Graphs with few Hamiltonian Cycles

Jan Goedgebeur[a,b,1], Barbara Meersman[a], Carol T. Zamfirescu[a,c,1]

[a]*Department of Applied Mathematics, Computer Science & Statistics*
Ghent University
Krijgslaan 281-S9
9000 Ghent, Belgium

[b]*Computer Science Department*
University of Mons
Place du Parc 20
7000 Mons, Belgium

[c]*Department of Mathematics*
Babeş-Bolyai University
Cluj-Napoca, Roumania

Abstract

We describe an algorithm for the exhaustive generation of non-isomorphic graphs with a given number $k \geq 0$ of hamiltonian cycles, which is especially efficient for small k. Our main findings, combining applications of this algorithm and existing algorithms with new theoretical results, revolve around graphs containing exactly one hamiltonian cycle (1H) or exactly three hamiltonian cycles (3H). Motivated by a classic result of Smith and recent work of Royle, we show that there exist nearly cubic 1H graphs of order n iff $n \geq 18$ is even. This gives the strongest form of a theorem of Entringer and Swart, and sheds light on a question of Fleischner originally settled by Seamone. We prove equivalent formulations of the conjecture of Bondy and Jackson that every planar 1H graph contains two vertices of degree 2, verify it up to order 16, and show that its toric analogue does not hold. We treat Thomassen's conjecture that every hamiltonian graph of minimum degree at least 3 contains an edge such that both its removal and its contraction yield hamiltonian graphs. We also verify up to order 21 the conjecture of Sheehan that there is no 4-regular 1H graph. Extending work of Schwenk, we describe all orders for which cubic 3H triangle-free graphs exist. We verify up to order 48 Cantoni's conjecture that every planar cubic 3H graph contains a triangle, and show that for every k that is 0 or at least 4 there exists a planar cyclically 4-edge-connected cubic graph with exactly k hamiltonian cycles. Finally, complementing work of Sheehan on 1H graphs of maximum size, we determine the maximum size of graphs containing exactly one hamiltonian path and give, for every order n, the exact number of such graphs on n vertices and of maximum size.

Keywords: Hamiltonian cycle, uniquely hamiltonian, uniquely traceable, Bondy-Jackson conjecture, cubic graph, girth, exhaustive generation

1. Introduction

In 1946, Smith showed that every edge in a cubic graph is contained in an even number of hamiltonian cycles; see Tutte's paper [34] or Berge's book [3, pp. 189–190]. Thus, a hamiltonian cubic graph contains at least three hamiltonian cycles, so among cubic graphs there exist no graphs with exactly one hamiltonian cycle, i.e. *uniquely hamiltonian* graphs. Strengthening Smith's result, Thomason proved in 1978 that in a graph containing only vertices of odd degree, every edge is contained in an even number of hamiltonian cycles [30]. Thus, uniquely hamiltonian graphs without vertices of even degree, and in particular k-regular uniquely hamiltonian graphs do not exist for odd k. What about even k? Using Lovász' Local Lemma, Thomassen [33] proved that k-regular uniquely hamiltonian graphs do not exist for even $k \geq 300$, and with a careful choice of parameters his theorems give 73 instead of 300. This was improved by Haxell, Seamone, and Verstraete [16] to $k \geq 23$. Sheehan conjectured that there are no 4-regular uniquely hamiltonian graphs [27]. By Petersen's 2-Factor Theorem, the truth of this conjecture would imply that cycles are the only regular uniquely hamiltonian graphs.

In another article [28], Sheehan studied the maximum size of uniquely hamiltonian graphs and proved that such a graph on n vertices contains at most $\lfloor \frac{n^2}{4} \rfloor + 1$ edges. He provides for each $n \geq 3$ an n-vertex uniquely hamiltonian graph of maximum size and states that these are the only uniquely hamiltonian graphs of this size. Barefoot and Entringer [2] proved that Sheehan erred for $n \geq 9$ by showing that for every $n \geq 7$ there exist exactly $2^{\lceil \frac{n}{2} \rceil - 4}$ uniquely hamiltonian graph of maximum size.

Thomason's aforementioned result implies that a uniquely hamiltonian graph must have at least two vertices of even degree. This relationship between a graph's degrees and whether or not it is uniquely hamiltonian raises some natural questions, for instance: are there uniquely hamiltonian graphs of minimum degree 3? Entringer and Swart answered this question affirmatively by describing an infinite family of *nearly cubic* graphs, i.e. graphs with exactly two vertices of degree 4 and all other vertices cubic [11]. Fleischner [12] recently showed that there exist uniquely hamiltonian graphs in which every vertex has degree 4 or 14.

Bondy and Jackson [4] proved that a uniquely hamiltonian graph of order n has at least one vertex of degree at most $c \log_2 8n + 3$, i.e. the minimum degree cannot be greater than this number, where $c \approx 2.41$. Abbasi and Jamshed [1] improved this to $c \log_2 n + 2$, where $c \approx 1.71$. In their article, Bondy and Jackson were particularly interested in *planar* uniquely hamiltonian graphs. They showed that such a graph must contain at least two vertices of degree 2 or 3, and conjectured that every planar uniquely hamiltonian graph contains at least two vertices of degree 2.

This paper is structured as follows. In Section 2 we describe our algorithm for the generation of graphs with "few" hamiltonian cycles—we emphasise that this includes the important class of non-hamiltonian graphs. Thereafter, in Section 3, we present new theoretical results which we combine with the conclusions derived from our implementation of the generation algorithm, as well as existing algorithms. More specifically, in Section 3.1 we determine all orders for which uniquely hamiltonian nearly cubic graphs exist, thereby giving the strongest

form of a theorem of Entringer and Swart [11]. This result extends work of Royle [24] and addresses a question of Fleischner [12] originally settled by Seamone [26]. We give equivalent formulations of the conjecture of Bondy and Jackson [4] mentioned above, verify it up to order 16, but also present a uniquely hamiltonian graph on the torus with exactly one 2-valent vertex. We also treat the conjecture of Thomassen [32] that every hamiltonian graph G of minimum degree at least 3 contains an edge e such that both $G - e$ (remove the edge but not its endpoints) and G/e (contract the edge) are hamiltonian. It is elementary to see that Thomassen's conjecture holds for all graphs with at least two hamiltonian cycles, but it is open for uniquely hamiltonian graphs. We present the orders of all hamiltonian 4-regular graphs up to order 21 with the minimum number of hamiltonian cycles, extending work of Haythorpe [17]. This verifies up to order 21 Sheehan's conjecture [27] that no 4-regular uniquely hamiltonian graph exists.

In Section 3.2, motivated by a classic result of Smith, we study cubic graphs with exactly three hamiltonian cycles. Note that these three hamiltonian cycles together cover each edge exactly twice and thus form a *cycle double cover*. There exist small such graphs of girth 3 in abundance—however, we show that up to order 32 there are only two cubic triangle-free graphs containing exactly three hamiltonian cycles (two generalised Petersen graphs), but that starting from order 34 every even order is covered. This extends a result of Schwenk [25]. Making use of a theorem of Thomason [30], we prove that a graph in which every vertex has odd degree and which has exactly p hamiltonian cycles, where p is prime, must be 3-connected. Thus, any counterexample to Cantoni's conjecture [35] stating that every planar cubic graph with exactly three hamiltonian cycles contains a triangle, is 3-connected. We verify this conjecture up to order 48. We also show that for every k that is 0 or at least 4 there exists a planar cyclically 4-edge-connected cubic graph with exactly k hamiltonian cycles, while by Thomason's result mentioned in the first paragraph, no such graph exists for $k \in \{1, 2\}$. For many of the above conjectures we establish significantly better bounds if a lower bound on the girth is imposed.

In Section 3.3, naturally complementing work of Sheehan [28] on the size of uniquely hamiltonian graphs, we give structural results on graphs containing exactly one hamiltonian path and determine their maximum size using a result of Barefoot and Entringer [2]. We also give the exact number of such graphs of maximum size. The paper ends with Section 3.4 in which we briefly comment on the implementation of our algorithm and on correctness testing. The total computational effort for this project amounted to 40 CPU years.

2. Generation of graphs with few hamiltonian cycles

We describe an algorithm to generate all pairwise non-isomorphic graphs of a given order n, containing, for a fixed non-negative integer k, exactly k hamiltonian cycles. More specifically, in Section 2.1 we present an algorithm for generating uniquely hamiltonian graphs, while in Section 2.2 we describe how this algorithm can be extended to generate graphs with $k \neq 1$ hamiltonian cycles efficiently. Our experiments indicate that our algorithm is significantly more efficient than previous algorithms for $k \leq 7$. To the best of our knowledge, all previously available methods to generate exhaustively graphs with exactly k hamiltonian cycles consisted in using a program such as *geng* [19, 21] to generate all graphs with a given order, to then use a separate program to count the number of hamiltonian

cycles of the generated graphs, and finally to filter the graphs with the desired number of hamiltonian cycles.

In Section 3 we present the computational results which we obtained with our implementation of this algorithm, together with new theoretical results. In Section 3.4 we report the running times of the algorithm and how we tested the correctness of our implementation.

2.1. Generation of uniquely hamiltonian graphs

To generate all uniquely hamiltonian graphs of a given order n, we start the algorithm from a cycle of order n and, in essence, recursively add edges to it in all possible ways as long as the graph stays uniquely hamiltonian. It is clear that all uniquely hamiltonian graphs of order n can be obtained in this way.

To guarantee that the algorithm does not output isomorphic copies, we use McKay's canonical construction path method [20]. In order to use this approach, we first have to define a *canonical reduction* which is unique up to isomorphism. An *expansion* is an operation which constructs a larger graph from a given graph, while the reverse operation is called a *reduction*. We call an expansion that is the inverse of a canonical reduction a *canonical expansion*. The two rules of the canonical construction path method are:

1. Only accept a graph if it was constructed by a canonical expansion.

2. For every graph G to which expansion operations are applied, only perform one expansion from each equivalence class of expansions of G.

The pseudocode of our algorithm to generate all uniquely hamiltonian graphs of order n can be found in Algorithm 1 (recall that we start the algorithm from a cycle of order n). We will now explain how we applied the canonical construction path method for the generation of uniquely hamiltonian graphs. In Theorem 2.1 we then prove that our algorithm indeed generates all pairwise non-isomorphic uniquely hamiltonian graphs of a given order n.

Algorithm 1 Construct(graph G)

 1: Output G
 2: Determine a list L of pairs of non-adjacent vertices in G
 3: Determine orbits of L
 4: **for** one representative vertex-pair $\{a, b\}$ in every orbit of L **do**
 5:     Add edge ab to G
 6:     **if** this expansion was canonical and G is still uniquely hamiltonian **then**
 7:         Construct(G)
 8:     **end if**
 9:     Remove edge ab from G
10: **end for**

In our case, there is only one expansion operation: to insert an edge between two non-adjacent vertices. We implement the second rule of the canonical construction path method by first computing the orbits of all pairs of non-adjacent vertices and only applying the expansion operation for one representative pair from each orbit. (We use the program

nauty [19, 21] to determine all generators of the automorphism group of a graph. Thereafter, we use a union-find algorithm to determine the orbits of pairs of non-adjacent vertices.)

A *reducible* edge is an edge e in G for which $G - e$ is still uniquely hamiltonian. So every edge of G is reducible, except for the n edges of the unique hamiltonian cycle of G. For the first rule of the canonical construction path method we first have to define a canonical reduction which is unique up to isomorphism. In order to do so efficiently, we assign a 9-tuple $(x_0, \ldots, x_8)$ to every reducible edge of a uniquely hamiltonian graph G and define a *canonical edge* as a reducible edge with the lexicographically maximal value for this 9-tuple. The canonical reduction is defined as the reduction of a canonical edge.

We denote with $\mathfrak{h}$ the hamiltonian cycle of G. For a reducible edge $e = ab$, the invariants $x_0, \ldots, x_6$ are invariants of increasing discriminating power and cost and are defined as follows:

- x_0 (x_1) is the maximum (minimum) of the degrees of a and b.

- x_2 is the negative of the minimum length between a and b on $\mathfrak{h}$.

- x_3 (x_4) is the negative of the maximum (minimum) of the sum of the degrees of the vertex preceding and succeeding a on $\mathfrak{h}$ and the sum of the degrees of the vertex preceding and succeeding b on $\mathfrak{h}$.

- x_5 is the number of common neighbours of a and b.

- x_6 is the negative of the number of vertices at distance at most 2 of a and b.

The values $x_0, \ldots, x_6$ are invariant under isomorphisms, but in principle two non-equivalent edges can have the same value for $(x_0, \ldots, x_6)$. Therefore we define $\{x_7, x_8\}$ as the lexicographically largest label of an edge which is in the same orbit as e in the canonical labelling of the graph. (We use the program *nauty* [19, 21] to compute a canonical labelling.)

For the correctness of the algorithm it would be sufficient only to compute the values of x_7 and x_8, but as computing a canonical labelling is computationally expensive, it is much more efficient to use the other invariants x_i, $0 \le i \le 6$, as well.

More specifically, we first compute the value of x_0 and x_1 for every reducible edge of the graph. Since we require a canonical edge to have maximal value for $(x_0, \ldots, x_8)$, we only need to compute x_{i+1} for the reducible edges which have maximal value for $(x_0, \ldots, x_i)$. Furthermore, if the edge e which was added by the last expansion operation is no longer in the list of reducible edges with maximal value for $(x_0, \ldots, x_i)$, we do not have to compute x_{i+1} as we already know that our last expansion was not canonical. Similarly, if e is the only edge with maximal value for $(x_0, \ldots, x_i)$, we do not have to compute x_{i+1} as we already know that our last expansion was canonical.

The discriminating power of the invariants $x_0, \ldots, x_6$ is sufficient to avoid the more expensive computation of x_7 and x_8. For example, this is so in $99.4\,\%$ of the cases for $n = 11$. Finally, note that the invariants x_0 and x_1 allow a look-ahead: in many cases it is easy to determine upfront that inserting an edge between two non-adjacent vertices $\{a, b\}$ cannot be canonical as there will be other edges with a larger degree vector.

The proof of Theorem 2.1 is analogous to the corresponding arguments in the proofs of other generation algorithms—nevertheless, we have chosen to include it for completeness' sake.

Theorem 2.1. *When Algorithm 1 is applied to an n-cycle, it outputs exactly one representative of every isomorphism class of uniquely hamiltonian graphs of order n.*

Proof. First we prove that at least one representative of every isomorphism class of uniquely hamiltonian graphs of order n is generated. Assume by induction on the number of edges m that every uniquely hamiltonian graph of order n with at most m edges is generated and accepted by the algorithm. Consider a uniquely hamiltonian graph G of order n with $m+1$ edges. Since $m+1 > m \geq n$, G contains a reducible edge, thus also a canonical edge $e = ab$. By definition $G - e$ is uniquely hamiltonian, so by induction a graph H isomorphic to $G - e$ was generated. Let γ be an isomorphism from $G - e$ to H. The graph H has a pair of non-adjacent vertices $e' = \{c, d\}$ which are in the same orbit of non-adjacent vertices as $\{\gamma(a), \gamma(b)\}$ to which the edge insertion operation is applied. This produces a graph $H + e'$ which is isomorphic to G and let γ' be an isomorphism from G to $H + e'$. e' is in the same orbit of edges as $\gamma'(e)$ under the action of the automorphism group of $H + e'$. This implies that e' has maximal value for $(x_0, \ldots, x_8)$, so $H + e'$ is accepted by the algorithm.

Now we show that at most one representative of every isomorphism class of uniquely hamiltonian graphs of order n is generated. Assume by induction on the number of edges m that every uniquely hamiltonian graph of order n with at most m edges is generated at most once by the algorithm. Let G_1 and G_2 be two isomorphic uniquely hamiltonian graphs of order n with $m+1$ edges that are both accepted by the algorithm. Let γ be an isomorphism from G_1 to G_2 and let $e_i = a_i b_i$ be the canonical edge from G_i which was added in the last step of the algorithm, for $i \in \{1, 2\}$ (so G_i was obtained from $G_i - e_i$ by adding e_i). Since e_1 and e_2 are both canonical edges, $\gamma(e_1)$ is in the same orbit of edges as e_2 under the action of the automorphism group of G_2. So there is an automorphism of G_2 which maps $\gamma(e_1)$ to e_2. But this automorphism induces an isomorphism γ' from $G_1 - e_1$ to $G_2 - e_2$. Thus, by our induction hypothesis, $G_1 - e_1$ and $G_2 - e_2$ are the same graph and hence γ' is an automorphism which maps $\{a_1, b_1\}$ to $\{a_2, b_2\}$. So $\{a_1, b_1\}$ is in the same orbit of non-adjacent vertex pairs as $\{a_2, b_2\}$, while our algorithm only inserts an edge for one representative of every orbit of non-adjacent vertex pairs. □

2.2. Extensions of the generation algorithm

2.2.1. Generation of graphs with $k > 1$ hamiltonian cycles

Our algorithm for uniquely hamiltonian graphs from Section 2.2 can be easily extended to generate (hamiltonian) graphs with at most or exactly $k > 1$ hamiltonian cycles. The modified algorithm still starts from a cycle of order n, but now on line 6 of Algorithm 1 we have to test if G has at most k hamiltonian cycles instead of testing if G is uniquely hamiltonian. (Note that some of the invariants in $x_0, \ldots, x_6$ have to be adapted or omitted as e.g. x_2 relies on the fact that the graph only contains one hamiltonian cycle.)

If we only want to generate hamiltonian graphs with exactly (instead of at most) k hamiltonian cycles, we nevertheless have to count the number of hamiltonian cycles before outputting the graphs. (Note that this method is not particularly efficient to generate graphs with exactly k hamiltonian graphs for large values of k.) It is also clear that our algorithm is not very efficient to generate all graphs with k hamiltonian cycles for large values of k—however, our experiments indicate that our algorithm is significantly more efficient than previous algorithms for $k \leq 7$.

The only other modification which is required is in the definition of reducible edge. Given a hamiltonian graph G with at most k hamiltonian cycles, an edge e of G is reducible if and only if $G - e$ is hamiltonian.

The proof of the following theorem is analogous to the proof of Theorem 2.1 and therefore omitted.

Theorem 2.2. *When the modified version of Algorithm 1 is applied to a cycle of order n, the algorithm outputs exactly one representative of every isomorphism class of hamiltonian graphs of order n with at most k hamiltonian cycles.*

2.2.2. Generation of non-hamiltonian graphs

Similarly as in Section 2.2.1, Algorithm 1 can be adapted to generate non-hamiltonian graphs of order n efficiently. To this end, one has to start the algorithm from a graph consisting of n isolated vertices (instead of a cycle of order n). Now every edge is reducible. One can also adjust this for a specialised algorithm for only generating connected non-hamiltonian graphs by starting the generation from all trees on n vertices and defining an edge of a connected non-hamiltonian graph G to be reducible if and only if $G - e$ is connected. However, as most non-hamiltonian graphs are connected, this will not be much faster than generating all non-hamiltonian graphs and filtering the connected graphs. (For example: more than $90\,\%$ of the non-hamiltonian graphs on 11 vertices are connected, and with increasing order this ratio increases as well.)

We close this section by mentioning that, since the algorithm only adds edges and never removes any edges, it is straightforward to extend it and restrict the generation to graphs with a given lower bound on the girth, planar graphs, graphs with an upper bound on the maximum degree, and many other interesting properties.

3. Results

3.1. Uniquely hamiltonian graphs

For a graph G, we shall denote by $h(G)$ the number of hamiltonian cycles it contains.

3.1.1. Nearly cubic uniquely hamiltonian graphs

Cubic uniquely hamiltonian graphs do not exist by Smith's theorem. Following Entringer and Swart [11], we call an n-vertex graph *nearly cubic* if exactly $n-2$ of its vertices are cubic, while the remaining two vertices are of degree 4. A uniquely hamiltonian graph contains at least two vertices of even degree, as we will show in the following lemma which essentially belongs to Thomason:

Lemma 3.1. *A graph G with $h(G) \in \{1, 2\}$ contains at least $3 - h(G)$ vertices of even degree.*

Proof. Thomason [30] showed that in a graph containing only vertices of odd degree, every edge is contained in an even number of hamiltonian cycles. Therefore, a hamiltonian graph containing only vertices of odd degree has at least three hamiltonian cycles. So in a graph with exactly one or two hamiltonian cycles there must be at least one vertex of even degree. Suppose there exists a uniquely hamiltonian graph G with exactly one vertex u of even degree. If the degree of u is not 2 (and thus at least 4), removing an edge uv which does not

lie on the hamiltonian cycle of G is a uniquely hamiltonian graph with all vertices of odd degree except for v. We iterate this procedure until the degree of the vertex of even degree, which we call w, is 2, and we denote the neighbours of w by w' and w''. Let G_1 and G_2 be disjoint copies of $G - w$, and w_i' and w_i'' the respective copies of w' and w''. Then $G_1 \cup G_2$ to which we add the edges $w_1'w_2'$ and $w_1''w_2''$ yields a uniquely hamiltonian graph in which all vertices have odd degree. However, this contradicts Thomason's theorem. Thus, a uniquely hamiltonian graph must contain at least two vertices of even degree. □

In contrast to Lemma 3.1, it was shown by Chia and Yu [10] that for every $k \geq 3$ there exists a planar cyclically 3-edge-connected cubic graph with precisely k hamiltonian cycles. The argument is short: The 1-skeleta of prisms, i.e. the cartesian product of the n-cycle with K_2, deal with all $k \neq 4$, while carefully combining two Tutte-fragments (first used in [34]) yields the $k = 4$ case (we note that this construction has cyclical edge-connectivity 3). Many non-hamiltonian planar 3-connected cubic graphs are known—take for instance Thomassen's infinite family [31]. In fact, Thomassen's graphs as well as all prisms excluding the triangular one are cyclically 4-edge-connected.

Using the program *plantri* [9] we generated all planar cyclically 4-edge-connected cubic graphs up to 48 vertices and tested if any such graph contains exactly four hamiltonian cycles. The result is as follows.

Observation 3.2. *The smallest planar cyclically 4-edge-connected cubic graph with exactly four hamiltonian cycles has* 38 *vertices and is shown in Figure 1. There are also exactly five such graphs on* 42 *vertices,* 32 *on* 46 *vertices and* 6 *on* 48 *vertices.*
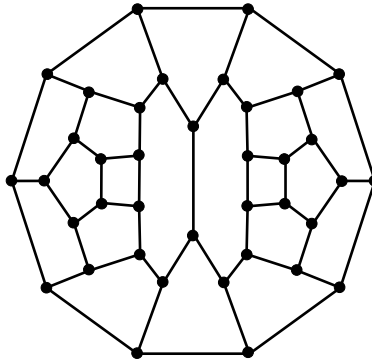

Figure 1: The smallest planar cyclically 4-edge-connected cubic graph with exactly four hamiltonian cyles. It has 38 vertices.

Drawings of the five planar cyclically 4-edge-connected cubic graphs with exactly four hamiltonian cycles on 42 vertices from Observation 1 can be found in Figure 6 in the Appendix. Hence, together with Lemma 3.1, we have:

Theorem 3.3. *For every non-negative integer $k \notin \{1, 2, 3\}$ there exists a planar cyclically 4-edge-connected cubic graph with exactly k hamiltonian cycles, while for $k \in \{1, 2\}$ there exist no such graphs with precisely k hamiltonian cycles.*

Thus only the case $k = 3$ remains open, which relates to a conjecture due to Cantoni on which we focus in Section 3.2. We remark that if one subdivides one of the quadrilaterals present in the graph depicted in Figure 1 into an odd number of quadrilaterals by adding an even number $2k$ of (parallel) edges—we have illustrated the result of this operation for $k = 1$ in Figure 6c in the Appendix—, it follows that there exist infinitely many planar cyclically 4-edge-connected cubic graphs with exactly four hamiltonian cycles.

Let us now return to nearly cubic graphs. By Euler's degree sum formula, every nearly cubic graph has even order. Entringer and Swart [11] showed that for all even $n \geq 22$ there exists a nearly cubic uniquely hamiltonian graph of order n. Recently, Royle presented a nearly cubic uniquely hamiltonian graph on 18 vertices [24], noting that this is the smallest such graph. By modifying our algorithm from Section 2 for nearly cubic graphs, we verified this independently, addressed the last remaining open case, and determined the exact counts of such graphs for the smallest orders for which they occur:

Theorem 3.4. *There exists a nearly cubic uniquely hamiltonian graph of order n if and only if n is even and $n \geq 18$. Royle's graph has girth 5 and is the only nearly cubic uniquely hamiltonian graph on 18 vertices. Furthermore, there are exactly 20 nearly cubic uniquely hamiltonian graphs of order 20, 337 of order 22, and 4592 of order 24. Finally, both the smallest nearly cubic uniquely hamiltonian graph of girth 3 as well as of girth 4 have order 20.*

Royle's nearly cubic uniquely hamiltonian graph on 18 vertices is shown in Figure 2a. There is precisely one nearly cubic uniquely hamiltonian graph of girth 4 on 20 vertices and it is shown in Figure 2b. There are exactly 17 such graphs of girth 3 on 20 vertices, one of which is shown in Figure 2c. The nearly cubic graphs up to 24 vertices can also be downloaded from the *House of Graphs* [6] at `http://hog.grinvin.org/UHG`.
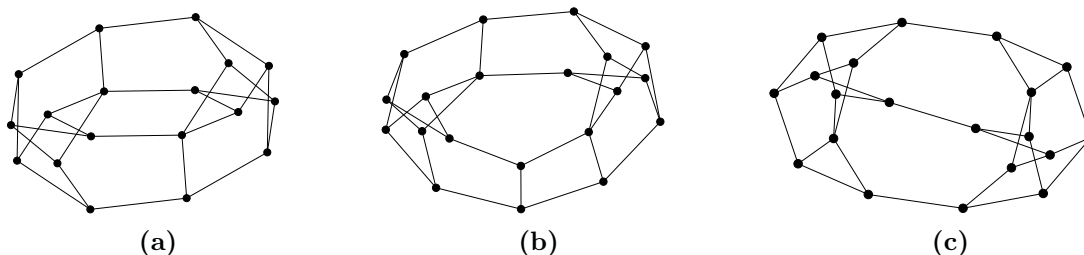

(a) (b) (c)

Figure 2: The smallest nearly cubic uniquely hamiltonian graphs of (a) girth 5, due to Royle, (b) girth 4, and (c) girth 3. Their orders are 18, 20, and 20, respectively.

Our focus on girth stems from a question of Fleischner [12, p. 176], who asked whether there exist uniquely hamiltonian graphs without 2-valent vertices and of girth > 3. Entringer and Swart's aforementioned approach [11] yields graphs containing exactly two triangles. Seamone provides in [26] a method to construct nearly cubic triangle-free uniquely hamiltonian graphs, thereby giving an affirmative answer to Fleischner's question. However, he does not discuss concrete examples, in particular small ones. The graph mentioned above due to Royle [24] provides such a concrete example and Royle showed that there is no smaller uniquely hamiltonian graph of minimum degree at least 3. Theorem 3.4 expands on this.

9

Conjecture (Bondy and Jackson [4]). *Every planar uniquely hamiltonian graph contains at least two vertices of degree* 2. $\qquad\qquad$ ($\mathfrak{B}$)

Using our implementation of Algorithm 1 from Section 2, we generated all planar uniquely hamiltonian graphs with girth at least 3, 4, and 5 up to certain orders, see Table 1. (We used Boyer and Myrvold's algorithm [5] to test if a graph is planar.)

Order	# planar UH graphs	girth ≥ 4	girth ≥ 5
3	1	0	0
4	2	1	0
5	3	1	1
6	12	2	1
7	49	3	1
8	460	11	3
9	4 994	33	4
10	68 234	178	8
11	997 486	1 011	23
12	15 582 567	6 816	91
13	253 005 521	47 669	317
14	4 250 680 376	352 901	1 353
15	73 293 572 869	2 680 512	6 473
16	1 293 638 724 177	20 939 433	30 834
17	?	166 713 951	148 907
18	?	1 352 143 860	768 178
19	?	11 129 922 982	3 987 517
20	?	?	20 767 030
21	?	?	110 819 167
22	?	?	599 311 836
23	?	?	3 256 610 004

Table 1: The number of all planar uniquely hamiltonian graphs, all such graphs with girth at least 4, and all such graphs with girth at least 5, respectively. Each contains at least two 2-valent vertices.

While performing these computations, we verified that none of the generated graphs is a counterexample to ($\mathfrak{B}$). That is:

Observation 3.5. *The conjecture of Bondy and Jackson* ($\mathfrak{B}$) *is true for graphs up to order* 16, *for graphs of girth at least* 4 *up to order* 19, *and for graphs of girth at least* 5 *up to order* 23.

If we relax the planarity condition in ($\mathfrak{B}$) to "having genus 1", we can give a counterexample on only 11 vertices—it is shown in Figure 3. More specifically, we determined that there are exactly two uniquely hamiltonian graphs with at most one vertex of degree 2 on

11 vertices, exactly 20 on 12 vertices, none on smaller orders, and that all of these 22 examples have genus 1. Furthermore, the smallest toric counterexample of girth 4 (girth 5) has order 13 (order 14). Using the aforementioned findings and observing that we can replace cubic vertices with triangles iteratively without altering neither the number of hamiltonian cycles nor the number of 2-valent vertices, nor the genus, we obtain the following result.

Observation 3.6. *There exists an n-vertex toric uniquely hamiltonian graph containing at most one 2-valent vertex if and only if $n \geq 11$.*
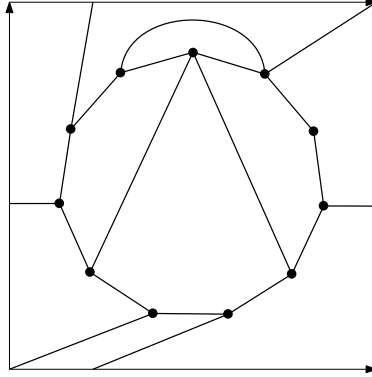

Figure 3: A toric uniquely hamiltonian graph containing exactly one vertex of degree 2. Its order is 11. There are no smaller uniquely hamiltonian graphs containing at most one 2-valent vertex, irrespective of genus.

What if we replace "uniquely hamiltonian graph" by "containing exactly two hamiltonian cycles" in the Bondy-Jackson conjecture? By subdividing an arbitrary edge once in a planar cubic graph with exactly three hamiltonian cycles—discussed in detail in Section 3.2—, we obtain infinitely many planar graphs with exactly two hamiltonian cycles having precisely one 2-valent vertex. The smallest such graph is K_4 with one subdivided edge. The next theorem addresses, among other things, the situation for minimum degree 3.

Theorem 3.7. *The following statements are equivalent.*

1. *There exists a counterexample to ($\mathfrak{B}$) of minimum degree at least 3.*

2. *There exists a counterexample to ($\mathfrak{B}$) containing exactly one vertex of degree 2.*

3. *There exist infinitely many counterexamples to ($\mathfrak{B}$).*

4. *There exists a planar graph with exactly two hamiltonian cycles and minimum degree at least 3.*

5. *There exists a planar non-hamiltonian n-vertex graph with exactly one $(n-1)$-cycle, no vertex of degree 0 or 1 and at most one vertex of degree 2.*

Proof. Let G be a counterexample to ($\mathfrak{B}$) of minimum degree at least 3. Consider an edge e on the unique hamiltonian cycle of G. Adding a vertex on e yields a counterexample to ($\mathfrak{B}$) containing exactly one 2-valent vertex. Now let G be a counterexample to ($\mathfrak{B}$) containing precisely one vertex of degree 2. We use the same idea as in the proof of Lemma 3.1 and obtain a counterexample to ($\mathfrak{B}$) of minimum degree at least 3. We have shown the equivalence of statements 1 and 2. These arguments immediately yield that if an n-vertex counterexample to ($\mathfrak{B}$) exists, then there are such counterexamples of order kn where k is any power of 2.

We now show the equivalence of the existence of a counterexample to ($\mathfrak{B}$) and statement 4. Suppose G is a planar graph with minimum degree at least 3 and containing exactly two hamiltonian cycles $\mathfrak{h}_1$ and $\mathfrak{h}_2$. We can then subdivide once an edge in $E(\mathfrak{h}_1) \setminus E(\mathfrak{h}_2)$ and obtain a uniquely hamiltonian graph G'. All vertices in G (which we see as a subgraph of G') have the same degree as in G', so G' contains exactly one vertex of degree 2, and clearly G' is planar. This means that we have obtained a counterexample to ($\mathfrak{B}$).

Assume now that there is a counterexample G to ($\mathfrak{B}$). Without loss of generality we may suppose that G has minimum degree at least 3. We denote the hamiltonian cycle in G by $\mathfrak{h}$. Consider, for adjacent vertices x, y in G such that xy lies on $\mathfrak{h}$, the graph

$$G' = (V(G) \cup \{v_1, v_2\}, E(G) \setminus \{xy\} \cup \{xv_1, xv_2, v_1v_2, v_1y, v_2y\}).$$

There are exactly two hamiltonian xy-paths in $G'[\{x, y, v_1, v_2\}]$ and there is exactly one hamiltonian xy-path in $G'[V(G)]$. Thus, G' is a planar graph with minimum degree 3 and containing exactly two hamiltonian cycles.

Let us prove the equivalence of the existence of a counterexample to ($\mathfrak{B}$) and statement 5. Let G be an n-vertex counterexample to ($\mathfrak{B}$), which by above discussion we may assume to contain a 2-valent vertex w with neighbours w', w''. Let G_1 and G_2 be disjoint copies of $G - w - w_1w_2$ (we remove the edge w_1w_2 only if it is present in G), and w_i' and w_i'' the respective copies of w' and w''. In $G_1 \cup G_2$, we identify w_1' with w_2' and w_1'' with w_2'', calling the respective resulting vertices v' and v''. We then add the edge between v' and v'' and a vertex v on this new edge. We obtain the graph G' of order $2n - 3$. G' is non-hamiltonian but contains exactly one $(2n - 4)$-cycle (which avoids v) and is clearly planar.

Let G be a graph satisfying the properties given in statement 5. We denote its unique $(n-1)$-cycle by $\mathfrak{c}$, and by v the vertex not contained in $\mathfrak{c}$. Let $v_1, \ldots, v_d$ be the neighbours of v. Observe that G may contain a vertex of degree 2 and that this can be v. This makes no difference in the arguments that follow. The graph $G - v = G'$ is uniquely hamiltonian and planar. If all but at most one vertices of G' have degree at least 3, then we have a counterexample to ($\mathfrak{B}$) and we are done.

The degrees of the vertices which are not $v_1, \ldots, v_d$ remain unchanged when we remove v from G. Thus, we must now take care of the degrees of $v_1, \ldots, v_d$. This is achieved by adding edges v_iv_{i+1} where necessary. Note that if $G' + v_iv_{i+1}$ contains a new hamiltonian cycle $\mathfrak{h}$, then this cycle must use v_iv_{i+1}; but then replacing in $\mathfrak{h}$ the edge v_iv_{i+1} with the path v_ivv_{i+1}, we obtain a hamiltonian cycle in G, a contradiction, as G was assumed to be non-hamiltonian. If the degrees of v_i and v_{i+1} in G' are already at least 3 then it is not necessary to add an edge. Otherwise the degrees of v_i and v_{i+1} are at least 2 since from each vertex only exactly one incident edge was removed. (In G, the vertex v_i could not have had degree 2 as then $\mathfrak{c}$ would have visited v.)

We still need to deal with the case when $v_iv_{i+1} \in E(G)$. We now prove that in this situation the degrees of v_i and v_{i+1} were already sufficiently large. If v_i and v_{i+1} are adjacent in G, then $v_iv_{i+1} \notin E(\mathfrak{c})$, as discussed above. Thus, each of v_i and v_{i+1} is incident with at least two further edges, namely the edges lying on $\mathfrak{c}$. Together with the edge v_iv_{i+1} we obtain that the degrees of v_i and v_{i+1} are at least 3 in G', and that it is not necessary to add an edge between these two vertices.

Therefore, we can modify the graph G' such that all of its vertices have degree at least 3, with the possible exception of at most one vertex of degree 2 already present in G. It is clear that the edges v_iv_{i+1} can be added such that the graph remains planar. This graph we have constructed is a counterexample to $(\mathfrak{B})$. □

The proof of the following observation consists of two parts. Firstly, consider the infinite family of planar non-hamiltonian graphs with a unique $(n-1)$-cycle and exactly two vertices of degree 2 from Figure 4, left-hand side. The maximum degree of this family of graphs is unbounded. The right-hand side of Figure 4 shows an infinite family of such graphs in which the maximum degree of each member is bounded above by 4. Secondly, using a computer, we verified that there exist no planar non-hamiltonian graphs of order $n < 10$ containing exactly one $(n-1)$-cycle, exactly two vertices of degree 2 and all other vertices of degree at least 3.

Observation 3.8. *There exists a planar non-hamiltonian graph of order n containing exactly one $(n-1)$-cycle, exactly two vertices of degree 2, and all other vertices of degree at least 3 if and only if $n \geq 10$.*
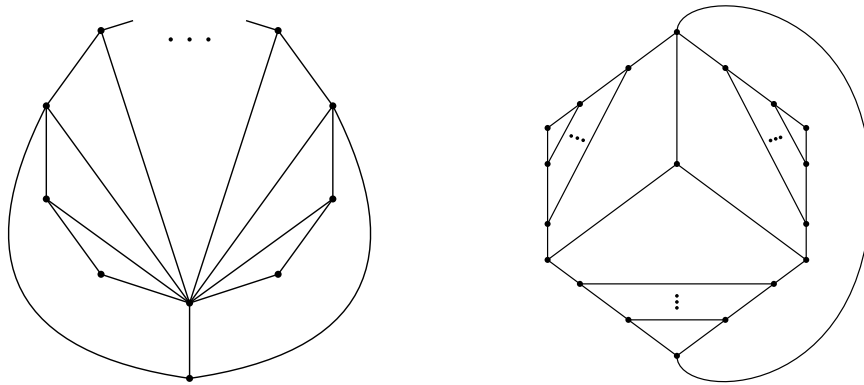

Figure 4: Two infinite families of planar non-hamiltonian graphs with a unique $(n-1)$-cycle and exactly two vertices of degree 2.

3.1.3. Thomassen's Conjecture and a Question of Royle

It is a natural question to ask for the smallest order of a uniquely hamiltonian graph with a certain minimum degree δ or connectivity κ. For $\delta = 2$ and $\kappa = 2$ the answer is trivial, namely K_3. There are many such graphs as presented in other sections of this article. For both $\delta = 3$ and $\kappa = 3$ the answer is 18 and was given by Royle [24] using a computer, see Figure 2a for an example. In [12], Fleischner describes uniquely hamiltonian graphs in which

every vertex has degree 4 or 14. His smallest example of connectivity 2 has 338 vertices, while the smallest graph of connectivity 3 he constructs has order 408. Thus, the smallest uniquely hamiltonian graph with $\delta = 4$ has order at least 18 and at most 338. To the best of our knowledge, no uniquely hamiltonian graph of minimum degree at least 5 is known.

Concerning $\kappa \geq 4$, Fleischner conjectured [12, p. 176] that every uniquely hamiltonian graph has connectivity at most 3. It seems that the first explicit construction of a 3-connected uniquely hamiltonian graph is due to Grinberg [15], and his example is of the same order (18) and only one edge larger than the smallest example there is, which was determined by Royle using a computer [24]. Aldred and Thomassen also described a 3-connected uniquely hamiltonian graph, see [18]. By replacing in these graphs cubic vertices (which they all contain) with triangles, we obtain infinitely many such graphs. Using Seamone's technique [26] we can render them triangle-free.

We now state the following intriguing conjecture of Thomassen concerning uniquely hamiltonian graphs with minimum degree at least 3, and discuss its connection with a recent question of Royle.

Conjecture (Thomassen [32]). *Every hamiltonian graph G of minimum degree at least 3 contains an edge e such that both $G - e$ and G/e are hamiltonian.* $(\mathfrak{T})$

(Recall that $G-e$ stands for removing the edge but not its endpoints and G/e denotes the graph obtained by contracting the edge e from G.) If G contains at least two hamiltonian cycles, then any edge contained in one but not the other hamiltonian cycle satisfies the condition from $(\mathfrak{T})$, so the conjecture's veracity is open exactly for the family of uniquely hamiltonian graphs. Let a graph G contain exactly one hamiltonian cycle $\mathfrak{h}$. Clearly, $G - e$ is non-hamiltonian for every $e \in E(\mathfrak{h})$, so candidate edges vw satisfying $(\mathfrak{T})$ must lie in $E(G) \setminus E(\mathfrak{h})$. For such an edge vw we have that G/vw is hamiltonian if and only if $G - v$ or $G - w$ is hamiltonian (possibly both).

Royle recently asked (personal communication) whether uniquely hamiltonian graphs of minimum degree 3 and without $(n-1)$-cycles exist. If such a graph would be found, it would constitute a counterexample to $(\mathfrak{T})$.

Recall that Royle [24] showed that the smallest uniquely hamiltonian graphs with minimum degree at least 3 have order 18. Royle's result implies the first statement of the following observation. The other statements were obtained using our implementation of Algorithm 1 combined with a separate program which tests if each generated graph G contains an edge e such that both $G - e$ and G/e are hamiltonian.

Observation 3.9. *Thomassen's conjecture $(\mathfrak{T})$ is true for graphs up to order 17, for graphs of girth at least 4 up to order 18, for graphs of girth at least 5 up to order 22, and for nearly cubic graphs up to order 24.*

The number of uniquely hamiltonian graphs with girth at least 5 and minimum degree at least 3 is shown in Table 2. These graphs can be downloaded from the *House of Graphs* [6] at `http://hog.grinvin.org/UHG`.

By extending our algorithm from Section 2 to only generate uniquely hamiltonian graphs without $(n - 1)$-cycles, we were also able to look into Royle's question which led to the following observation.

Order	< 18	18	19	20	21	22
Number of graphs	0	2	1	2	25	33

Table 2: The number of uniquely hamiltonian graphs with girth at least 5 and minimum degree at least 3. All of these graphs have girth 5 and minimum degree 3. We also verified that there are no uniquely hamiltonian graphs with girth 4 and minimum degree at least 3 on 18 (or less) vertices.

Observation 3.10. *There are no uniquely hamiltonian graphs of minimum degree at least* 3 *without* $(n-1)$*-cycles up to order* 18. *Furthermore, there are no uniquely hamiltonian graphs of minimum degree at least* 3 *and girth at least* 4 *(resp.* 5*) without* $(n-1)$*-cycles up to order* 19 *(resp.* 23*).*

3.1.4. Sheehan's conjecture

We recall a conjecture of Sheehan from 1974:

Conjecture (Sheehan [27]). *There is no uniquely hamiltonian* 4*-regular graph.* (𝔖)

Petersen's 2-Factor Theorem [23] states that every $(2k)$-regular graph can be decomposed into k edge-disjoint 2-factors. Hence, if (𝔖) is true, then the only regular uniquely hamiltonian graphs are cycles.

In [17] Haythorpe computationally determined the minimum number of hamiltonian cycles among all hamiltonian k-regular graphs of a given small order for $4 \le k \le 7$. In particular, he determined the minimum non-zero number of hamiltonian cycles in 4-regular hamiltonian graphs up to 16 vertices and in 4-regular hamiltonian graphs of connectivity 2 up to 18 vertices.

Using the generator for regular graphs *genreg* [22] we generated all 4-regular graphs up to 21 vertices and determined the minimum number of hamiltonian cycles for each order. The results, together with the counts for girth at least 4 and 5, are listed in Table 3. Note the striking drop in the minimum non-zero number of hamiltonian cycles for 4-regular graphs of girth at least 4 for orders 12 and 20.

Our results from Table 3 imply the following.

Observation 3.11. *Sheehan's conjecture* (𝔖) *is true for graphs on up to* 21 *vertices. Furthermore, the conjecture is true for* 4*-regular graphs with girth at least* 5 *up to* 26 *vertices.*

Haythorpe [17] describes an infinite family of 4-regular graphs based on K_5 and the 1-skeleton of the octahedron. With a few minor additions we obtain the following result, which essentially belongs to him:

Observation 3.12. *There exists a* 4*-regular graph of order* n *with exactly* $36 \cdot 2^{\lfloor \frac{n}{5} \rfloor - 2}$ *hamiltonian cycles for every* $n \in \{10, 15, 16, 20, 21, 22, 25, 26, 27, 28\}$ *and all* $n \ge 30$.

3.2. Cubic graphs with exactly three hamiltonian cycles

It follows from Smith's result that a hamiltonian cubic graph contains at least three hamiltonian cycles. We devote this section to the study of the extremal case of cubic graphs

15

Order	Minimum number of hamiltonian cycles		
	girth ≥ 3	girth ≥ 4	girth ≥ 5
5	12 (1)	-	-
6	16 (1)	-	-
7	23 (1)	-	-
8	29 (1)	72	-
9	36 (1)	-	-
10	36 (1)	96	-
11	48 (2)	145	-
12	60 (2)	142	-
13	72 (3)	250	-
14	72 (1)	323	-
15	72 (2)	460	-
16	72 (1)	604	-
17	96 (2)	936	-
18	108 (1)	1 024	-
19	144 (21)	1 512	2 688
20	144 (18)	1 024	2 716
21	144 (13)	1 600	3 657
22	?	?	5 589
23	?	?	8 382
24	?	?	12 412
25	?	?	18 906
26	?	?	25 299

Table 3: The minimum number of hamiltonian cycles among all 4-regular hamiltonian graphs of girth at least 3, 4, and 5 for that order. The symbol "-" indicates that no 4-regular graphs of that order and girth exist. The numbers in parentheses indicate how many graphs of that order have the minimum non-zero number of hamiltonian cycles. For the columns regarding girth at least 4 and 5, respectively, there is in each case only one graph with the minimum non-zero number of hamiltonian cycles.

containing precisely three hamiltonian cycles, the smallest of which is K_4. Note that these three hamiltonian cycles together cover each edge exactly twice and thus form a *cycle double cover* with the smallest possible number of cycles. For a brief overview of results on cubic graphs with exactly three hamiltonian cycles, see [18]. In such a graph G, for each vertex v in G any pair of distinct edges incident with v is traversed by exactly one hamiltonian cycle of G. By starting with K_4 and replacing vertices by triangles one can construct for every $k \geq 2$ a polyhedral cubic graph of order $n = 2k$ with exactly three hamiltonian cycles.

These graphs clearly have girth 3. Are there triangle-free graphs with exactly three hamiltonian cycles? (Note that we do not require planarity at this point.) In order to address this question we make use of ideas of Chia and Yu [10], which we now briefly introduce. Clearly, between any two vertices of a triangle T in a cubic graph there is exactly one path visiting every vertex of the triangle, and T is separated from the rest of the graph by a 3-edge-cut. Thus, we can replace T with a so-called "tup", a graph introduced in [10]:

A *tup* is a graph in which all but three vertices are cubic, the three exceptional vertices have degree 2, and between any two such vertices there is exactly one hamiltonian path.

One can make from any cubic graph with three hamiltonian cycles a tup by removing one of the graph's vertices. Given a cubic graph G and a vertex v in G, one can replace v with a tup H, where each neighbour of v in G is connected by an edge to one of the 2-valent vertices of H. The resulting graph will have the same number of hamiltonian cycles as G. In fact $G - v$ is a tup. More generally, when we consider two tups H and H' and join their 2-valent vertices by edges using a bijection, we say that we *merge* H and H' and write $H \circ H'$ for the graph we obtain. When a vertex v and a tup H are merged, we are referring to the graph $H \circ v = (V(H) \cup \{v\}, E(H) \cup \{vx_1, vx_2, vx_3\})$, where x_1, x_2, x_3 are the 2-valent vertices of H.

Chia and Yu [10] proved that a cubic graph G has exactly three hamiltonian cycles if and only if G is the merger of two tups, or of a vertex and a tup. We shall use the family of *generalised Petersen graphs*

$$\mathrm{GP}(n,k) = \left(\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}, \bigcup_{i=0}^{n-1} \{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\} \right),$$

with indices mod n and $k < n/2$.

Schwenk [25] proved that $h(\mathrm{GP}(n,2)) = 3$ if and only if $n \equiv 3$ mod 6. These graphs have girth 5, so they yield an infinite family of triangle-free cubic graphs with exactly three hamiltonian cycles. The question now is whether there are other such graphs—the answer is *yes, but no small ones*:

Theorem 3.13. *The only triangle-free cubic graphs of order at most* 32 *with exactly three hamiltonian cycles are* $\mathrm{GP}(9,2)$ *and* $\mathrm{GP}(15,2)$. *However, for every* $k \geq 19$ *there exists a cubic graph of order* $2k$ *and girth* 4 *having exactly three hamiltonian cycles, and for every* $\ell \geq 17$ *there exists a cubic graph of order* 2ℓ *and girth* 5 *with exactly three hamiltonian cycles.*

Proof. For the first statement we ran the program *snarkhunter* [7, 8] to generate all cubic graphs up to 32 vertices and used a separate program to count the number of hamiltonian cycles of the generated graphs. This allowed us to determine all cubic graphs with exactly 0, 3, or > 3 hamiltonian cycles up to 32 vertices, see Tables 7–9 in the Appendix.

We now show the third statement. For $k \equiv 3$ mod 6 we know that $\mathrm{GP}(k,2)$ is a cubic graph of order $2k$ and girth 5 which has exactly three hamiltonian cycles. For a graph G we denote with G^* the removal of an arbitrary vertex of G. Removing a vertex v from $\mathrm{GP}(9,2)$ yields the tup G_1 of order 17 with 2-valent vertices x, y, z. Moreover, consider the tups

$$G_2 = (V(G_1) \cup \{v_1, v_2\}, E(G_1) \cup \{xv_1, v_1v_2, v_2y\}),$$

$$G_3 = (V(G_2) \cup \{v_3, v_4\}, E(G_2) \cup \{v_2v_3, v_3v_4, v_4z\}),$$

and

$$G_4 = (V(G_3) \cup \{v_5, v_6\}, E(G_3) \cup \{v_4v_5, v_5v_6, v_6v_1\}).$$

These graphs have order 19, 21, and 23, respectively.

Merging the tup G_1 with the tups G_1, G_2, G_3, G_4 yields graphs of order $34, 36, 38, 40$. For order 42, consider $\mathrm{GP}(21, 2)$. For order 44, merge G_3 and G_4 carefully (to ensure that the

17

resulting graph has girth 5): if v_1, v_3, v_4 are the exceptional 2-valent vertices of G_3 and v_3', v_5', v_6' are the exceptional 2-valent vertices of G_4, then join v_1 with v_6', v_3 with v_3', and v_4 with v_5'. For order 46, merge two copies G_4: if v_3, v_5, v_6 are the exceptional 2-valent vertices of the first copy and v_3', v_5', v_6' are the exceptional 2-valent vertices of the second copy, then join v_3 with v_5', v_5 with v_3', and v_6 with v_6'. For order 48, consider $GP(15, 2)$, remove a vertex, and apply the same operation as above with which we obtained G_2 from G_1. We obtain a 31-vertex tup which we merge with G_1, which has order 17, yielding our desired graph. We have described cubic graphs T_n of girth 5 containing exactly three hamiltonian cycles—the straightforward verification of the details, in particular concerning the girth requirement, are left to the reader—of all even orders n in $34, \ldots, 48$.

For orders ≥ 50, consider

$$\{GP(k, 2)^* \circ T_n^* : k \equiv 3 \bmod 6, \ n \in \{34, 36, 38, 40, 42, 44\}\}.$$

Similar techniques yield the theorem's second statement. $\qquad\square$

It remains an open question whether there exist cubic graphs of girth 4 with exactly three hamiltonian cycles that have order 34 or 36.

Using the program *snarkhunter* [7, 8] we also verified that there are no cubic graphs with exactly three hamiltonian cycles of girth 6 up to order 36, none of girth 7 up to order 40, none of girth 8 up to order 46, and none of girth 9 up to order 64.

Theorem 3.14. *Let G be a graph in which all vertices have odd degree, and containing exactly p hamiltonian cycles, where p is a prime number. Then G is 3-connected.*

Proof. Since $p \geq 2$, G is hamiltonian, so it must be 2-connected. Suppose G has connectivity 2. Then it contains a 2-cut $X = \{x, y\}$ whose removal from G yields exactly two components C, C'. (If more than two components are present, we obtain a contradiction to the hamiltonicity of G by a simple toughness argument.) Denote the degrees of x and y in $F = G[V(C) \cup X]$ by d_x and d_y, respectively. We can assume that either F or $G[V(C') \cup X]$, say F, contains exactly one hamiltonian xy-path: if they would both contain more than one such path, say $k \geq 2$ and $k' \geq 2$ respectively, then G would contain kk' hamiltonian cycles, but this is impossible as G contains a prime number of hamiltonian cycles. Three situations can occur:

CASE 1. *Both d_x and d_y are even.* Consider the graph

$$G_1 = (V(F) \cup \{v_1, v_2, v_3, v_4\}, E(F) \cup \{xv_1, v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_4y\}).$$

All vertices in G_1 have odd degree and G_1 contains exactly two hamiltonian cycles, as F contains exactly one hamiltonian xy-path and $G_1[\{x, y, v_i\}_{i=1}^4]$ contains exactly two hamiltonian xy-paths.

CASE 2. *d_x is even while d_y is odd.* Consider a copy F' of $F - xy$ and denote the copy of x (y) in F' by x' (y'). In $(F - xy) \cup F'$, identify x with y' and y with x'. Denote the graph we obtain by G_2. All vertices in G_2 have odd degree and G_2 is uniquely hamiltonian.

CASE 3. *Both d_x and d_y are odd.* Let

$$G_3 = (V(F) \cup \{v_1, v_2\}, E(F) \cup \{xv_1, xv_2, v_1v_2, v_1y, v_2y\}).$$

18

The graph G_3 contains only vertices of odd degree and G_3 contains exactly two hamiltonian cycles.

Each of the above cases leads to a contradiction, since by Thomason's theorem a hamiltonian graph in which all vertices are of odd degree contains at least three hamiltonian cycles. $\qquad\square$

Conjecture (Cantoni [35]). *Every cubic planar graph with exactly three hamiltonian cycles contains a triangle.* ($\mathfrak{C}$)

Every cubic planar graph G with exactly three hamiltonian cycles has connectivity 3 by above theorem. Furthermore, every vertex-deleted subgraph of G must be hamiltonian—in particular, G cannot be bipartite—by Thomason's Corollary 1.5 in [30] which states that in any cubic graph H we have $h(H - v) = h(H) \bmod 2$ for every vertex v in H.

It was shown by Fowler (see page 30 of [13]) that if G is a cubic planar graph with exactly three hamiltonian cycles, then the statement "G contains a triangle", i.e. ($\mathfrak{C}$), is equivalent to the statement "G is uniquely edge-3-colourable." For further definitions and details we refer to Fowler's thesis.

Using the program *plantri* [9] we generated all planar 3-connected triangle-free cubic graphs up to 48 vertices and tested if any such graph contains exactly three hamiltonian cycles. This resulted in the following observation.

Observation 3.15. *Cantoni's conjecture ($\mathfrak{C}$) is true up to at least 48 vertices. Furthermore, we also verified that there are no planar 3-connected cubic graphs of girth 5 with exactly three hamiltonian cycles up to at least 62 vertices.*

3.3. Uniquely traceable graphs

Motivated by results on uniquely hamiltonian graphs, we now study a natural variation of the concept and call a graph *uniquely traceable* if it contains exactly one hamiltonian path. Sheehan's [28, Theorem 1] states that a uniquely hamiltonian graph of order $n \geq 3$ has size at most $\lfloor \frac{n^2}{4} \rfloor + 1$. This upper bound is sharp as Sheehan constructed for every $n \geq 3$ an n-vertex uniquely hamiltonian graph of maximum size. He claims in [28, Theorem 2] that these are the only graphs reaching the bound. However, Barefoot and Entringer showed that for $n \geq 7$ there are exactly $2^{\lceil \frac{n}{2} \rceil - 4}$ such graphs [2], so for uniquely hamiltonian graphs of order at most 8 Sheehan's Theorem 2 does hold, but for $n \geq 9$ it does not. Sheehan writes [28]: "An analogous result to Theorem 1 can easily be obtained when instead of Hamiltonian circuits we consider Hamiltonian paths. In this case $h(n)$ [the maximum size of an n-vertex uniquely hamiltonian graph] is replaced by $[n^2/4] - [n/2] + 1$, and the extremal graph is again unique." In the following we shall confirm Sheehan's size bound, but disprove his uniqueness claim.

Observation 3.16. *Let G be a uniquely traceable graph. Then G has exactly two vertices of degree 1, namely the starting point and endpoint of the hamiltonian path. In particular, G has connectivity 1. Furthermore, identifying the endpoints of the hamiltonian path of G, we obtain a uniquely hamiltonian graph. Conversely, if a uniquely hamiltonian graph contains a vertex of degree 2, then splitting this vertex into two vertices of degree 1 yields a uniquely traceable graph.*

Proof. We denote the unique hamiltonian path of G by $v_1 \ldots v_n$. Assume that the degree of v_1 is at least 2. Then v_1 is adjacent to v_i for some $i \geq 3$. Thus $v_{i-1}v_{i-2} \ldots v_1 v_i v_{i+1} \ldots v_n$ is a second hamiltonian path in G, a contradiction. So v_1 and analogously v_n must have degree 1.

Denote the graph resulting from the identification of v_1 and v_n by G', and let v be the vertex in G' obtained by identifying v_1 and v_n. Combining that $G' - v$ contains exactly one hamiltonian $v_2 v_{n-1}$-path (otherwise G would not be uniquely traceable) with the fact that, due to the argument above, v has degree 2, the graph G' is uniquely hamiltonian. The converse argument is very similar. $\qquad\square$

Theorem 3.17. *Let $n \geq 2$. Then the following hold.*

1. *A uniquely traceable graph of order n has at most $s(n) = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ edges.*

2. *For every m such that $n - 1 \leq m \leq s(n)$ there exists a uniquely traceable graph of order n and size m.*

3. *There exist exactly $\max \left\{ 1, 2^{\lceil \frac{n-1}{2} \rceil - 3} \right\}$ uniquely traceable graphs of order n and size $s(n)$.*

Proof. Barefoot and Entringer [2] showed that for every $n \geq 7$ there exist uniquely hamiltonian graphs of order n and size $s(n + 1)$, and no such graphs of greater size. Statement 1 now follows: for $n < 8$ it is elementary to verify, so consider henceforth $n \geq 8$. Assume that there exists an n-vertex uniquely traceable graph G of size greater than $s(n)$. Using Observation 3.16, if we identify the 1-valent vertices of G, we obtain a uniquely hamiltonian graph of order $n - 1$ and size greater than $s(n)$. This however contradicts the result of Barefoot and Entringer.

They also describe the following crucial property of uniquely hamiltonian graphs of order n and size $s(n+1)$: there is always a 5-vertex subpath of the hamiltonian cycle such that the degrees of these vertices are $2, n-1, 2, n-2, 3$, respectively, and all other vertices have degree more than 2 and less than $n - 2$. Thus, if we use the two vertices of degree 2 to produce uniquely traceable graphs (see Observation 3.16), we obtain two non-isomorphic graphs by a valency argument. Two uniquely traceable graphs that are constructed from non-isomorphic uniquely hamiltonian graphs of order n and size $s(n+1)$ must themselves be non-isomorphic. Barefoot and Entringer showed that there are exactly $2^{\lceil \frac{n}{2} \rceil - 4}$ uniquely hamiltonian graphs of order n and maximum size. We thus obtain $t(n + 1) = 2 \cdot 2^{\lceil \frac{n}{2} \rceil - 4}$ uniquely traceable graphs of order $n + 1$ and maximum size. Consider such a uniquely traceable graph G of maximum size. In G, we call edges not belonging to its unique hamiltonian path *chords*. Removing chords from G one-by-one yields statement 2.

For statement 3, suppose that there are more than $t(n)$ uniquely traceable graphs of order n. This means that there exists a uniquely traceable graph G of order n and size $s(n)$ which was not obtained from an $(n-1)$-vertex uniquely hamiltonian graph G' of size $s(n)$ by splitting a vertex of degree 2 into two 1-valent vertices. However, identifying in G its 1-valent vertices, by Observation 3.16 we obtain a uniquely hamiltonian graph of order $n - 1$ and, since identifying did not change the size, exactly $s(n)$ edges, a contradiction. Hence, each uniquely traceable graph of maximum size can be constructed from a uniquely hamiltonian

graph of maximum size, and thus we find all uniquely traceable graphs of order $n \geq 8$ and maximum size from the uniquely hamiltonian graphs of order $n - 1$.

Finally, for $n \leq 7$ the unique uniquely traceable graphs of order n and size $s(n)$ are the paths for $n \leq 4$ and the graphs shown in Figure 5 for $n \in \{5, 6, 7\}$. (These can be obtained from the unique uniquely hamiltonian graphs of order at most 6 and of maximum size.) □
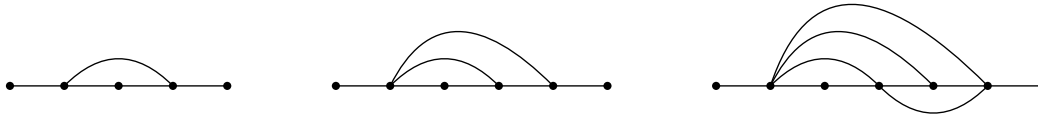

Figure 5: The unique uniquely traceable graphs of maximum size and order 5, 6, and 7.

3.4. Remarks on computational results

We implemented the algorithm from Section 2 in C. Our program is called *generateUHG* and its source code can be downloaded from [14]. We used this program to generate complete lists of all non-isomorphic graphs with exactly k hamiltonian cycles of a given order (and a given lower bound on the girth) for various values of k. The main computational results for uniquely hamiltonian graphs are listed in this section. (The counts of planar uniquely hamiltonian graphs were already reported in Table 1 from Section 3.1 in the context of the Bondy-Jackson conjecture.) Additional tables with counts of graphs with exactly k hamiltonian cycles for $k \neq 1$ and counts for cubic graphs can be found in the Appendix.

Table 4 lists the counts of uniquely hamiltonian graphs with girth at least 3, 4 and 5. The running times of our program are reported in Table 5. The code was compiled using gcc and was performed on Intel Xeon E5-2680 CPU's at 2.60GHz. The running times for the larger orders include a small overhead due to parallelisation. The total computational effort for this project amounted to 40 CPU years and the most time-consuming computations were the generation of all uniquely hamiltonian graphs on 15 vertices and the test if any cubic graphs on 32 vertices contain exactly three hamiltonian cycles (which required 11 and 20 CPU years, respectively).

We also compared the running times of our program *generateUHG* to the approach of using the program *geng* [19, 21] to generate all graphs and then filtering the uniquely hamiltonian graphs (which was, as far as we are aware, up until now the only available method in the literature to generate all uniquely hamiltonian graphs of a given order). Our program is about 200 times faster than the filter approach for generating uniquely hamiltonian graphs of order 11. For order 12 it is 600 times faster than the filter approach and for order 13 about 3000 times faster.

The graphs from Tables 1 and 4 can be downloaded from the *House of Graphs* [6] at `http://hog.grinvin.org/UHG`.

Correctness testing

It is important to independently verify computational results to minimise the chance of programming errors. The counts of all (connected) non-hamiltonian graphs up to 12 vertices

Order	# UH graphs	girth ≥ 4	girth ≥ 5
3	1	0	0
4	2	1	0
5	3	1	1
6	12	2	1
7	49	3	1
8	482	11	3
9	6 380	38	4
10	135 252	250	10
11	3 939 509	2 171	32
12	166 800 470	25 518	167
13	9 739 584 172	388 854	899
14	818 717 312 364	7 283 110	6 470
15	95 353 226 103 276	171 355 621	55 815
16	?	4 915 591 680	549 981
17	?	174 203 813 967	6 155 795
18	?	7 526 329 299 531	78 520 177
19	?	?	1 123 544 810
20	?	?	18 005 054 988
21	?	?	322 434 738 089
22	?	?	6 427 598 615 569

Table 4: The number of uniquely hamiltonian graphs with girth at least 3, 4 and 5, respectively.

Order	time UH graphs	time girth ≥ 4	time girth ≥ 5
10	< 1	< 1	< 1
11	8.2	< 1	< 1
12	383	< 1	< 1
13	25 944	1	< 1
14	2 487 313	20.2	< 1
15	358 436 755	527	< 1
16		19 375	1.7
17		739 167	21
18		36 974 877	292
19			5 242
20			89 701
21			1 722 169
22			34 513 677

Table 5: Running time (in seconds) of our algorithm for the generation of uniquely hamiltonian graphs with girth at least 3, 4 and 5, respectively.

were already on the On-Line Encyclopedia of Integer Sequences [29] (i.e. sequence A126149) and are in complete agreement with our results from Table 6.

We also used the program *geng* [19, 21] to generate all graphs up to 13 vertices and used

a separate program to count the number of hamiltonian cycles of the generated graphs (cf. Table 6 in the Appendix). Also here the results were in complete agreement with the counts we obtained using our generator for graphs with exactly k hamiltonian cycles. Furthermore we also used *geng* to compute all uniquely hamiltonian graphs of girth at least 4 up to 16 vertices and all uniquely hamiltonian graphs of girth at least 5 up to 20 vertices. Also these counts were in complete agreement with the results from Table 4.

Similarly, we used the program *snarkhunter* [8] to generate all cubic graphs up to 32 vertices and used a separate program to count the number of hamiltonian cycles of the generated graphs (cf. Table 7). By restricting the maximum degree to three in our generator for graphs with k hamiltonian cycles, we were able to verify these results up to 20 vertices.

Acknowledgements

References

[1] S. Abbasi and A. Jamshed. A degree constraint for uniquely hamiltonian graphs. *Graphs and Combinatorics*, 22(4):433–442, 2006.

[2] C.A. Barefoot and R.C. Entringer. A census of maximum uniquely Hamiltonian graphs. *Journal of Graph Theory*, 5(3):315–321, 1981.

[3] C. Berge. *Graphs and hypergraphs*. North-Holland Publishing Company, 1973.

[4] J.A. Bondy and B. Jackson. Vertices of small degree in uniquely hamiltonian graphs. *Journal of Combinatorial Theory, Series B*, 74(2):265–275, 1998.

[5] J.M. Boyer and W.J. Myrvold. On the Cutting Edge: Simplified $O(n)$ Planarity by Edge Addition. *Journal of Graph Algorithms and Applications*, 8(2):241–273, 2004.

[6] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot. House of Graphs: a database of interesting graphs. *Discrete Applied Mathematics*, 161(1-2):311–314, 2013. Available at http://hog.grinvin.org/.

[7] G. Brinkmann and J. Goedgebeur. Generation of cubic graphs and snarks with large girth. *Journal of Graph Theory*, 86(2):255–272, 2017.

[8] G. Brinkmann, J. Goedgebeur, and B.D. McKay. Generation of cubic graphs. *Discrete Mathematics and Theoretical Computer Science*, 13(2):69–80, 2011.

[9] G. Brinkmann and B. D. McKay. Fast generation of planar graphs. *MATCH Commun. Math. Comput. Chem.*, 58(2):323–357, 2007.

[10] G. Chia and Q.R. Yu. On the number of hamilton cycles in cubic graphs. *Congressus Numerantium*, pages 13–32, 1995.

[11] R.C. Entringer and H. Swart. Spanning cycles of nearly cubic graphs. *Journal of Combinatorial Theory, Series B*, 29(3):303–309, 1980.

[12] H. Fleischner. Uniquely Hamiltonian graphs of minimum degree 4. *Journal of Graph Theory*, 75(2):167–177, 2014.

[13] T.G. Fowler. *Unique Coloring of Planar Graphs*. PhD thesis, Georgia Institute of Technology, 1998.

[14] J. Goedgebeur, B. Meersman, and C. T. Zamfirescu. Homepage of genhypohamiltonian: `http://caagt.ugent.be/uhg/`.

[15] E. Grinberg. Three-connected graphs with exactly one Hamiltonian cycle (in Russian). *Republican Foundation of Algorithms and Programmes. Computing centre, P. Stutschka University, Riga, USSR*, 1986.

[16] P. Haxell, B. Seamone, and J. Verstraete. Independent dominating sets and Hamiltonian cycles. *Journal of Graph Theory*, 54(3):233–244, 2007.

[17] M. Haythorpe. On the minimum number of hamiltonian cycles in regular graphs. *Experimental mathematics*, pages 1–5, 2017.

[18] D. Holton and R.E.L. Aldred. Planar graphs, regular graphs, bipartite graphs and hamiltonicity. *Australasian Journal of Combinatorics*, 20:111–132, 1999.

[19] B.D. McKay. nauty User's Guide (Version 2.5). Technical Report TR-CS-90-02, Department of Computer Science, Australian National University. The latest version of the software is available at `http://cs.anu.edu.au/~bdm/nauty`.

[20] B.D. McKay. Isomorph-free exhaustive generation. *Journal of Algorithms*, 26(2):306–324, 1998.

[21] B.D. McKay and A. Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60:94–112, 2014.

[22] M. Meringer. Fast generation of regular graphs and construction of cages. *Journal of Graph Theory*, 30(2):137–146, 1999.

[23] J. Petersen. Die theorie der regulären graphs. *Acta Mathematica*, 15(1):193, 1891.

[24] G.F. Royle. The smallest uniquely hamiltonian graph with minimum degree at least 3. `https://mathoverflow.net/questions/255784/what-is-the-smallest-uniquely-hamiltonian-graph-with-minimum-degree-at-least-3/`, 2017.

[25] A.J. Schwenk. Enumeration of hamiltonian cycles in certain generalized petersen graphs. *Journal of Combinatorial Theory, Series B*, 47(1):53–59, 1989.

[26] B. Seamone. On uniquely hamiltonian claw-free and triangle-free graphs. *Discussiones Mathematicae Graph Theory*, 35(2):207–214, 2015.

[27] J. Sheehan. The multiplicity of hamiltonian circuits in a graph. *Recent advances in graph theory*, pages 477–480, 1975.

[28] J. Sheehan. Graphs with exactly one hamiltonian circuit. *Journal of Graph Theory*, 1(1):37–43, 1977.

[29] N. Sloane. The on-line encyclopedia of integer sequences: `http://oeis.org/`.

[30] A.G. Thomason. Hamiltonian cycles and uniquely edge colourable graphs. In *Annals of Discrete Mathematics*, volume 3, pages 259–268. Elsevier, 1978.

[31] C. Thomassen. Planar cubic hypohamiltonian and hypotraceable graphs. *Journal of Combinatorial Theory, Series B*, 30(1):36–44, 1981.

[32] C. Thomassen. On the number of hamiltonian cycles in bipartite graphs. *Combinatorics, Probability and Computing*, 5(4):437–442, 1996.

[33] C. Thomassen. Independent dominating sets and a second hamiltonian cycle in regular graphs. *Journal of Combinatorial Theory, Series B*, 72(1):104–109, 1998.

[34] W.T. Tutte. On hamiltonian circuits. *Journal of the London Mathematical Society*, 1(2):98–101, 1946.

[35] W.T. Tutte. Hamiltonian circuits. *Colloquio Internazionale sulle Teorie Combinatorie. Atti dei Convegni Lincei 17, Accad. Naz. Lincei, Roma I*, pages 193–199, 1976.

Appendix

Order	0	1	2	3	> 3
2	1	0	0	0	0
3	1	1	0	0	0
4	3	2	0	1	0
5	13	3	2	0	3
6	64	12	11	3	22
7	470	49	75	17	242
8	4 921	482	740	283	4 691
9	83 997	6 380	10 692	5 069	154 942
10	2 411 453	135 252	229 068	132 345	8 808 453
11	123 544 541	3 939 509	7 005 022	4 451 059	867 760 434
12	11 537 642 646	166 800 470	305 866 545	209 875 768	151 839 645 047
13	2 013 389 528 672	9 739 584 172	18 868 736 922	13 364 007 134	48 280 546 012 319
14	?	818 717 312 364	?	?	?
15	?	95 353 226 103 276	?	?	?

Table 6: Number of connected graphs which contain exactly 0, 1, 2, 3 or > 3 hamiltonian cycles, respectively.

Order	0	3	> 3
4	0	1	0
6	0	1	1
8	0	1	4
10	2	3	14
12	5	7	73
14	35	24	450
16	219	93	3 748
18	1 666	435	39 200
20	14 498	2 112	493 879
22	148 790	11 019	7 159 638
24	1 768 732	58 833	116 112 970
26	24 029 714	322 733	2 070 128 417
28	366 939 032	1 799 413	40 128 399 566
30	6 213 299 362	10 185 443	839 256 743 264
32	115 388 854 837	58 344 442	18 826 074 985 311

Table 7: Number of connected cubic graphs which contain exactly 0, 3 or > 3 hamiltonian cycles, respectively.

Order	0	3	> 3
4	0	0	0
6	0	0	1
8	0	0	2
10	1	0	5
12	0	0	22
14	2	0	108
16	8	0	784
18	59	1	7 745
20	425	0	97 121
22	3 862	0	1 431 858
24	41 293	0	23 739 521
26	518 159	0	432 239 409
28	7 398 734	0	8 535 072 760
30	117 963 348	1	181 374 174 463
32	2 069 516 990	0	4 125 007 626 872

Table 8: Number of connected cubic graphs with girth at least 4 which contain exactly 0, 3 or > 3 hamiltonian cycles, respectively.

Order	0	3	> 3
4	0	0	0
6	0	0	0
8	0	0	0
10	1	0	0
12	0	0	2
14	0	0	9
16	0	0	49
18	3	1	451
20	15	0	5 768
22	110	0	90 828
24	1 130	0	1 619 349
26	15 444	0	31 463 140
28	239 126	0	656 544 764
30	4 073 824	1	14 617 797 379
32	75 458 941	0	345 900 189 621

Table 9: Number of connected cubic graphs with girth at least 5 which contain exactly 0, 3 or > 3 hamiltonian cycles, respectively.
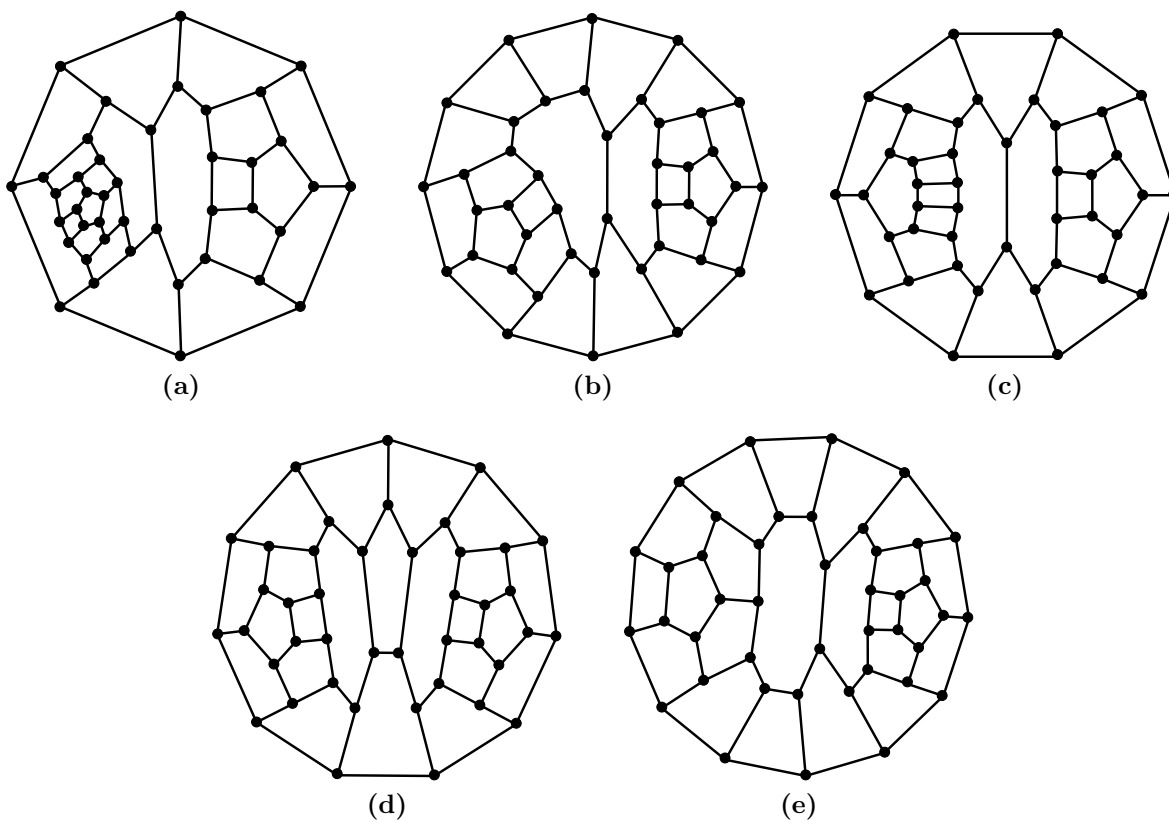

Figure 6: The five planar cyclically 4-edge-connected cubic graphs with exactly four hamiltonian cycles on 42 vertices from Observation 1.