

On finding all positive integers a, b such that $b \pm a$ and ab are palindromic

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Abstract

It is proven that the only integer solutions (a, b) such that $a + b$ and ab are palindromic are $(2, 5 \cdot 10^k - 3)$, $(3, 24)$ and $(9, 9)$, and in a similar fashion, $b - a$ and ab are only palindromic at $(a, b) = (3, 147 \cdot 10^{4(k+1)} + 5247 \sum_{i=0}^k 10^{4i})$, $(3, 161\,247 \cdot 10^{4k+7} + 5247 \sum_{i=0}^k 10^{4i+3} + 387)$, $(3, 147)$ and $(3, 161\,247\,387)$ for $k = 0, 1, 2, \dots$. Note $a \leq b$ without loss of generality.

1 Introduction

The challenge to determine all positive integers a, b such that $b + a$ and ab are palindromic has been explored by a few people, but none has yet provided a rigorous proof of all the solutions. In 2009, the conjectured solutions were posted on OEIS by Mark Nandor [2]. More recently, there was a question on Quora [3] asking this, and we have even done so on Mathematics Stack Exchange where a user Michael Lugo [4] conjectured the same as Nandor. In this paper, we will prove their claims, and will also generalise this to the case where $b - a$ and ab are palindromic.

Definition:

Two integers are palindromic if the digits of one integer are the same as the reverse of the digits of the other, and if they both have the same number of digits.

That is, if one integer a has digit representation $a_n a_{n-1} \dots a_1$ and the other integer b has digit representation $b_n b_{n-1} \dots b_1$, then they are palindromic if and only if

$$a_1 = b_n, \quad a_2 = a_{n-1}, \quad \dots, \quad a_{n-1} = b_2, \quad a_n = b_1$$

Furthermore, $a_2, \dots, a_{n-1}, b_2, \dots, b_{n-1} \in \{0, 1, 2, \dots, 9\}$ but $a_1, a_n, b_1, b_n \in \{1, 2, \dots, 9\}$. This is so that the the first and last digits of a and b are not zero; otherwise, they would not have the same number of digits.

2 When $a + b$ and ab are palindromic

2.1 Formulating the problem

Without loss of generality assume that $a \leq b$. Then $\frac{ab}{a+b} > \frac{ab}{2b} = \frac{a}{2}$ and this must be less than 10. Hence $a < 20$.

For $n > 1$, let b have the above digit representation, and let

$$a + b = c_n c_{n-1} \dots c_2 c_1 = 10^{n-1} c_n + 10^{n-2} c_{n-1} + \dots + 10 c_2 + c_1 \quad (1)$$

$$ab = c_1 c_2 \dots c_{n-1} c_n = 10^{n-1} c_1 + 10^{n-2} c_2 + \dots + 10 c_{n-1} + c_n \quad (2)$$

Substituting b from (1) into (2), we get

$$(a \cdot 10^{n-1} - 1) c_n + (a \cdot 10^{n-2} - 10) c_{n-1} + \dots + (10a - 10^{n-2}) c_2 + (a - 10^{n-1}) c_1 = a^2 \quad (3)$$

Note that to preserve the same number of digits in each expression, $a c_n < 10$, except for extreme cases highlighted in §2.3.

2.2 Finding the solutions

In this section it will be assumed that $c_n = b_n$. Exceptions to this are also discussed in §2.3.

2.2.1 Solutions when $a = 2$

If $n = 1$, we can immediately solve the equation $2 + b = 2b \implies b = 2$ so $(2, 2)$ is a trivial solution.

From (3), the equation becomes

$$(2 \cdot 10^{n-1} - 1)c_n + (2 \cdot 10^{n-2} - 10)c_{n-1} + \dots + (20 - 10^{n-2})c_2 + (2 - 10^{n-1})c_1 = 4 \quad (4)$$

Clearly c_n is even, and since $2 \cdot 6 > 10$, $c_n = 2, 4$. If c_n is the former, then $b_n = 2$ so the first digit of $2b$ is $c_1 = 4$ or 5 . If $c_1 = 4$, then $b_1 = c_1 - 2 = 2$ so the last digit of $2b$ must be $4 = c_n$ which is a contradiction. Similarly, if $c_1 = 5$, then $b_1 = 3$ so the last digit of $2b$ must be $6 = c_n$, again, a contradiction. Therefore $c_n = 4$, meaning that $2b$ starts with either $c_1 = 8, 9 \implies b_1 = 6, 7$. If $b_1 = 6$ then the last digit of $2b$ is 2 contradicting the fact that the first digit of b is 4 . Hence $b_1 = 7 \implies c_1 = 9$.

Dividing equation (4) by 2, the RHS is still even, so to fulfill that on the LHS, we must have that $-5c_{n-1} + c_1$ is even since all other terms on that side have at least one even factor. Since c_1 is odd, so is c_{n-1} . Notice that $c_1 = 2c_n + 1$, implying that there is carrying. This narrows c_{n-1} down to either being 5, 7 or 9.

If $c_{n-1} = 5$, then $2b$ ends in the digits 54 so b must end in the digits 27 or 77, implying that $c_2 = 2, 7$. However the second digit of $2b$ is $c_2 = 2c_{n-1} = 0, 1$ which is a contradiction. Similarly, if $c_{n-1} = 7$, then $2b$ ends in the digits 74 so b must end in the digits 37 or 87, implying that $c_2 = 3, 8$. However the second digit of $2b$ is $c_2 = 2c_{n-1} = 4, 5$ which is a contradiction. Finally, if $c_{n-1} = 9$, then $2b$ ends in the digits 94 so b must end in the digits 47 or 97, implying that $c_2 = 4, 9$. However the second digit of $2b$ is $c_2 = 2c_{n-1} = 8, 9$ which forces $c_2 = 9$.

We have now arrived at $b = 49 \dots 97 \implies 2 + b = 49 \dots 99$ and $2b = 99 \dots 94$ so $c_3 \geq 5$. Again, we previously showed that c_1 is odd so c_{n-2} is also odd. This is a cycle, so the only solutions when $a = 2$ are $b = 2, 47, 497, 4997, \dots$ which can be generalised to $5 \cdot 10^k - 3$ for $k = 0, 1, 2, \dots$.

2.2.2 Solutions when $a = 3$

In this section, equation (3) will be used for $n > 2$. That said, the cases $n = 1, 2$ will firstly be considered. Of course, $3 + b = 3b$ gives no integer solutions so this eliminates the first one. If $n = 2$, equation (3) can be modified to give $29c_2 - 7c_1 = 9$ which is a standard Diophantine equation. Solving using the Euclidean Algorithm gives the general solution $(c_1, c_2) = (9 + 7t, 36 + 29t)$ for an integer t . But since $c_1, c_2 < 10$, the only possible solution is when $t = -1$, so $(c_1, c_2) = (2, 7) \implies b = 24$.

For $n > 2$, we have

$$1 + (3 \cdot 10^{n-1} - 1)c_n + (3 \cdot 10^{n-2} - 10)c_{n-1} + \dots + (30 - 10^{n-2})c_2 + (3 - 10^{n-1})c_1 = 10$$

so $1 - c_n + 3c_1$ is a multiple of 10. As $3c_n < 10$, c_n is restricted to 1, 2, 3.

If $c_n = 1$, 10 divides $1 - 1 + 3c_1$ which is impossible.

If $c_n = 2$, 10 divides $3c_1 - 1$. This can be achieved only if $c_1 = 7$, so $3b$ starts with 7. Since the first digit of $3 + b$ is 2, this indicates carrying, and in particular, that $c_{n-1} = 3, 4, 5, 6$. If $c_{n-1} = 3$, $3b$ ends in 32 so b ends in 44. This means that $b + 3$ ends in 47 and in turn, $3b$ ends in 74. A contradiction arises as $3 \cdot 23 < 74$. If $c_{n-1} = 4$, $3b$ ends in 42 so b ends in 14. This means that $b + 3$ ends in 17 and in turn, $3b$ starts with 71. However this is impossible as $3 \cdot 24 > 71$. If $c_{n-1} = 5$, $3b$ ends in 52 so b ends in 84. This means that $b + 3$ ends in 87 and in turn, $3b$ starts with 78. Again this is contradictory since $3 \cdot 25 < 78$. Finally, if $c_{n-1} = 6$, $3b$ ends in 62 so b

ends in 54. This means that $b + 3$ ends in 57 and in turn, $3b$ starts with 75 which is impossible as $3 \cdot 26 > 75$. No solutions exist in this category.

If $c_n = 3$, 10 divides $3c_1 - 2$. This can be achieved only if $c_1 = 4$. Now $3c_n = 9 = c_1$ only since carrying will increase the number of digits so immediately there is a contradiction. Therefore the only solution when $a = 3$ is $b = 24$.

2.2.3 Solutions when $a = 4, 5$

If $n = 1$ then it is easy to show that no solutions exist for $a = 4, 5$.

For $a = 4$, $c_n = 1, 2$ and we have that

$$(4 \cdot 10^{n-1} - 1)c_n + (4 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (40 - 10^{n-2})c_2 + (4 - 10^{n-1})c_1 = 16$$

and clearly c_n must be even to keep the parities consistent on both sides of the equation. Thus if $c_n = 2$, $c_1 = 8, 9$ including carrying leading to $b_1 = 4, 5$. However, $4b_1$ ends in 6, 0 respectively, not $c_n = 2$, which is a contradiction.

Similarly, for $a = 5$, $c_n = 1$ and we have that

$$(5 \cdot 10^{n-1} - 1) + (5 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (50 - 10^{n-2})c_2 + (5 - 10^{n-1})c_1 = 25$$

but the LHS is not divisible by 5. This is again a contradiction.

2.2.4 Solutions when $a = 7, 9$

When $a = 7$, $c_n = 1$ so that $ac_n < 10$ so equation (3) becomes

$$(7 \cdot 10^{n-1}) + (7 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (70 - 10^{n-2})c_2 + (7 - 10^{n-1})c_1 = 50$$

Now for $n > 2$, every term on the LHS is divisible by 10 except $7c_1$ so c_1 must be divisible by 10. This is a contradiction as c_1 must be a single digit and cannot be zero. The case where $n = 1$ trivially gives no solutions and if $n = 2$, this requires the solution of the Diophantine equation $(7 \cdot 10 - 1)c_2 - (7 - 10)c_1 = 7^2 \implies 3(23c_2 - c_1) = 49$, but the RHS is not divisible by 3. Hence no solutions exist and this completes $a = 7$.

Similarly, when $a = 9$, $c_n = 1$ so

$$(9 \cdot 10^{n-1}) + (9 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (90 - 10^{n-2})c_2 + (9 - 10^{n-1})c_1 = 82$$

For $n > 2$, every term on the LHS is even except $9c_1$ so c_1 must be even. This means that c_n is also even which is a contradiction. The case where $n = 1$ trivially also gives no solutions and this case where $n = 2$ requires the solution of the Diophantine equation $(9 \cdot 10 - 1)c_2 + (9 - 10)c_1 = 9^2 \implies 89c_2 - c_1 = 81$, which is $(c_1, c_2) = (t, -81 + 89t)$. As $0 < c_1, c_2 < 10$, the solution occurs when $t = 1$, so $(c_1, c_2) = (1, 8)$, implying that $b = 18 - 9 = 9$. Hence the only solution when $a = 9$ is $b = 9$.

2.2.5 Solutions for the remaining a

If a is even, so is the RHS of (3). The LHS is also even as a and 10 are divisible by 2, except for the term $-c_n$. This means that c_n is even. However, for even $a > 5$, we must set $c_n = 1$ so that $ac_n < 10$ which is again a contradiction. If a is odd; that is, $a = 11, 13, \dots, 19$, then clearly $ac_n > 10$ so all that is left is to check the extreme cases in the next section, where, for example, $b = 10^n - a$ so that $a + b$ and ab have the same number of digits.

2.3 Checking the extreme cases

2.3.1 When $a = 2, 3, 4, 5, 7, 9$

There were instances in §2.2 where it was assumed that $c_n = b_n$. While this may be true for the majority of the values of b , there are still some exceptions. For example, consider the case $a = 2$. If $b = 19 \cdots 98$ or $19 \cdots 99$ then c_n becomes 2 not 1. However, by inspection it is apparent that $2 + b$ and $2b$ are not palindromic for these b , since $2 + b$ starts with 2 but $2b$ ends in 6 or 8. Similarly, for the other five values of a , we can check from $b = 10^{n-1}(b_n + 1) - a$ to $10^{n-1}(b_n + 1) - 1$ - since here, $c_n = b_n + 1$, but this yields no solutions either for all $n > 1$.

2.3.2 When $a = 11, 13, 15, 17, 19$

For these values of a , $b_n = 9$ so that $a + b$ and ab have the same number of digits. This means that we need only check from $b = 10^n - a$ to $10^n - 1$ and it can be easily verified (by comparing the first and last digits of each of $a + b$ and ab) that no solutions exist here either.

3 When $b - a$ and ab are palindromic

3.1 Formulating the problem

This is very similar to section 2.1. The LHS of equation (1) in §2.1 will be replaced by $b - a$ but the rest of (1) and (2) remain the same. We get a near equivalent equation to (3); the only difference is due to the negative sign on the RHS:

$$(a \cdot 10^{n-1} - 1)c_n + (a \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (10a - 10^{n-2})c_2 + (a - 10^{n-1})c_1 = -a^2 \quad (5)$$

Note that $a < 10$; otherwise, ab will have at least one more digit than $b - a$ as subtraction of a positive integer cannot increase the value of the expression. Again, the criterion (bar exceptions) that $ac_n < 10$ still holds.

3.2 Finding the solutions

As in §2.2, assume that $c_n = b_n$ in this section.

3.2.1 Solutions when $a = 2, 4, 5$

We will start with $a = 5$. We have two cases: $c_n = 0$ and $c_n = 5$, as $5b$ ends in c_n . Of course, $c_n = 0$ is trivially false, and $c_n = 5$ implies that $5b$ has more digits than $b - 5$. Both cases lead to contradictions, so there are no solutions.

We will now consider $a = 4$, first assuming $n = 1$ we get $b - 4 = 4b$, and this implies no solutions. Now we can apply (5) and get:

$$(4 \cdot 10^{n-1} - 1)c_n + (4 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (40 - 10^{n-2})c_2 + (4 - 10^{n-1})c_1 = -16$$

Therefore c_n is even, and to keep $ac_n < 10$ we also get $c_n < 3 \implies c_n = 2$. Multiplying by 4 we get $c_1 = 8, 9$ (which is the last digit of $b - 4$) and we need only consider the carrying of 1 because beyond that we will get a new digit. Now $b = 2 \cdots \cdots 2$ or $b = 2 \cdots \cdots 3$, and multiplying the possible last digits by 4 we get $c_n = 8$ for the former and $c_n = 2$ in the latter. Thus, if solutions exists, $c_n = 2$ and $c_1 = 3$.

This gives $b = 2 \cdots \cdots 3 \implies b - 4 = 2 \cdots \cdots 9 \implies 4b = 9 \cdots \cdots 2$; in other words, there is a carrying of 1 from c_{n-1} , so we get two possible values: $c_{n-1} = 3, 4$.

If $c_n c_{n-1} = 23$ then $4b$ ends with 32, dividing 32 by 4 yields $b_1 = 8$, which contradicts the fact that $b_1 = 3$.

If $c_n c_{n-1} = 24$ then $4b$ ends with 42 which is not divisible by 4 so we have another contradiction.

So for $a = 4$ there are no solutions.

For $a = 2$, $n = 1$ gives the equation $2a = a - 2$ which yields no results, and if $n \geq 2$, we can start by noticing that $c_n = 2, 4$ because

$$(2 \cdot 10^{n-1} - 1)c_n + (2 \cdot 10^{n-2} - 10)c_{n-1} + \dots + (20 - 10^{n-2})c_2 + (2 - 10^{n-1})c_1 = 4$$

If $c_n = 4$, c_1 is either 8 or 9, but c_1 cannot be 8 as $8 + 2$ ends in 0. Thus $c_1 = 9 \implies b_1 = 1$, but $2b$ ends in 4, which is a contradiction.

If $c_n = 2$, c_1 is 4 or 5, but we know that $2b$ ends in $c_n = 2$. Hence b ends in either 1 or 6, so $b - 2$ ends with 9 or 4 which is c_1 . Now the only common number between 4, 5 and 4, 9 is 4, so $c_1 = 4$. It is easy to see that $24 + 2 = 26$ is not a solution. So the possible values of c_{n-1} are 0, 1, 2, 3, 4.

If $c_{n-1} = 0, 2, 4$ then $2b$ ends with $d2$ where $d = 0, 2, 4$. This is a contradiction as for a positive integer k we have $2 \cdot (k \cdot 10 + 6)$ ends in $t2$ where t is odd.

If $c_{n-1} = 1$ then $b - 2 = 21 \dots 4 \implies 2b = 4 \dots 12$, which means that b ends in 06 or 56, so $c_2 = 0, 5$ which is a contradiction to the fact that b starts with 21. We are just left with $c_{n-1} = 3$; if $b - 2 = 23 \dots 4$ then $2b = 4 \dots 32$, and $c_{n-1} = 3$ also means that $c_2 = 6, 7$. Hence $b = 23 \dots 66$ or $b = 23 \dots 76$, but $76 \cdot 2 = 152$ does not end in 32 so $c_2 = 6$.

Claim:

Let $3_k = \underbrace{3 \dots 3}_{k \text{ times}}$, $6_k = \underbrace{6 \dots 6}_{k \text{ times}}$. Then $b = 2(3_k) \dots 6_k$ implies that $b = 2(3_{k+1}) \dots 6_{k+1}$.

Proof:

To prove this claim it will be shown that $b = 2(3_k)6_k$ is not a solution, and indeed there exists the digit 7 in $2b$ but not in $b - 2$.

If 23_k6_k is not a solution, there exists more digits, namely c_{n-k-1}, c_{k+1} , that are not in the 3_k or 6_k , so we need to check $c_{n-k-1} = 0, 1, 2, 3, 4$.

For even c_{n-k-1} we get an easy contradiction just like at the start: for an even digit d , $b - 2 = 2(3_k)d \dots 6_{k-1}4 \implies 2b = 4(6_{k-1}) \dots d3_k2$, but this is impossible as the last 6 has a carrying which makes the next digit odd, and d is even.

For $c_{n-k-1} = 1$ we get $b - 2 = 2(3_k)1 \dots 6_{k-1}4 \implies 2b = 4(6_{k-1}) \dots 13_k2$ that means b ends in 06_{k-1} or 56_{k-1} and just like we did in the case of $c_{n-1} = 1$, it is a contradiction.

This forces $c_{n-k-1} = 3$, so $b - 2 = 2(3_{k+1}) \dots 6_{k-1}4 \implies 2b = 4(6_{k-1}) \dots 3_{k+1}2$, and $c_{n-k-1} = 3 \implies c_{k+1} = 6, 7$. If $c_{k+1} = 7$ we get 5 in the place where there should be 3, so $c_{k+1} = 3$. We are done, because $6 \neq 3$ and the number of digits is ever growing (infinite), but every integer is finite, so there are no solutions. \square

3.2.2 Solutions when $a = 6, 7, 8, 9$

From (5), we get that when $a = 6, 8$, c_n is even, but if $c_n > 1$ then $ac_n > 10$, so for $a = 6, 8$ there are no solutions.

For $a = 7$ we get $c_n = 1$, and $c_1 = 7, 8, 9$. If $c_1 = 7$ then $b = 1 \dots 4$ so $7b = 7 \dots 8$, but $8 \neq 1$ is a contradiction. For $c_1 = 8$ we get $b = 1 \dots 5$, then $7b$ ends in 5 and not 1, again, a contradiction. Lastly, if $c_1 = 9$ then $b = 1 \dots 6$, so $7b$ ends in $2 \neq 1$. So there are no solutions. For $a = 9$ we have $c_n = 1$ and $c_1 = 9$, so $b = 1 \dots 8$ and $9b$ ends in 2 and not in 9, so again no solutions.

3.2.3 Solutions when $a = 3$

For $n \leq 3$, it is possible to solve the respective Diophantine equations, and no solutions exist when $n = 1, 2$. When $n = 3$, the Diophantine equation becomes

$$(3 \cdot 10^2 - 1)c_3 + (30 - 10)c_2 + (3 - 100)c_1 = -3^2 \implies 299c_3 + 20c_2 - 97c_1 = -9$$

and the general solutions are $(c_1, c_2, c_3) = (t, -135 - 299s + 259t, 9 + 20s - 17t)$ for integers s, t . Since $c_1 = t < 10$, there are only nine cases to cover. For $t = 1, 2, 3, 7, 8, 9$, $c_3 \geq 10$ and for $t = 5, 6$, $c_2 \gg 10$. However, when $t = 4$, $c_2 = 4$ and $c_3 = 1$ so $b - 3 = 144 \implies b = 147$ is the only solution for $n \leq 3$.

For the rest of this section, $n > 3$. Firstly, we know that $c_n = 1, 2, 3$. Plugging $a = 3$ and subtracting one from both sides of (5), the equation becomes

$$-1 + (3 \cdot 10^{n-1} - 1)c_n + (3 \cdot 10^{n-2} - 10)c_{n-1} + \dots + (30 - 10^{n-2})c_2 + (3 - 10^{n-1})c_1 = -10 \quad (6)$$

and it can be seen that 10 divides $-1 - c_n + 3c_1$. However, if $c_n = 2$ then $3(c_1 - 1)$ is divisible by 10 which is impossible as c_1 is a single-digit positive integer. If $c_n = 3$ this forces $c_1 = 8$, so $b - 3$ (and hence b) begins with 3 and $3b$ begins with 8 due to their palindromicity. This is contradictory since $3 \cdot 3 = 9 > 8$, so the only case left is $c_n = 1$.

If $c_n = 1$ this forces $c_1 = 4$ since 10 must divide $3c_1 - 2$. Now b starts with 1 and $3b$ starts with 4 so there is a carrying of 1 from the second digit to the first digit - implying that $c_{n-1} = 4, 5, 6$. If $c_{n-1} = 5$, $b - 3 = 15 \dots \dots 4 \implies b = 15 \dots \dots 7$ and $3b = 4 \dots \dots 51$. This forces $c_2 = 1$ since $17 \cdot 3 = 51$ so $3b$ starts with 41. We reach a contradiction as $15 \cdot 3 > 41$. This leaves us with $c_{n-1} = 4, 6$.

3.2.3.1 When $c_{n-1} = 4$

If $c_{n-1} = 4$, $b = 14 \dots \dots 47 \implies b - 3 = 14 \dots \dots 44 \implies 3b = 44 \dots \dots 41$ and since there is a carrying of 2 from the third digit, $c_{n-2} = 7, 8, 9$. If $c_{n-2} = 9$ then $b_3 = 6 = c_3$ since $647 \cdot 3 = 1941$. Thus b starts with 149 and $3b$ with 446 which is a contradiction as $149 \cdot 3 > 446$. Similarly, if $c_{n-2} = 8$ then $b = 3 = 9 = c_3$ since $947 \cdot 3 = 2841$. Thus b starts with 148 and $3b$ with 449 which is again a contradiction as $148 \cdot 3 = 444 < 449$.

If $c_{n-2} = 7$, $b_3 = 2 = c_3$ since $247 \cdot 3 = 741$. Thus b starts with 147 and $3b$ with 442 so there is a carrying of 1 from the fourth digit. This implies that $c_{n-3} = 4, 5, 6$.

If $c_{n-3} = 4$, $b = 1474 \dots 247 \implies 3b = 442 \dots 4741$ which forces $c_4 = 8$ as $8247 \cdot 3 = 24741$. Hence $3b$ starts with 4428 which is contradictory since $1474 \cdot 3 = 4422$. If $c_{n-3} = 6$, $b = 1476 \dots 247 \implies 3b = 442 \dots 6741$ which forces $c_4 = 2$ as $2247 \cdot 3 = 6741$. Hence $3b$ starts with 4422 which is contradictory yet again since $1476 \cdot 3 = 4428$. In a similar argument, it can be shown that $c_{n-3} = 5 = c_4 (*)$. In the next five paragraphs we will demonstrate that a pattern shows up.

Suppose that b has seven digits; that is, $b = 147x247$ where x is a digit from 0 to 9. If this is multiplied by 3, $3b = 441(3x)741$. But $3b$ starts with 442 due to palindromicity, so $3x = 12, 15, 18$ or that $x = 4, 5, 6$. This forces $x = 5$ since the last digit of $3x$ is the same as x itself. Hence $b = 1475247$ is a solution.

Suppose that b has eight digits; that is, $b = 147xy247$ where x, y are single digits. From similar reasoning to the above, we deduce that $3(xy) = yx + 100$ (concatenation, not multiplication of xy) due to carrying, so $29x - 7y = 100$. The general solution to this is $(x, y) = (100 + 7t, 400 + 29t)$ for an integer t , but it is impossible for y to be positive and less than 10. No solutions exist.

Suppose that b has nine digits; that is, $b = 147xyz247$. The Diophantine equation this time is $299x + 20y - 97z = 100$ with general solution $(x, y, z) = (20 + 20s - 17t, -249 - 299s + 259t, t)$. For $t = 1, 4, 5, 6$, $x \geq 10$ and for $t = 2, 3, 7, 8, 9$, $s = 1, 2, 5, 6, 7$ respectively, so $y \gg 10$. No solutions exist.

Suppose that b has ten digits; that is, $b = 147wxyz247$. The Diophantine equation is $2999w + 290x - 70y - 997z = 10000$ and it can be verified from [1] that all solutions produced have at least one variable that is no less than 10. No solutions exist.

Now suppose that b has eleven digits; that is, $b = 147vwxyz247$. The Diophantine equation is $2990w + 200x - 970y - 9997z = 100000 - 29999v$. Going through v from 1 to 9 and solving this gives us that only $v = 5$ provides one solution with all variables less than 10. The unique solution is $(v, w, x, y, z) = (5, 2, 4, 7, 5)$ so $b = 14752475247$.

In particular, we have that for b of any length greater than eleven, $b = 1475247 \cdots 5247$. But we have arrived at exactly the same situation as in (*). This means that the cycle repeats, and thus b is equal to 147 followed by blocks of 5247. Formally, $b = 147 \cdot 10^{4(k+1)} + 5247 \sum_{i=0}^k 10^{4i}$ and 147 for $k = 0, 1, 2, \dots$, and this completes $c_{n-1} = 4$.

3.2.3.2 When $c_{n-1} = 6$

It can be verified that for $n \leq 3$, no solutions exist. This can be done by employing linear Diophantine equations. From (6), since $c_n = 1$ and $c_1 = 4$,

$$-2 + 3 \cdot 10^{n-1} + (3 \cdot 10^{n-2} - 10)c_{n-1} + \cdots + (30 - 10^{n-2})c_2 + 12 - 4 \cdot 10^{n-1} = -10$$

Combining terms and dividing by 10 gives (for $n > 3$)

$$\begin{aligned} -10^{n-2} + (3 \cdot 10^{n-3} - 1)c_{n-1} + \cdots + (3 - 10^{n-3})c_2 &= -2 \\ -8 - 10^{n-2} + (3 \cdot 10^{n-3} - 1)c_{n-1} + \cdots + (3 - 10^{n-3})c_2 &= -10 \end{aligned}$$

This means that for $c_{n-1} = 6$, $-8 - 6 + 3c_2$ is divisible by 10; in other words, $c_2 = 8$. Hence $b - 3 = 16 \cdots 84 \implies b = 16 \cdots 87$ and $3b = 48 \cdots 61$. In turn, it can be implied that $c_{n-2} = 0, 1, 2, 3$ as $16 \cdot 3 = 48$.

If $c_{n-2} = 0$, $3b$ ends in 061 forcing $c_3 = 6$ since $684 \cdot 3 = 2061$ which is a contradiction as b starts with 160 and $3b$ starts with 486. If $c_{n-2} = 2$, $3b$ ends in 261 forcing $c_3 = 0$ since $84 \cdot 3 = 261$ which is contradictory as b starts with 162 and $3b$ starts with 480. Furthermore, if $c_{n-2} = 3$, $3b$ ends in 361 forcing $c_3 = 7$ since $787 \cdot 3 = 2361$ which is again a contradiction as b starts with 163 and $3b$ starts with $487 < 163 \cdot 3$. There is no contradiction when $c_{n-2} = 1$ because $c_3 = 3$ works so we can continue down this route.

We are now left with $b = 161 \cdots 387 \implies b - 3 = 161 \cdots 384 \implies 3b = 483 \cdots 161$ so again, $c_{n-3} = 0, 1, 2, 3$. If $c_{n-3} = 0$, b starts with 1610 so $c_4 = 0, 1, 2$ as the fourth digit of $3b$. Now $0387 \cdot 3 = 1161$, $1387 \cdot 3 = 4161$ and $2387 \cdot 3 = 7161$ which is a contradiction as $1, 4, 7 \neq 0$ which is what we assumed for c_{n-3} in this case. Similarly, if $c_{n-3} = 1$, b starts with 1611 so $c_4 = 3, 4, 5$ as the fourth digit of $3b$. Now $3387 \cdot 3 = 10161$, $4387 \cdot 3 = 13161$ and $5387 \cdot 3 = 16161$ which is a contradiction as $0, 3, 6 \neq 1$ which is what we assumed for c_{n-3} in this case. If $c_{n-3} = 3$, this forces $c_4 = 9$ since $1613 \cdot 3 = 4839$ which in turn forces $c_{n-3} = 8$ as $9384 \cdot 3 = 28161$. The contradiction is apparent. This means that $b = 1612 \cdots 7387$ as it turns out that if $c_{n-3} = 2$, $c_4 = 7$ using the same method as above. (†)

Suppose that b has nine digits; that is, $b = 1612x7387$ just like in the previous subsection. Then $3b = 4836(3x + 2)2161$ which means that $20 > 3x + 2 \geq 10 \implies 3x + 2 = 11, 14, 17 \implies x = 3, 4, 5$. The last digits of $3x + 2$ are 1, 4, 7 and since 4 is the only digit such that the last

digit of $3x + 2$ is the same as x , the only solution is $b = 161247387$.

Suppose that b has ten digits; that is, $b = 1612xy7387$. From similar reasoning to the above, we deduce that $3(xy) + 2 = yx + 100$. This means that in algebraic terms, $29x - 7y = 98$. This forces $x = 7$ as $29x = 7(y + 14)$, but this gives $y = 15 > 10$ so no solutions exist.

Suppose that b has eleven digits; that is, $b = 1612xyz7387$. The Diophantine equation this time is $299x + 20y - 97z = 998$ and it can be verified from [1] that there are no solutions such that $x, y, z < 10$.

Suppose that b has twelve digits; that is, $b = 1612wxyz7387$. The Diophantine equation is $2999w + 290x - 70y - 997z = 9998$ and similarly it can be verified from [1] that there are no solutions such that $w, x, y, z < 10$.

Now suppose that b has thirteen digits; that is, $b = 1612vwxyz7387$. The Diophantine equation is $2990w + 200x - 970y - 9997z = 99998 - 29999v$. Going through v from 1 to 9 and solving this gives us that only $v = 4$ provides one solution with all variables less than 10. The unique solution is $(v, w, x, y, z) = (4, 7, 5, 2, 4)$ so $b = 161\,247\,5247\,387$.

In particular, we have that for b of any length greater than thirteen, $b = 161\,247\,5247 \cdots 387 \implies b = 161\,247\,52 \cdots 387$. But this is exactly where we had arrived at previously in (†). This means that the cycle repeats, and thus b is equal to 161247 followed by blocks of 5247, followed by 387 at the end. Formally, $b = 161247 \cdot 10^{4k+7} + 5247 \sum_{i=0}^k 10^{4i+3} + 387$ and 161247387 for $k = 0, 1, 2, \dots$, and this completes $c_{n-1} = 6$.

4 References

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