PRIMARY PSEUDOPERFECT NUMBERS, ARITHMETIC PROGRESSIONS, AND THE ERDŐS-MOSER EQUATION

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Abstract. A *primary pseudoperfect number* (PPN) is an integer $K > 1$ satisfying the equation
$$\frac{1}{K} + \sum_{p\,|\,K} \frac{1}{p} = 1,$$
where p denotes a prime. PPNs arise in studying perfectly weighted graphs and singularities of algebraic surfaces, and are related to Sylvester's sequence, Giuga numbers, Znám's problem, the inheritance problem, and Curtiss's bound on solutions of a unit fraction equation.

Here we show $K \equiv 6 \pmod{6^2}$ if $6 \mid K$, and uncover a remarkable 7-term arithmetic progression of residues modulo $6^2 \cdot 8$ in the sequence of known PPNs. On that basis, we pose a conjecture which leads to a conditional proof of the new record lower bound $k > 10^{3.99\times 10^{20}}$ on any non-trivial solution to the Erdős-Moser Diophantine equation $1^n + 2^n + \cdots + k^n = (k+1)^n$.

1. INTRODUCTION.

In 1922 Curtiss [10] proved Kellogg's [15] conjectured bound on solutions to a *unit fraction* equation
$$\sum_{i=1}^{n} \frac{1}{x_i} = 1 \quad \Longrightarrow \quad \max_{1\,\le\, i\,\le n} x_i \le S_n - 1 \tag{1}$$
where *Sylvester's sequence* [1, 25, 27], [22, A000058],
$$S_n = 2, 3, 7, 43, 1807, 3263443, 10650056950807, 113423713055421844361000443, \dots, \tag{2}$$
is defined by the recurrence $S_n = S_1 S_2 \cdots S_{n-1} + 1$, with $S_1 = 2$.

The equation in (1) also appears in finite group theory. Suppose we have a finite group G, and assume it has conjugacy classes $C_1, \dots, C_n$. The number of elements of C_i divides the order N of G, so we can write $\#C_i = N/m_i$ with m_i an integer and
$$N = \#C_1 + \cdots + \#C_n = \frac{N}{m_1} + \cdots + \frac{N}{m_n}.$$
It follows that $1 = \sum_i 1/m_i$. Curtiss's result now says that the number of groups with a prescribed number n of conjugacy classes is finite. For more on this, see Landau [16] or Lenstra [17].

The present article is concerned with the particular unit fraction equation
$$\frac{1}{K} + \sum_{p\,|\,K} \frac{1}{p} = 1. \tag{3}$$
Here and throughout the paper, p denotes a prime. Equation (3) is related to perfectly weighted graphs [8] and singularities of algebraic surfaces [6]. The companion equation
$$\frac{-1}{L} + \sum_{p\,|\,L} \frac{1}{p} = 1$$

occurs in the study of *Giuga numbers* [4, 24], [13, A17], [22, A007850], and a generalization of (3),

$$\prod_{i=1}^{r} \frac{1}{x_i} + \sum_{i=1}^{r} \frac{1}{x_i} = 1,$$

arises in *Znám's problem* [7, 9], [22, A075461] and *the inheritance problem* [1]. See also [2] for recent work on the equation in (1).

In Section 2, we summarize the known facts about solutions to the unit fraction equation (3). In Section 3, we reduce the solutions modulo 288 and uncover a remarkable 7-term arithmetic progression of residues, leading to two conjectures. In the final section, we relate solutions of (3) to possible solutions of the *Erdős-Moser Diophantine equation*

$$1^n + 2^n + \cdots + (k-1)^n + k^n = (k+1)^n. \tag{4}$$

Assuming a weak form of one of our conjectures, we give a conditional proof of a new record lower bound on any non-trivial solution of (4).

2. PRIMARY PSEUDOPERFECT NUMBERS.

Recall that a positive integer is called *perfect* if it is the sum of *all* of its proper divisors, and *pseudoperfect* if it is the sum of *some* of its proper divisors [13, B1, B2], [22, A000396, A005835].

Definition 1 (Butske, Jaje, and Mayernik [8]). A *primary pseudoperfect number* (PPN for short) is an integer $K > 1$ that satisfies the unit fraction equation (3). See [20, 26, 27] and [22, A054377]. Note that, just as 1 is not a prime number, so too 1 is not a PPN.

Multiplying equation (3) by K gives the equivalent integer condition

$$1 + \sum_{p \mid K} \frac{K}{p} = K. \tag{5}$$

For example, $42 = 2 \cdot 3 \cdot 7$ is a PPN, because $42/2 = 21,\ 42/3 = 14,\ 42/7 = 6$, and $1+21+14+6 = 42$. From (5), we see that *all PPNs are square-free*, and that *every PPN except* 2 *is pseudoperfect*. As with perfect numbers, it is unknown whether there are infinitely many PPNs or any odd ones.

Notation. For an integer $r \geq 1$, we denote by K_r any PPN with exactly r (distinct) prime factors.

Remarkably, there exists precisely one K_r for each positive integer $r \leq 8$. This was conjectured by Ke and Sun [14] and Cao, Liu, and Zhang [9], and then verified in [8] (see also Anne [1]) using computational search techniques. Table 1 lists all known PPNs and their prime factors.

r	K_r	Prime Factorization
1	2	2
2	6	$2 \cdot 3$
3	42	$2 \cdot 3 \cdot 7$
4	1806	$2 \cdot 3 \cdot 7 \cdot 43$
5	47058	$2 \cdot 3 \cdot 11 \cdot 23 \cdot 31$
6	2214502422	$2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \cdot 47059$
7	52495396602	$2 \cdot 3 \cdot 11 \cdot 17 \cdot 101 \cdot 149 \cdot 3109$
8	8490421583559688410706771261086	$2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \cdot 47059 \cdot 2217342227 \cdot 1729101023519$

TABLE 1. The primary pseudoperfect numbers with $r \leq 8$ prime factors

Here are five related observations on Table 1 and Sylvester's sequence (2).

(a). $K_1 = 2$, $K_2 = 2 \cdot 3 = 6$, $K_3 = 6 \cdot 7 = 42$, and $K_4 = 42 \cdot 43 = 1806$, but $K_5 \neq 1806 \cdot 1807$.
(b). $K_5 = 47058$ and $K_6 = 47058 \cdot 47059 = 2214502422$, but $K_7 \neq 2214502422 \cdot 2214502423$.
(c). $K_6 = 2214502422$ and $K_8 = 2214502422 \cdot 2217342227 \cdot 1729101023519$.

(d). $K_1, K_2, K_3, K_4 = 2, 6, 42, 1806$ are each 1 less than the terms $S_2, S_3, S_4, S_5 = 3, 7, 43, 1807$.
(e). $K_r < S_{r+1}$, for $r = 1, 2, \ldots, 8$.

These patterns can all be explained.

Proposition 1. *For any integer K, set $K' := K(K+1)$.*

(i). *Assume that $K+1$ is prime. Then K is a PPN if and only if K' is also a PPN.*
(ii). *Assume that we can factor $K^2+1 = (p-K)(q-K)$, for some primes $p > K$ and $q > K$. Then K is a PPN if and only if $K \cdot p \cdot q$ is also a PPN.*
(iii). *If $K+1 = S_n$ is a term in Sylvester's sequence, then $K'+1 = S_{n+1}$ is the next term in it.*
(iv). *The inequality $K_r \le S_{r+1} - 1$ holds for* any *PPN with $r \ge 1$ prime factors.*

Proof. (i). This follows easily from Definition 1 and the relation $\frac{1}{K'} = \frac{1}{K} - \frac{1}{K+1}$.
(ii). The proof is similar; for details, see Brenton and Hill's more general Proposition 12 in [6], as well as [1, Lemma 2] and [8, Lemma 4.1].
(iii). Sylvester's sequence satisfies $S_{n+1} = (S_n - 1)S_n + 1$. Setting $S_n = K+1$ gives (iii).
(iv). This follows directly from Curtiss's bound (1). □

Now, as $3, 7, 43, 47059$ are prime, but $1807 = 13 \cdot 139$ and $2214502423 = 7^2 \cdot 45193927$ are composite, and as the numbers 2217342227 and 1729101023519 in the factorization

$$2214502422^2 + 1 = (2217342227 - 2214502422)(1729101023519 - 2214502422)$$

are prime, the observations (a), (b), (c), (d), and (e) are explained.

Analogs of (i) and (ii) for $K-1$ and K^2-1, involving PPNs and Giuga numbers, are given in [24, Theorem 8].

3. PPNs AND ARITHMETIC PROGRESSIONS.

According to Table 1, the PPNs having $r = 2, 3, 4, 5, 6, 7, 8$ prime factors, *i.e.,*

$$K_r = 6, 42, 1806, 47058, 2214502422, 52495396602, 8490421583559688410706771261086,$$

are all multiples of $2 \cdot 3 = 6$:

$$\frac{K_r}{6} = 1, 7, 301, 7843, 369083737, 8749232767, 1415070263926614735117795210181.$$

Proposition 2. *Let K be* any *PPN divisible by* 6. *Then $K \equiv 6 \pmod{6^2}$.*

Proof. Denote by μ (≥ 0) the number of prime factors of K congruent to -1 modulo 6. Since $6 \mid K$ and K is square-free, $\frac{K}{6} \equiv (-1)^\mu \pmod 6$. Now, reducing equation (5) modulo 6 gives

$$1 + \frac{K}{2} + \frac{K}{3} + \sum_{3<p\,|\,K} \frac{K}{p} = K \implies 1 + 3(-1)^\mu + 2(-1)^\mu \equiv 0 \pmod 6 \tag{6}$$

and hence μ is even. This proves the proposition. □

In particular, for $r = 2, 3, 4, 5, 6, 7, 8$ we find respectively that

$$\frac{K_r - 6}{6^2} = 0, 1, 50, 1307, 61513956, 1458205461, 235845043987769122519632535030.$$

Let us write $N \,(\mathrm{mod}\ M) = R$ if the remainder upon division of N by M is R, so that both the congruence $N \equiv R \pmod M$ and the inequalities $0 \le R < M$ hold. In light of Proposition 2 and the values $(K_2, K_3) = (6, 42)$, one might predict that if we divide $K_2, \ldots, K_8$ by some number M, the remainders will form the arithmetic progression (AP for short)

$$K_r \,(\mathrm{mod}\ M) = 6, 42, 78, 114, 150, 186, 222, \text{ for } r = 2, 3, 4, 5, 6, 7, 8, \tag{7}$$

respectively. This requires M to exceed 222 and to divide each of the differences

$$\begin{aligned}
1806 - 78 = 1728 &= 2^6 \cdot 3^3, \\
47058 - 114 = 46944 &= 2^5 \cdot 3^2 \cdot 163, \\
2214502422 - 150 = 2214502272 &= 2^7 \cdot 3^2 \cdot 89 \cdot 21599, \\
52495396602 - 186 = 52495396416 &= 2^6 \cdot 3^2 \cdot 47 \cdot 1939103, \\
8490421583559688410706771261086 - 222 &= 8490421583559688410706771260864 \\
&= 2^6 \cdot 3^2 \cdot 338293 \cdot 4357262860668095873923.
\end{aligned}$$

Since their greatest common divisor is $2^5 \cdot 3^2 = 288 > 222$, and no proper factor of 288 exceeds 222, the choice $M = 288 = 6^2 \cdot 8$ is both necessary and sufficient. This establishes a remarkable property of these PPNs.

Proposition 3. *Upon division of the primary pseudoperfect numbers K_2, K_3, K_4, K_5, K_6, K_7, K_8 by $M = 288$, the remainders form the 7-term arithmetic progression* (7)*, that is,*

$$K_r \pmod{6^2 \cdot 8} = 6 + 6^2(r-2) \text{ for } r = 2,3,4,5,6,7,8. \tag{8}$$

Moreover, no other modulus will do.

Notice that the inequalities

$$6 + 6^2 \cdot (9-2) = 258 < 288 < 294 = 6 + 6^2 \cdot (10-2)$$

hold. Thus, the remainder pattern in (8) might persist for $r = 9$ (assuming that a K_9 exists), but cannot for $r \geq 10$. Throwing caution to the wind, we therefore make the following prediction.

Conjecture 1. There exists exactly one primary pseudoperfect number K_9 with nine prime factors, and $K_9 \pmod{6^2 \cdot 8} = 258$ holds. No further PPNs exist.

Anyone thinking of settling Conjecture 1 by computation should be aware that Curtiss's upper bound for a ninth PPN is $K_9 < S_{10}$, a 106-digit number.

In case all or part of Conjecture 1 fails, we also predict a strengthening of Proposition 2 for all PPNs divisible by 6, including those with more than eight prime factors, if any.

Conjecture 2. For all $r \geq 2$, if $6 \mid K_r$, then $K_r \equiv 6 + 6^2(r-2) \pmod{6^2 \cdot 8}$. Equivalently (by Proposition 2), if $K_r > 2$, then K_r is a multiple of 6 and

$$\frac{K_r - 6}{6^2} \equiv r - 2 \pmod{8}.$$

Note that the case $r = 9$ here is weaker than Conjecture 1. Note also that the quantity $r-2$ equals the number of prime factors of K_r different from 2 and 3. Thus, each such factor conjecturally contributes 1 to $(K_r - 6)/6^2$ modulo 8 in some variant of the relation (6).

Although the modulus $6^2 \cdot 8$ cannot be changed in Proposition 3, other moduli provide interesting APs for subsets of the PPNs. For example, we have APs of complementary subsequences K_2, K_4, K_6, $K_8 \pmod{128} = 6, 14, 22, 30$ and K_3, K_5, $K_7 \pmod{128} = 42, 82, 122$, so that

$$K_r \pmod{2^7} = \begin{cases} 6 + 4(r-2) & \text{for } r = 2,4,6,8, \\ 42 + 20(r-3) & \text{for } r = 3,5,7. \end{cases} \tag{9}$$

Finally, we give a way to generate triples of PPNs congruent modulo $6^3 \cdot 4 = 864$ to 3-term APs.

Proposition 4. *Let K be a PPN such that $K+1$ and K^2+K+1 are prime. Then the products $K' := K(K+1)$ and $K'' := K'(K'+1)$ are also PPNs, and*

$$K \equiv 0 \pmod{6} \implies K, K', K'' \equiv K, K+6^2, K+6^2 \cdot 2 \pmod{6^3 \cdot 4}, \tag{10}$$

respectively.

Proof. Since $K+1$ and $K'+1 = K^2+K+1$ are prime, Proposition 1 part (i) implies that K' and K'' are also PPNs. As $6 \mid K$, Proposition 2 gives $K = 6 + 6^2 n$, for some n. Now, we can write

$$K' - K = K^2 = 6^2 + 6^2 \cdot 4 \cdot 3n(3n+1) \equiv 6^2 \pmod{6^3 \cdot 4},$$

because $3n(3n+1)$ is even. In the same way we get $K'' - K' \equiv 6^2 \pmod{6^3 \cdot 4}$, and (10) follows. □

The only known example of Proposition 4 is with $K = 6$. The primary pseudoperfect numbers K, K', K'' are then $6, 42, 1806$, whose remainders modulo $6^3 \cdot 4$ form the 3-term arithmetic progression $6, 42, 78$. Compare to Proposition 3 for $r = 2, 3, 4$.

It would be interesting to find explanations and extensions to all PPNs, analogous to the statements and proofs of Propositions 1, 2, and 4, for the APs of certain K_r modulo $6^2 \cdot 8$ and 2^7 in (8) and (9), respectively.

4. THE ERDŐS-MOSER CONJECTURE AND A CONDITIONAL RABBIT.

Erdős and Moser (EM for short) studied equation (4) around 1953 and made the following prediction.

Conjecture 3 (EM). The only solution to the EM equation (4) in positive integers is the trivial solution $1^1 + 2^1 = 3^1$.

Moser proved the following result toward Conjecture 3.

Theorem 1 (Moser [19]). *If (k, n) is a non-trivial solution of (4), then $k > 10^{10^6}$.*

This bound was improved to $k > 10^{1.485 \times 9321155}$ in [8], and to $k > 10^{10^9}$ by Gallot, Moree, and Zudilin [12] (see also [5, Chapter 8]). On the other hand, it is not even known whether the number of solutions is finite. See the surveys [13, D7] and [18].

In [23] the authors approximated the EM equation by the *EM congruence*

$$1^n + 2^n + \cdots + (k-1)^n + k^n \equiv (k+1)^n \pmod{k}, \tag{11}$$

as well as by the supercongruence modulo k^2, and proved the following connection with PPNs.

Proposition 5. *The EM congruence (11) holds if and only if the inclusion*

$$\frac{1}{k} + \sum_{p \mid k} \frac{1}{p} \in \mathbb{Z} \tag{12}$$

is true and $p \mid k$ implies $(p-1) \mid n$. In particular, every primary pseudoperfect number K provides a solution $k := K$ to (11) with exponent $n := \operatorname{lcm}\{p - 1 : p \mid K\}$.

Part of this is implicit in [19]: Moser's work shows that (4) implies (12); see [8, p. 409].

In [18] Moree wrote, "In order to improve on [Theorem 1] by Moser's approach one needs to find additional rabbit(s) in the top hat. The interested reader is wished good luck in finding these elusive animals!" Moree's top hat is a von Staudt-Clausen type theorem. Instead, we find a conditional rabbit in a hypothesis weaker than Conjecture 1.

Proposition 6. *If there are no primary pseudoperfect numbers K_r with $r \geq 33$, and if the Erdős-Moser equation (4) has a non-trivial solution (k, n), then $k > 10^{3.99 \times 10^{20}}$.*

Proof. In [12, Section 5.1] it is shown that if (k, n) is a solution of (4) with $n > 1$, then the number of distinct prime factors of k is at least 33. Thus if no K_r exists with $r \geq 33$, then by Proposition 5 the left-hand side of (12) cannot equal 1 and so, being a positive integer, must be ≥ 2. In the analysis of Moser's proof, this leads now to the inequality

$$\frac{1}{m-1} + \frac{2}{m+1} + \frac{2}{2m-1} + \frac{4}{2m+1} + \sum_{p \mid M} \frac{1}{p} \geq 4\frac{1}{6} \tag{13}$$

(instead of $\geq 3\frac{1}{6}$ as in [18, equation (14)] and [19, equation (19)]), where $m-1=k$ and $M=(m^2-1)(4m^2-1)/12$. Now, $m-1=k>2^{33}>8\times 10^9$ and so (13) implies

$$\sum_{p\,|\,M}\frac{1}{p}>4.166666. \tag{14}$$

From (14) *it follows that* $M>\prod_{p\leq x}p$ *if* $\sum_{p\leq x}\frac{1}{p}<4.166666$. We show that the last inequality in turn holds if $x=x_0:=3.6769\times 10^{21}$. First, recall that the theorem of Mertens states that $\lim_{x\to\infty}(\sum_{p\leq x}\frac{1}{p}-\log\log x)=B_1$, where $B_1=0.261497\ldots$ is Mertens's constant [22, A077761]. Now, with $x=x_0$ compute Dusart's explicit form of Mertens's theorem [11, Theorem 6.10], namely,

$$\left|\sum_{p\,\leq\,x}\frac{1}{p}-\log\log x-B_1\right|\leq\frac{1}{10\log^2 x}+\frac{4}{15\log^3 x}\qquad(x\geq 10372). \tag{15}$$

In [11, Theorem 5.2] Dusart also proved that

$$\sum_{p\,\leq\,x}\log p>\left(1-\frac{1}{\log^3 x}\right)x\qquad(x\geq 89967803).$$

Hence

$$\log M>\log\prod_{p\,\leq\,x_0}p=\sum_{p\,\leq\,x_0}\log p>\left(1-\frac{1}{\log^3 x_0}\right)x_0>3.6768\times 10^{21}.$$

Now, $3M<m^4=(k+1)^4$, so $\log(k+1)>(\log 3+\log M)/4>9.192\times 10^{20}$. Therefore $k>e^{9.19\times 10^{20}}>10^{3.99\times 10^{20}}$. This proves the proposition. $\square$

Remark. If we assume the Riemann Hypothesis, then we may replace (15) with Schoenfeld's conditional inequality [21]

$$\left|\sum_{p\,\leq\,x}\frac{1}{p}-\log\log x-B_1\right|\leq\frac{3\log x+4}{8\pi\sqrt{x}}\qquad(x\geq 13.5)$$

(see [3, equation (7.1)]), and infer that $\sum_{p\leq x_1}\frac{1}{p}<4.166666$ if $x_1:=3.6847\times 10^{21}$. Using x_1 in place of x_0 in the rest of the proof, we arrive at the slightly better, but doubly conditional bound $k>10^{4\times 10^{20}}$.

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