TWO-LAYERED NUMBERS

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Abstract. In this paper, first, I introduce two-layered numbers. Two-layered numbers are positive integers that their positive divisors except 1 can be partitioned into two disjoint subsets. Similarly, I defined a half-layered number as a positive integer n that its proper positive divisors excluding 1 can be partitioned into two disjoint subsets. I also investigate the properties of two-layered and half-layered numbers and their relation with practical numbers and Zumkeller numbers.

0. Introduction

A perfect number is a positive integer n that equals the sum of its proper positive divisors. Generalizing the concept of perfect numbers, Zumkeller in [1] published a sequence of integers that their divisors can be partitioned into two disjoint subsets with equal sum. Clark et al. in [2] called such integers Zumkeller numbers and investigated some of their properties, and also suggested some conjectures about them. Peng and Bhaskara Rao in [3] introduced half-Zumkeller numbers and provided interesting results about Zumkeller numbers.

In the present paper, I define two-layered numbers based on the concept of perfect numbers and Zumkeller numbers. A two-layered number is a positive integer n that its positive divisors excluding 1 can be partitioned into two disjoint subsets of an equal sum. A partition $\{A, B\}$ of the set of positive divisors of n except 1 is a two-layered partition if each of A and B has the same sum.

In the first section, I investigate the properties of two-layered numbers. For a two-layered number n, that sum of its divisors is $\sigma(n)$, the following statements hold (See Proposition 1.4):

Let $\sigma(n)$ be the sum of all positive divisors of n. If n is a two-layered number, then

(1) $\sigma(n)$ is odd.
(2) Powers of all odd prime factors of n should be even.
(3) $\sigma(n) \geq 2n + 1$, so n is abundant.

After that, In theorem 1.5, I prove that The integer n is a two-layered number if and only if $\frac{\sigma(n)-1}{2} - n$ is a sum of distinct proper positive divisors of n excluding 1. I also introduce two methods of generating new two-layered numbers from known two-layered numbers. Suppose that n is a two-layered number and p is a prime number with $(n, p) = 1$, then np^{α} is a two-layered number for any even positive integer α (See Theorem 1.7). We can also generate two-layered numbers in another

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way. Let n be a two-layered number and $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ be the prime factorization of n. Then for any nonnegative integers $\alpha_1,\dots\alpha_m$, the integer

$$p_1^{k_1+\alpha_1(k_1+1)}p_2^{k_2+\alpha_2(k_2+1)}\dots p_m^{k_m+\alpha_m(k_m+1)}$$

is a two-layered number (See Theorem 1.8).

In the second section of the present paper, I generalize the concept of practical numbers and define semi-practical numbers. A practical number is a positive integer n that every positive integer less than n can be represented as a sum of distinct positive divisors of n [5]. A positive integer n is a semi-practical number if every positive integer x where $1 < x < n$ can be represented as a sum of distinct positive divisors of n excluding 1 (See Definition 2.2).

I investigate some properties of semi-practical numbers and their relations with two-layered numbers. For example, every semi-practical number is divisible by 12 (See Proposition 2.3). I also proved that a positive integer n is is a semi-practical number if and only if every positive integer x where $1 < x < \sigma(n)$, is a sum of distinct positive divisors of n excluding 1 (See Theorem 2.4). The most important relation between semi-practical numbers and two-layered numbers is that a semi-practical number n is two-layered if and only if $\sigma(n)$ is odd (See Proposition 2.5).

In section 3, I define a half-layered number. A positive integer n is said to be a half-layered number if the proper positive divisors of n excluding 1 can be partitioned into two disjoint non-empty subsets of an equal sum (See Definition 3.5). A half-layered partition for a half-layered number n is a partition $\{A, B\}$ of the set of proper positive divisors of n excluding 1 so that each of A and B sums to the same value (See Definition 3.2).

After these definitions, I investigate the properties of half-layered numbers. For example, A positive integer n is half-layered if and only if $\frac{\sigma(n)-n-1}{2}$ is the sum of some distinct positive proper positive divisors of n (See Proposition 3.3). A positive even integer n is half-layered if and only if $\frac{\sigma(n)-2n-1}{2}$ is the sum (possibly empty sum) of some distinct positive divisors of n excluding n, $\frac{n}{2}$, and 1 (See Theorem 3.5). If n is an odd half-layered number, then at least one of the powers of prime factors of n should be even (See Proposition 3.7).

Using the definition of half-Zumkeller numbers, we can derive some of the interesting properties of half-layered numbers. A positive integer n is said to be a half-Zumkeller number if the proper positive divisors of n can be partitioned into two disjoint non-empty subsets of an equal sum. A half-Zumkeller partition for a half-Zumkeller number n is a partition $\{A, B\}$ of the set of proper positive divisors of n so that each of A and B sums to the same value (Definition 3 in [3]). Based on these definition, I prove that if m and n are half-layered numbers with $(m, n) = 1$, then mn is half-layered (See Proposition 3.9).

After that, I investigate some relations between half-layered and two-layered numbers. For example, let n be even. Then n is half-layered if and only if n admits a two-layered partition such that n and $\frac{n}{2}$ are in distinct subsets. Therefore, if n is an even half-layered number then n is two-layered (See Proposition 3.10). It is also proved that if n is an even two-layered number and If $\sigma(n) < 3n$, then n is half-layered (See Theorem 3.11). Let n be even. Then, n is two-layered if and only if either n is half-layered or $\frac{\sigma(n)-3n-1}{2}$ is a sum (possibly an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1 (See Proposition 3.12).

If 6 divides n, n is two-layered, and $\sigma(n) < \frac{10n}{3}$, then n is half-layered (See Proposition 3.13). If n is two-layered, then $2n$ is half-layered (See Proposition 3.14). Let n be an even half-layered number and p be a prime with (n, p) = 1. Then np^{ℓ} is half- layered for any positive integer ℓ (See Proposition 3.16). Let n be an even half-layered number and the prime factorization of n be $p_1^{k_1}p_2^{k_2}/dotsp_m^{k_m}$ Then for nonnegative integers $\ell_1, \dots, \ell_m$, the integer

$$p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2+\ell_2(k_2+1)} \dots p_m^{k_m+\ell_m(k_m+1)}$$

is half-layered (See Theorem 3.18).

1. TWO-LAYERED NUMBERS

Definition 1.1. A positive integer n is a two-layered number if the positive divisors of n excluding 1 can be partitioned into two disjoint subsets of an equal sum.

Definition 1.2. A two-layered partition for a two-layered number n is a partition $\{A, B\}$ of the set of positive divisors of n excluding 1 so that each of A and B sums to the same value.

Example 1.3. The number 36 is a two-layered number and its two-layered partition is $\{A, B\}$, where $A = \{2, 3, 4, 36\}$ and $B = \{6, 9, 12, 18\}$. You can check that each of A and B has the sum of 45. The numbers $72, 144$, and 200 are also two-layered. You can find the sequence of two-layered numbers in [4].

Proposition 1.4. *Let $\sigma(n)$ be the sum of all positive divisors of n. If n is a two-layered number, then*

(1) $\sigma(n)$ is odd.
(2) Powers of all odd prime factors of n should be even.
(3) $\sigma(n) \geq 2n + 1$, so n is abundant.

Proof. (1) : If $\sigma(n)$ is even, then $\sigma(n) - 1$ is odd, so it is impossible to partition the positive divisors of n into two subset of equal sum.

(2) : using (1), the number of odd positive divisors of n is odd. Suppose that the prime factorization of n is $2^{k_0}p_1^{k_1}p_2^{k_2} \dots p_m^{k_m}$. The number of odd positive divisors of n is $(k_1 + 1)(k_2 + 1) \dots (k_m + 1)$. All of k_i should be even in order to make the product $(k_1 + 1)(k_2 + 1) \dots (k_m + 1)$ odd.

(3) : Let n be a two-layered number with two-layered partition $\{A, B\}$. Without loss of generality we may assume that $n \in A$, so the sum in A is at least n and we can conclude $\sigma(n) - 1 \geq 2n$. □

Theorem 1.5. *The integer n is a two-layered number if and only if $\frac{\sigma(n)-1}{2} - n$ is a sum of distinct proper positive divisors of n excluding 1.*

Proof. Let n be a two-layered number and its two-layered partition is $\{A, B\}$. Without loss of generality we assume that $n \in A$, so the sum of the remaining elements of A is $\frac{\sigma(n)-1}{2} - n$.

Conversely, if we have a set of proper divisors of n excluding 1 that its sum is $\frac{\sigma(n)-1}{2} - n$, we can augment this set with n to construct a set of positive divisors of n summing to $\frac{\sigma(n)-1}{2}$. The complementary set of positive divisors of n sums to the same value, and so these two sets form a two-layered partition for n. □

With the help of the next two theorems, we can generate some new two-layered numbers by knowing a two-layered number.

Definition 1.6 (Definition 1 in [3])**.** A positive integer n is said to be a Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number n is a partition $\{A, B\}$ of the set of positive divisors of n so that each of A and B sums to the same value.

Theorem 1.7. *Let n be a two-layered number and p be a prime number with $(n, p) = 1$, then np^α is a two-layered number for any even positive integer α.*

Proof. Suppose that $\{A, B\}$ is a Zumkeller partition of n. Then $\{(A \setminus \{1\}) \cup (pA) \cup (p^2A) \cup \cdots \cup (p^\alpha A), (B \setminus \{1\}) \cup (pB) \cup (p^2B) \cup \cdots \cup (p^\alpha B)\}$ is a two-layered partition of np^α. □

Theorem 1.8. *Suppose that n is a two-layered number and $p_1^{k_1}p_2^{k_2} \dots p_m^{k_m}$ is the prime factorization of n. Then for any nonnegative even integers $\alpha_1, \dots \alpha_m$, the integer*

$$p_1^{k_1+\alpha_1(k_1+1)}p_2^{k_2+\alpha_2(k_2+1)} \dots p_m^{k_m+\alpha_m(k_m+1)}$$

is a two-layered number.

Proof. If we show that $p_1^{k_1+\alpha_1(k-1+1)}p_2^{k_2} \dots p_m^{k_m}$ the proof will be completed. Suppose that $\{A, B\}$ is a Zumkeller partition of n. If D is the set of positive divisors of n, then $(D \setminus \{1\}) \cup (p_1^{k_1+1}D) \cup (p_1^{2(k_1+1)}D) \cup \cdots \cup (p_1^{\alpha_1(k_1+1)}D))$ is the set of positive divisors of $p_1^{k_1+\alpha_1(k-1+1)}p_2^{k_2} \dots p_m^{k_m}$ excluding 1. Therefore a two-layered partition for $p_1^{k_1+\alpha_1(k-1+1)}p_2^{k_2} \dots p_m^{k_m}$ is $\{A \setminus \{1\} \cup (p_1^{k_1+1}A) \cup (p_1^{2(k_1+1)}A) \cup \cdots \cup (p_1^{\alpha_1(k_1+1)}A), B \setminus \{1\} \cup (p_1^{k_1+1}B) \cup (p_1^{2(k_1+1)}B) \cup \cdots \cup (p_1^{\alpha_1(k_1+1)}B)\}$ and the proof is complete. □

2. Semi-practical numbers and two-layered numbers

Practical numbers have been introduced by Srinivasan in 1948 as what follows:

Definition 2.1. A positive integer n is a practical number if every positive integer less than n can be represented as a sum of distinct positive divisors of n.[5]

Because of the structure of two-layered number, if we change the definition of practical numbers and call them semi-practical numbers, we can drive some useful relation between them and two-layered numbers, so I define semi-practical numbers as what follows:

Definition 2.2. A positive integer n is practical if every positive integer x where $1 < x < n$ can be represented as a sum of distinct positive divisors of n excluding 1.

Proposition 2.3. *Every semi-practical number is divisible by* 12*.*

Proof. Since we can not write $2, 3,$ and 4 as sums of more than one positive integer greater than 1, they should be divisors of our semi-practical number. □

Theorem 2.4. *A positive integer n is is a semi-practical number if and only if every positive integer x where $1 < x < \sigma(n)$, is a sum of distinct positive divisors of n excluding* 1*.*

Proof. Suppose that n is a semi-practical number. I introduce an algorithm for writing all positive integer x between n and $\sigma(n)$ as sum of distinct positive divisors of n excluding 1.

First, let x be $n+1$. Since n is semi-practical, by Propositin 2.3, it is divisible by $n/2$ and $n/3$. Hence, $n+1 = n/2 + n/3 + r$, where r is a positive integer. By Proposition 2.3, $n > 6$, so $n+1-n/2-n/3 < n/3$. On the other hand, since n is a semi-practical number and $r < n/3 < n$, r is equal to some of distinct divisors of n which are less than $n/3$ and greater than 1.

For $n+1 < x < \sigma(n)$, let the positive divisors of n which are greater than 1 be written in increasing order as $m_1 < m_2 < \cdots < m_k$. Now we can write $x = \sum_{i=\ell}^{k} m_i + r$ where $1 \leq \ell \leq k$ and $0 \leq r < m_{\ell-1}$. If $r = 0$ then x is a sum of distinct divisors of n. If $1 < r < m_{\ell-1}$, since n is semi-practical and $r < n$, then we can write r as a sum of distinct divisors of n which are less than $m_{\ell-1}$, so x is a sum of distinct divisors of n. If $r = 1$, then we can write $x = \sum_{i=\ell+1}^{k} + r_1$ where $1 < r_1 < m_\ell$. since n is semi-practical and $r < n$, then r_1 is sum of distinct divisors of n which are less than m_ℓ, so x is a sum of distinct divisors of n.

Conversely, if every positive integer less than $\sigma(n)$ excluding 1, is a some of distinct positive divisors of n excluding 1, it is clear that n is semi-practical.

□

Proposition 2.5. *A semi-practical number n is two-layered if and only if $\sigma(n)$ is odd.*

Proof. If n is two-layered number, then $\sigma(n)$ is odd by Proposition 1.4. Conversely, if $\sigma(n)$ is odd, then $\frac{\sigma(n)-1}{2}$ is a positive integer smaller than $\sigma(n)$. Since n is a semi-practical number, using Proposition 2.4. □

Theorem 2.6. *Let n be a positive integer and p be a prime with $(n,p) = 1$. Let D be the set of all positive divisors of n including 1. The following conditions are equivalent:*

(1) np is two-layered.
(2) There exist two partitions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of $D \setminus \{1\}$ such that

$$p(\sum_{d\in D_1} d - \sum_{d\in D_2} d) = (\sum_{d\in D_3} d - \sum_{d\in D_4} d).$$

(3) There exists a partition $\{D_1, D_2\}$ of $D \setminus \{1\}$ and subsets $A_1 \subseteq D_1$ and $A_2 \subseteq D_2$ such that

$$\frac{p+1}{2}(\sum_{d\in D_1} d - \sum_{d\in D_2} d) = (\sum_{d\in A_1} d - \sum_{d\in A_2} d).$$

Proof. It is clear that $(pD) \cup (D \setminus \{1\})$ is the set of all positive divisors of np excluding 1.

$(1) \Rightarrow (2)$. Suppose that np is two-layered. Hence, there is a two-layered partition $\{A, B\}$ of $(pD) \cup (D \setminus \{1\})$. Let $D_1 = \frac{1}{p}(A \cap (pD))$, $D_2 = \frac{1}{p}(B \cap (pD))$, $D_3 = B \cap (D \setminus \{1\})$, $A \cap (D \setminus \{1\})$, then

$$p\sum_{d\in D_1} d + \sum_{d\in D_4} d = p\sum_{d\in D_2} d + \sum_{d\in D_3} d.$$

and the proof is complete.

$(2) \Rightarrow (3)$. Let $A_1 = D_1 \cap D_3$ and $A_2 = D_2 \cap D_4$. We have

$$\begin{aligned}\frac{p+1}{2}(\sum_{d\in D_1} d-\sum_{d\in D_2} d) &= \frac{1}{2}[p(\sum_{d\in D_1} d-\sum_{d\in D_2} d)+(\sum_{d\in D_1} d-\sum_{d\in D_2} d)]\\ &= \frac{1}{2}[\sum_{d\in D_3} d-\sum_{d\in D_4} d+\sum_{d\in D_1} d-\sum_{d\in D_2} d]\\ &= \frac{1}{2}[2(\sum_{d\in D_1\cap D_3} d)-2(\sum_{d\in D_2\cap D_4} d)]\\ &= \sum_{d\in A_1} d-\sum_{d\in A_2} d.\end{aligned}$$

(3) $\Rightarrow$ (1). We can rewrite the equation in (3) as follows:

$$\frac{p}{2}\sum_{d\in D_1} d+\frac{1}{2}\sum_{d\in A_2}+\frac{1}{2}\sum_{D_1\setminus A_1} d=\frac{p}{2}\sum_{d\in D_2} d+\frac{1}{2}\sum_{d\in A_1} d+\frac{1}{2}\sum_{d\in D_2\setminus A_2} d.$$

By multiplying this by 2, we obtain the two-layered partition $\{(pD_1)\cup A_2\cup(D_1-A_1),(pD_2)\cup A_1\cup(D_2-A_2)\}$ for np, so np is a two-layered number. □

Proposition 2.7. *Let the positive divisors of n excluding 1 be written in increasing order as follows: $a_1<a_2<\cdots<a_k=n$. If $a_{i+1}<2a_i$ for all $1\le i<k$ and $\sigma(n)$ is odd, then n is two-layered.*

Proof. Let $b_i=a_i$ or a_i for each i. I will explain how to chose the sign of b_i precisely. Then I show that $\sum_{i=1}^k b_k=0$. Hence, it will imply that $\sigma(n)-1$ can be partitioned into two equal-summed subsets.

Let $b_k=a_k=n$ and let $b_{k1}=a_{k1}$. Note that $0<b_k+b_{k1}<a_{k1}$ since $a_k<2a_{k1}$. Since the current sum b_k+b_{k1} is positive, we assign the negative sign to b_{k2}. Then $b_{k2}<b_k+b_{k1}+b_{k2}<a_{k1}a_{k2}<a_{k2}$ since $a_{k1}<2a_{k2}$. If $b_k+b_{k1}+b_{k2}\ge 0$, we assign the negative sign to b_{k3}; Otherwise, we assign the positive sign to b_{k3}. Let s_i be $\sum_{j=1}^k b_j$. In general, the sign assigned to b_{i1} is the opposite of the sign of s_i . Let us show inductively that $|s_i|<a_i$ for $1\le i\le k$. It is true for $i=k$. Assume that $|s_{i+1}|<a_{i+1}$. Since the sign of b_i is opposite of the sign of s_{i+1}, $|s_i|=||s_{i+1}|a_i|$. Note that $a_i<|s_{i+1}|a_i<a_{i+1}a_i<a_i$ since $a_{i+1}<2a_i$. Therefore $|s_i|<a_i$. So $|s_1|<a_1=1$. Since $\sigma(n)-1$ is even, s_1, which is obtained by assigning a positive or negative sign to each of the terms in $\sigma(n)-1$ is even as well. So $s_1=0$. This implies that $\sigma(n)-1$ can be partitioned into two equal-summed subsets. Hence it is two-layered. □

Proposition 2.8 (Proposition 1 in [3]). *Let the prime factorization of n be $\prod_{i=1}^m p_i^{k_i}$. Then*

$$\sigma(n)=\prod_{i=1}^m \frac{p_i^{k_i+1}-1}{p_i-1}$$

and

$$\frac{\sigma(n)}{n}=\prod_{i=1}^m \frac{p_i^{k_i+1}-1}{p_i^{k_i}(p_i-1)}<\prod_{i=1}^m \frac{p_i}{p_i-1}$$

Proposition 2.9. *Let the prime factorization of an odd number n be $p_1^k p_2^k \dots p_m^{k_m}$, where $3 \leq p_1 < p_2 < \cdots < p_m$. If n is two-layered, then*

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} > 2,$$

and m is at least 3. In particular:

(1) *If $m \leq 6$, then $p_1 = 3$, $p_2 = 5$, 7 or 11.*
(2) *If $m \leq 4$, then $p_1 = 3$, $p_2 = 5$ or 7.*
(3) *If $m = 3$, then $p_1 = 3$, $p_2 = 5$, and $p_3 = 7$ or 11 or 13.*

Proof. If n is two-layered, then by Propositions 1.4 and 2.8,

$$2p_1^{k_1} p_2^{k_2} \dots p_m^{k_m} = 2n < \sigma(n) = \prod_{i=1}^{m} (\sum_{j=0}^{k_i} p_i^j).$$

Dividing both sides by $p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, we get

$$2 < \prod_{i=1}^{m} (\sum_{j=0}^{k_i} p_i^{j-k_i}) < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

If $m \leq 2$, then

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{5}{4} < 2$$

Therefore $m \geq 3$. The parts of $1-3$ follows by verifying the condition $\prod_{i=1}^{m} \frac{p_i}{p_i - 1} > 2$ directly as given below.

1. Let $m \leq 6$. If $p_1 \neq 3$, then $p_1 \geq 5$ and

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{5}{4} \times \frac{7}{6} \times \frac{11}{10} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{18} < 2.$$

Therefore, $p_1 = 3$. If $p_2 > 11$, then $p2 \geq 13$ and

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{18} \times \frac{23}{22} \times \frac{29}{28} < 2.$$

Hence, $p_2 \leq 11$. This implies that $p_2 = 5$, 7 or 11.

2. Let $m \leq 4$. By 1, $p_1 = 3$. If $p_2 > 7$, then $p_2 \geq 11$, so

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{11}{10} \times \frac{13}{12} \times \frac{17}{16} < 2.$$

Therefore, $p_2 \leq 7$. This implies that $p_2 = 5$ or 7.

3. Let $m = 3$. By 1, $p_1 = 3$. If $p_2 \neq 5$, then $p_2 \geq 7$ and $p3 \geq 11$. So

$$\prod_{i=1}^{3} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{7}{6} \times \frac{11}{10} < 2.$$

Hence $p_2 = 5$.

If $p_3 \geq 17$, then

$$\prod_{i=1}^{3} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{5}{4} \times \frac{17}{16} < 2.$$

Hence, $p_3 < 17$ and consequently $p_3 = 7$, 11 or 13. $\square$

3. HALF-LAYERED NUMBERS

Definition 3.1. A positive integer n is said to be a half-layered number if the proper positive divisors of n excluding 1 can be partitioned into two disjoint non-empty subsets of equal sum.

Definition 3.2. A half-layered partition for a half-layered number n is a partition $\{A, B\}$ of the set of proper positive divisors of n excluding 1 so that each of A and B sums to the same value.

Proposition 3.3. *A positive integer n is half-layered if and only if $\frac{\sigma(n)-n-1}{2}$ is the sum of some distinct positive proper positive divisors of n.*

Example 3.4. In Example 1.3, we saw that 36 was a two-layered number. It is also a half-layered number and its half-layered partition is $\{A, B\}$, where $A = \{2, 3, 4, 18\}$ and $B = \{6, 9, 12\}$. You can check that each of A and B has the sum of 27. The numbers $72, 105$, and 144 are also half-layered. You can find the sequence of half-layered numbers in [6].

Theorem 3.5. *A positive even integer n is half-layered if and only if $\frac{\sigma(n)-2n-1}{2}$ is the sum (possibly empty sum) of some distinct positive divisors of n excluding n, $\frac{n}{2}$, and 1.*

Proof. An even number n is half-layered if and only if there exists a which is the sum (possibly empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1 such that

$$\frac{n}{2} + a = \frac{\sigma(n) - n - 1}{2}.$$

Therefore, $a = \frac{\sigma(n)-2n-1}{2}$. □

Example 3.6. The number $3^4 \times 2^4$ is a half-layered number, since

$$\frac{\sigma(3^4 \times 2^4) - 2(3^4 \times 2^4) - 1}{2} = 579 = 432 + 108 + 36 + 3$$

is a sum of positive divisors of $3^4 \times 2^4$ excluding $3^4 \times 2^4$, $3^4 \times 2^3$, and 1. Hence, by Theorem 3.5, it is a half-layered number.

Proposition 3.7. *If n is an odd half-layered number, then at least one of the powers of prime factors of n should be even.*

Proof. If n is odd and half-layered, then $\sigma(n)n - 1$ must be even and $\sigma(n)$ must be even. Let the prime factorization of n be $\prod_{i=1}^{m} p_i^{k_i}$. Then $\sigma(n) = \prod_{i=1}^{m}(\sum_{j=0}^{k_i} p_i^j)$. If $\sigma(n)$ is odd, then there exists one $k - i$ which is odd. □

Definition 3.8 (Definition 3 in [3])**.** A positive integer n is said to be a half-Zumkeller number if the proper positive divisors of n can be partitioned into two disjoint non-empty subsets of an equal sum. A half-Zumkeller partition for a half-Zumkeller number n is a partition $\{A, B\}$ of the set of proper positive divisors of n so that each of A and B sums to the same value.

Proposition 3.9. *If m and n are half-layered numbers with $(m, n) = 1$, then mn is half-layered.*

Proof. Let M be the set of proper positive divisors of m and let $\{M_1, M_2\}$ be a half-Zumkeller partition for m. Let Nbe the set of proper positive divisors of n and let $\{N1, N2\}$ be a half-Zumkeller partition for n. Since $(m, n) = 1$, then the set of proper positive divisors of mn is $(MN) \cup (nM) \cup (mN)$. Observe that $\{(M_1N \setminus \{1\}) \cup (mN_1) \cup (nM_1), (M_2N \setminus \{1\}) \cup (mN_2) \cup (nM_2)\}$ is a half-layered partition for mn. Therefore mn is half-layered. □

Proposition 3.10. *Let n be even. Then n is half-layered if and only if n admits a two-layered partition such that n and $\frac{n}{2}$ are in distinct subsets. Therefore, if n is an even half-layered number then n is two-layered.*

Proof. Let n be even. Let D be the set of all positive divisors of n excluding 1. The number n is half-layered if and only if there exists $A \subset D \setminus \{n, \frac{n}{2}\}$ such that

$$\frac{n}{2} + \sum_{a \in A} a = \sum_{b \in D, b \notin \{n, \frac{n}{2}\} \cup A} b.$$

That is,

$$n + \sum_{a \in A} a = \frac{n}{2} + \sum_{b \in D, b \notin \{n, \frac{n}{2}\} \cup A} b.$$

This is equivalent to saying that n admits a two-layered partition such that n and $\frac{n}{2}$ are in distinct subsets.

□

Theorem 3.11. *Let n be an even two-layered number. If $\sigma(n) < 3n$, then n is half-layered.*

Proof. Since n and $\frac{n}{2}$ together sum to more than $\frac{\sigma(n)}{2}$, they must be in different subsets in any two-layered partition for n. Therefore, by Proposition 3.10, n is half-layered. □

Proposition 3.12. *Let n be even. Then, n is two-layered if and only if either n is half-layered or $\frac{\sigma(n)-3n-1}{2}$ is a sum (possibly an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1.*

Proof. Let n be even. If n is two-layered but not half-layered, then by Proposition 3.10, any two-layered partition of the positive divisors of n must have n and $\frac{n}{2}$ in the same subsets. In other words, there exists a which is a sum (possibly an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1 such that

$$2(n + \frac{n}{2} + a) = \sigma(n) - 1$$

So, $a = \frac{\sigma(n)-3n-1}{2}$. Therefore, the number $\frac{\sigma(n)-3n-1}{2}$ is a sum (possibly an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1.

If n is half-layered, then n is two-layered by Proposition 3.10. If $\frac{\sigma(n)-3n-1}{2}$ is a sum (possibly an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1, then

$$\frac{\sigma(n) - 2n - 1}{2} = \frac{\sigma(n) - 3n - 1}{2} + \frac{n}{2}$$

is a sum of some positive divisors of n excluding n, and 1. By Theorem 1.5, the number n is two-layered. □

Proposition 3.13. *If* 6 *divides* n*,* n *is two-layered, and* $\sigma(n) < \frac{10n}{3}$ *, then* n *is half-layered.*

Proof. If n is not half-layered, by Proposition 3.12, $\frac{\sigma(n)-3n-1}{2}$ is a sum (might be an empty sum) of some positive divisors of n excluding n, $\frac{n}{2}$, and 1. Then,

$$\frac{\sigma(n)-2n-1}{2} = \frac{\sigma(n)-3n-1}{2} + \frac{n}{3} + \frac{n}{6}.$$

Since $\sigma(n)/n < \frac{10}{3}$ we have that $\frac{\sigma(n)-3n-1}{2} < \frac{n}{6}$. Hence $\frac{\sigma(n)-2n-1}{2}$ is a sum of some positive divisors of n excluding n, $\frac{n}{2}$, and 1. By Proposition 3.3, n is half layered. This is a contradiction. □

Proposition 3.14. *If* n *is two-layered, then* $2n$ *is half-layered.*

Proof. Let $n = 2^k L$ with k a nonnegative integer and L an odd number, be a two-layered number. Then all positive divisors of n excluding 1 can be partitioned into two disjoint equal-summed subsets D_1 and D_2. Observe that every positive divisor of $2n$ which is not a positive divisor of n can be written as $2^{k+1}\ell$ where ℓ is a positive divisor of L. Observe that $2^k\ell$ is either in D_1 or D_2. Without loss of generality, assume that $2^k\ell$ is in D_1. In this case, we move $2^k\ell$ to D_2 and add $2^{k+1}\ell$ to D_1. Perform this procedure to all positive divisors of $2n$ which are not positive divisors of n except $2n$ itself. This procedure will yield an equal-summed partition of all positive divisors of $2n$ except $2n$ itself. This shows that $2n$ is half-Zumkeller. □

Corollary 3.15. *Let* n *be even and the prime factorization of* n *be* $2^k p_1^{k_1} \dots p_m^{k_m}$*. If* n *is two-layered but not half- layered, then* $2^i p_1^{k_1} \dots p_m^{k_m}$ *is not two-layered for any* $i \leq k-1$*, and* $2^i p_1^{k_1} \dots p_m^{k_m}$ *is half-layered for any* $i \geq k+1$*.*

Proposition 3.16. *Let* n *be an even half-layered number and* p *be a prime with* $(n, p) = 1$*. Then* np^ℓ *is half- layered for any positive integer* ℓ*.*

Proof. Since n is an even half-layered number, the set of all positive divisors of n, excluding 1, denoted by D_0 can be partitioned into two disjoint subsets A_0 and B_0 so that the sums of the two subsets are equal and n and $\frac{n}{2}$ are in distinct subsets (by Proposition 3.10).

Group the positive divisors of np^ℓ except 1 into $\ell+1$ groups $D_0, D_1, \dots D_\ell$ according to how many positive divisors of p they admit, i.e., D_i consists of all positive divisors of np^ℓ admitting i positive divisors of p. Then each D_i can be partitioned into two disjoint subsets so that the sums of the two subsets are equal and np^i and $\frac{np^i}{2}$ are in distinct subsets according to the two-layered partition of the set D_0. Therefore all positive divisors of np^ℓ excluding 1 can be partitioned into two disjoint subsets so that the sum of these two subsets equal and np^ℓ and $\frac{np^\ell}{2}$ are in distinct subsets. By Proposition 3.10, np^ℓ is half- layered. □

Corollary 3.17. *If* n *is an even half-layered number and* m *is a positive integer with* $(n, m) = 1$*, then* nm *is half-layered.*

Theorem 3.18. *Let* n *be an even half-layered number and the prime factorization of* n *be* $p_1^{k_1} p_2^{k_2} /dotsp_m^{k_m}$ *Then for nonnegative integers* $\ell_1, \dots, \ell_m$*, the integer*

$$p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2+\ell_2(k_2+1)} \dots p_m^{k_m+\ell_m(k_m+1)}$$

is half-layered.

Proof. It is sufficient to show that $p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}$ is half-layered if $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ is an even half-layered number. Assume that $p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$ is even and half-layered, then the set of all positive divisors of n excluding 1, denoted by D_0 can be partitioned into two disjoint subsets A_0 and B_0 so that the sums of the two subsets are equal and n and $\frac{n}{2}$ are in distinct subsets (by Proposition 3.10). Note that the positive divisors of $p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}$ excluding 1 can be partitioned into ℓ_1+1 disjoint groups $D_i, 0\le i\le \ell_1$, where elements in D_i are obtained by multiplying $p_1^{i(k_1+1)}$ with elements in D_0. Using the partition A_0, B_0 of $D0$ we can partition every D_i into two disjoint subsets A_i and B_i so that the sums of the corresponding subsets are equal and $np_1^{i(k_1)+1}$ and $\frac{np_1^{i(k_1)+1}}{2}$ are in distinct subsets. Therefore, the set of all positive divisors of $p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}$ excluding 1 can be partitioned into two disjoint equal-summed subsets and $p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}$ and $\frac{p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}}{2}$ are in distinct subsets. By Proposition 3.10, $p_1^{k_1+\ell_1(k_1+1)}p_2^{k_2}\dots p_m^{k_m}$ is half-layered. □

Theorem 3.19. *Let n be an even integer and p be a prime with $(n,p)=1$. Let D be the set of all positive divisors of n excluding 1. Then the following conditions are equivalent:*

(1) *np is half-layered.*
(2) *There exist two partitions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of D such that n is in D_1, $\frac{n}{2}$ is in D_2 and*

$$p(\sum_{d\in D_1} d - \sum_{d\in D_2} d) = \sum_{d\in D_3} d - \sum_{d\in D_4} d.$$

(3) *There exists a partition $\{D_1, D_2\}$ of D and subsets $A_1\subseteq D_1$and $A_2\subseteq D_2$ such that n is in D_1, $\frac{n}{2}$ is in D_2 and*

$$\frac{p+1}{2}(\sum_{d\in D_1} d - \sum_{d\in D_2} d) = \sum_{d\in A_1} d - \sum_{d\in A_2} d.$$

Proof. By Proposition 3.10, np is half-layered if and only if there is a two-layered partition $\{A, B\}$ of $(pD)\cup D$such that $n\in A$ and $\frac{n}{2}\in B$. The rest of the proof follows along the lines of the proof of Theorem 2.6. □

Proposition 3.20. *If $a_1 < a_2 < \cdots < a_k = n$ are all positive divisors of an even number n excluding 1 with $a_{i+1} < 2a_i$ for all i and $\sigma(n)$ is odd, then n is half-layered.*

Proof. Note that in the proof of Proposition 2.7, $b_k = n$ and $b_{k1} = -\frac{n}{2}$ have different signs. So we get a two-layered partition of n such that n and $\frac{n}{2}$ are in distinct subsets. By Proposition 3.10, n is half-layered. □

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