-a}{b}}\chi(n),$$

and so

$$\sum_{n=0}^{M-1} \zeta^{\frac{n^2-a}{b}}\chi(n) = 0.$$

For positive integers s and N, consider the partial sum

$$F(q; N) := \sum_{n=0}^{N} f_n(q)(q; q)_n \tag{1.11}$$

and its s-dissection

$$F(q; N) = \sum_{i=0}^{s-1} q^i A_{F,s}(N, i, q^s).$$

Define $S_{a,b,\chi}(s) \subseteq \{0, 1, \ldots, s - 1\}$ by

$$S_{a,b,\chi}(s) := \left\{ \frac{n^2 - a}{b} \pmod{s} \quad : \quad \chi(n) \neq 0 \right\}.$$

Our first main result is the following.

Theorem 1.2. *Suppose that F is a function as in* (1.6) *and that $P_{a,b,\chi}^{(\nu)}$ is a function as in* (1.8)*. Suppose that for each root of unity ζ we have the asymptotic expansion*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim F(\zeta e^{-t}) \quad \textit{as} \quad t \to 0^+. \tag{1.12}$$

Suppose that s and N are positive integers and that $i \notin S_{a,b,\chi}(s)$. Then we have

$$(q; q)_{\lambda(N,s)} \mid A_{F,s}(N, i, q).$$

Analogously, for positive integers s and N with s odd, consider the partial sum

$$G(q;N) := \sum_{n=0}^{N} g_n(q)(q;q^2)_n \tag{1.13}$$

and its s-dissection

$$G(q;N) = \sum_{i=0}^{s-1} q^i A_{G,s}(N,i,q^s).$$

Then the $A_{G,s}(N,i,q^s)$ also enjoy strong divisibility properties. Define

$$\mu(N,k,s) = \left\lfloor \frac{N}{s(2k-1)} + \frac{1}{2} \right\rfloor. \tag{1.14}$$

Theorem 1.3. *Suppose that G is a function as in* (1.7) *and that $P^{(\nu)}_{a,b,\chi}$ is a function as in* (1.8)*. Suppose that for each root of unity ζ of odd order we have*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim G(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

Suppose that s and N are positive integers with s odd and that $i \notin S_{a,b,\chi}(s)$. Then we have

$$(q;q^2)_{\mu(N,1,s)} \mid A_{G,s}(N,i,q).$$

We illustrate Theorem 1.3 with an example from Ramanujan's lost notebook. Consider the q-series

$$\mathcal{G}(q) = \sum_{n\geq 0} (q;q^2)_n q^n.$$

From [3, Entry 9.5.2] we have the identity

$$\sum_{n\geq 0} (q;q^2)_n q^n = \sum_{n\geq 0} (-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

which may be written as

$$\sum_{n\geq 0} (q;q^2)_n q^n = \sum_{n\geq 0} \chi_6(n) q^{(n^2-1)/3},$$

where

$$\chi_6(n) := \begin{cases} 1, & \text{if } n \equiv 1,2 \pmod 6, \\ -1, & \text{if } n \equiv 4,5 \pmod 6, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for each odd-order root of unity ζ we find that

$$P^{(0)}_{1,3,\chi_6}(\zeta e^{-t}) \sim \mathcal{G}(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

Since χ_6 is odd, it satisfies conditions (1.9) and (1.10). Thus, from Theorem 1.3, we find that for $i \notin S_{1,3,\chi_6}(s)$ we have

$$(q;q^2)_{\lfloor \frac{N}{s}+\frac{1}{2} \rfloor} \mid A_{\mathcal{G},s}(N,i,q). \tag{1.15}$$

For example, when $s=5$ we have $S_{1,3,\chi_6}(5) = \{0,1,3\}$. For $N=8$ we have

$$A_{\mathcal{G},5}(8,2,q) = q^2(q;q^2)_2(1+q^2-q^3+2q^4-q^5+2q^6+q^8)$$

and

$$A_{\mathcal{G},5}(8,4,q) = -q(q;q^2)_2(1-q+q^2)(1+q+q^2+q^4+q^6),$$

as predicted by (1.15), while the factorizations of $A_{\mathcal{G},5}(8,i,q)$ into irreducible factors for $i \in \{0,1,3\}$ are

$$\begin{aligned}
A_{\mathcal{G},5}(8,0,q) &= (1-q)(1+q^4-2q^5+q^6-2q^7+2q^8-3q^9+q^{10}-2q^{11}+q^{12}),\\
A_{\mathcal{G},5}(8,1,q) &= 1+2q^3-q^4+2q^5-3q^6+5q^7-5q^8+4q^9-5q^{10}+4q^{11}-2q^{12}+q^{13}-q^{14},\\
A_{\mathcal{G},5}(8,3,q) &= q(-1+q^2-2q^3+2q^4-5q^5+5q^6-4q^7+5q^8-4q^9+3q^{10}-2q^{11}+q^{12}).
\end{aligned}$$

The rest of the paper is organized as follows. In the next section we discuss power series expansions of F and G at roots of unity, and in Section 3 we discuss the asymptotic expansions of partial theta functions. In Section 4 we prove the main theorems. In Section 5 we give two further examples—one generalizing (1.5) and one generalizing (1.15). We close with some remarks on congruences for the coefficients of $F(1-q)$ and $G(1-q)$.

2. Power series expansions of F and G

Let $F(q)$ be a function as in (1.6) and $G(q)$ be a function as in (1.7). Here we collect some facts which allow us to meaningfully define $F(\zeta e^{-t})$ and $G(\zeta e^{-t})$ as formal power series.

Lemma 2.1. *Let $F(q;N)$ be as in* (1.11)*, and let $G(q;N)$ be as in* (1.13)*. Suppose that ζ is a kth root of unity.*

1. *The values $\left(q\frac{d}{dq}\right)^\ell F(q;N)\big|_{q=\zeta}$ are stable for $N \geq (\ell+1)k-1$.*
2. *If k is odd then the values $\left(q\frac{d}{dq}\right)^\ell G(q;N)\big|_{q=\zeta}$ are stable for $2N \geq (2\ell+1)k$.*

Proof. For each positive integer k we have

$$\begin{aligned}
(1-q^k)^{\ell+1} \mid (q;q)_N \quad &\text{for} \quad N \geq (\ell+1)k,\\
(1-q^{2k-1})^{\ell+1} \mid (q;q^2)_N \quad &\text{for} \quad 2N \geq (2\ell+1)(2k-1)+1.
\end{aligned}$$

It follows that for $0 \leq j \leq \ell$ we have

$$\begin{aligned}
\left(\tfrac{d}{dq}\right)^j (q;q)_N\big|_{q=\zeta} = 0 \quad &\text{for} \quad N \geq (\ell+1)k,\\
\left(\tfrac{d}{dq}\right)^j (q;q^2)_N\big|_{q=\zeta} = 0 \quad &\text{for odd } k \text{ and} \quad 2N \geq (2\ell+1)k+1.
\end{aligned}$$

The lemma follows since for any polynomial $f(q)$, the polynomial $\left(q\frac{d}{dq}\right)^\ell f(q)$ is a linear combination (with polynomial coefficients) of $\left(\frac{q}{dq}\right)^j f(q)$ with $0 \leq j \leq \ell$ (see for example [4, Lemma 2.2]). $\square$

For any polynomial $f(q)$, any ζ and any $\ell \geq 0$ we have [4, Lemma 2.3]

$$\left(\frac{d}{dt}\right)^\ell f(\zeta e^{-t})\big|_{t=0} = (-1)^\ell \left(q\frac{d}{dq}\right)^\ell f(q)\big|_{q=\zeta}. \tag{2.1}$$

Let $F(q)$ be as in (1.6) and let ζ be a kth root of unity. The last fact together with Lemma 2.1 allows us to define

$$\left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0} := \left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t}; N)\big|_{t=0} \qquad \text{for any } N \geq k(\ell+1)-1.$$

We therefore have a formal series expansion

$$F(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0}}{\ell!} t^{\ell}. \tag{2.2}$$

Similarly, if $G(q)$ is a function as in (1.7) and ζ is a kth root of unity with odd k, then we can define

$$\left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0} := \left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t}; N)\big|_{t=0} \qquad \text{for any } 2N \geq k(2\ell+1), \tag{2.3}$$

using (2.1) and Lemma 2.1. Thus, we have a formal series expansion

$$G(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0}}{\ell!} t^{\ell}. \tag{2.4}$$

3. The asymptotics of $P^{(\nu)}_{a,b,\chi}$

In this section we discuss the asymptotic expansion of the partial theta functions $P^{(\nu)}_{a,b,\chi}(q)$ defined in (1.8). Recall that

$$P^{(\nu)}_{a,b,\chi}(q) := \sum_{n\geq 0} n^{\nu}\chi(n) q^{\frac{n^2-a}{b}},$$

where $\nu \in \{0,1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi : \mathbb{Z} \to \mathbb{C}$ is a function satisfying properties (1.9) and (1.10).

The properties which we describe in the next proposition are more or less standard (see for example [10, p. 98]). For convenience and completeness we sketch a proof of the following:

Proposition 3.1. *Suppose that $P^{(\nu)}_{a,b,\chi}(q)$ is as in (1.8). Let ζ be a root of unity and let N be a period of the function $n \mapsto \zeta^{\frac{n^2-a}{b}}\chi(n)$. Then we have the asymptotic expansion*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim \sum_{n=0}^{\infty} \gamma_n(\zeta) t^n, \qquad t \to 0^+,$$

where

$$\gamma_n(\zeta) = \sum_{\substack{1\leq m\leq N \\ \chi(m)\neq 0}} a(m,n,N)\zeta^{\frac{m^2-a}{b}} \tag{3.1}$$

with certain complex numbers $a(m,n,N)$.

We begin with a lemma. For $n \geq 0$ let $B_n(x)$ denote the nth Bernoulli polynomial. In the rest of this section we use s for a complex variable since there can be no confusion with the parameter s used above.

Lemma 3.2. *Let $C : \mathbb{Z} \to \mathbb{C}$ be a function with period N and mean value zero, and let*

$$L(s, C) := \sum_{n=1}^{\infty} \frac{C(n)}{n^s}, \qquad \mathrm{Re}(s) > 0.$$

Then $L(s, C)$ has an analytic continuation to $\mathbb{C}$, and we have

$$L(-n, C) = \frac{-N^n}{n+1} \sum_{m=1}^{N} C(m) B_{n+1}\left(\frac{m}{N}\right) \qquad \textit{for } n \geq 0. \tag{3.2}$$

Proof. Let $\zeta(s, \alpha)$ denote the Hurwitz zeta function, whose properties are described for example in [5, Chapter 12]. We have

$$L(s, C) = N^{-s} \sum_{m=1}^{N} C(m) \zeta\left(s, \tfrac{m}{N}\right). \tag{3.3}$$

The lemma follows using the fact that each Hurwitz zeta function has only a simple pole with residue 1 at $s = 1$ and the formula for the value of each function at $s = -n$ [5, Thm. 12.13]. □

Proof of Proposition 3.1. It is enough to prove the proposition for the function

$$f(t) := e^{-\frac{at}{b}} P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) = \sum_{n \geq 1} n^{\nu} \chi(n) \zeta^{\frac{n^2 - a}{b}} e^{-\frac{n^2 t}{b}}, \qquad t > 0.$$

Setting

$$C(n) := \zeta^{\frac{n^2 - a}{b}} \chi(n), \tag{3.4}$$

we have the Mellin transform

$$\int_0^{\infty} f(t) t^{s-1}\, dt = b^s \Gamma(s) L(2s - \nu, C), \qquad \mathrm{Re}(s) > \frac{1}{2}.$$

Inverting, we find that

$$f(t) = \frac{1}{2\pi i} \int_{x=c} b^s \Gamma(s) L(2s - \nu, C) t^{-s}\, ds,$$

for $c > \frac{1}{2}$, where we write $s = x + iy$. Using (3.3), the functional equation for the Hurwitz zeta functions, and the asymptotics of the Gamma function, we find that, for fixed x, the function $L(s, C)$ has at most polynomial growth in $|y|$ as $|y| \to \infty$. Shifting the contour to the line $x = -R - \frac{1}{2}$ we find that for each $R \geq 0$ we have

$$f(t) = \sum_{n=0}^{R} \frac{(-1)^n}{b^n n!} L(-2n - \nu, C) t^n + O\left(t^{R + \frac{1}{2}}\right),$$

from which

$$f(t) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{b^n n!} L(-2n - \nu, C) t^n.$$

The proposition follows from (3.4) and (3.2). □

4. Proof of Theorems 1.2 and 1.3

We begin with a lemma. The first assertion is proved in [4, Lemma 2.4], and the second, which is basically equation (2.4) in [1], follows by extracting an arithmetic progression using orthogonality. (We note that there is an error in the published version of [1] which is corrected below; in that version the operators $\frac{d}{dq}$ and $q\frac{d}{dq}$ are conflated in the statement of (2.3) and (2.4). This does not affect the truth of the rest of the results.)

Let $C_{\ell,i,j}(s)$ be the array of integers defined recursively as follows:

(1) $C_{0,0,0}(s) = 1$,
(2) $C_{\ell,i,0}(s) = i^\ell$ and $C_{\ell,i,j}(s) = 0$ for $j \geq \ell + 1$ or $j < 0$,
(3) $C_{\ell+1,i,j}(s) = (i + js)C_{\ell,i,j}(s) + sC_{\ell,i,j-1}(s)$ for $1 \leq j \leq \ell$.

Lemma 4.1. *Suppose that s is a positive integer and that*

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

with polynomials $A_s(i, q)$. Then the following are true:

(1) For all $\ell \geq 0$ we have

$$\left(q\frac{d}{dq}\right)^\ell h(q) = \sum_{j=0}^{\ell}\sum_{i=0}^{s-1} C_{\ell,i,j}(s) q^{i+js} A_s^{(j)}(i, q^s).$$

(2) Let ζ_s be a primitive sth root of unity. Then for $\ell \geq 0$ and $i_0 \in \{0, \ldots, s-1\}$ we have

$$\sum_{j=0}^{\ell} C_{\ell,i_0,j}(s) q^{i_0+js} A_s^{(j)}(i_0, q^s) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-ki_0}\left(\left(q\tfrac{d}{dq}\right)^\ell h(q)\right)\Big|_{q\to\zeta_s^k q}. \tag{4.1}$$

Proof of Theorem 1.2. Suppose that $F(q)$ and $P_{a,b,\chi}(q)$ are as in the statement of the theorem. Suppose that s and k are positive integers, that $i \notin S_{a,b,\chi}(s)$ and that ζ_k is a primitive kth root of unity. Let $\Phi_k(q)$ be the kth cyclotomic polynomial. Recall the definition (1.4) of $\lambda(N, s)$ and note that since

$$(q; q)_n = \pm \prod_{k=1}^{n} \Phi_k(q)^{\lfloor \frac{n}{k} \rfloor} \tag{4.2}$$

and

$$\left\lfloor \frac{\lfloor \frac{x}{s} \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{ks} \right\rfloor,$$

we have

$$(q; q)_{\lambda(N,s)} = \pm \prod_{k=1}^{\lambda(N,s)} \Phi_k(q)^{\lambda(N,ks)}.$$

Therefore, Theorem 1.2 will follow once we show for each $\ell \geq 0$ that

$$A_{F,s}^{(\ell)}(N, i, \zeta_k) = 0 \quad \text{for} \quad N \geq (\ell + 1)ks - 1,$$

since this implies that $\Phi_k(q)^{\lambda(N,ks)} \mid A_{F,s}(N, i, q)$ for $1 \leq k \leq \lambda(N, s)$.

From the definition we find that

$$A_{F,s}(N,i,q) = \sum_{j=0}^{k-1} q^j A_{F,ks}(N,i+js,q^k).$$

If $i \notin S_{a,b,\chi}(s)$, then $i+js \notin S_{a,b,\chi}(ks)$. It is therefore enough to show that for all s, k, and ℓ, and for $i \notin S_{a,b,\chi}(ks)$, we have

$$A^{(\ell)}_{F,ks}(N,i,1) = 0 \qquad \text{for } N \geq (\ell+1)ks - 1.$$

After replacing ks by s, it is enough to show that for all s and ℓ, and for $i \notin S_{a,b,\chi}(s)$, we have

$$A^{(\ell)}_{F,s}(N,i,1) = 0 \qquad \text{for } N \geq (\ell+1)s - 1. \tag{4.3}$$

We prove (4.3) by induction on ℓ. For the base case $\ell = 0$, assume that $N \geq s-1$. Using (4.1) with $q = 1$ gives

$$A_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=0}^{s-1} \zeta_s^{-ji} F(\zeta_s^j; N).$$

By (1.12), (2.1), Lemma 2.1, and Proposition 3.1 we find that

$$A_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=1}^{s} \zeta_s^{-ji} \gamma_0(\zeta_s^j).$$

By (3.1) and orthogonality (recalling that $i \notin S_{a,b,\chi}(s)$), we find that $A_{F,s}(N,i,1) = 0$.

For the induction step, suppose that $N \geq (\ell+1)s - 1$, that $i \notin S_{a,b,\chi}(s)$, and that (4.3) holds with ℓ replaced by j for $1 \leq j \leq \ell - 1$. By (4.1) and the induction hypothesis we have

$$C_{\ell,i,\ell}(s) A^{(\ell)}_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=1}^{s} \zeta_s^{-ji} \left(q\frac{d}{dq}\right)^{\ell} F(q;N)\Big|_{q=\zeta_s^j}.$$

Using Proposition 3.1, (2.2), (3.1), and orthogonality, we find as above that

$$C_{\ell,i,\ell}(t) A^{(\ell)}_{F,s}(N,i,1) = 0.$$

This establishes (4.3) since $C_{\ell,i,\ell}(s) > 0$. Theorem 1.2 follows. □

Proof of Theorem 1.3. Suppose that s and k are positive integers with s odd, that $i \notin S_{a,b,\chi}(s)$ and that ζ_{2k-1} is a $(2k-1)$th root of unity. Recall the definition (1.14) of $\mu(N,k,s)$. In analogy with (4.2), we have

$$(q;q^2)_n = \pm \prod_{k=1}^{n} \Phi_{2k-1}(q)^{\lfloor \frac{(2n-1)}{2(2k-1)} + \frac{1}{2} \rfloor},$$

and as above we obtain

$$(q;q^2)_{\mu(N,1,s)} = \pm \prod_{k=1}^{\mu(N,1,s)} \Phi_{2k-1}(q)^{\mu(N,k,s)}.$$

Therefore, Theorem 1.3 follows once we show for each $\ell \geq 0$ that

$$A^{(\ell)}_{G,s}(N,i,\zeta_{2k-1}) = 0 \quad \text{for} \quad 2N \geq (2\ell+1)(2k-1)s.$$

The rest of the proof is similar to that of Theorem 1.2 (we require s to be odd because $G(q)$ has a series expansion only at odd-order roots of unity). Arguing as above, we show that for each odd s we have

$$A^{(\ell)}_{G,s}(N,i,1)=0 \quad \text{for} \quad 2N\geq(2\ell+1)s,$$

and the result follows. $\square$

5. Examples

In this section we illustrate Theorems 1.2 and 1.3 with two families of examples.

5.1. **The generalized Kontsevich-Zagier functions.** In a study of quantum modular forms related to torus knots and the Andrews-Gordon identities, Hikami [9] defined the functions

$$X^{(\alpha)}_m(q):=\sum_{k_1,k_2,\dots,k_m\geq 0}(q;q)_{k_m}q^{k_1^2+\cdots+k_{m-1}^2+k_{\alpha+1}+\cdots+k_{m-1}}\left(\prod_{\substack{i=1\\ i\neq\alpha}}^{m-1}\begin{bmatrix}k_{i+1}\\ k_i\end{bmatrix}\right)\begin{bmatrix}k_{\alpha+1}+1\\ k_\alpha\end{bmatrix}, \tag{5.1}$$

where m is a positive integer and $\alpha\in\{0,1,\dots,m-1\}$. Here we have used the usual q-binomial coefficient (or Gaussian polynomial)

$$\begin{bmatrix}n\\ k\end{bmatrix}:=\begin{bmatrix}n\\ k\end{bmatrix}_q:=\begin{cases}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leq k\leq n,\\ 0, & \text{otherwise.}\end{cases}$$

The simplest example

$$X^{(0)}_1(q)=\sum_{n\geq 0}(q;q)_n$$

is the Kontsevich-Zagier function. From (5.1) we can write

$$X^{(\alpha)}_m(q)=\sum_{k_m\geq 0}(q;q)_{k_m}f^{(\alpha)}_{k_m}(q),$$

with polynomials $f^{(\alpha)}_{k_m}(q)$.

Hikami's identity [9, eqn (70)] implies that for each root of unity ζ we have

$$P^{(1)}_{(2m-2\alpha-1)^2,8(2m+1),\chi^{(\alpha)}_{8m+4}}(\zeta e^{-t})\sim X^{(\alpha)}_m(\zeta e^{-t})$$

as $t\to 0^+$, where $\chi^{(\alpha)}_{8m+4}(n)$ is defined by

$$\chi^{(\alpha)}_{8m+4}(n)=\begin{cases}-1/2, & \text{if } n\equiv 2m-2\alpha-1 \text{ or } 6m+2\alpha+5 \pmod{8m+4},\\ 1/2, & \text{if } n\equiv 2m+2\alpha+3 \text{ or } 6m-2\alpha+1 \pmod{8m+4},\\ 0, & \text{otherwise.}\end{cases} \tag{5.2}$$

The function $\chi^{(\alpha)}_{8m+4}(n)$ satisfies condition (1.9). For (1.10) we record a short lemma.

Lemma 5.1. *Suppose that $\chi^{(\alpha)}_{8m+4}(n)$ is as defined in (5.2) and that ζ is a root of unity of order M. Define*

$$\psi(n)=\zeta^{\frac{n^2-(2m-2\alpha-1)^2}{8(2m+1)}}\chi^{(\alpha)}_{8m+4}(n).$$

Then

$$\sum_{n=1}^{M(8m+4)} \psi(n) = 0.$$

Proof. Note that ψ is supported on odd integers, so we assume in what follows that n is odd. From the definition, we have

$$\chi_{8m+4}^{(\alpha)}(n + M(4m+2)) = (-1)^M \chi_{8m+4}^{(\alpha)}(n). \tag{5.3}$$

The exponent in the ratio of the corresponding powers of ζ is $mM^2 + \frac{M^2+Mn}{2}$. So the ratio of these powers of ζ is

$$\zeta^{\frac{M^2+Mn}{2}}.$$

If M is odd then this becomes $\zeta^{M\left(\frac{M+n}{2}\right)} = 1$, while if M is even then this becomes $\zeta^{\frac{M^2}{2}}\zeta^{\frac{M}{2}n} = -1$ (since M is the order of ζ and n is odd). Therefore the ratio in either case is $(-1)^{M+1}$. Combining this with (5.3) gives

$$\psi(n + M(4m+2)) = -\psi(n),$$

from which the lemma follows. $\square$

Therefore $X_m^{(\alpha)}(q)$ satisfies the conditions of Theorem 1.2, and we obtain the following.

Corollary 5.2. *If s is a positive integer and $i \notin S_{(2m-2\alpha-1)^2, 8(2m+1), \chi_{8m+4}^{(\alpha)}}(s)$, then*

$$(q;q)_{\lambda(N,s)} \big| A_{X_m^{(\alpha)},s}(N,i,q),$$

where $A_{X_m^{(\alpha)},s}(N,i,q)$ are the coefficients in the s-dissection of the partial sums (in k_m) of $X_m^{(\alpha)}(q)$.

For example, when $s = 3$ we have $S_{9,40,\chi_{20}^{(0)}}(3) = \{0,1\}$ and $S_{1,40,\chi_{20}^{(1)}}(3) = \{0,2\}$. For $N = 8$ we have

$$A_{X_2^{(0)},3}(8,2,q) = (q;q)_3(1+q)(1+q+q^2)(1-q+\cdots-q^{25}+q^{26})$$

and

$$A_{X_2^{(1)},3}(8,1,q) = (q;q)_3(1+q)(1-q+q^2)(1+q+q^2)(1+2q+\cdots-q^{26}+q^{27}),$$

as predicted by Corollary 5.2, while

$$\begin{aligned} A_{X_2^{(0)},3}(8,0,q) &= (1-q+q^2)(9+9q+\cdots+q^{33}+q^{34}), \\ A_{X_2^{(0)},3}(8,1,q) &= -8-7q+\cdots+q^{34}-q^{35}, \\ A_{X_2^{(1)},3}(8,0,q) &= 9-7q+\cdots+2q^{36}+q^{39}, \end{aligned}$$

and

$$A_{X_2^{(1)},3}(8,2,q) = -7+3q^3-\cdots+q^{36}-q^{38}$$

are not divisible by $(q;q)_3$.

5.2. **An example with $\nu = 0$.** For $k \geq 1$ let $\mathcal{G}_k(q)$ denote the q-series

$$\mathcal{G}_k(q) = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} q^{n_k + 2n_{k-1}^2 + 2n_{k-1} + \cdots + 2n_1^2 + 2n_1} (q;q^2)_{n_k} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2}.$$

Then we have the identity

$$\mathcal{G}_k(q) = \sum_{n \geq 0} (-1)^n q^{(2k+1)n^2 + 2kn} (1 + q^{2n+1}), \tag{5.4}$$

which follows from Andrews' generalization [2] of the Watson-Whipple transformation

$$\begin{aligned} &\sum_{m=0}^{N} \frac{(1-aq^{2m})}{(1-a)} \frac{(a, b_1, c_1, \ldots, b_k, c_k, q^{-N})_m}{(q, aq/b_1, aq/c_1, \ldots, aq/b_k, aq/c_k, aq^{N+1})_m} \left(\frac{a^k q^{k+N}}{b_1 c_1 \cdots b_k c_k} \right)^m \\ &= \frac{(aq, aq/b_k c_k)_N}{(aq/b_k, aq/c_k)_N} \sum_{N \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k, c_k)_{n_{k-1}} \cdots (b_2, c_2)_{n_1}}{(q;q)_{n_{k-1} - n_{k-2}} \cdots (q;q)_{n_2 - n_1} (q;q)_{n_1}} \\ &\qquad \times \frac{(aq/b_{k-1}c_{k-1})_{n_{k-1} - n_{k-2}} \cdots (aq/b_2 c_2)_{n_2 - n_1} (aq/b_1 c_1)_{n_1}}{(aq/b_{k-1}, aq/c_{k-1})_{n_{k-1}} \cdots (aq/b_1, aq/c_1)_{n_1}} \\ &\qquad \times \frac{(q^{-N})_{n_{k-1}} (aq)^{n_{k-2} + \cdots + n_1} q^{n_{k-1}}}{(b_k c_k q^{-N}/a)_{n_{k-1}} (b_{k-1} c_{k-1})^{n_{k-2}} \cdots (b_2 c_2)^{n_1}}. \end{aligned}$$

Here we have extended the notation in (1.1) to

$$(a_1, a_2, \ldots, a_k)_n := (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n.$$

To deduce (5.4), we set $q = q^2, a = q^2, b_k = q$, and $c_k = q^2$ and then let $N \to \infty$ along with all other b_i, c_i.

The identity (5.4) may be written as

$$\mathcal{G}_k(q) = \sum_{n \geq 0} \chi_{4k+2}(n) q^{\frac{n^2 - k^2}{2k+1}},$$

where

$$\chi_{4k+2}(n) := \begin{cases} 1, & \text{if } n \equiv k, k+1 \pmod{4k+2}, \\ -1, & \text{if } n \equiv -k, -k-1 \pmod{4k+2}, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that for each odd-order root of unity ζ, we have

$$P^{(0)}_{k^2, 2k+1, \chi_{4k+2}}(\zeta e^{-t}) \sim G_k(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

The function $\chi_{4k+2}(n)$ satisfies conditions (1.9) and (1.10) (see the remark following (1.10)), so Theorem 1.3 gives

Corollary 5.3. *Suppose that k and N are positive integers, that s is a positive odd integer, and that $i \notin S_{k^2, 2k+1, \chi_{4k+2}}(s)$. Then*

$$(q;q^2)_{\lfloor \frac{N}{s} + \frac{1}{2} \rfloor} \mid A_{\mathcal{G}_k, s}(N, i, q).$$

6. Remarks on congruences

Congruences for the coefficients of the functions $F(q)$ and $G(q)$ in Theorems 1.2 and 1.3 can be deduced from the results of [7]. In closing we mention another approach. Theorems 1.2 and 1.3 guarantee that many of the coefficients in the s-dissection are divisible by high powers of $1-q$, and the congruences follow from this fact when $s=p^r$ together with an argument as in [1, Section 3].

For example, let $\mathcal{G}_k$ be the function defined in the last section and define $\xi_{\mathcal{G}_k}(n)$ by

$$\mathcal{G}_k(1-q)=\sum_{n\geq 0}\xi_{\mathcal{G}_k}(n)q^n.$$

Consider the expansions

$$\mathcal{G}_1(1-q)=\sum_{n\geq 0}\xi_{\mathcal{G}_1}(n)q^n=1+q+2q^2+6q^3+25q^4+135q^5+\cdots,$$
$$\mathcal{G}_2(1-q)=\sum_{n\geq 0}\xi_{\mathcal{G}_2}(n)q^n=1+2q+6q^2+28q^3+189q^4+1680q^5+\cdots.$$

Then we have such congruences as

$$\xi_{\mathcal{G}_1}(5^rn-1)\equiv 0 \pmod{5^r},$$
$$\xi_{\mathcal{G}_1}(7^rn-1)\equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_1}(13^rn-\beta)\equiv 0 \pmod{13^r}$$

for $\beta\in\{1,2,3,4\}$, and

$$\xi_{\mathcal{G}_2}(7^rn-1)\equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_2}(11^rn-1)\equiv 0 \pmod{11^r}.$$

References

[1] S. Ahlgren and B. Kim, *Dissections of a "strange" function*, Int. J. Number Theory **11** (2015), 1557–1562.
[2] G.E. Andrews, *Problems and prospects for basic hypergeometric series*, in: R. Askey, Theory and Application of Special Functions, Academic Press, New York 1975, 191–224.
[3] G.E. Andrews and B.C. Berndt, *Ramanujan's lost notebook. Part I*, Springer, New York, 2005.
[4] G.E. Andrews and J. Sellers, *Congruences for the Fishburn numbers*, J. Number Theory **161** (2016), 298–310.
[5] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.
[6] F. Garvan, *Congruences and relations for r-Fishburn numbers*, J. Combin. Theory Ser. A **134** (2015), 147–165.
[7] P. Guerzhoy, Z. Kent, and L. Rolen, *Congruences for Taylor expansions of quantum modular forms*, Res. Math. Sci. **1** (2014), Art. 17, 17 pp.
[8] K. Habiro, *Cyclotomic completions of polynomial rings*, Publ. Res. Inst. Math. Sci. **40** (2004) no. 4, 1127–1146.
[9] K. Hikami, *q-series and L-functions related to half-derivatives of the Andrews-Gordon identity*, Ramanujan J. **11** (2006), no. 2, 175–197.
[10] R. Lawrence and D. Zagier, *Modular forms and quantum invariants of* 3*-manifolds*, Asian J. Math. **3** (1999), no. 1, 93–107. MR1701924
[11] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 2010, Sequence A022493.
[12] A. Straub, *Congruences for Fishburn numbers modulo prime powers*, Int. J. Number Theory **11** (2015), no. 5, 1679–1690.

[13] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), 945–960.

[14] D. Zagier, *Quantum modular forms*, Quanta of Maths: Conference in honor of Alain Connes. Clay Mathematics Proceedings 11. AMS and Clay Mathematics Institute 2010, 659–675.

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